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Master's thesis

Solutions to the Nonsteady Heat Problem with a Nonlinear Nonlocal Condition

Nestacionaraus šilumos laidumo uždavinio su netiesine nelokalia sąlyga sprendiniai

Vytautas Bačianskas

Supervisor : Doc. Dr. Kristina Kaulakytė

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Summary

In this thesis, an inverse heat conduction problem is studied, where functions E and u_0 are prescribed and the goal is to find a pair (u, f), satisfying the differential equation $u_t(x,t) - \Delta u(x,t) = f(x,t)$, initial condition $u(x,0) = u_0(x)$, the Dirichlet boundary condition $u|_{\partial\Omega\times[0,T]} = 0$ as well as an additional nonlinear nonlocal condition $\int_\Omega |u(x,t)|^2 dx = E(t)$ for all $t \in [0,T]$. When $E \in W^{1,2}(0,T)$, we formulate the definition of a weak solution for this problem and prove that there exists at least one such solution. In the case when E is only from the space $L^2(0,T)$, we formulate the definition of a very weak solution and prove that there exists at least one such solution.

Keywords: Sobolev spaces, Weak solution, Very weak solution, Heat problem, Nonlinear condition.

Santrauka

Šiame darbe nagrinėjamas atvirkštinis šilumos laidumo uždavinys, kuriame yra duotos funkcijos E ir u_0, o tikslas yra surasti porą (u, f), tenkinančią diferencialinę lygtį $u_t(x,t) - \Delta u(x,t) = f(x,t)$, pradinę sąlygą $u(x,0) = u_0(x)$, Dirichlė kraštinę sąlygą $u|_{\partial\Omega\times[0,T]} = 0$ bei papildomą netiesinę nelokalią sąlygą $\int_\Omega |u(x,t)|^2 dx = E(t)$ visiems $t \in [0,T]$. Kai $E \in W^{1,2}(0,T)$, yra suformuluojamas minėto uždavinio silpno sprendinio apibrėžimas bei įrodomas šio sprendinio egzistavimas. Atveju, kai E yra tik iš erdvės $L^2(0,T)$, yra suformuluojamas minėto uždavinio labai silpno sprendinio apibrėžimas bei įrodomas šio sprendinio egzistavimas.

Raktiniai žodžiai: Sobolevo erdvės, silpnas sprendinys, labai silpnas sprendinys, šilumos lygtis, netiesinė sąlyga.

Contents

Introduction

The heat equation is an important and well-known partial differential equation that arises in problems involving diffusion. For instance, it can be used to model the heat problem which describes how temperature changes over time within a specified region Ω. There are many formulations of the heat problem depending on the choice of initial and boundary conditions. However, we will be focusing only on the following formulation. Suppose we are prescribed a source function $f(x,t)$, an initial condition $u(x,0) = u_0(x)$ as well as a homogeneous Dirichlet boundary condition $u|_{\partial\Omega\times[0,T]} = 0$. The goal of the classical heat problem is to determine the temperature distribution $u(x,t)$, that satisfies the initial and boundary conditions as well as the differential equation $u_t - \Delta u = f$ on the cylinder $\Omega \times (0,T)$. The existence and uniqueness of solutions for this problem are well-established (see, e.g., [1, 10]). However, we will be interested in a modified problem where the source function f is also unknown and has to be found as part of the solution. This type of problem is sometimes called the inverse heat conduction problem and it requires an additional condition to ensure that it is well-posed. We will focus on the following inverse problem. Suppose that $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded domain with Lipschitz boundary $\partial\Omega$. The goal is to find a pair of functions (u,f), such that

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ u\Big|_{\partial\Omega\times[0,T]} = 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

and such that u satisfies the side condition

$$\int_\Omega |u(x,t)|^2 dx = E^2(t) \quad \forall t \in [0,T], \text{ with } E(0) = \|u_0\|_{L^2(\Omega)}, \tag{2}$$

where E and u_0 are given functions. Usually, instead of the nonlinear condition (2), inverse problems are studied with the linear integral condition

$$\int_\Omega u(x,t) dx = F(t) \quad \forall t \in [0,T], \text{ with } F(0) = \int_\Omega u_0(x) dx, \tag{3}$$

where the function F is given. It is important to note that these inverse problems are usually studied with weak solutions in mind rather than classical ones. Problem (1) together with condition (3) has been studied in different forms by several authors (see [4, 5, 9, 11, 12]). This specific problem has been analyzed in two cases: when F is from the Sobolev space $W^{1,2}(0,T)$ in [11] and when F is only from the space $L^2(0,T)$ in [12]. In these cases both uniqueness and existence were established. However, there have also been a few papers where condition (2) was used instead, such as [2] or in a slightly different context in [3]. Our goal will be to prove the existence of weak solutions to problem (1) with the nonlinear side condition (2) when $u_0 \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ in two cases just like it was done with condition (3): when E is from the Sobolev space $W^{1,2}(0,T)$ and when E is only from the space $L^2(0,T)$. While this exact problem was studied in [2] in the case when $E \in W^{1,2}(0,T)$, the function u that was found lacks the weak differentiability in terms of x. As such, one of the goals of this thesis will be to improve this result.

1 Notations and important results

1.1 Function spaces and useful results

In this section we will mention some of the definitions and notation used in the further sections. For any open set $U \subset \mathbb{R}^n$ we will denote the set of all infinitely differentiable functions (also called smooth functions) on U as $C^\infty(U)$ and the set of all compactly supported smooth functions as $C_c^\infty(U)$. Throughout this thesis we will use two different notations to represent the classical and the weak derivatives. Thus, u_t and $\frac{\partial u}{\partial t}$ will both refer to the (weak) partial derivative of the function u by the variable t. If u is a one variable function, the (weak) derivative will be denoted as u'. Also, given a normed space X, the norm of an element $u \in X$ will be denoted as $\|u\|_X$ and, in the case when $\mathbf{u} = (u_1, u_2, \ldots, u_k) \in X^k$ for some $k \in \mathbb{N}$, we will also define $\|\mathbf{u}\|_X := \left(\sum_{i=1}^k \|u_i\|_X^2\right)^{\frac{1}{2}}$. This will be useful when talking about the norm of the weak gradient of a function. Let $L^2(U)$ denote the space of all square integrable functions which form a Hilbert space with the following inner product

$$\langle u, v \rangle_{L^2(U)} = \int_U u(x)v(x)dx$$

and let $W^{1,2}(U)$ denote the Sobolev space which is also a Hilbert space with the following inner product

$$\langle u, v \rangle_{W^{1,2}(U)} = \int_U \left(u(x)v(x) + \nabla u(x) \cdot \nabla v(x)\right) dx,$$

where ∇u denotes the weak gradient $(u_{x_1}, u_{x_2}, \ldots, u_{x_n})$. Additionally, $\mathring{W}^{1,2}(U)$ will refer to the closure of the space $C_c^\infty(U)$ under the $W^{1,2}(U)$ norm. We will also need the space $W^{2,2}(U)$ which is the Sobolev space of functions that are twice weakly differentiable together with the inner product

$$\langle u, v \rangle_{W^{2,2}(U)} = \int_U \left(u(x)v(x) + \nabla u(x) \cdot \nabla v(x) + \sum_{i=1}^n \sum_{j=1}^n u_{x_i x_j}(x) v_{x_i x_j}(x) \right) dx.$$

Suppose that $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and that the boundary $\partial\Omega$ is a Lipschitz boundary (see [7]). Also, suppose that $0 < T < \infty$ is a fixed number and define the cylinder $Q_T := \Omega \times (0,T)$. We will require the following spaces of functions that are defined on Q_T:

- $L^2(0,T; L^2(\Omega))$ is just different notation for the space $L^2(Q_T)$. A different definition of this space can be found in [6].
- $L^2(0,T; W^{1,2}(\Omega)) = \{u \in L^2(0,T; L^2(\Omega)) : u_{x_i} \in L^2(0,T; L^2(\Omega)), i = 1,2,\ldots,d\}$ is a

Hilbert space with the inner product

$$\langle u, v\rangle_{L^2(0,T;W^{1,2}(\Omega))} = \int_0^T \int_\Omega (u(x,t)v(x,t) + \nabla u(x,t) \cdot \nabla v(x,t))\, dx.$$

We will also use the subspace

$$L^2(0,T;\mathring{W}^{1,2}(\Omega)) = \{u \in L^2(0,T;W^{1,2}(\Omega)) : u|_{\partial\Omega\times[0,T]} = 0\},$$

where $u|_{\partial\Omega\times[0,T]} = 0$ is understood in the trace sense. As the boundary of Ω is Lipschitz, the trace is well defined. This space will be equipped with the subspace topology.

- $\hat{W}^{1,2}(Q_T) = \{u \in W^{1,2}(Q_T) : u|_{\partial\Omega\times[0,T]} = 0\}$ is a Hilbert subspace where $u|_{\partial\Omega\times[0,T]} = 0$ is understood in terms of traces.

- $W^{1,2}(0,T;L^2(\Omega)) = \{u \in L^2(0,T;L^2(\Omega)) : u_t \in L^2(0,T;L^2(\Omega))\}$ is the Hilbert space with the inner product defined as

$$\langle u, v\rangle_{W^{1,2}(0,T;L^2(\Omega))} = \int_0^T \int_\Omega (u(x,t)v(x,t) + u_t(x,t)v_t(x,t))\, dxdt.$$

There is a very important embedding theorem that we will use, the proof of which can be found in [1], [6] and [10].

Theorem 1.1. *Suppose that a function $u \in L^2(0,T;L^2(\Omega))$ has a weak derivative $u_t \in L^2(0,T;L^2(\Omega))$. Then for almost all $0 < t_1 < t_2 < T$ and almost all $x \in \Omega$ the Newton–Leibniz formula is true*

$$u(x,t_2) - u(x,t_1) = \int_{t_1}^{t_2} u_t(x,t)dt. \tag{4}$$

However, it is possible to redefine the function u on a set of measure 0 such that equation (4) is true for all $0 \leqslant t_1 < t_2 \leqslant T$ and for almost all $x \in \Omega$. Thus, u is absolutely continuous with respect to t and we have the embedding $W^{1,2}(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$.

Moreover, we have the inclusion $W^{1,2}(0,T) \subset C[0,T]$. In the case when a function u is only from the space $L^2(0,T;L^2(\Omega))$, we cannot define its value at any point t. For that reason, we will generalize this definition based on the definition of a Lebesgue point of a function.

Definition 1.2. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a locally integrable function. The point $x \in \mathbb{R}$ is then called a Lebesgue point of g if

$$\lim_{r\to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} |g(y) - g(x)|dy = 0.$$

The more general version of this definition can be found in [8]. Based on this definition, if $E \in L^2(0,T)$, then we will say that $E(0) = L$ if

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t |E(\tau) - L|d\tau = 0.$$

Similarly, if $u \in L^2(0,T;L^2(\Omega))$, then we will define $u(\cdot,0) = g$ in terms of Lebesgue points to mean that

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t \|u(\cdot,\tau) - g\|_{L^2(\Omega)} d\tau = 0.$$

Now suppose that H is an arbitrary Hilbert space. We have the following important facts which can be found in most analysis books (see ,e.g., [8]).

Theorem 1.3 (Riesz representation theorem)**.** *For every bounded functional F from the dual space H^*, there exists an element $y \in H$ such that $F(x) = \langle x, y\rangle$ for all $x \in H$. In fact, this mapping between H^* and H itself is an isomorphism which shows that H and H^* are isomorphic.*

Theorem 1.4. *If a sequence $\{u_k\}_{k=1}^{\infty} \subset H$ is bounded, then there exists a weakly convergent subsequence $\{u_{k_l}\}_{l=1}^{\infty}$.*

Theorem 1.5 (Pythagorean theorem)**.** *If $x_1, x_2, \ldots x_N \in H$ and $\langle x_i, x_j\rangle = 0$ if $i \neq j$, then*

$$\left\|\sum_{k=1}^{N} x_i\right\|_H^2 = \sum_{k=1}^{N} \|x_i\|_H^2.$$

Theorem 1.6 (Parseval's Identity)**.** *Suppose that $\{v_k\}_{k=1}^{\infty}$ is a complete orthonormal system, also called an orthonormal basis. Then for any element $x \in H$ we have that*

$$\|x\|_H^2 = \sum_{k=1}^{\infty} |\langle x, v_k\rangle_H|^2.$$

In order to prove inequalities, we will mainly be using the Cauchy-Schwarz inequality, though only on Hilbert spaces like L^2.

Theorem 1.7 (Cauchy-Schwarz inequality)**.** *For all $x, y \in H$ the following inequality holds*

$$|\langle x, y\rangle_H| \leqslant \|x\|_H \|y\|_H.$$

Additionally, we will require the following theorem from functional analysis. The proof can be found in [8].

Theorem 1.8. *If X and Y are Banach spaces and T is a bounded linear operator between them, then $T(U)$ is open for all open sets $U \subset X$.*

Specifically, we will need the following corollary of this theorem.

Corollary 1.9. *If X and Y are Banach spaces and T is a bijective bounded linear operator between them, then T is an isomorphism, thus T^{-1} is also a bounded linear operator.*

Finally, these integration by parts formulas will be helpful when defining the weak and very weak solutions to problem (1), (2). The proofs can be found in [10] and [8].

Proposition 1.10. *Suppose that the functions u and v are absolutely continuous on the interval $[a, b]$, where $-\infty < a < b < \infty$. Then*

$$\int_a^b u'(t)v(t)dt = u(b)v(b) - u(a)v(a) - \int_a^b u(t)v'(t)dt.$$

Proposition 1.11. *Suppose that $u \in W^{1,2}(\Omega)$ and that $v \in \mathring{W}^{1,2}(\Omega)$. Then*

$$\int_\Omega u_{x_i}(x)v(x)dx = -\int_\Omega u(x)v_{x_i}(x)dx,$$

for all $i = 1, 2, \ldots d$.

1.2 Laplace operator eigenvalues and eigenfunctions

Let $v_k \in \mathring{W}^{1,2}(\Omega)$ and $\lambda_k \in \mathbb{R}$, $k \in \mathbb{N}$, be the eigenfunctions and eigenvalues of the Laplace operator

$$\begin{cases} -\Delta v_k(x) = \lambda_k v_k(x), \quad x \in \Omega, \\ v_k\Big|_{\partial\Omega} = 0, \end{cases}$$

which satisfies the following integral identity for the weak solutions

$$\forall \eta \in \mathring{W}^{1,2}(\Omega): \quad \int_\Omega \nabla v_k(x) \cdot \nabla \eta(x)dx = \lambda_k \int_\Omega v_k(x)\eta(x)dx. \tag{5}$$

According to [6], the eigenvalues λ_k are real, positive and $\lambda_k \to \infty$, when $k \to \infty$. Also, we can choose the eigenvalues in such a way that they are orthogonal and form a basis of not only $\mathring{W}^{1,2}(\Omega)$ but $L^2(\Omega)$ as well. We can also assume that the eigenfunctions are normalized so that $\|v_k\|_{L^2(\Omega)} = 1$. Notice that if we take the integral identity (5) and substitute $\eta = v_k$, we get

$$\int_\Omega |\nabla v_k(x)|^2 dx = \lambda_k,$$

whereas if we substitute $\eta = v_l$ where $k \neq l$, then

$$\int_\Omega \nabla v_k(x) \cdot \nabla v_l(x)dx = \lambda_k \int_\Omega v_k(x)v_l(x)dx = 0.$$

From this it follows that if $k \neq l$, then v_k and v_l are orthogonal in $\mathring{W}^{1,2}(\Omega)$. Thus, we have the following identities

$$\int_\Omega v_k(x)v_l(x)dx = \begin{cases} 1, \text{ if } k = l, \\ 0, \text{ if } k \neq l, \end{cases} \quad \int_\Omega \nabla v_k(x) \cdot \nabla v_l(x)dx = \begin{cases} \lambda_k, \text{ if } k = l, \\ 0, \text{ if } k \neq l. \end{cases} \tag{6}$$

2 Weak solution

2.1 Definition of a weak solution

Assume that (u, f) is a classical solution of problem (1). In the case when $E \in W^{1,2}(0,T)$, we will derive the definition of a weak solution in the following way. Multiply equation $(1)_1$ by a test function $\eta \in L^2(0,T;L^2(\Omega))$, integrate by x over Ω and then integrate by t over the interval $(0,T)$ to get

$$\int_0^T \int_\Omega u_t(x,t)\eta(x,t)dxdt - \int_0^T \int_\Omega \Delta u(x,t)\eta(x,t)dxdt = \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt. \tag{7}$$

For the next step, we will additionally assume that $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$. Applying integration by parts on the second integral in (7) with respect to x and using the fact that $\eta(\cdot,t) \in \mathring{W}^{1,2}(\Omega)$ for almost all $t \in (0,T)$ together with Proposition 1.11, we get

$$\begin{aligned}\int_0^T \int_\Omega \Delta u(x,t)\eta(x,t)dxdt &= \sum_{i=1}^d \int_0^T \int_\Omega \frac{\partial^2 u(x,t)}{\partial x_i^2}\eta(x,t)dxdt \\ &= \sum_{i=1}^d \int_0^T \left(-\int_\Omega \frac{\partial u(x,t)}{\partial x_i}\frac{\partial \eta(x,t)}{\partial x_i}dx \right) dt \\ &= -\int_0^T \int_\Omega \nabla u(x,t) \cdot \nabla \eta(x,t) dxdt. \end{aligned} \tag{8}$$

Substituting (8) into (7) we get the integral identity

$$\int_0^T \int_\Omega u_t(x,t)\eta(x,t)dxdt + \int_0^T \int_\Omega \nabla u(x,t) \cdot \nabla\eta(x,t)dxdt = \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt \tag{9}$$

for every $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$. We will use the following definition of a weak solution to problem (1) subject to an additional condition (2).

Definition 2.1. Suppose that $E \in W^{1,2}(0,T)$ and $u_0 \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$. Also, suppose that E satisfies the compatibility condition $E(0) = \|u_0\|_{L^2(\Omega)}$. A pair of functions (u, f) such that $u \in \hat{W}^{1,2}(Q_T)$ and $f \in L^2(0,T;L^2(\Omega))$ is called a weak solution of problem (1), (2) if the integral identity (9) is true for all test functions $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$ and if the initial and side conditions are satisfied

$$u(x,0) = u_0(x) \text{ for a.e. } x \in \Omega, \tag{10}$$

$$\int_\Omega |u(x,t)|^2 dx = E^2(t), \quad \text{for all } t \in [0,T]. \tag{11}$$

Our first goal will be to prove the following theorem.

Theorem 2.2. *Suppose that $u_0 \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ and $E \in W^{1,2}(0,T)$ are given functions and that E satisfies the compatibility condition $E(0) = \|u_0\|_{L^2(\Omega)}$. Then there exists at least one weak solution (u, f) to problem* (1), (2).

For the rest of the next section we will assume that u_0 is not identically equal to 0, so that $\|u_0\|_{L^2(\Omega)} > 0$. The case when $u_0 \equiv 0$ is treated very similarly so it will only be mentioned at the end.

2.2 Approximate solution

For every $N \in \mathbb{N}$ we will find approximate solutions $(u^{(N)}, f^{(N)})$ that are represented in the following forms

$$\begin{aligned} u^{(N)}(x,t) &= \sum_{k=1}^{N} w_k(t) v_k(x), \\ f^{(N)}(x,t) &= \sum_{k=1}^{N} q_k(t) v_k(x), \end{aligned} \tag{12}$$

where v_k are eigenfunctions defined in Section 1.2. We will also require them to satisfy the following system of equations for every $t \in [0,T]$ and $k \leqslant N$

$$\begin{cases} \int_\Omega (u^{(N)})_t(x,t) v_k(x) dx + \int_\Omega \nabla u^{(N)}(x,t) \cdot \nabla v_k(x) dx = \int_\Omega f^{(N)}(x,t) v_k(x) dx, \\ u^{(N)}(x,0) = \sum_{k=1}^{N} \beta_k v_k(x), \quad x \in \Omega, \\ \int_\Omega \left| u^{(N)}(x,t) \right|^2 dx = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 E^2(t), \end{cases} \tag{13}$$

where β_k, $k \in \mathbb{N}$, are the Fourier coefficients of the function u_0 in terms of the orthonormal basis $\{v_k\}_{k=1}^\infty$ in $L^2(\Omega)$. Substituting the expressions (12) into $(13)_1$ and using the fact that different eigenfunctions are orthogonal we get an ordinary differential equation

$$w_k'(t) + \lambda_k w_k(t) = q_k(t). \tag{14}$$

If we substitute $(12)_1$ into $(13)_3$ and use Theorem 1.5, we get

$$\begin{aligned} \int_\Omega |u^{(N)}(x,t)|^2 dx &= \left\| \sum_{k=1}^{N} w_k(t) v_k(x) \right\|^2_{L^2(\Omega)} = \sum_{k=1}^{N} \|w_k(t) v_k(x)\|^2_{L^2(\Omega)} \\ &= \sum_{k=1}^{N} (w_k(t))^2 = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 E^2(t), \end{aligned} \tag{15}$$

where the last equality follows from $(13)_3$. Since we require equation (15) to hold for all $N \in \mathbb{N}$, we get that for all $t \in [0,T]$

$$w_k(t) = \frac{\beta_k}{\|u_0\|_{L^2(\Omega)}} E(t). \tag{16}$$

In order to satisfy equation (14) we will set

$$q_k := w_k' + \lambda_k w_k \in L^2(0,T). \tag{17}$$

Since the function E is from the space $W^{1,2}(0,T)$, it is also continuous. Therefore, $u^{(N)}$ is from the space $C([0,T];L^2(\Omega))$ and $u^{(N)}(\cdot,0)$ is well-defined. With this in mind as well as the fact that we have assumed $E(0) = \|u_0\|_{L^2(\Omega)}$, we also have that

$$u^{(N)}(x,0) = \sum_{k=1}^{N} w_k(0)v_k(x) = \frac{1}{\|u_0\|_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k E(0) v_k(x) = \sum_{k=1}^{N} \beta_k v_k(x).$$

With this, the pair of functions $(u^{(N)}, f^{(N)})$ satisfies the system (13). Next, we will get some estimates for $u^{(N)}$ and $f^{(N)}$. Recall that β_k are Fourier coefficients of u_0, so $\sum_{k=1}^{\infty} \beta_k^2 = \|u_0\|^2_{L^2(\Omega)}$. Thus

$$\|u^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega |u^{(N)}(x,t)|^2 dxdt \overset{(16)}{=} \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \int_0^T E^2(t)dt \leqslant \|E\|^2_{L^2(0,T)}. \tag{18}$$

Similarly, by using identities (6) we obtain

$$\begin{aligned}
\|\nabla u^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega |\nabla u^{(N)}(x,t)|^2 dxdt = \int_0^T \int_\Omega \left| \sum_{k=1}^{N} w_k(t) \nabla v_k(x) \right|^2 dxdt \\
&= \int_0^T \left\| \sum_{k=1}^{N} w_k(t) \nabla v_k \right\|^2_{L^2(\Omega)} dt = \int_0^T \sum_{k=1}^{N} |w_k(t)|^2 \lambda_k dt \\
&= \frac{\|E\|^2_{L^2(0,T)}}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \lambda_k. \qquad (19)
\end{aligned}$$

For the time derivative $u_t^{(N)}$ we have that

$$\begin{aligned}
\|u_t^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \left\| \sum_{k=1}^{N} (w_k^{(N)})'(t) v_k \right\|^2_{L^2(\Omega)} dt = \int_0^T \sum_{k=1}^{N} ((w_k^{(N)})'(t))^2 dt = \int_0^T \sum_{k=1}^{N} \frac{\beta_k^2 |E'(t)|^2}{\|u_0\|^2_{L^2(\Omega)}} dt \\
&= \sum_{k=1}^{N} \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^T |E'(t)|^2 dt \leqslant \|E\|^2_{W^{1,2}(0,T)}. \qquad (20)
\end{aligned}$$

Lastly, we will get an estimate for $f^{(N)}$ in the space $L^2(0,T;L^2(\Omega))$ using equation (17). Notice that for any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\int_0^T |q_k^{(N)}(t)|^2 dt &= \int_0^T \left| (w_k^{(N)})'(t) + \lambda_k w_k^{(N)}(t) \right|^2 dt \leqslant 2 \int_0^T \left(\left| (w_k^{(N)})'(t) \right|^2 + \left| \lambda_k w_k^{(N)}(t) \right|^2 \right) dt \\
&= 2 \int_0^T \left| \frac{\beta_k E'(t)}{\|u_0\|_{L^2(\Omega)}} \right|^2 dt + 2 \int_0^T \left| \frac{\lambda_k \beta_k E(t)}{\|u_0\|_{L^2(\Omega)}} \right|^2 dt \\
&= \frac{2\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \left(\|E'\|^2_{L^2(0,T)} + \lambda_k^2 \|E\|^2_{L^2(0,T)} \right) \\
&\leqslant \frac{2\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} (1 + \lambda_k^2) \|E\|^2_{W^{1,2}(0,T)}. \qquad (21)
\end{aligned}$$

From this inequality it follows that

$$\|f^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega \left| \sum_{k=1}^N q_k^{(N)}(t) v_k(x) \right|^2 dxdt = \int_0^T \left\| \sum_{k=1}^N q_k^{(N)}(t) v_k \right\|^2_{L^2(\Omega)} dt$$
$$= \sum_{k=1}^N \int_0^T |q_k^{(N)}(t)|^2 dt \overset{(21)}{\leqslant} \sum_{k=1}^N \frac{2\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} (1+\lambda_k^2) \|E\|^2_{W^{1,2}(0,T)}$$
$$\leqslant 2\|E\|^2_{W^{1,2}(0,T)} + \frac{2\|E\|^2_{W^{1,2}(0,T)}}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^N \beta_k^2 \lambda_k^2. \tag{22}$$

Let us now show that bounds (19) and (22) are independent of N. To do this, we will bound the series $\sum_{k=1}^\infty \beta_k^2 \lambda_k^2$ using the fact that u_0 is from the space $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$. First notice that since $u_0 \in W^{2,2}(\Omega)$, the weak Laplacian of u_0 is from the space $L^2(\Omega)$. Since $\{v_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$, we know that there exist coefficients γ_k such that

$$\Delta u_0 = \sum_{k=1}^\infty \gamma_k v_k = \sum_{k=1}^\infty \langle \Delta u_0, v_k \rangle_{L^2(\Omega)} v_k \quad \text{ in } L^2(\Omega). \tag{23}$$

However, using the fact that v_k and u_0 are from the space $\mathring{W}^{1,2}(\Omega)$ and by applying integration by parts, we obtain

$$\int_\Omega \Delta u_0(x) \cdot v_k(x) dx = -\int_\Omega \nabla u_0(x) \cdot \nabla v_k(x) dx \overset{(5)}{=} -\lambda_k \int_\Omega u_0(x) v_k(x) dx = -\lambda_k \beta_k. \tag{24}$$

The last equality holds because the Fourier coefficients β_k of u_0 can be written as $\langle u_0, v_k \rangle_{L^2(\Omega)}$. So, we see that $\gamma_k = -\lambda_k \beta_k$ for all $k \in \mathbb{N}$. Then, Parseval's identity (Theorem 1.6) gives us that

$$\|\Delta u_0\|^2_{L^2(\Omega)} = \sum_{k=1}^\infty |\gamma_k|^2 \implies \sum_{k=1}^\infty \lambda_k^2 \beta_k^2 = \|\Delta u_0\|^2_{L^2(\Omega)} \leqslant \|u_0\|^2_{W^{2,2}(\Omega)}. \tag{25}$$

From (22) and the inequality (25) we get that f is bounded in the space $L^2(0,T;L^2(\Omega))$

$$\|f\|^2_{L^2(0,T;L^2(\Omega))} \leqslant 2\|E\|^2_{W^{1,2}(0,T)} \left(1 + \frac{\|u_0\|^2_{W^{2,2}(\Omega)}}{\|u_0\|^2_{L^2(\Omega)}} \right) = C_1 \|E\|^2_{W^{1,2}(0,T)}, \tag{26}$$

where $C_1 = 2 + 2 \cdot \frac{\|u_0\|^2_{W^{2,2}(\Omega)}}{\|u_0\|^2_{L^2(\Omega)}} > 0$ is a constant that depends on u_0.

From (25) it follows that we can bound the series $\sum_{k=1}^\infty \lambda_k \beta_k^2$ because $\lambda_k \geqslant \frac{1}{C_2}$ for some positive constant $C_2 > 0$ and for all $k \in \mathbb{N}$. This is because $\lambda_k \to \infty$ as $k \to \infty$ and $\lambda_k > 0$ for all $k \in \mathbb{N}$. Thus

$$\sum_{k=1}^\infty \lambda_k \beta_k^2 \leqslant \sum_{k=1}^\infty C_2 \lambda_k^2 \beta_k^2 \leqslant C_2 \|u_0\|^2_{W^{2,2}(\Omega)}. \tag{27}$$

Using (18), (19) and (20) together with (27) we can bound u in the space $\hat{W}^{1,2}(Q_T)$ like this

$$\begin{aligned}
\|u\|^2_{\hat{W}^{1,2}(Q_T)} &= \|u\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla u\|^2_{L^2(0,T;L^2(\Omega))} + \|u_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leqslant \|E\|^2_{W^{1,2}(0,T)} + C_2 \frac{\|E\|^2_{W^{1,2}(0,T)}}{\|u_0\|^2_{L^2(\Omega)}} \|u_0\|^2_{W^{2,2}(\Omega)} + \|E\|^2_{W^{1,2}(0,T)} \\
&\leqslant C_3 \|E\|^2_{W^{1,2}(0,T)},
\end{aligned} \tag{28}$$

for some large enough constant C_3 that depends on the function u_0. By using these estimates, we have that the sequence $\{u^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $\hat{W}^{1,2}(Q_T)$ while the sequence $\{f^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $L^2(0,T;L^2(\Omega))$.

2.3 Convergence

From the last section we know that the sequence $\{u^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $\hat{W}^{1,2}(Q_T)$ while the sequence $\{f^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $L^2(0,T;L^2(\Omega))$. Therefore, there exists a subsequence $\{N_l\}_{l=1}^{\infty}$ such that $u^{(N_l)}$ weakly converges to a function $u \in \hat{W}^{1,2}(Q_T)$ and $f^{(N_l)}$ weakly converges to some $f \in L^2(0,T;L^2(\Omega))$. We will require the following lemma.

Lemma 2.3. *Suppose a sequence g_N converges weakly to g in the space $\hat{W}^{1,2}(Q_T)$. Then $g_N \rightharpoonup g$, $\nabla g_N \rightharpoonup \nabla g$ and $\frac{\partial g_N}{\partial t} \rightharpoonup \frac{\partial g}{\partial t}$ in $L^2(0,T;L^2(\Omega))$.*

Proof. Take an arbitrary functional $\varphi \in \Big(L^2(0,T;L^2(\Omega))\Big)^*$. We need to show that $\varphi(g_N) \to \varphi(g)$. By the Riesz representation theorem (Theorem 1.3) this functional can be written in the following way

$$\varphi(v) = \int_0^T \int_\Omega v(x,t)\eta(x,t)dxdt$$

for some function $\eta \in L^2(0,T;L^2(\Omega))$. Notice that it is also a bounded linear functional when restricted to $\hat{W}^{1,2}(Q_T)$ because by the Cauchy-Schwarz inequality

$$\begin{aligned}
|\varphi(v)| = \left|\int_0^T \int_\Omega v(x,t)\eta(x,t)dxdt\right| &\leqslant \|\eta\|_{L^2(0,T;L^2(\Omega))}\|v\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant \|\eta\|_{L^2(0,T;L^2(\Omega))}\|v\|_{\hat{W}^{1,2}(Q_T)}.
\end{aligned}$$

Thus, since φ is from the dual space of $\hat{W}^{1,2}(Q_T)$ and g_N converges weakly to g in $\hat{W}^{1,2}(Q_T)$, we get that $\varphi(g_N) \to \varphi(g)$, $N \to \infty$. This implies that $g_N \rightharpoonup g$ in $L^2(0,T;L^2(\Omega))$.

For the gradient, we need to show that for each $i = 1, 2, \dots, d$ the sequence $\frac{\partial g_N}{\partial x_i}$ converges weakly to $\frac{\partial g}{\partial x_i}$ in $L^2(0,T;L^2(\Omega))$. Just like before, if $\varphi \in (L^2(0,T;L^2(\Omega)))^*$, then there exists an $\eta \in L^2(0,T;L^2(\Omega))$ such that

$$\varphi(v) = \int_0^T \int_\Omega v(x,t)\eta(x,t)dxdt$$

This time, consider a functional from $\hat{W}^{1,2}(Q_T)$ to $\mathbb{R}$ defined by

$$v \mapsto \int_0^T \int_\Omega \frac{\partial v(x,t)}{\partial x_i} \eta(x,t) dx dt.$$

This functional is bounded because by the Cauchy-Schwarz inequality

$$\begin{aligned} \left| \int_0^T \int_\Omega \frac{\partial v(x,t)}{\partial x_i} \eta(x,t) dx dt \right| &\leqslant \|\eta\|_{L^2(0,T;L^2(\Omega))} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(0,T;L^2(\Omega))} \\ &\leqslant \|\eta\|_{L^2(0,T;L^2(\Omega))} \|v\|_{\hat{W}^{1,2}(Q_T)}. \end{aligned}$$

Thus, since $g_N \rightharpoonup g$ in $\hat{W}^{1,2}(Q_T)$, we see that

$$\varphi\left(\frac{\partial g_N}{\partial x_i}\right) = \int_0^T \int_\Omega \frac{\partial g_N(x,t)}{\partial x_i} \eta(x,t) dx dt \to \int_0^T \int_\Omega \frac{\partial g(x,t)}{\partial x_i} \eta(x,t) dx dt = \varphi\left(\frac{\partial g}{\partial x_i}\right).$$

As this is true for all $i = 1, 2, \ldots, d$, we have that $\nabla g_N \rightharpoonup \nabla g$ in $L^2(0,T;L^2(\Omega))$. The same reasoning also shows that $\frac{\partial g_N}{\partial t} \rightharpoonup \frac{\partial g}{\partial t}$ in $L^2(0,T;L^2(\Omega))$. □

From Lemma 2.3 we get that $u^{(N_l)} \rightharpoonup u$, $u_t^{(N_l)} \rightharpoonup u_t$ and $\nabla u^{(N_l)} \rightharpoonup \nabla u$ in $L^2(0,T;L^2(\Omega))$. Next, consider arbitrary functions $d_k \in C^\infty[0,T]$, $k \in \mathbb{N}$. Multiply equations $(13)_1$ by d_k and then sum over all $k \in \{1, 2, \ldots, M\}$, where $M \leqslant N$. If we also denote $\eta(x,t) = \sum_{k=1}^M d_k(t) v_k(x)$ and replace N by N_l in the result, we get the following identity

$$\int_\Omega u_t^{(N_l)}(x,t)\eta(x,t)dx + \int_\Omega \nabla u^{(N_l)}(x,t) \cdot \nabla \eta(x,t) dx = \int_\Omega f^{(N_l)}(x,t)\eta(x,t)dx.$$

By integrating this equation by t from 0 to T, we get

$$\int_0^T\int_\Omega u_t^{(N_l)}(x,t)\eta(x,t)dxdt + \int_0^T\int_\Omega \nabla u^{(N_l)}(x,t) \cdot \nabla \eta(x,t) dxdt = \int_0^T\int_\Omega f^{(N_l)}(x,t)\eta(x,t)dxdt. \quad (29)$$

For the first integral in (29), notice that it can be written as an inner product in the space $L^2(0,T;L^2(\Omega))$. Then, by using the fact that $u_t^{(N_l)} \rightharpoonup u_t$ in this space, we get

$$\int_0^T\int_\Omega u_t^{(N_l)}(x,t)\eta(x,t)dxdt = \langle u_t^{(N_l)}, \eta \rangle_{L^2(0,T;L^2(\Omega))} \xrightarrow{l\to\infty} \langle u_t, \eta \rangle_{L^2(0,T;L^2(\Omega))} = \int_0^T\int_\Omega u_t(x,t)\eta(x,t)dxdt.$$

For the other integrals, we can similarly write them as inner products in the space $L^2(0,T;L^2(\Omega))$ and use weak convergence to get that

$$\begin{aligned} &\int_0^T \int_\Omega \nabla u^{(N_l)}(x,t) \cdot \nabla \eta(x,t) dxdt \xrightarrow{l\to\infty} \int_0^T \int_\Omega \nabla u(x,t) \cdot \nabla \eta(x,t) dxdt, \\ &\int_0^T \int_\Omega f^{(N_l)}(x,t)\eta(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt. \end{aligned}$$

Thus, after passing to the limit as $l \to \infty$ in (29), we get integral identity (9) for the pair of functions (u, f), however only for those η that are finite linear combinations of functions v_k. However, it is well known (see [1]) that these linear combinations are dense in the space $L^2(0,T;\mathring{W}^{1,2}(\Omega))$. Also, since $N_l \to \infty$ when $l \to \infty$, the number M in the sum $\eta(x,t) = \sum_{k=1}^{M} d_k(t)v_k(x)$ can be arbitrarily large. Therefore, by approximation we have that our constructed solution, which is the pair (u,f), satisfies the integral identity (9) for all $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$.

Side condition

Next, we need to show that the function u satisfies the side condition

$$\int_\Omega |u(x,t)|^2 dx = E^2(t)$$

for all $t \in [0,T]$. Recalling the definition of $u^{(N)}$ (see (12)) and eigenfunction identities (6), notice that if $N, M \in \mathbb{N}$, $N \leqslant M$, then

$$\begin{aligned}\int_\Omega u^{(N)}(x,t)u^{(M)}(x,t)dx &= \int_\Omega \left(\sum_{k=1}^{N} w_k(t)v_k(x)\right)\left(\sum_{k=1}^{M} w_k(t)v_k(x)\right)dx \\ &= \sum_{k=1}^{N} w_k^2(t) \overset{(16)}{=} \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 E^2(t).\end{aligned}$$

Integrate this equality from 0 to an arbitrary $t \in [0,T]$ to get

$$\int_0^t \int_\Omega u^{(N)}(x,\tau)u^{(M)}(x,\tau)dxd\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \int_0^t E^2(\tau)d\tau.$$

If we also introduce an indicator function, the left side of this equation can be written as an inner product in $L^2(0,T;L^2(\Omega))$

$$\int_0^T \int_\Omega u^{(N)}(x,\tau)u^{(M)}(x,\tau)\mathbb{1}_{\{\tau\leqslant t\}}dxd\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \int_0^t E^2(\tau)d\tau.$$

Now, we can replace M by the subsequence N_l we got earlier and take the limit as $l \to \infty$. For any $N \in \mathbb{N}$, the inequality $N \leqslant N_l$ holds for large enough l. Also, $u^{(N_l)} \rightharpoonup u$ in $L^2(0,T;L^2(\Omega))$. Thus, after passing to the limit, we obtain

$$\int_0^T \int_\Omega u^{(N)}(x,\tau)u(x,\tau)\mathbb{1}_{\{\tau\leqslant t\}}dxd\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \int_0^t E^2(\tau)d\tau.$$

This equation is true for all $N \in \mathbb{N}$ so we can replace N by N_l and take the limit as $l \to \infty$. Recalling that $\sum_{k=1}^{\infty} \beta_k^2 = \|u_0\|_{L^2(\Omega)}$, we get that

$$\int_0^T \int_\Omega u(x,\tau)u(x,\tau)\mathbb{1}_{\{\tau\leqslant t\}}dxd\tau = \int_0^t E^2(\tau)d\tau,$$

which can be rewritten as

$$\int_0^t \|u(\cdot,\tau)\|^2_{L^2(\Omega)}dxd\tau = \int_0^t E^2(\tau)d\tau. \tag{30}$$

The function u is from the space $C([0,T];L^2(\Omega))$ by Theorem 1.1, therefore $\|u(\cdot,\tau)\|_{L^2(\Omega)}$ is a continuous function. The same can be said about the function E, because it is from $W^{1,2}(0,T) \subset C[0,T]$. Hence, we can differentiate both sides of equation (30) by t and obtain

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} = E^2(t) \tag{31}$$

for all $t \in [0,T]$, which is equivalent to (11), i.e. $\int_\Omega |u(x,t)|^2 dx = E^2(t)$ for all $t \in [0,T]$.

Initial condition

Denote $u_0^{(N)} = \sum_{k=1}^N \beta_k v_k$ to be the partial sums of the Fourier series for u_0 in $L^2(\Omega)$. Then it is clear that $u_0^{(N)} \to u_0$ in $L^2(\Omega)$ and since T is finite we have that $u_0^{(N)} \to u_0$ in $L^2(0,T;L^2(\Omega))$ as well, where we interpret both functions to be constant in time. Suppose that $N, M \in \mathbb{N}$ and that $N \leqslant M$. Then from (12) and from the fact that v_k are orthonormal, we obtain

$$\begin{aligned}
&\int_\Omega \left(u^{(N)}(x,t) - u_0^{(N)}(x)\right)\left(u^{(M)}(x,t) - u_0^{(M)}(x)\right) dx \\
&= \int_\Omega \left(\sum_{k=1}^N w_k(t)v_k(x) - \sum_{k=1}^N \beta_k v_k(x)\right)\left(\sum_{k=1}^M w_k(t)v_k(x) - \sum_{k=1}^M \beta_k v_k(x)\right) dx \\
&\overset{(16)}{=} \int_\Omega \left(\sum_{k=1}^N \frac{\beta_k v_k(x)}{\|u_0\|_{L^2(\Omega)}}\left(E(t) - \|u_0\|_{L^2(\Omega)}\right)\right)\left(\sum_{k=1}^M \frac{\beta_k v_k(x)}{\|u_0\|_{L^2(\Omega)}}\left(E(t) - \|u_0\|_{L^2(\Omega)}\right)\right) dx \\
&= \sum_{k=1}^N \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}}\left(E(t) - \|u_0\|_{L^2(\Omega)}\right)^2.
\end{aligned}$$

Integrating the left and right sides of this equality from 0 to an arbitrary $t \in [0,T]$ and writing one of the integrals with an indicator function gives us

$$\begin{aligned}
&\int_0^T \int_\Omega \left(u^{(N)}(x,\tau) - u_0^{(N)}(x)\right)\left(u^{(M)}(x,\tau) - u_0^{(M)}(x)\right) \mathbb{1}_{\{\tau \leqslant t\}} dx d\tau \\
&= \sum_{k=1}^N \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^t (E(\tau) - \|u_0\|_{L^2(\Omega)})^2 d\tau.
\end{aligned} \tag{32}$$

Now, we have that $u^{(N_l)} \rightharpoonup u$ in $L^2(0,T;L^2(\Omega))$, therefore $u^{(N_l)} - u_0^{(N_l)} \rightharpoonup u - u_0$ in the space $L^2(0,T;L^2(\Omega))$. So, if we replace M by N_l in (32) and pass to the limit as $l \to \infty$, we get that

$$\begin{aligned}
&\int_0^T \int_\Omega \left(u^{(N)}(x,\tau) - u_0^{(N)}(x)\right)(u(x,\tau) - u_0(x)) \mathbb{1}_{\{\tau \leqslant t\}} dx d\tau \\
&= \sum_{k=1}^N \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^t (E(\tau) - \|u_0\|_{L^2(\Omega)})^2 d\tau.
\end{aligned}$$

Similarly, if we replace N by N_l and pass to the limit again as $l \to \infty$, we get

$$\int_0^T \int_\Omega (u(x,\tau) - u_0(x))(u(x,\tau) - u_0(x)) \mathbb{1}_{\{\tau \leqslant t\}} dx d\tau = \int_0^t (E(\tau) - \|u_0\|_{L^2(\Omega)})^2 d\tau,$$

where we have used the fact that $\sum_{k=1}^{\infty}\beta_k^2 = \|u_0\|^2_{L^2(\Omega)}$. Rewriting the final expression we obtain

$$\int_0^t \|u(\cdot,\tau) - u_0\|^2_{L^2(\Omega)} d\tau = \int_0^t (E(\tau) - \|u_0\|_{L^2(\Omega)})^2 d\tau. \tag{33}$$

Just as before, because the integrands are continuous functions, we can differentiate this equation by t. This shows that for all $t \in [0, T]$:

$$\|u(\cdot,t) - u_0\|^2_{L^2(\Omega)} = (E(t) - \|u_0\|_{L^2(\Omega)})^2. \tag{34}$$

Specifically, if we set $t = 0$ in (34), then noting the fact that we have originally assumed that $E(0) = \|u_0\|_{L^2(\Omega)}$, we finally obtain

$$\|u(\cdot,0) - u_0\|^2_{L^2(\Omega)} = 0 \tag{35}$$

which is equivalent to (10), i.e. $u(x,\ 0) = u_0(x)$ for almost all $x \in \Omega$.

Remark 2.4. In the case when $u_0 \equiv 0$, we need to modify (13) slightly. Instead of coefficients β_k, which are Fourier coefficients of u_0, we can replace them by some rapidly decreasing to zero sequence γ_k, such that $\sum_{k=1}^{\infty}\gamma_k^2 = 1$ and $\sum_{k=1}^{\infty}\lambda_k^2\gamma_k^2 < \infty$. Also, instead of condition $(13)_3$ we will require

$$\int_\Omega |u^{(N)}(x,t)|^2 dx = \sum_{k=1}^{N}\gamma_k^2 E^2(t).$$

From there, most of the steps are the same. The only difference is that we get the side condition directly from the initial condition. This also shows that in this case there exist infinitely many weak solutions, provided that E is not identically 0.

3 Very weak solution

3.1 Definition of a very weak solution

Now we will derive the definition of a very weak solution of problem (1), (2) in the case when the function E does not possess a weak derivative and is only from the space $L^2(0,T)$. In order to do so, we have to first generalize the compatibility condition $E(0) = \|u_0\|_{L^2(\Omega)}$ from (2). This is because in general, evaluating an L^2 function at a point is ill-defined. The condition $E(0) = \|u_0\|_{L^2(\Omega)}$ will be understood to mean that the value $\|u_0\|_{L^2(\Omega)}$ is a Lebesgue point of E at $t = 0$ and thus $\lim\limits_{t\to 0^+} \frac{1}{t}\int_0^t |E(\tau) - \|u_0\|_{L^2(\Omega)}|d\tau = 0$.

For the derivation of the very weak solution we will also need a generalized version of a primitive function. For any function $g \in L^2(0,T)$, define its primitive $S_g(t)$ by $S_g(t) = \int_0^t g(\tau)d\tau$. It is well known that S_g is differentiable almost everywhere with both its regular and weak derivatives equal to g almost everywhere, and it is clear that $S_g(0) = 0$.

We will extend this notion to a larger class of functions in the same way it is done in [12]. Define $W_T^{1,2}(0,T;L^2(\Omega))$ to be the subspace $\{\varphi \in W^{1,2}(0,T;L^2(\Omega)) : \varphi(\cdot,\ T) = 0\}$, where we interpret a function from $W^{1,2}(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$ as the continuous representative so that $\varphi(\cdot,\ T)$ is well-defined. Also, denote the dual of $W_T^{1,2}(0,T;L^2(\Omega))$ as $W_T^{-1,2}(0,T;L^2(\Omega))$. We have the following representation of this space.

Lemma 3.1 ([12])**.** *If h is from the space $W_T^{-1,2}(0,T;L^2(\Omega))$, then there exists a unique $H \in L^2(0,T;L^2(\Omega))$, such that*

$$\langle h, \eta\rangle = \int_0^T \int_\Omega H(x,t)\eta_t(x,t)dxdt \quad \forall \eta \in W_T^{1,2}(0,T;L^2(\Omega)). \tag{36}$$

Proof. The functional given in (36) is bounded, because if $H \in L^2(0,T;L^2(\Omega))$, then

$$|\langle h,\eta\rangle| \leqslant \|H\|_{L^2(0,T;L^2(\Omega))}\|\eta_t\|_{L^2(0,T;L^2(\Omega))} \leqslant \|H\|_{L^2(0,T;L^2(\Omega))}\|\eta\|_{W_T^{1,2}(0,T;L^2(\Omega))}.$$

Next, consider the derivative operator $D : W_T^{1,2}(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$, defined by $D\eta = \eta_t$. This operator is surjective, because if $\varphi \in L^2(0,T;L^2(\Omega))$, then after defining

$$\psi(x,t) := -\int_t^T \varphi(x,\tau)d\tau,$$

we have that $\psi \in W_T^{1,2}(0,T;L^2(\Omega))$ and since ψ is absolutely continuous with respect to t:

$$\psi_t(x,t) = \left(\int_0^t \varphi(x,\tau)d\tau - \int_0^T \varphi(x,\tau)d\tau\right)_t = \varphi(x,t).$$

This operator is injective, because if $\eta_1, \eta_2 \in W_T^{1,2}(0,T;L^2(\Omega))$ and $(\eta_1)_t = (\eta_2)_t$, then we have that $(\eta_1 - \eta_2)_t = 0$. Applying Theorem 1.1, specifically the Newton–Leibniz formula, shows that $\eta_1(x,t) - \eta_2(x,t)$ is constant in time. However, by definition of the space $W_T^{1,2}(0,T;L^2(\Omega))$ we know that $\eta_1(x,T) = \eta_2(x,T) = 0$ for almost all $x \in \Omega$. This implies that $\eta_1(x,t) = \eta_2(x,t)$ for almost all $x \in \Omega$ and all $t \in [0,T]$ and thus $\eta_1 = \eta_2$. Operator D is clearly bounded, hence by Corollary 1.9 it is an isomorphism between $W_T^{1,2}(0,T;L^2(\Omega))$ and $L^2(0,T;L^2(\Omega))$

Now, suppose that h is from $W_T^{-1,2}(0,T;L^2(\Omega))$. Define a functional F on $L^2(0,T;L^2(\Omega))$ by the formula $F(\varphi) = \langle h, D^{-1}\varphi\rangle = (h \circ D^{-1})(\varphi)$. Since D^{-1} is bounded, F is a bounded functional. Therefore, by the Riesz representation theorem on $L^2(0,T;L^2(\Omega))$, there exists a unique $H \in L^2(0,T;L^2(\Omega))$ such that

$$F(\varphi) = \langle h, D^{-1}\varphi\rangle = \int_0^T \int_\Omega H(x,t)\varphi(x,t)dxdt. \tag{37}$$

For any $\eta \in W_T^{1,2}(0,T;L^2(\Omega))$, $\eta = D^{-1}\eta_t$, which implies that

$$F(\eta_t) = \langle h,\eta\rangle = \int_0^T \int_\Omega H(x,t)\eta_t(x,t)dxdt. \tag{38}$$

□

If $h \in W_T^{-1,2}(0,T;L^2(\Omega))$, we will define $S_h := -H$, where H is the $L^2(0,T;L^2(\Omega))$

function which represents the functional $h \in W_T^{-1,2}(0,T;L^2(\Omega))$ according to Lemma 3.1. This definition is an extension of the operator S, because if $\tilde{h}$ is the functional, associated by the Riesz representation theorem with a function $h \in L^2(0,T;L^2(\Omega))$ as $\langle \tilde{h}, \eta \rangle = \int_0^T\int_\Omega h(x,t)\eta(x,t)dxdt$, then after applying Proposition 1.10 on S_h and η, we get

$$\langle \tilde{h}, \eta \rangle = \int_\Omega \left(S_h(x,t)\eta(x,t)\Big|_{t=0}^{t=T} - \int_0^T S_h(x,t)\eta_t(x,t)dt \right) dx.$$

By definition $S_h(x,0) = 0$ and $\eta(x,T) = 0$, hence we have that

$$\langle \tilde{h}, \eta \rangle = -\int_0^T \int_\Omega S_h(x,t)\eta_t(x,t)dxdt,$$

which implies that $H = -S_h$ in this case.

Let us now derive the integral identity for the very weak solution. Suppose that the pair (u, f) is a classical solution of problem (1). Multiply equation $(1)_1$ by an arbitrary function $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$ and integrate with respect to x over Ω and then integrate with respect to t over the interval $(0,T)$ to get

$$\int_0^T \int_\Omega u_t(x,t)\eta(x,t)dxdt - \int_0^T \int_\Omega \Delta u(x,t)\eta(x,t)dxdt = \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt.$$

Integrating by parts with respect to x on the second integral yields

$$\int_0^T \int_\Omega u_t(x,t)\eta(x,t)dxdt + \int_0^T \int_\Omega \nabla u(x,t) \cdot \nabla\eta(x,t)dxdt = \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt, \tag{39}$$

which is the same as (9). Next, similarly to how it is done in [12], we will use integration by parts with respect to t on the first and third integrals in (39). In order to do this, we will additionally assume that $\eta_t \in L^2(0,T;L^2(\Omega))$ and $\eta(\,\cdot\,,T) \equiv 0$. This guarantees that the integrals exist, gets rid of one of the boundary terms and also allows us to define $\eta(\cdot, 0)$ using Theorem 1.1. Integrating the first integral in (39) by parts (applying Proposition 1.10) gives us

$$\begin{aligned}
\int_0^T \int_\Omega u_t(x,t)\eta(x,t)dxdt &= \int_\Omega \int_0^T u_t(x,t)\eta(x,t)dtdx \\
&= \int_\Omega \left(u(x,t)\eta(x,t)\Big|_{t=0}^{t=T} - \int_0^T u(x,t)\eta_t(x,t)dt \right) dx \\
&\overset{(1)_3}{=} \int_\Omega \left(-u_0(x)\eta(x,0) - \int_0^T u(x,t)\eta_t(x,t)dt \right) dx \\
&= -\int_\Omega u_0(x)\eta(x,0)dx - \int_0^T \int_\Omega u(x,t)\eta_t(x,t)dxdt.
\end{aligned} \tag{40}$$

For the third integral in (39), again using integration by parts with respect to t and the facts that $\frac{\partial S_f(x,t)}{\partial t} = f(x,t)$, $S_f(\,\cdot\,,0) \equiv 0$, $\eta(\,\cdot\,,T) \equiv 0$, we obtain

$$\begin{aligned}\int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt &= \int_\Omega \int_0^T \frac{\partial S_f(x,t)}{\partial t}\eta(x,t)dtdx \\ &= \int_\Omega \left(S_f(x,t)\eta(x,t)\Big|_{t=0}^{t=T} - \int_0^T S_f(x,t)\eta_t(x,t)dt \right) dx \\ &= -\int_0^T \int_\Omega S_f(x,t)\eta_t(x,t)dxdt. \end{aligned} \tag{41}$$

Substituting (40) and (41) into (39) we get the following integral identity

$$\begin{aligned}\int_\Omega u_0(x)\eta(x,0)dx + \int_0^T \int_\Omega u(x,t)\eta_t(x,t)dxdt \\ - \int_0^T \int_\Omega \nabla u(x,t)\cdot\nabla\eta(x,t)dxdt = \int_0^T \int_\Omega S_f(x,t)\eta_t(x,t)dxdt,\end{aligned} \tag{42}$$

where $\eta(x,t)$ is any function such that $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$, $\eta_t \in L^2(0,T;L^2(\Omega))$ and $\eta(\,\cdot\,,T) \equiv 0$. Hence, we get the following definition.

Definition 3.2. Suppose that $E \in L^2(0,T)$ and $u_0 \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$. Also suppose that $E(0) = \|u_0\|_{L^2(\Omega)}$ in terms of Lebesgue points. A pair of functions (u,f) such that $u \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$ and $f \in W_T^{-1,2}(0,T;L^2(\Omega))$, is called a very weak solution of problem (1), (2) if the integral identity (42) is true for all test functions $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$, $\eta_t \in L^2(0,T;L^2(\Omega))$, $\eta(\,\cdot\,,T) \equiv 0$ and if the initial and side conditions are satisfied

$$u(\cdot,\ 0) = u_0 \text{ in terms of Lebesgue points,} \tag{43}$$

$$\int_\Omega |u(x,t)|^2 dx = E^2(t), \quad \text{for a.e. } t \in [0,T]. \tag{44}$$

Our second goal will be to prove the following theorem.

Theorem 3.3. *Suppose that $u_0 \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ and $E \in L^2(0,T)$ are given functions and that E satisfies the compatibility condition $E(0) = \|u_0\|_{L^2(\Omega)}$ in terms of Lebesgue points. Then there exists at least one very weak solution (u,f) to problem* (1), (2).

For the rest of the next section we will assume that u_0 is not identically equal to 0, so that $\|u_0\|_{L^2(\Omega)} > 0$. The case when $u_0 \equiv 0$ is treated very similarly so it will only be mentioned at the end.

3.2 Approximate solution

Choose any sequence of functions $E_n \in W^{1,2}(0,T)$ which converges to E in the space $L^2(0,T)$ such that $E_n(0) = \|u_0\|_{L^2(\Omega)}$. This should be understood in the usual sense because of the embedding $W^{1,2}(0,T) \subset C[0,T]$. Since we know that $C_c^\infty(0,T)$ is dense in $L^2(0,T)$ (see [6]) and $C_c^\infty(0,T) \subset W^{1,2}(0,T)$, we can choose a sequence of smooth and compactly supported functions. For every $N \in \mathbb{N}$ and $n \in \mathbb{N}$ we will find approximate solutions $(u_n^{(N)}, f_n^{(N)})$ that are represented in the following forms

$$\begin{aligned} u_n^{(N)}(x,t) &= \sum_{k=1}^{N} w_{k,n}(t) v_k(x), \\ f_n^{(N)}(x,t) &= \sum_{k=1}^{N} q_{k,n}(t) v_k(x). \end{aligned} \tag{45}$$

We will also require them to satisfy the following system of equations for every $t \in [0,T]$ and $k \leqslant N$, which is almost identical to (13)

$$\begin{cases} \int_\Omega (u_n^{(N)})_t(x,t) v_k(x) dx + \int_\Omega \nabla u_n^{(N)}(x,t) \cdot \nabla v_k(x) dx = \int_\Omega f_n^{(N)}(x,t) v_k(x) dx, \\ u_n^{(N)}(x,0) = \sum_{k=1}^{N} \beta_k v_k(x), \quad x \in \Omega, \\ \int_\Omega \left| u_n^{(N)}(x,t) \right|^2 dx = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 E_n^2(t), \end{cases} \tag{46}$$

where β_k, $k \in \mathbb{N}$, are the Fourier coefficients of the function u_0 in terms of the orthonormal basis $\{v_k\}_{k=1}^\infty$ in $L^2(\Omega)$. In the same way as in Section 2.2, we get the following expressions for $w_{k,n}$ and $q_{k,n}$, namely

$$w_{k,n}(t) = \frac{\beta_k}{\|u_0\|_{L^2(\Omega)}} E_n(t), \tag{47}$$

$$q_{k,n} = (w_{k,n})' + \lambda_k w_{k,n} \in L^2(0,T). \tag{48}$$

Since the function E_n is from the space $W^{1,2}(0,T)$, it is also continuous. Therefore, $u_n^{(N)}$ is from the space $C([0,T]; L^2(\Omega))$ and $u_n^{(N)}(x,0)$ is well-defined. With this in mind as well as the fact that we have assumed $E_n(0) = \|u_0\|_{L^2(\Omega)}$, we have that

$$u_n^{(N)}(x,0) = \sum_{k=1}^{N} w_{k,n}(0) v_k(x) = \frac{1}{\|u_0\|_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k E_n(0) v_k(x) = \sum_{k=1}^{N} \beta_k v_k(x).$$

With this, the pair of functions $(u_n^{(N)}, f_n^{(N)})$ satisfies the system of equations (46). Next, we will get suitable estimates for $u_n^{(N)}$ and $f_n^{(N)}$. The bounds for $u_n^{(N)}$ and $\nabla u_n^{(N)}$ we get almost identically to the ones in the previous section, (see (18) and (19)):

$$\|u_n^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \|E_n\|^2_{L^2(0,T)}, \tag{49}$$

$$\|\nabla u_n^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \frac{\|E_n\|^2_{L^2(0,T)}}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \lambda_k. \tag{50}$$

For the sequence $\{f_n^{(N)}\}_{N=1}^\infty$, we will not be able to bound it in the space $L^2(0,T;L^2(\Omega))$ because estimate (22) involves the derivative of E. Instead, we will get an estimate for $S_{f_n^{(N)}}$

in the space $L^2(0,T;L^2(\Omega))$. First, integrate equation (48) from 0 to t to get

$$\begin{aligned} S_{q_{k,n}}(t) &= \int_0^t (w_{k,n})'(\tau)d\tau + \lambda_k \int_0^t w_{k,n}(\tau)d\tau = w_{k,n}(t) - \overbrace{w_{k,n}(0)}^{\beta_k} + \lambda_k \int_0^t w_{k,n}(\tau)d\tau \\ &\overset{(47)}{=} \frac{\beta_k}{\|u_0\|_{L^2(\Omega)}} E_n(t) - \beta_k + \frac{\lambda_k \beta_k}{\|u_0\|_{L^2(\Omega)}} \int_0^t E_n(\tau)d\tau. \end{aligned} \tag{51}$$

We will also use the following inequality that follows from Cauchy-Schwarz inequality

$$\begin{aligned} &\int_0^T |E_n(t)|dt \leqslant \left(\int_0^T E_n^2(t)dt\right)^{\frac{1}{2}} \left(\int_0^T dt\right)^{\frac{1}{2}} = \|E_n\|_{L^2(0,T)}\sqrt{T} \\ \implies &\int_0^T \left|\int_0^t E_n(\tau)d\tau\right|^2 dt \leqslant \int_0^T \left(\int_0^T |E_n(\tau)|d\tau\right)^2 dt \leqslant T^2 \|E_n\|^2_{L^2(0,T)}. \end{aligned}$$

Thus, together with the inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, $a,b,c \in \mathbb{R}$, which is true by Jensen's inequality[1], we have

$$\begin{aligned} \|S_{q_{k,n}}\|^2_{L^2(0,T)} &= \int_0^T |S_{q_{k,n}}(t)|^2 dt = \int_0^T \left(\frac{\beta_k}{\|u_0\|_{L^2(\Omega)}} E_n(t) - \beta_k + \frac{\lambda_k \beta_k}{\|u_0\|_{L^2(\Omega)}} \int_0^t E_n(\tau)d\tau\right)^2 dt \\ &\leqslant 3\left(\frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^T E_n^2(t)dt + \beta_k^2 T + \frac{\lambda_k^2 \beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^T \left|\int_0^t E_n(\tau)d\tau\right|^2 dt\right) \\ &\leqslant \frac{3\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \|E_n\|^2_{L^2(0,T)} + 3\beta_k^2 T + \frac{3\lambda_k^2 \beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} T^2 \|E_n\|^2_{L^2(0,T)} \\ &\leqslant C_4(\beta_k^2 \|E_n\|^2_{L^2(0,T)} + \beta_k^2 \lambda_k^2 \|E_n\|^2_{L^2(0,T)} + \beta_k^2), \end{aligned}$$

for some constant $C_4 > 0$ that depends on T and u_0. Therefore, by the last inequality

$$\begin{aligned} \|S_{f_n^{(N)}}\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \left\|\sum_{k=1}^N S_{q_{k,n}}(\tau)v_k\right\|^2_{L^2(\Omega)} d\tau = \sum_{k=1}^N \int_0^T |S_{q_{k,n}}(\tau)|^2 d\tau \\ &\leqslant \sum_{k=1}^\infty C_4(\beta_k^2 \|E_n\|^2_{L^2(0,T)} + \beta_k^2 \lambda_k^2 \|E_n\|^2_{L^2(0,T)} + \beta_k^2) \\ &= C_4 \|u_0\|^2_{L^2(\Omega)} \|E_n\|^2_{L^2(0,T)} + C_4 \|u_0\|^2_{L^2(\Omega)} + C_4 \|E_n\|^2_{L^2(0,T)} \sum_{k=1}^\infty \beta_k^2 \lambda_k^2. \end{aligned}$$

Inequality (25) gives the following bound

$$\begin{aligned} \|S_{f_n^{(N)}}\|^2_{L^2(0,T;L^2(\Omega))} &\leqslant C_4 \|u_0\|^2_{L^2(\Omega)} \|E_n\|^2_{L^2(0,T)} + C_4 \|u_0\|^2_{L^2(\Omega)} + C_4 \|E_n\|^2_{L^2(0,T)} \|u_0\|^2_{W^{2,2}(\Omega)} \\ &\leqslant C_5(\|E_n\|^2_{L^2(0,T)} + 1) \end{aligned} \tag{52}$$

for a large enough constant $C_5 > 0$ that depends on the function u_0. Combining equations

[1] If $g : \mathbb{R} \to \mathbb{R}$ is a convex function, $x_1, x_2, \ldots, x_k \in \mathbb{R}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$ are coefficients such that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$, then $g\left(\sum_{i=1}^k \lambda_i x_i\right) \leqslant \sum_{i=1}^k \lambda_i g(x_i)$. In this particular case, we set $k = 3$, $\lambda_i = 1/3$ and $g(x) = x^2$.

(49), (50) and (27) we have

$$\begin{aligned}\|u_n^{(N)}\|^2_{L^2(0,T;\mathring{W}^{1,2}(\Omega))} &= \|u_n^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla u_n^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} \\ &\leqslant \|E_n\|^2_{L^2(0,T)} + \frac{\|E_n\|^2_{L^2(0,T)}}{\|u_0\|^2_{L^2(\Omega)}}\|u_0\|^2_{W^{2,2}(\Omega)} \\ &\leqslant C_6\|E_n\|^2_{L^2(0,T)} \end{aligned} \tag{53}$$

for some large enough constant $C_6 > 0$. By using these estimates, we have that the sequence $\{u_n^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $L^2(0,T;\mathring{W}^{1,2}(\Omega))$ while the sequence $\{S_{f_n^{(N)}}\}_{N=1}^{\infty}$ is bounded in the space $L^2(0,T;L^2(\Omega))$.

3.3 Convergence

From the last section we know that the sequence $\{u_n^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $L^2(0,T;\mathring{W}^{1,2}(\Omega))$ while the sequence $\{S_{f_n^{(N)}}\}_{N=1}^{\infty}$ is bounded in the space $L^2(0,T;L^2(\Omega))$ which also means that $\{f^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $W_T^{-1,2}(0,T;L^2(\Omega))$ by Lemma 3.1. Therefore, for each fixed $n \in \mathbb{N}$ there exists a subsequence $\{N_l\}_{l=1}^{\infty}$, such that $u_n^{(N_l)}$ weakly converges to a function $u_n \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$ and that $f_n^{(N_l)}$ weakly converges to $f_n \in W_T^{-1,2}(0,T;L^2(\Omega))$. We will require the following lemma.

Lemma 3.4. *Suppose a sequence $g_N \rightharpoonup g$ in the space $L^2(0,T;W^{1,2}(\Omega))$. Then $g_N \rightharpoonup g$ and $\nabla g_N \rightharpoonup \nabla g$ in $L^2(0,T;L^2(\Omega))$.*

Proof. The proof is omitted as it is very similar to the proof of Lemma 2.3. □

From Lemma 3.4 we get that $u_n^{(N_l)} \rightharpoonup u_n$ and $\nabla u_n^{(N_l)} \rightharpoonup \nabla u_n$ in $L^2(0,T;L^2(\Omega))$. As for the functions $f_n^{(N_l)}$, according to Lemma 3.1 and using the extended notion of the primitive function we have

$$\begin{aligned}\langle f_n^{(N_l)}, \eta\rangle &= -\int_0^T\int_\Omega S_{f_n^{(N_l)}}(x,t)\eta_t(x,t)dxdt, \\ \langle f_n, \eta\rangle &= -\int_0^T\int_\Omega S_{f_n}(x,t)\eta_t(x,t)dxdt.\end{aligned}$$

Given that $f_n^{(N_l)} \rightharpoonup f_n$ in $W_T^{-1,2}(0,T;L^2(\Omega))$ and by using the inclusion of the space $W_T^{1,2}(0,T;L^2(\Omega))$ to its double dual, we have that for all $\eta \in W_T^{1,2}(0,T;L^2(\Omega))$

$$-\int_0^T\int_\Omega S_{f_n^{(N_l)}}(x,t)\eta_t(x,t)dxdt \to -\int_0^T\int_\Omega S_{f_n}(x,t)\eta_t(x,t)dxdt. \tag{54}$$

But this also implies that $S_{f_n^{(N_l)}} \rightharpoonup S_{f_n}$ in $L^2(0,T;L^2(\Omega))$, since the derivative operator D, defined in Lemma 3.1, is surjective.

Next, consider arbitrary functions $d_k \in C^\infty[0,T]$ with $d_k(T) = 0$, $k \in \mathbb{N}$. Multiply equations $(46)_1$ by d_k and then sum over all $k \in \{1,2,\dots,M\}$, where $M \leqslant N$. If we also denote $\eta(x,t)=\sum_{k=1}^{M} d_k(t)v_k(x)$ and integrate by t from 0 to T, we get the following identity

$$\int_0^T\!\!\int_\Omega (u_n^{(N)})_t(x,t)\eta(x,t)dxdt+\int_0^T\!\!\int_\Omega \nabla u_n^{(N)}(x,t)\cdot\nabla\eta(x,t)dxdt=\int_0^T\!\!\int_\Omega f_n^{(N)}(x,t)\eta(x,t)dxdt. \quad (55)$$

Then, similarly to how we derived the weak formulation, we can use integration by parts on the first and third integrals in equation (55). Recalling that $u_n^{(N)}(x,0)=\sum_{k=1}^{N}\beta_k v_k(x)$ and that $\eta(\cdot,\ T)\equiv 0$, we get the following identity

$$\begin{aligned}\int_\Omega\left(\sum_{k=1}^{N}\beta_k v_k(x)\right)\eta(x,0)dx+\int_0^T\int_\Omega u_n^{(N)}(x,t)\eta_t(x,t)dxdt\\ -\int_0^T\int_\Omega \nabla u_n^{(N)}(x,t)\cdot\nabla\eta(x,t)dxdt=\int_0^T\int_\Omega S_{f_n^{(N)}}(x,t)\eta_t(x,t)dxdt.\end{aligned} \quad (56)$$

We can write the last three integrals in (56) as inner products in the space $L^2(0,T;L^2(\Omega))$ and use weak convergence to get that

$$\begin{aligned}&\int_0^T\int_\Omega u_n^{(N_l)}(x,t)\eta_t(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T\int_\Omega u_n(x,t)\eta_t(x,t)dxdt,\\ &\int_0^T\int_\Omega \nabla u_n^{(N_l)}(x,t)\cdot\nabla\eta(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T\int_\Omega \nabla u_n(x,t)\cdot\nabla\eta(x,t)dxdt,\\ &\int_0^T\int_\Omega S_{f_n^{(N_l)}}(x,t)\eta_t(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T\int_\Omega S_{f_n}(x,t)\eta_t(x,t)dxdt.\end{aligned}$$

As for the first integral in (56), it is an inner product in $L^2(\Omega)$ and it is easy to see that $\sum_{k=1}^{N}\beta_k v_k\to u_0,\ N\to\infty$, in the same space. Thus, after substituting N_l for N in equation (13) and passing to the limit as $l\to\infty$, we get equation (42) for the pair of functions (u_n,f_n) for all $n\in\mathbb{N}$:

$$\begin{aligned}\int_\Omega u_0(x)\eta(x,0)dx+\int_0^T\int_\Omega u_n(x,t)\eta_t(x,t)dxdt\\ +\int_0^T\int_\Omega \nabla S_{u_n}(x,t)\cdot\nabla\eta_t(x,t)dxdt=\int_0^T\int_\Omega S_{f_n}(x,t)\eta_t(x,t)dxdt\end{aligned} \quad (57)$$

for the case when η is a linear combination $\sum_{k=1}^{M} d_k(t)v_k(x)$. The number M in the sum can be arbitrarily large, because $N_l\to\infty$ when $l\to\infty$. It is well known (see [1]) that the set of all these linear combinations is dense in the space

$$V=\{\eta:\eta\in L^2(0,T;\mathring{W}^{1,2}(\Omega)),\ \eta_t\in L^2(0,T;L^2(\Omega)),\ \eta(\cdot,T)\equiv 0\}.$$

Thus, by approximation we get that the identity (57) is true for all $\eta\in L^2(0,T;\mathring{W}^{1,2}(\Omega))$, $\eta_t\in L^2(0,T;L^2(\Omega))$ and $\eta(\,\cdot\,,T)\equiv 0$.

To pass to the next limit, notice that for each $n\in\mathbb{N}$, since $u_n^{(N_l)}\rightharpoonup u_n$ in $L^2(0,T;L^2(\Omega))$, we have the following inequality $\|u_n\|_{L^2(0,T;\mathring{W}^{1,2}(\Omega))}\leqslant \liminf\limits_{l\to\infty}\|u_n^{(N_l)}\|_{L^2(0,T;\mathring{W}^{1,2}(\Omega))}$. Also, $E_n\to E$ in $L^2(0,T)$, so there exists a positive constant C such that $\|E_n\|^2_{L^2(0,T)}\leqslant C$ for all $n\in\mathbb{N}$.

Therefore, we can pass to the limit as l goes to infinity on the bound (53):

$$\|u_n\|^2_{L^2(0,T;\mathring{W}^{1,2}(\Omega))} \leqslant \liminf_{l\to\infty} \|u_n^{(N_l)}\|^2_{L^2(0,T;\mathring{W}^{1,2}(\Omega))} \leqslant C_6\|E_n\|^2_{L^2(0,T)} \leqslant C_6 C. \tag{58}$$

Similarly with the bound (52) involving functions $f_n^{(N_l)}$:

$$\|S_{f_n}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \liminf_{l\to\infty} \|S_{f_n^{(N_l)}}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C_5(\|E_n\|^2_{L^2(0,T)}+1) \leqslant C_5(C+1). \tag{59}$$

Just like before, since $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;\mathring{W}^{1,2}(\Omega))$ and $\{S_{f_n}\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;L^2(\Omega))$, there exists a subsequence $\{n_l\}_{l=1}^{\infty}$, such that $u_{n_l} \rightharpoonup u$ and $f_{n_l} \rightharpoonup f$ for some $u \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$ and $f \in W_T^{-1,2}(0,T;L^2(\Omega))$. Thus, after passing to the limit in (57) as $l \to \infty$, we get

$$\begin{aligned}\int_\Omega u_0(x)\eta(x,0)dx + \int_0^T\int_\Omega u(x,t)\eta_t(x,t)dxdt \\ -\int_0^T\int_\Omega \nabla u(x,t)\cdot\nabla\eta(x,t)dxdt = \int_0^T\int_\Omega S_f(x,t)\eta_t(x,t)dxdt\end{aligned}$$

for all $\eta \in L^2(0,T;\mathring{W}^{1,2}(\Omega))$, $\eta_t \in L^2(0,T;L^2(\Omega))$ and $\eta(\,\cdot\,,T)\equiv 0$ which is the integral identity we wanted to prove, namely (42).

Side condition

Going back to the definition of $u_n^{(N)}$ (see $(45)_1$), notice that if $N, M \in \mathbb{N}$, $N \leqslant M$, then since eigenfunctions v_k are orthonormal, we have the following identity for all $t \in [0,T]$

$$\begin{aligned}\int_\Omega u_n^{(N)}(x,t)u_m^{(M)}(x,t)dx &= \int_\Omega \left(\sum_{k=1}^N w_{k,n}(t)v_k(x)\right)\left(\sum_{k=1}^M w_{k,m}(t)v_k(x)\right)dx \\ &= \sum_{k=1}^N w_{k,n}(t)w_{k,m}(t) = \frac{1}{\|u_0\|^2_{L^2(\Omega)}}\sum_{k=1}^N \beta_k^2 E_n(t)E_m(t).\end{aligned}$$

Integrate this last equality from 0 to an arbitrary $t \in [0,T]$ to get

$$\int_0^t\int_\Omega u_n^{(N)}(x,\tau)u_m^{(M)}(x,\tau)dxd\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}}\sum_{k=1}^N \beta_k^2\int_0^t E_n(\tau)E_m(\tau)d\tau.$$

If we also introduce indicator functions, this can be written in terms of inner products as such

$$\int_0^T\int_\Omega u_n^{(N)}(x,\tau)u_m^{(M)}(x,\tau)\mathbb{1}_{\{\tau\leqslant t\}}dxd\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}}\sum_{k=1}^N \beta_k^2\int_0^T E_n(\tau)E_m(\tau)\mathbb{1}_{\{\tau\leqslant t\}}d\tau. \tag{60}$$

Now, because there is a subsequence M_l such that $u_m^{(M_l)} \rightharpoonup u_m$ in $L^2(0,T;L^2(\Omega))$, if we replace M by M_l in (60) and pass to the limit as $l \to \infty$ (as $N \leqslant M_l$ for large enough l, this is valid):

$$\int_0^T\int_\Omega u_n^{(N)}(x,\tau)u_m(x,\tau)\mathbb{1}_{\{\tau\leqslant t\}}dxd\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}}\sum_{k=1}^N \beta_k^2\int_0^T E_n(\tau)E_m(\tau)\mathbb{1}_{\{\tau\leqslant t\}}d\tau \tag{61}$$

for all $N \in \mathbb{N}$. Similarly, there is a subsequence N_l such that $u_n^{(N_l)} \rightharpoonup u_n$ in $L^2(0,T;L^2(\Omega))$, so if we replace N by N_l in (61) and pass to the limit as $l \to \infty$:

$$\int_0^T \int_\Omega u_n(x,\tau) u_m(x,\tau) \mathbb{1}_{\{\tau \leqslant t\}} dx d\tau = \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^{\infty} \beta_k^2 \int_0^T E_n(\tau) E_m(\tau) \mathbb{1}_{\{\tau \leqslant t\}} d\tau$$
$$= \int_0^T E_n(\tau) E_m(\tau) \mathbb{1}_{\{\tau \leqslant t\}} d\tau, \qquad (62)$$

because $\sum_{k=1}^{\infty} \beta_k^2 = \|u_0\|^2_{L^2(\Omega)}$. Next, we will take the limit as $n, m \to \infty$ in a similar way. It does not matter in which order we take the limit as the result will be the same. First, we already have that there is a subsequence n_l such that $u_{n_l} \rightharpoonup u$ in $L^2(0,T;L^2(\Omega))$. Also, because strong convergence implies weak convergence, we have that $E_{n_l} \rightharpoonup E$ in $L^2(0,T)$. Thus, if we replace n by n_l in (62) and pass to the limit as $l \to \infty$:

$$\int_0^T \int_\Omega u(x,\tau) u_m(x,\tau) \mathbb{1}_{\{\tau \leqslant t\}} dx d\tau = \int_0^T E(\tau) E_m(\tau) \mathbb{1}_{\{\tau \leqslant t\}} d\tau. \qquad (63)$$

Lastly, if we replace m by n_l in (63) and pass to the limit as $l \to \infty$:

$$\int_0^T \int_\Omega u(x,\tau) u(x,\tau) \mathbb{1}_{\{\tau \leqslant t\}} dx d\tau = \int_0^T E(\tau) E(\tau) \mathbb{1}_{\{\tau \leqslant t\}} d\tau$$

Thus

$$\int_0^t \int_\Omega |u(x,\tau)|^2 dx d\tau = \int_0^t E^2(\tau) d\tau.$$

Differentiating this equality by t, we see that for almost all $t \in [0,T]$ the side condition (44) is satisfied:

$$\int_\Omega |u(x,t)|^2 dx = E^2(t).$$

Initial condition

Denote $u_0^{(N)} = \sum_{k=1}^{N} \beta_k v_k$ to be the partial sums of the Fourier series for u_0 in $L^2(\Omega)$. Then it is clear that $u_0^{(N)} \to u_0$ in $L^2(\Omega)$ and since T is finite we have that $u_0^{(N)} \to u_0$ in $L^2(0,T;L^2(\Omega))$ as well, where we interpret both functions to be constant in time. Suppose that $N, M \in \mathbb{N}$, $n, m \in \mathbb{N}$ and that $N \leqslant M$. Then from (45) and from the fact that v_k are orthonormal, we obtain

$$\int_\Omega \left(u_n^{(N)}(x,t) - u_0^{(N)}(x)\right)\left(u_m^{(M)}(x,t) - u_0^{(M)}(x)\right) dx$$
$$= \int_\Omega \left(\sum_{k=1}^{N} w_{k,n}(t) v_k(x) - \sum_{k=1}^{N} \beta_k v_k(x)\right)\left(\sum_{k=1}^{M} w_{k,m}(t) v_k(x) - \sum_{k=1}^{M} \beta_k v_k(x)\right) dx$$
$$= \int_\Omega \left(\sum_{k=1}^{N} \frac{\beta_k v_k(x)}{\|u_0\|_{L^2(\Omega)}} \left(E_n(t) - \|u_0\|_{L^2(\Omega)}\right)\right)\left(\sum_{k=1}^{M} \frac{\beta_k v_k(x)}{\|u_0\|_{L^2(\Omega)}} \left(E_m(t) - \|u_0\|_{L^2(\Omega)}\right)\right) dx$$
$$= \sum_{k=1}^{N} \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} (E_n(t) - \|u_0\|_{L^2(\Omega)})(E_m(t) - \|u_0\|_{L^2(\Omega)}).$$

Integrating the last identity from 0 to an arbitrary t and writing the integrals with indicator functions gives us

$$\begin{aligned}&\int_0^T \int_\Omega \left(u_n^{(N)}(x,\tau) - u_0^{(N)}(x)\right)\left(u_m^{(M)}(x,\tau) - u_0^{(M)}(x)\right) \mathbb{1}_{\{\tau\leqslant t\}} dx d\tau \\ &= \sum_{k=1}^N \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^T (E_n(\tau) - \|u_0\|_{L^2(\Omega)})(E_m(\tau) - \|u_0\|_{L^2(\Omega)}) d\tau.\end{aligned} \tag{64}$$

Now, pick a subsequence M_l, $l \in \mathbb{N}$, such that $u_m^{(M_l)} \rightharpoonup u_m$ in $L^2(0,T;L^2(\Omega))$. We also know that $u_0^{(M_l)}$ strongly converges to u_0 in $L^2(0,T;L^2(\Omega))$, thus it also weakly converges. Using this, we can replace M with M_l in equation (64) and pass to the limit as $l \to \infty$ to get

$$\begin{aligned}&\int_0^T \int_\Omega (u_n^{(N)}(x,\tau) - u_0^{(N)}(x))(u_m(x,\tau) - u_0(x)) \mathbb{1}_{\{\tau\leqslant t\}} dx d\tau \\ &= \sum_{k=1}^N \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \int_0^T (E_n(\tau) - \|u_0\|_{L^2(\Omega)})(E_m(\tau) - \|u_0\|_{L^2(\Omega)}) d\tau.\end{aligned} \tag{65}$$

Similarly, we can find a subsequence N_l, $l \in \mathbb{N}$, such that $u_n^{(N_l)} \rightharpoonup u_n$ in $L^2(0,T;L^2(\Omega))$. Then $u_n^{(N_l)} - u_0^{(N_l)} \rightharpoonup u_n - u_0$ in $L^2(0,T;L^2(\Omega))$. Replace N with N_l in (65) and pass to the limit as $l \to \infty$ to get

$$\int_0^T \int_\Omega (u_n(x,\tau) - u_0(x))(u_m(x,\tau) - u_0(x)) \mathbb{1}_{\{\tau\leqslant t\}} dx d\tau \tag{66}$$

$$= \int_0^T (E_n(\tau) - \|u_0\|_{L^2(\Omega)})(E_m(t) - \|u_0\|_{L^2(\Omega)}) \mathbb{1}_{\{\tau\leqslant t\}} d\tau. \tag{67}$$

Finally, choose a subsequence n_l, $l \in \mathbb{N}$, such that $u_{n_l} \rightharpoonup u$ in $L^2(0,T;L^2(\Omega))$. Then since we also have that $E_{n_l} \rightharpoonup E$ in $L^2(0,T)$, we can replace n with n_l and pass to the limit:

$$\begin{aligned}&\int_0^T \int_\Omega (u(x,\tau) - u_0(x))(u_m(x,\tau) - u_0(x)) \mathbb{1}_{\{\tau\leqslant t\}} dx d\tau \\ &= \int_0^T (E(\tau) - \|u_0\|_{L^2(\Omega)})(E_m(t) - \|u_0\|_{L^2(\Omega)}) \mathbb{1}_{\{\tau\leqslant t\}} d\tau.\end{aligned}$$

Similarly replacing m by n_l and passing to the limit we get

$$\begin{aligned}&\int_0^T \int_\Omega (u(x,\tau) - u_0(x))(u(x,\tau) - u_0(x)) \mathbb{1}_{\{\tau\leqslant t\}} dx d\tau \\ &= \int_0^T (E(\tau) - \|u_0\|_{L^2(\Omega)})(E(t) - \|u_0\|_{L^2(\Omega)}) \mathbb{1}_{\{\tau\leqslant t\}} d\tau,\end{aligned}$$

which can be rewritten as

$$\int_0^t \int_\Omega |u(x,\tau) - u_0(x)|^2 dx d\tau = \int_0^t |E(\tau) - \|u_0\|_{L^2(\Omega)}|^2 d\tau. \tag{68}$$

Differentiating this equality by t and taking the square root of both sides shows that

$$\|u(\cdot,t) - u_0\|_{L^2(\Omega)} = |E(t) - \|u_0\|_{L^2(\Omega)}| \tag{69}$$

for almost all $t \in [0,T]$. Recall that we have assumed that $E(0) = \|u_0\|_{L^2(\Omega)}$ in terms of Lebesgue points, thus

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t |E(\tau) - \|u_0\|_{L^2(\Omega)}|d\tau = 0. \tag{70}$$

Hence, if we integrate equation (69) from 0 to an arbitrary but positive t, divide both sides by t and take the limit as $t \to 0^+$

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t \|u(\cdot,\tau) - u_0\|_{L^2(\Omega)}d\tau = \lim_{t\to 0^+} \frac{1}{t}\int_0^t |E(\tau) - \|u_0\|_{L^2(\Omega)}|d\tau = 0. \tag{71}$$

This shows that the initial condition (43) is satisfied in terms of Lebesgue points.

Remark 3.5. In the case when $u_0 \equiv 0$, we need to modify (46) slightly, just like in Remark 2.4. Just like before, all of the steps are mostly the same and we get the side condition directly from the initial condition. This also shows that in this case there exist infinitely many very weak solutions, provided that E is not identically 0.

Results and conclusions

In this thesis we have defined the weak and very weak solutions for the inverse heat problem with an unknown source function $f = f(x,t)$, subject to a nonlinear nonlocal condition $\int_\Omega |u(x,t)|^2 dx = E^2(t)$ for all $t \in [0,T]$. We then proved that in the case, when E is from the space $W^{1,2}(0,T)$, there exists at least one weak solution while in the case when E is only from $L^2(0,T)$, we proved the existence of at least one very weak solution.

References and sources

[1] A. Ambrazevicius, A. Domarkas. *Matematinės fizikos lygtys 2 dalis*. Aldorija, 1999.

[2] T. Belickas, K. Kaulakytė, G. Puriuškis. "Nonstationary heat equation with nonlinear side condition." In: *Mathematical Modeling and Analysis* 30 (1 2025), pages 109–119.

[3] T. Buckmaster, V. Vicol. "Nonuniqueness of weak solutions to the Navier-Stokes equation." In: *Annals of Mathematics* 189.1 (2019), pages 101–144.

[4] J. R. Cannon. "The solution of the heat equation subject to the specification of energy." In: *Quarterly of Applied Mathematics* 21.2 (1963), pages 155–160.

[5] J. R. Cannon. *The One-Dimensional Heat Equation*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1984.

[6] L. C. Evans. *Partial Differential Equations*. 2nd. Volume 19. American Mathematical Society, 2010.

[7] L. C. Evans. *Measure theory and fine properties of functions*. 1st. CRC Press, 2015.

[8] G. B. Folland. *Real Analysis: Modern Techniques and Their Applications*. John Wiley & Sons, 1999.

[9] G. P. Galdi, K. Pileckas, A. L. Silvestre. "On the unsteady Poiseuille flow in a pipe." In: *Zeitschrift für angewandte Mathematik und Physik* 58.6 (2007), pages 994–1007.

[10] O. A. Ladyzhenskaya. *The Boundary Value Problems of Mathematical Physics*. Springer, 1985.

[11] K. Pileckas. *Navjė-Stokso lygčių matematinė teorija*. MII, Vilnius, 2007.

[12] K. Pileckas, R. Čiegis. "Existence of nonstationary Poiseuille-type solutions under minimal regularity assumptions." In: *Zeitschrift für angewandte Mathematik und Physik* 71, 192 (2020).