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MATHEMATICS

Master Thesis

Baire Functions

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Contents

1 Introduction

The study of functions, their continuity, differentiability has always be of great interest for mathematicians.
Baire functions were introduced by the French mathematician **René-Louis Baire** in his doctoral thesis in 1899. Baire's idea was to classify functions that could be constructed as limit of some other functions iteratively and starting from continuous functions.
Before his work, mathematicians focused on continuous functions but many important functions such as discontinuous solutions of differential equations could not be described in a good way only with continuity.
For this purpose, Baire introduced a hierarchy of functions to get a more general frame and extend the notions of **"well behaved"** functions
The first level is Baire class 0, consisting on continuous functions. Baire class 1 consists of functions that are pointwise limit of continuous functions. By taking pointwise limit of Baire class 1 functions we have Baire class 2 functions. This iterative process allows to define Baire class n functions for arbitrary n.
In this thesis we explore elementary aspects of Baire functions. After we defining them, we give some basic properties of those functions and characterization theorems and deal with some concrete functions
The original component of this work consists in the study of several functions to determine their respective Baire classes. To achieve this, depending on the case, we analyzed their set on continuity, employed characterization theorems, or directly constructed these functions as limit of others.

2 Definition and basic properties

Let X be an interval from $\mathbb{R}$.

2.0.1 Definition. Let $f_n\colon X \to \mathbb{R}$ a real valued sequence of functions defined on X and let $f\colon X \to \mathbb{R}$. We say that $\{f_n\}$ converges pointwise to f if for every $x \in X$, $f_n(x)$ converges to $f(x)$:

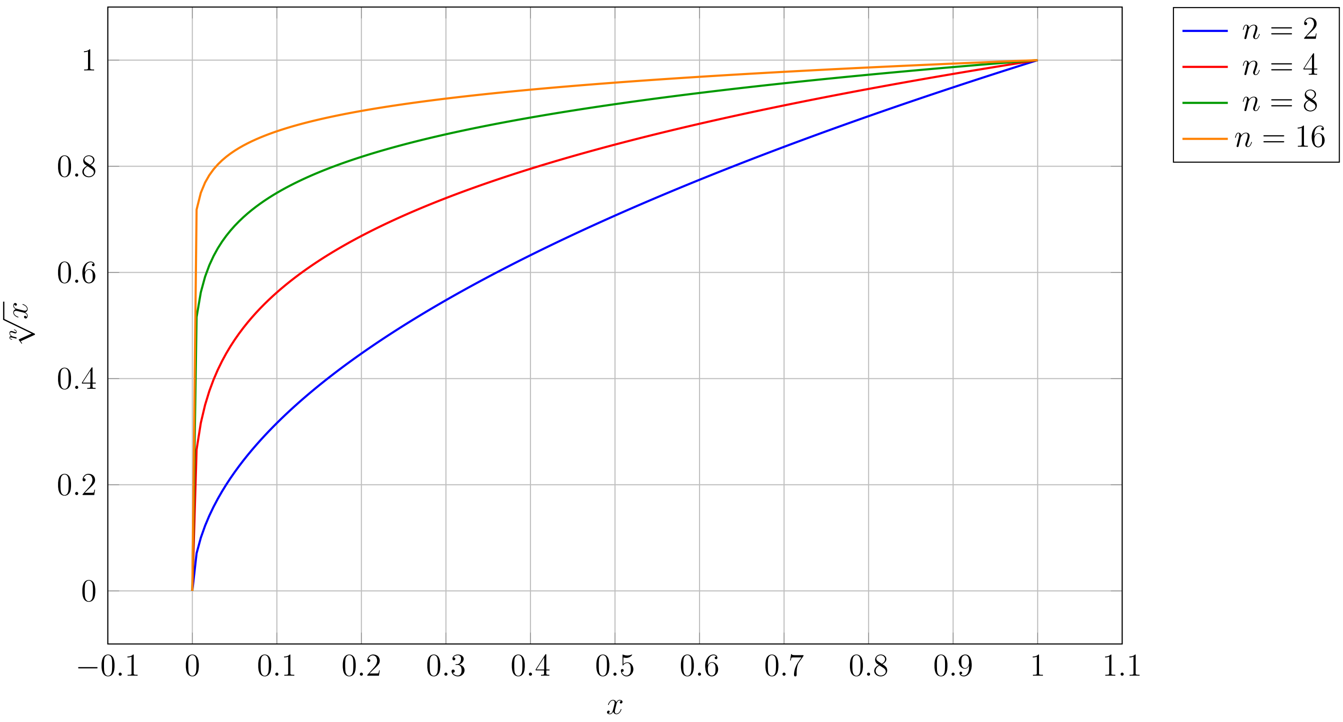
$$\lim_{n\to+\infty} f_n(x) = f(x)$$

for every x from X.

2.0.2 Example. We define the sequence of functions f_n on the interval $[0,1]$ to be the n-th root of x. We see that f_n converges pointwise to the function f:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Graph of $\sqrt[n]{x}$ for different n



2.0.3 Definition. Let $f\colon X \to \mathbb{R}$ a real valued function defined on X.
f is said Baire one if there exist a sequence of continuous function $\{f_n\}$ from X that converges pointwise to f.

2.0.4 Remark. Baire 0 functions are continuous functions.

Now we can define Baire n functions for every natural number n.

2.0.5 Definition. Let $f\colon X \to \mathbb{R}$ a real valued function defined on X
f is said Baire n if there exist a sequence of Baire $n-1$ function $\{f_n\}$ from X that converges pointwise to f.

2.0.6 Remark.
1. A continuous function f is a Baire 1 function. Indeed, by taking $f_n = f$ for every n we have of course $\lim\limits_{n\to+\infty} f_n(x) = f(x)$
2. By denoting sets of Baire n functions B_n, we have that $B_n \subset B_{n+1}$ for every natural number n
3. More generally, we have: $C \subset B_1 \subset B_2 \subset ... \subset B_n \subset ...$

2.0.7 Proposition. *Let f be a function that is discontinuous at a finite number of points and continuous everywhere else. Then f is Baire one function.*

Proof. Let $x_1 < x_2 < ... < x_n$ be points of discontinuity of the function f. f is continuous on $(-\infty,x_1 - 1/n]$.Then we set f_n to be f on $(-\infty,x_1 - 1/n]$. Then f_n will be the line between $f(x_1 - 1/n)$ and $f(x1)$. More formally:

$$f(x) = \begin{cases} f(x) & \text{if } x \in (-\infty,x_1 - 1/n], \\ n*(x - x_1 + 1/n)*(f(x_1) - f(x_1 - 1/n)) + +f(x - 1/n) & \text{if } x \in [x_1 - 1/n,x_1] \end{cases}$$

Same thing from the right side of $x1$ and same thing for the other points of discontinuity. □

2.0.8 Example. Let's define the function f as follow:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1 \text{and } x \ne 0.5 \\ 2 & \text{if } x = 0.5 \end{cases}$$

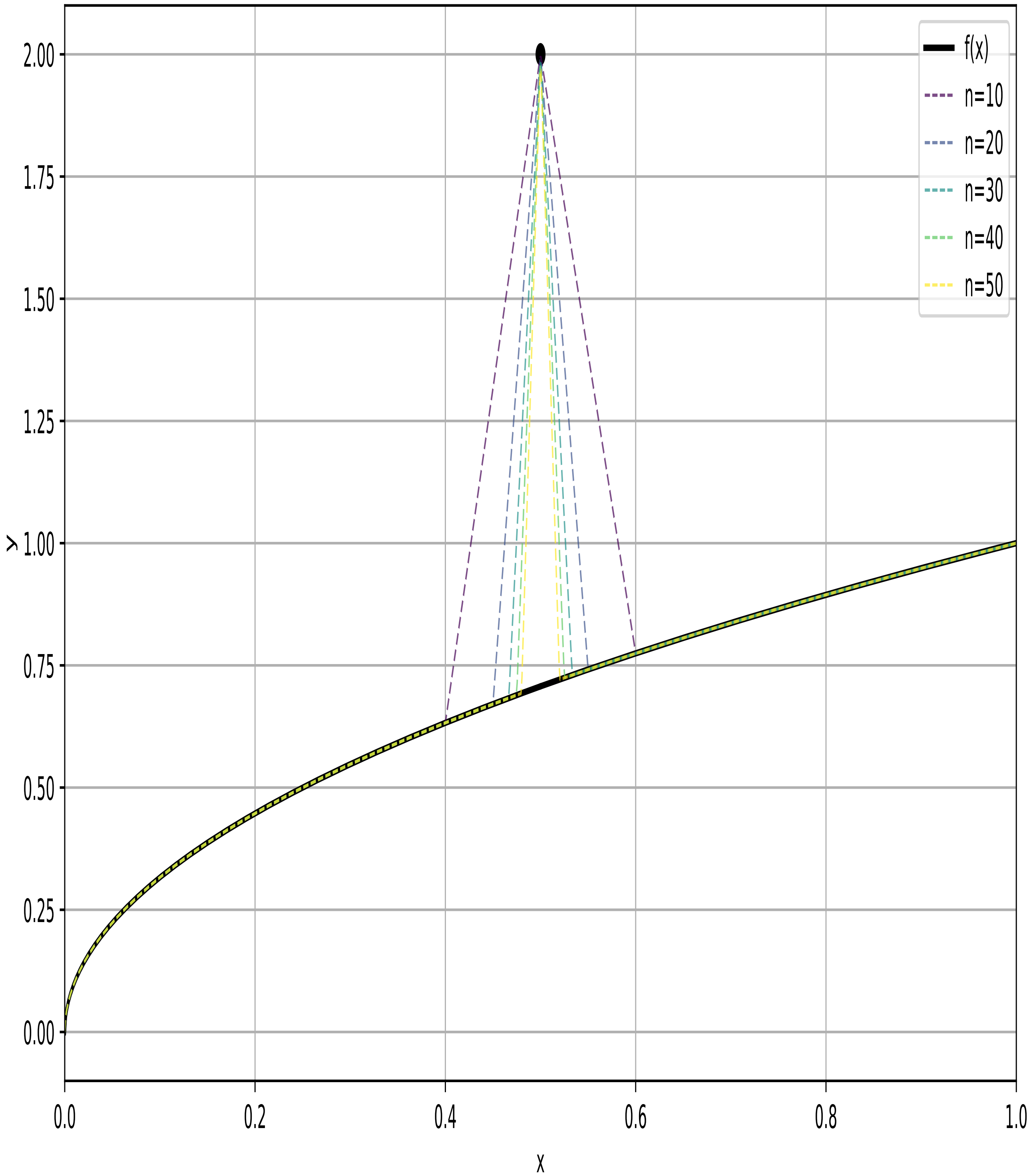
2.0.9 Remark. The previous example shows that there are Baire 1 functions that are not continuous functions. That is to say set of continuous functions B_0 is a proper subset of set of Baire 1 functions.

2.0.10 Proposition. *Let$f$ and g be Baire 1 functions. Then $f+g$ and fg are also Baire 1 functions.*

Proof. Let f and g be Baire 1 functions, then there exists sequence of continuous functions f_n (respectively g_n) that converges to f (respectively to g). Then $f_n + g_n$ is a sequence of continuous function that converges to $f+g$.
Same way, $f_n * g_n$ is a sequence of continuous functions that converges to $f*g$ □

2.0.11 Proposition. *Let$f$ and g be Baire n functions. Then $f+g$ and fg are also Baire n functions.*

Proof. This is true for Baire 1 functions according to previous proposition.
Suppose it is true for some $n \in \mathbb{N}$ and let f and g be Baire $n+1$ functions.
Then there exist f_n (respectively g_n) sequences of Baire n functions that converges to f and g then $(f_n + g_n)$ is Baire n function and converges to $f+g$.
Therefore $f+g$ is Baire $n+1$ function.
The same reasoning allows us to prove the second point. □



1 figure. *Function f and it's approximation by Baire class 1 functions*

2.0.12 Definition. Let $\{f_n\}$ be a sequence of functions defined on X and let f be a function defined on X.
We say that $\{f_n\}$ converges uniformly to f if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in X$. We note it:

$$f_n \xrightarrow[n\to+\infty]{\text{C.V.U.}} f$$

2.0.13 Example. Let's consider the function

$$f_n(x) = \frac{sin(nx)}{n}$$

.

f_n is pointwise convergent since for every fixed x,

$$|\frac{sin(nx)}{n}| \leq \frac{1}{n}$$

and $\frac{1}{n}$ is converging to zero.
Therefore, $f_n(x)$ converges pointwise to the zero function
We have that

$$\|f_n - f\|_\infty = \sup_{x\in\mathbb{R}} \left|\frac{\sin(nx)}{n} - 0\right| = \sup_{x\in\mathbb{R}} \frac{|\sin(nx)|}{n} \leq \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$$

Therefore f_n converges uniformly.

2.0.14 Proposition. *Let f_n a sequence of functions converging uniformly to f then f_n converges pointwise to f*

2.0.15 Remark. The converse is not true.
To see this, let's consider the sequence of functions $f_n = x^n$ defined on the interval $[0,1]$.
We see earlier that f_n converges pointwise to the function:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\|f_n - f\|_\infty = \sup_{x\in)0,1[} |x^n - 1| = \xrightarrow[n\to+\infty]{} 1$$

This proves that the sequence of functions f_n does not converge uniformly.

2.0.16 Proposition. *The uniform limit of a Baire class 1 functions is a Baire class 1 function.*

Proof. [**2**] Let $f_n \xrightarrow[n\to+\infty]{\text{C.V.U.}} f$, where each f_n function is a Baire class 1 function.
We can find a subsequence $g_k = f_{n_k}$ such that $|g_k - f| \leq \frac{1}{2^k}$.
We can rewrite $f(x) = g_1(x) + \sum_{n=2}^{\infty}(g_n - g_{n-1})(x)$
We have that $|g_n - g_{n-1}| = |g_n - f + f - g_{n-1}| \leq |g_n - f| + |g_{n-1} - f| = \frac{1}{2^n} + \frac{1}{2^{n-1}}$

Finally, $|g_n - g_{n-1}| \leq \frac{3}{2^n}$ for every k

We will prove that $\sum_{n=2}^{\infty}(g_n - g_{n-1})$ is Baire class 1 function, this will prove that f is Baire class 1 function.

Let ϕ_{n_k} a continuous function converging to $(g_n - g_{n-1})$ when $k \to \infty$.

We can assume that $|\phi_{n_k}| \leq \frac{3}{2^n}$. Otherwise we consider

$$\widetilde{\phi}_{n_k} = \phi_{n_k}\chi_{\{x:|\phi_{n_k}(x)| \leq \frac{3}{2^n}\}} + \frac{3}{2^n}\phi_{n_k}\chi_{\{x:\phi_{n_k}(x) > \frac{3}{2^n}\}} - \frac{3}{2^n}\phi_{n_k}\chi_{\{x:\phi_{n_k}(x) < \frac{3}{2^n}\}}$$

Then we define $h_k = \sum_{n=2}^{\infty}\phi_{n_k}$ which is uniformly convergent by the Weierstrass M-test.

Let's fix x and let's ϵ arbitrary.

We can find N_1 such that $\sum_{N_1}^{\infty}|g_n - g_{n-1}| \leq \epsilon$ (a convergent sequence is a Cauchy sequence)

We can find N_2 such that $\sum_{N_2}^{\infty}|\phi_{n_k}| \leq \epsilon$

Taking the maximum of N_1 and N_2, let's note it N, both conditions are satisfied.Then,

$$\begin{aligned}
&|h_k(x) - \sum_{2}^{\infty}(g_n - g_{n-1})(x)| \\
&\leq \sum_{n=N}^{\infty}(g_n - g_{n-1})(x) + \sum_{n=N}^{\infty}|\phi_{n_k}| \\
&+|\sum_{n=N}^{\infty}\phi_{n_k} - \sum_{n=2}^{N-1}(g_n - g_{n-1})(x)| \\
&\leq 2\epsilon \quad \text{when } k \to \infty
\end{aligned}$$

Finally, we obtain when $\epsilon \to 0$ that $\sum_{n=2}^{\infty}(g_n - g_{n-1})$ is Baire class 1 function, from where the result. $\square$

2.0.17 Definition. A subset $X \subseteq \mathbb{R}$ is called F_σ set if is a countable union of closed sets:

$$A = \bigcup_{n=1}^{\infty} C_n$$

, where C_n are closed sets.

2.0.18 Proposition. *Open sets are F_σ sets*

Proof. Let (a,b) an open set. We can write it:

$$\bigcup_{n=n_0}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$$

for n_0 sufficiently large.

Since every open sets is union of intervals, then every open set is F_σ set. $\square$

2.0.19 Definition. A subset $X \subseteq \mathbb{R}$ is called G_σ set if is a countable intersection of open sets:

$$A = \bigcap_{n=1}^{\infty} O_n$$

, where O_n are open sets.

2.0.20 Proposition. *Suppose that* $[a, b] = \bigcup_{k=1}^{n} B_k$ *with* B_k F_σ *sets and pairwise disjoint. Then:*

$$f(x) = \sum_{k=1}^{n} c_k * \chi_{B_k}(x)$$

for $x \in [a,b]$ *is a Baire* 1 *function.*

2.0.21 Example. The **Cantor set** C is the subset of the interval $[0,1]$ obtained by iteratively removing the open middle third from each remaining interval.
Formally, it can be defined as:

$$C = \bigcap_{n=1}^{\infty} C_n,$$

where:

- $C_0 = [0,1]$,
- $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$ for $n \geq 1$.

Equivalently, C consists of all real numbers in $[0,1]$ that can be written in base 3 without the digit 1:

$$C = \left\{ x \in [0,1] \;\middle|\; x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,2\} \right\}.$$

The **characteristic function** of the Cantor set C, denoted $\chi_C \colon [0,1] \to \{0,1\}$, is defined by:

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

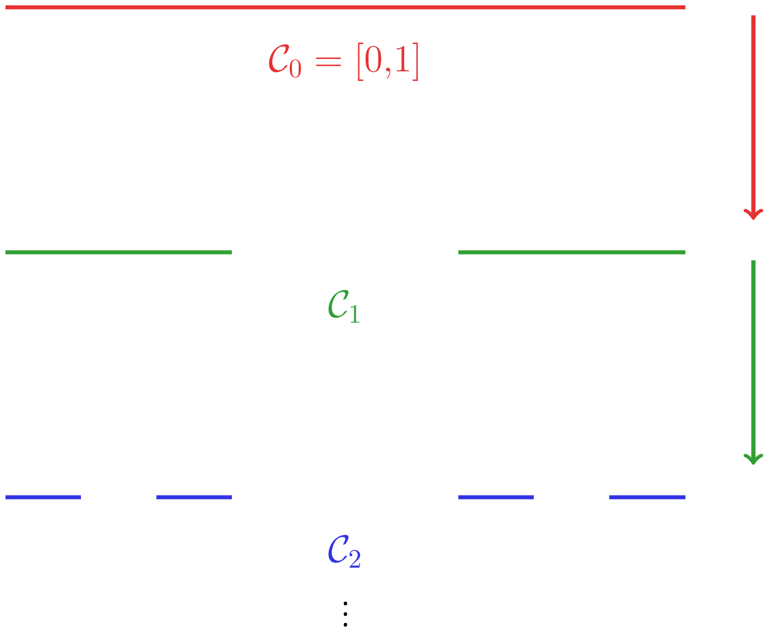
We can write $\chi_{C_n} = \bigcup_{k=1}^{p} c_k \chi_{B_k}$ where $p = (n * 2) + 1$, $c_k = 1$ for every k and

$$\chi_{B_k} = \begin{cases} 1 & \text{if k is odd }, \\ 0 & \text{otherwise.} \end{cases}$$

Using the last proposition we see that χ_{C_n} is Baire 1 function for every n. And we have that

$$\lim_{n \to +\infty} \chi_{C_n} = \chi_C$$

Therefore χ_C, the characteristic function of the Cantor set, is Baire 2 function.
We will see later that it is indeed Baire 1 function.



Iterative construction of Cantor set

2.0.22 Theorem (Characterization of Baire One Functions via F_σ Sets)**.** *A function $f : [a,b] \to \mathbb{R}$ is Baire one if and only if for every real number r, the following sets are F_σ:*

$$\{x \in [a,b] : f(x) < r\} \quad \text{and} \quad \{x \in [a,b] : f(x) > r\}.$$

Proof. [**1**] We divide the proof into two parts.
Part 1: Necessity (f is Baire one $\implies$ the sets are F_σ).

Let $\{f_n\}$ be a sequence of continuous functions converging pointwise to f on $[a,b]$. Fix $r \in \mathbb{R}$. We show that $\{x \in [a,b] : f(x) < r\}$ is F_σ. Observe that:

$$\{x \in [a,b] : f(x) < r\} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{x \in [a,b] : f_n(x) \le r - \frac{1}{k}\right\}.$$

Since each f_n is continuous, the set $\{x : f_n(x) \le r - \frac{1}{k}\}$ is closed. Thus, the countable union of intersections of closed sets is F_σ.
The proof for $\{x : f(x) > r\}$ is analogous by considering $-f$, which is also Baire one.
Part 2: Sufficiency (The sets are $F_\sigma \implies f$ is Baire one).

Assume f is bounded (the unbounded case reduces to the bounded case via a continuous transformation).
For each n, we partition $[a,b]$ into subintervals using points $y_k = -M + \frac{2Mk}{n}$ for $k = 0, \ldots, n$, where M bounds $|f|$.
Then we define:

$$A_k = \{x \in [a,b] : y_{k-1} < f(x) < y_{k+1}\}.$$

By hypothesis, A_k is F_σ. We express $[a,b] = \bigcup_{k=1}^{n-1} B_k$, where $B_k \subseteq A_k$ are pairwise disjoint F_σ sets.

Then we construct a sequence of Baire one functions:

$$f_n(x) = \sum_{k=1}^{n-1} y_k \cdot \chi_{B_k}(x).$$

Then $|f_n(x) - f(x)| < \frac{2M}{n}$ for all x, so $\{f_n\}$ converges uniformly to f.
By Theorem 2.0.16 (uniform limits of Baire one functions are Baire one), f is Baire one.

For the unbounded case, we compose f with a homeomorphism $h : \mathbb{R} \to (0,1)$.
Then $h \circ f$ is bounded and satisfies the F_σ condition, so it is Baire one.
Since $f = h^{-1} \circ (h \circ f)$ and h^{-1} is continuous, f is Baire one. □

2.0.23 Theorem. *The set of Baire class n functions has the cardinality of the continuum for every $n \in \mathbb{N}$*

Proof. We know that the set of continuous functions has the cardinality of continuum.
Every Baire class 1 function is pointwise limit of continuous functions. Thus every Baire class 1 function is determined by $\{f_1, f_2, f_3, ...\}$ where f_i are continuous functions.
Cardinality of $\{f_1, f_2, f_3, ...\}$ is at most $\mathbb{N} * \mathbb{R}$ which is the $\mathbb{R}$ (continuum). This shows that Baire class 1 function has the cardinality of continuum.
By processing the same, we prove that Baire class n function has the cardinality of continuum for every $n \in \mathbb{N}$ □

2.0.24 Definition. A set $A \subset \mathbb{R}$ is a meager set if it can be written as a countable union of nowhere dense sets of X:

$$A = \bigcup_{n=1}^{\infty} A_n$$

, where each A_n is nowhere dense set.

2.0.25 Example.

- The set of rational numbers $\mathbb{Q}$ is a meager set. Indeed we can write it as a countable union of singletons:

  $$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

  Each singleton $\{q\}$ is nowhere dense set in $\mathbb{R}$ (it's closure $\{q\}$ has empty interior).
  Therefore, $\mathbb{Q}$ is a countable union of nowhere dense sets, then it is a meager set.

- The Cantor set is a meager set. During construction, at each step n, the lenght of intervals that belongs to the Cantor set is devided by 3. At the end we will have that $int(C) = \emptyset$, meaning it contains no non-empty open intervals.
  One can write:

  $$C = \bigcup_{n=1}^{\infty} A_n$$

  where $A_1 = C$ and $A_n = \emptyset$ for $n \geq 2$. Therefore Cantor set is a meager set.

2.0.26 Theorem. *Let f: $X \to \mathbb{R}$ a real valued function defined on X. Then f is Baire class 1 function if and only in the sets of discontinuity of f is a meager set.*

3 Case studies

In this section, we will study three functions and determine to which class they belongs. We will do this, depending on cases, by using characterization theorems or directly constructs those functions as limits of some other functions.

3.0.1 Example. Let's consider the indicator function of the rational numbers $\chi_{\mathbb{Q}}$:

$$\chi_{\mathbb{Q}} = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We will prove that it is a Baire 2 function.
We will construct this function as a double limit. Let's consider the sequence of functions $f_{m,n}$ defined as follow:

$$f_{m,n}(x) = (cos(m!\pi x))^{2n}$$

Let's define for every $m \in \mathbb{N}$:

$$g_m(x) = \lim_{n\to\infty} (cos(m!\pi x))^{2n}$$

This limit exist because:

- If x is rational ($x = \frac{p}{q}$ with p and q coprime) then for $m > \frac{q}{p}$, $cos(m!\pi x) = \pm 1$. Therefore $(cos(m!\pi x))^{2n} = 1$ for every $n \in \mathbb{N}$. This gives us $g_m(x) = 1$
- If x is irrational then $m!x$ is not an integer then $|cos(m!\pi x)| < 1$. Therefore, $(cos(m!\pi x))^{2n} = 0$. This gives us $g_m(x) = 0$

Functions $f_{m,n}$ are continuous functions and g_m are pointwise limit of those functions. This shows that g_m are Baire class 1 functions.
Let's consider the function

$$h(x) = \lim_{m\to\infty} g_m(x)$$

- If x is rational and for sufficiently large m, $h(x) = 1$
- If x is irrational $h(x) = 1$

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

We expressed the Dirichlet function as a pointwise limit of Baire class 1 functions, so it is a Baire class 2 function. In order to prove that it is not Baire class 1 we will show that $g_m(x) = \lim_{n\to\infty}(cos(m!\pi x))^{2n}$ is not continuous.
We have that $g_m(x) = 1$ if x is a multiple of $\frac{1}{m!}$ and $g_m(x) = 0$ otherwise.
Let's take $x_0 = \frac{1}{m!}$ then $g_m(x_0) = 1$. In any neighborhood of x_0 there exist points not in $(\frac{1}{m!}\mathbb{Z})$. For example, $x = x_0 + \sigma$ for sufficiently small σ not making x a multiple of $\frac{1}{m!}$.
In this case $g_m(x) = 0$, so $\lim_{x\to x_0} g_m(x)$ is not equal $g_m(x_0)$.

Thus g_m is discontinuous at every point in $(\frac{1}{m!}\mathbb{Z})$.
If x_0 is not in $(\frac{1}{m!}\mathbb{Z})$, $g_m(x) = 0$. Since $(\frac{1}{m!}\mathbb{Z})$ is discrete, there is a neighborhood $(x_0 - \sigma, x_0 + \sigma)$ where $g_m(y) = 0$ for all y in this interval. Thus g_m is continuous.
Of course, to be continuous, g_m must be continuous at every points. Thus g_m is not continuous
This proves that Dirichlet function can not be Baire class 1 function.

3.0.2 Example. Let's consider the function g defined as follow:

$$g(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, (p,q) = 1 \\ 0 & , x \notin \mathbb{Q}. \end{cases}$$

We will first show that this function is discontinuous at all rationals.
Let take $x_0 \in \mathbb{Q}$, $x_0 = \frac{p}{q}$ with p and q coprime. In this case according to the definition, we have: $g(x_0) = \frac{1}{q}$.
Let's take $\epsilon > \frac{1}{q}$. For every $\sigma > 0$, the interval $(x_0 - \sigma, x_0 + \sigma)$ contains some irrational number y because irrationals are dense in $\mathbb{R}$. Therefore, $g(y) = 0$. So we obtain:

$$|g(x_0) - g(y)| = |\frac{1}{q}| = \frac{1}{q} > \epsilon.$$

This proves that the function g is discontinuous at x_0.
Now, we will prove that the function g is continuous on irrationals.
Let's take x_0 irrational and let's take $\epsilon > 0$. We take N strictly positive integer such that $N > \frac{1}{\epsilon}$
The set of rationals $\frac{p}{q}$ such that $q \leq N$ in the interval $[x_0 - 1, x_0 + 1]$ is finite. Let's denote it F, $F = \{r_1, r_2, ..., r_m\}$
Now we have to choose σ such that:

- $\sigma \leq 1$ in order to have $(x_0 - \sigma, x_0 + \sigma) \subset [x_0 - 1, x_0 + 1]$.
- $(x_0 - \sigma, x_0 + \sigma) \cap F = \emptyset$.

Now for every $x \in (x_0 - \sigma, x_0 + \sigma)$:

- If x is irrational then : $|g(x) - g(x_0)| = 0 < \epsilon$
- If x is rational, $x = \frac{p}{q}$, so $q > N$ because $x \notin F$. Finally, we get $|g(x) - g(x_0)| = |\frac{1}{q}| = \frac{1}{q} < \frac{1}{N} < \epsilon$

The set of continuity of this function is $\mathbb{Q}$ which is a countable set.
We can write $\mathbb{Q} = \bigcap_{n=1} \infty\{q_n\}$. Singleton is of course nowhere dense set.
The set of discontinuity is a meager set (countable union of nowhere dense sets), thus the function g is Baire class 1 function.
Another way to prove this is to construct a sequence of continuous functions g_n converging to g.
Let's define function $h(x) = max(|p|, q)$ for every rational $x = \frac{p}{q}$ and let define S_n sets as follow:

$$S_n = \{x \in \mathbb{Q}, h(x) \leq n\}$$

According definition $|p| \leq n$ and $q \leq n$ for every rational. We have also that couples (p,q) with p and q coprime are finite for every fixed n. Thus S_n is finite for every $n \in \mathbb{N}$.
We have to choose σ_n such that :

- $\sigma_n < \frac{1}{n^2}$
- $\sigma_n < \frac{1}{2} min\{(x - x')x,x' \in S_n, x \neq x'\}$

Thus intervals $x - \sigma_n, x + \sigma_n$ are disjoints. We define functions $f_n : \mathbb{R} \to \mathbb{R}$:

- If $x \notin \bigcup_{x \in S_n} [x - \sigma_n, x + \sigma_n]$ then $g_n(x) = 0$
- If $x \in [x - \sigma_n, x + \sigma_n]$ for some $x \in S_n$, then:
  - $g_n(x) = g(x) = \frac{1}{q}$.
  - $g_n(x - \sigma_n) = g_n(x + \sigma_n) = 0$
  - g_n is linear on interval $[x - \sigma_n, x]$ from 0 to $g(x)$.
  - g_n is linear on interval $[x,x + \sigma_n]$ from $g(x)$ to 0.

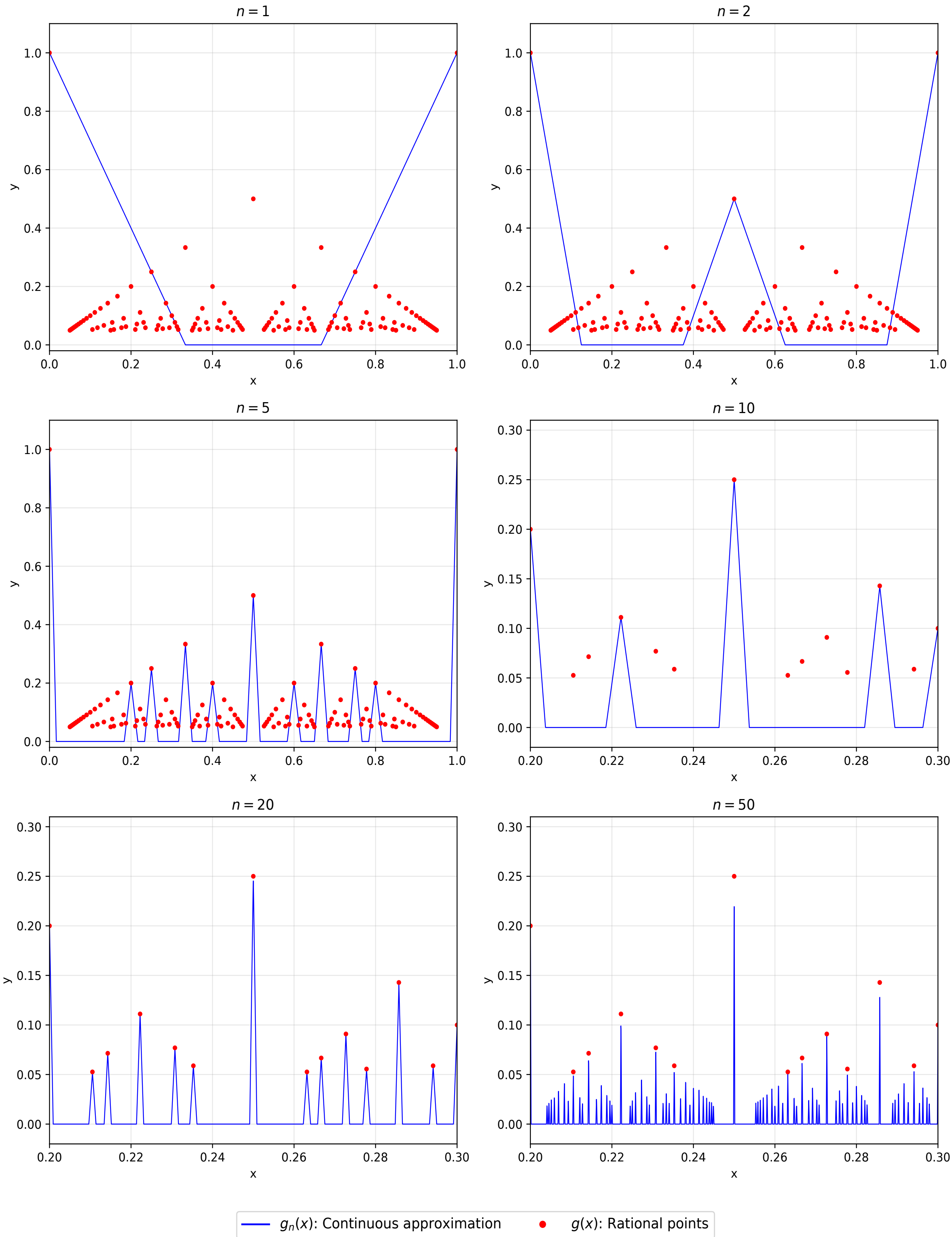
As defined, g_n is continuous on intervals $[x - \sigma_n, x + \sigma_n]$ and constant elsewhere. Intervals are disjoints. Thus g_n is continuous on $\mathbb{R}$. Now we will show that $g_n \to g$ for every $x \in \mathbb{R}$.

- $x = \frac{p}{q}$ in irreducible form. Then $g(x) = \frac{1}{q}$. For every $n \geq h(x)$, $x \in S_n$.
  Then by construction $g_n(x) = g(x)$. Thus for every $n \geq h(x)$, $|g_n(x) - g(x)| = 0$. Thus $lim_{n\to\infty} g_n(x) = g(x)$.
- When x is irrational or $x = 0$:
  For $x = 0$, $g(0) = 1$, then $h(0) = 1$. For $n \geq 1$, $0 \in S_n$ so $g_n(0) = g(0) = 1$. Thus $\lim n \to \infty g_n(0) = g(0) = 1$.
  Now for x irrational $g(x) = 0$. Let $\epsilon > 0$. We must find some $N \in \mathbb{N}$ such that for every $n \geq N$, $|g_n(x)| < \epsilon$.
  Let $K = [x - 1, x + 1]$ and let A_ϵ be the set of rationals x such that $g(x) \geq \epsilon$, otherwise, such that $q \leq \frac{1}{\epsilon}$. For every $q \leq \frac{1}{\epsilon}$ there is finite number of p such that $x \in \mathbb{Q}$. Thus $A_\epsilon \bigcap K$ is finite.
  Let $B_\epsilon = A_\epsilon \bigcap K$. B_ϵ is finite and x is irrational, then $x \notin B_\epsilon$. Thus distance $\sigma_\epsilon = min_{x \in S_n} |y - x|$ is strictly positive.
  Let $M = max\{h(x), x \in B_\epsilon\}$. We choose N such that for every $n \geq N$:
  - $\sigma_n < \frac{\sigma_\epsilon}{2}$
  - $n \geq M$

  For $n \geq N$, if $g_n(x) \neq 0$ then $x \in [x' - \sigma_n, x' + \sigma_n]$, $x' \in S_n$. Thus $|x - x'| < \sigma_n < \frac{\sigma_\epsilon}{2}$.
  As $|x - x'| < \sigma_n < 1$, we have that $x' \in K$. Furthermore $n \geq M$ so $x' \in B_\epsilon$, thus $|x - x'| < \sigma_n < \frac{\sigma_\epsilon}{2}$ but $\sigma_\epsilon \leq |x - x'|$. This is the contradiction.
  So, $x' \notin B_\epsilon$. Because $x' \in K$ we have that $x' \notin A_\epsilon$, thus $g(x') < \epsilon$. Then, by construction, $|g_n(x)| \leq g(x') < \epsilon$

If $g_n(x) = 0$, $|g_n(x)| < \epsilon$ also.
We proved that, for $n \geq N$, $|g_n(x)| < \epsilon$. Thus $\lim_{n\to\infty} g_n(x) = g(x)$.

The figure below illustrates the function g approximation by g_n continuous functions.



2 figure. *Approximation of function g by continuous functions*

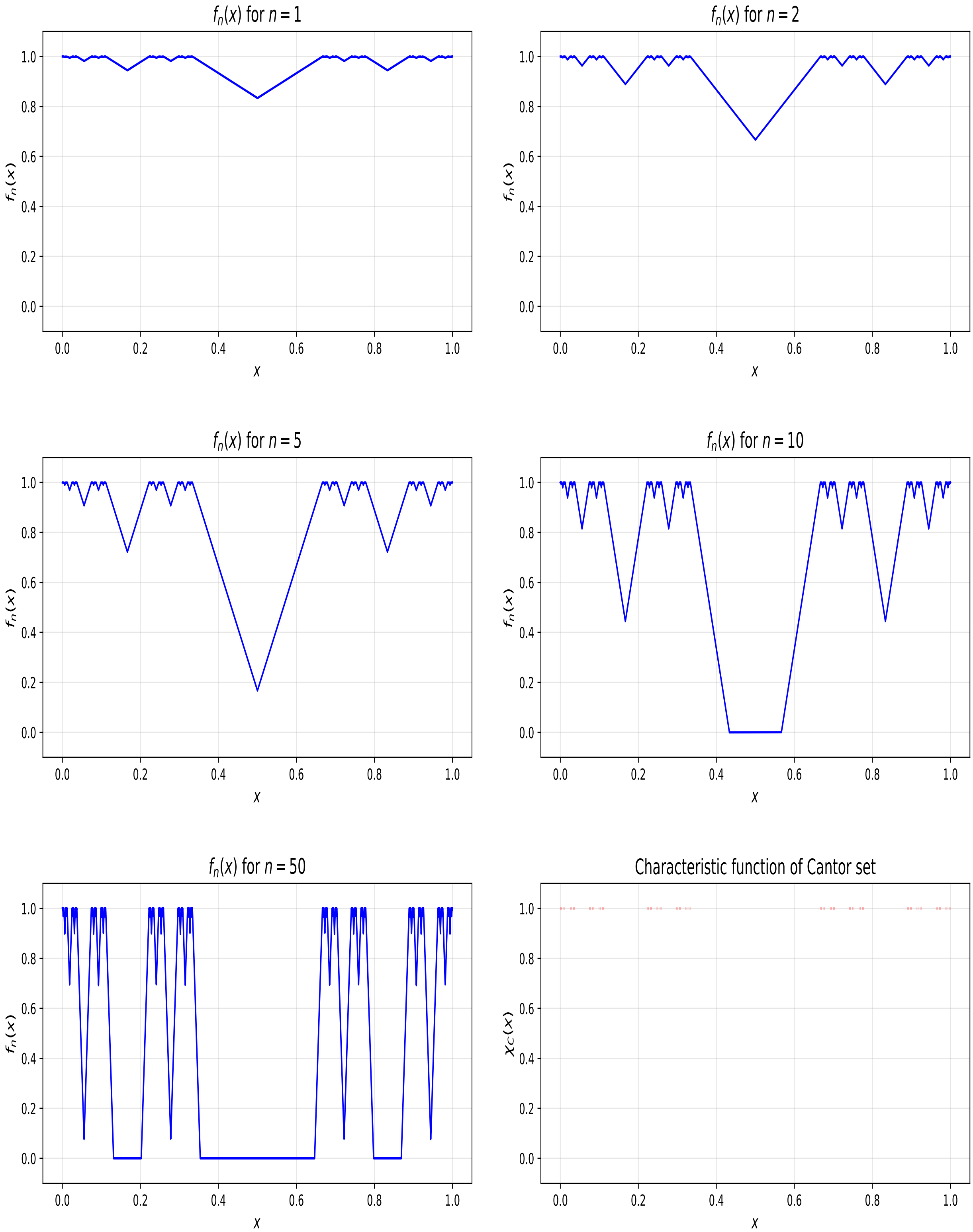
3.0.3 Example. Let's consider the characteristic function of Cantor set.
We have already proved that it is a Baire 2 function.
We will prove that it is also a Baire 1 function, first by using the the previous theorem of characterization then by constructing a sequence of continuous function that converges to it.
Let's consider the sequence of functions:

$$f_n = max(1 - n * d(x,C), 0)$$

where $d(x,C) = \inf_{y \in C} |x - y|$
Functions f_n are continuous for every $n \in \mathbb{N}$ (the distance is a continuous function and $max(f,g)$ where f and g are continuous is continuous)
We will prove that f_n converges to the characteristic function of the Cantor set.
Let's consider two cases:

- $x \in C$: $d(x,C) = 0$ then $f_n(x) = 1$.
- $x \notin C$: We have that $d(x,C) > 0$ because C is closed.
  We have also that $n * d(x, C) \geq 1$ for $n \geq \frac{1}{d(x,C)}$.
  This gives us $\lim_{y \to x} f_n(y) = 0$.

Therefore the sequence of functions f_n converges pointwise to χ_C.
This shows that the characteristic function of Cantor set χ_C is Baire 1 function.
The figure 3 below illustrates this convergence.



3 figure. *Characteristic Function of Cantor Set and its Approximation by Baire Class One Functions*

Let's prove the same fact using the previous theorem. We need to prove that sets

$$\{x \in [a,b] : \chi_C(x) < r\} \quad \text{and} \quad \{x \in [a,b] : \chi_C(x) > r\}.$$

are F_σ sets.
Let's distinguish three cases:

1. $r \leq 0$:

   - $\{x \in [a,b] : \chi_C(x) < r\} = \emptyset$
     The empty set is closed. Therefore it is a F_σ set.

   - $\{x \in [a,b] : \chi_C(x) > r\} = [0,1]$. This set is closed. Therefore it is a F_σ

2. $0 < r \leq 1$:

   - $\{x \in [a,b] : \chi_C(x) < r\} = \overline{C}$.
     The Cantor set is closed. Then it's complement is open and open sets are F_σ set.
   - $\{x \in [a,b] : \chi_C(x) > r\} = C$ if $r < 1$.
     $\{x \in [a,b] : \chi_C(x) > r\} = \emptyset$ if $r = 1$.
     The empty set and the Cantor set are F_σ sets.

3. $r > 1$:

   - $\{x \in [a,b] : \chi_C(x) < r\} = [0,1]$ (F_σ set).
   - $\{x \in [a,b] : \chi_C(x) > r\} = \emptyset$ (F_σ set)

   Therefore using the previous theorem the characteristic function of the Cantor set is a Baire class 1 function.

3.0.4 Example. Let's consider another example of Baire 2 class function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^m} \text{ for some integer } k \in \mathbb{Z}, m \in \mathbb{N}, \\ 0 & \text{else} \end{cases}$$

The dyadic rationals are the set $D = \left\{\frac{k}{2^m} : k, m \in \mathbb{Z}\right\}$
(where $f = 1$) . So, this function is the characteristic function of the dyadic rationals set. We will construct the function f via double limit in order to show that it is a Baire class 2 function.
Let's define for $m \in \mathbb{N}$, $D_m = \left\{\frac{k}{2^m} : k \in \mathbb{Z}\right\}$
D_m is discrete and periodic of period $\frac{1}{2^m}$
Now, let's define the continuous function $h_{m,n} : \mathbb{R} \to \mathbb{R}$ for each $m, n \in \mathbb{N}$ as follow:

$$h_{m,n}(x) = \max_{d \in D_m} \{1 - n \cdot |x - d|, 0\}$$

For fixed $m \in \mathbb{N}$, the minimum distance between two points in D_m is $\frac{1}{2^m}$, so this function is continuous.
When $n > 2^m$ the supports $\{x : |x - d| < 1/n\}$ for $d \in D_m$ are disjoints. Thus $h_{m,n}$ is piecewise linear.
Now, we fix m and consider $\lim_{n\to\infty} h_{m,n}(x)$:

- If $x \in D_m$, then $h_{m,n}(x) = 1$ for every $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} h_{m,n}(x) = 1$
- If $x \notin D_m$, then by taking $n > \frac{1}{d(x,D_m)}$, we get $h_{m,n}(x) = 0$. Thus $\lim_{n\to\infty} h_{m,n}(x) = 0$

We proved that $\lim_{n\to\infty} h_{m,n}(x) = \mathbf{1}_{D_m}(x)$.
Now let's consider $\lim_{m\to\infty} \mathbf{1}_{D_m}(x)$.

- If $x \in D$, $x = \frac{k}{2^{m_0}}$ for some $m_0 \in \mathbb{N}$, then for every $m > m_0$, $x \in D_m$ because $\frac{k}{2^{m_0}} = \frac{k \cdot 2^{m-m_0}}{2^m} \in D_m$. Thus $\mathbf{1}_{D_m}(x) = 1$.
- If $x \notin D$, then $x \notin D_m$ for every $m \in \mathbb{N}$. Thus $\mathbf{1}_{D_m}(x) = 0$

We proved that:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} h_{m,n}(x)\right) = \lim_{m\to\infty} \mathbf{1}_{D_m}(x) = \mathbf{1}_D(x) = f(x).$$

So, we expressed f as pointwise limit of the function $\mathbf{1}_{D_m}$ and $\mathbf{1}_{D_m}$ is pointwise limit of functions $h_{m,n}$, which are continuous functions. Thus function f is a Baire class 2 function.
Now, we have to prove that it is not a Baire class 1 function. We will study the continuity of f to determine the set of it's discontinuity .

- If $x \in D$, then $f(x) = 1$. However every neighborhood of x contains:
  - Irrational, so where $f = 0$.
  - Rationals that are not in D , then $f = 0$. We proved that f is discontinuous at every point $x \in D$
- If $x \notin D$, then $f(x) = 0$. However every neighborhood of x contains elements from D (since D is dense in $\mathbb{R}$), where $f = 1$.
  Thus $\limsup_{y\to x} f(y) = 1$. We have here $f(x) = 0$. Thus $\limsup_{y\to x} f(y) \neq f(x)$.
  We proved that f is discontinuous at every point $x \notin D$.

Thus, the functions f is discontinuous everywhere in $\mathbb{R}$.
The set is discontinuity of the function f is the empty set, which is not,of course, a meager set. According to theorem 2.0.26 the function f is not a Baire class 1 function.

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