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The Approximation of Analytic Functions Using Shifts of the Lerch Zeta-Function in Short Intervals

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Abstract: In this paper, we obtain approximation theorems of classes of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$ of the Lerch zeta-function for $\tau \in [T, T + H]$ where $H \in [T^{27/82}, T^{1/2}]$. The cases of all parameters, $\lambda, \alpha \in (0, 1]$, are considered. If the set $\{\log(m + \alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$, then every analytic function in the strip $\{s = \sigma + it \in \mathbb{C} : \sigma \in (1/2, 1)\}$ is approximated by the above shifts.

Keywords: Hurwitz zeta-function; Lerch zeta-function; Mergelyan theorem; short intervals; universality; weak convergence of probability measures

MSC: 11M35; 60B10

1. Introduction

Let $s = \sigma + it$ be a complex variable and $\lambda, \alpha \in (0, 1]$ fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined in the half-plane $\sigma > 1$ by the series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1 for $\lambda = 1$. Notice that $\lambda \in \mathbb{R}$ can be arbitrary, but, by the virtue of the periodicity of $e^{2\pi i \lambda m}$, it suffices to consider only the case $\lambda \in (0, 1]$.

The function $L(\lambda, \alpha, s)$ was introduced independently by M. Lerch [1] and R. Lipschitz [2]. Clearly, for $\lambda = 1$, $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1,$$

and

$$L(1, 1, s) = \zeta(s)$$

is the Riemann zeta-function. Moreover, the equalities

$$L\left(\frac{1}{2}, 1, s\right) = \zeta(s)\left(1 - 2^{1-s}\right)$$

and

$$L\left(1, \frac{1}{2}, s\right) = \zeta(s)(2^s - 1)$$

are valid. These remarks show that the Lerch zeta-function is a generalization of the classical zeta-functions $\zeta(s, \alpha)$ and $\zeta(s)$.

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The function $L(\lambda, \alpha, s)$, as other zeta-functions, satisfies the functional equation. We denote by $\widetilde{\lambda}$ the fractional part of λ. Then, for all $s \in \mathbb{C}$,

$$L(\lambda, \alpha, 1-s) = (2\pi)^{-s}\Gamma(s) \exp\Big\{2\pi i\Big(\frac{s}{4} - \alpha\lambda\Big)\Big\}L(-\alpha, \lambda, s) + \exp\Big\{2\pi i\Big(-\frac{s}{4} + \alpha(1-\widetilde{\lambda})\Big)\Big\}L(\alpha, 1-\widetilde{\lambda}, s).$$

Here $\Gamma(s)$ denotes the Gamma-function, and $\exp\{a\} = e^a$.

The above equation was proven by Lerch in [1]. Other proofs of this equation are given by T.M. Apostol [3], F. Oberhettinger [4], M. Mikolás [5], and B.C. Berndt [6].

A generalization of $L(\lambda, \alpha, s)$,

$$\Phi(z, \alpha, s) = \sum_{m=0}^{\infty} \frac{z^m}{(m+\alpha)^s}$$

with complex $|z| \leqslant 1$ and $\alpha \neq -m$, $m \in \mathbb{N}_0$, was introduced and investigated in [7]. In [8], the function $\Phi(z, \alpha, s)$ was studied as a function of the complex variables z, α, and s. This was continued by J. Lagarias and W-C.W. Li in a series of works [9–12].

Dependence on two parameters ensures a certain advantage for the function $L(\lambda, \alpha, s)$ compared with other similar functions defined by Dirichlet series. The arithmetic of the parameters λ and α has a significant influence for analytic properties, and $L(\lambda, \alpha, s)$ is considered as an interesting function useful in various branches of mathematics. Therefore, the Lerch zeta-function is widely studied by many mathematicians. Numerous papers are devoted to the problem of the approximation of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$ with $\tau \in \mathbb{R}$. Recall that the latter approximation property for the Riemann zeta-function $\zeta(s) = L(1, 1, s)$ was discovered by S.M. Voronin [13], was successfully extended for other zeta-functions, and has found applications in some natural sciences; see the informative survey paper [14].

The first result on the approximation of analytic functions by shifts of the function $L(\lambda, \alpha, s)$ is given in [15]. Let $\mathfrak{D} = \{s \in \mathbb{C} : \sigma \in (1/2, 1)\}$. Suppose that the parameter α is transcendental, K is a compact subset of the strip $\mathfrak{D}$ with a connected complement, and $f(s)$ is a continuous function on K which is analytic inside of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{m}\Big\{\tau \in [0, T] : \sup_{s\in K} |f(s) - L(\lambda, \alpha, s + i\tau)| < \varepsilon\Big\} > 0. \quad (1)$$

Here and below, $\mathfrak{m}\{\bullet\}$ stands for a Lebesgue measure on the real line.

The proof of the latter theorem is based on a probabilistic limit theorem on weakly convergent probability measures in the space of analytic functions. Such a method was proposed in [16]. Inequality (1) shows that there are infinitely many shifts $L(\lambda, \alpha, s + i\tau)$ approximating a given function $f(s)$. Since $f(s)$ is an arbitrary continuous function on K and analytic inside of K, we have that the whole class of analytic functions is approximated by shifts in one and the same function $L(\lambda, \alpha, s)$. In this sense, the function $L(\lambda, \alpha, s)$ with transcendental α is universal.

It is not difficult to see that the transcendence of α in (1) can be replaced by the linear independence over the field of rational numbers $\mathbb{Q}$ for the set $\mathfrak{L}(\alpha) \stackrel{\text{def}}{=} \{\log(m+\alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$.

In general, a crucial role for universality theorems of the Lerch zeta-functions is played by the parameter α and, more precisely, by the arithmetic nature of α. An universality theorem for $L(\lambda, \alpha, s)$ is also known with the rational parameter α. We denote by (c, d) the greatest common divisor of $c, d \in \mathbb{N}$. Then in [17], the following result is contained. Let

$\alpha = a/b$, $a < b$, $(a,b) = 1$, $a,b \in \mathbb{N}$, $\alpha \neq 1/2$, $\lambda = r/q$, $r < q$, $(r,q) = 1$, $r,q \in \mathbb{N}$, and $(bl + a, bq) = 1$ for all $l = 0,1,\dots q-1$. Let K and $f(s)$ be as in (1). Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{m}\left\{\tau \in [0,T] : \sup_{s\in K}\left|f(s) - L\left(\frac{r}{q}, \frac{a}{b}, s+i\tau\right)\right| < \varepsilon\right\} > 0.$$

The latter result follows from a more general theorem for the periodic Hurwitz zeta-function

$$\zeta(s;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \quad \sigma > 1,$$

where $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers with the minimal period q. The universality of $L(\lambda,\alpha,s)$ with algebraic irrational α is the most complicated case and remains an open problem. This problem for the Hurwitz zeta-function $L(1,\alpha,s)$ with a certain effectively described finite set of α for disks was solved in [18].

In [19], a certain approximation to universality of the function $L(\lambda,\alpha,s)$ with arbitrary λ,α indicating good approximation properties of shifts $L(\lambda,\alpha,s+i\tau)$ was proposed. We recall the result of [19].

Suppose that the parameters $\lambda \in (0,1)$ and $\alpha \in (0,1)$ are arbitrary numbers, and $\mathcal{H}(\mathfrak{D})$ is the space of analytic functions on $\mathfrak{D}$ equipped with the topology of uniform convergence on compacta. Then there is a closed non-empty set, $\mathfrak{F}_{\lambda,\alpha} \subset \mathcal{H}(\mathfrak{D})$, such that, for every compact set $K \subset D$, $f(s) \in \mathfrak{F}_{\lambda,\alpha}$, and $\varepsilon > 0$, Inequality (1) is valid. Moreover, the lower density in (1) can be replaced by the density, i.e., the limit

$$\lim_{T\to\infty} \frac{1}{T}\mathfrak{m}\left\{\tau \in [0,T] : \sup_{s\in K}|f(s) - L(\lambda,\alpha,s+i\tau)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The above-mentioned universality theorems and other results are useful; however, they are not effective in the sense that any concrete approximating shift, $L(\lambda,\alpha,s+i\tau)$, is not known. The cited results deal with a density of approximating shifts in intervals of the length T as $T \to \infty$. More informative approximation theorems are related to the density of approximating shifts in narrow intervals. This observation leads to universality theorems for zeta-functions in so-called short intervals, i.e., intervals of the length $o(T)$ as $T \to \infty$. For the Riemann zeta-function, this was performed in [20] and improved in [21]. The purpose of this paper is to prove universality of the function $L(\lambda,\alpha,s)$ in short intervals. We denote by $\mathfrak{K}$ the set of compact subsets of the strip D with connected complements and by $H(K)$ with $K \in \mathfrak{K}$ the set of continuous functions on K that are analytic inside of K. We will prove the following theorems.

Theorem 1. *Suppose that the set $\mathfrak{L}(\alpha)$ is linearly independent over $\mathbb{Q}$, $\lambda \in (0,1]$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Let $K \in \mathfrak{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H}\mathfrak{m}\left\{\tau \in [T,T+H] : \sup_{s\in K}|f(s) - L(\lambda,\alpha,s+i\tau)| < \varepsilon\right\} > 0. \tag{2}$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{H}\mathfrak{m}\left\{\tau \in [T,T+H] : \sup_{s\in K}|f(s) - L(\lambda,\alpha,s+i\tau)| < \varepsilon\right\} \tag{3}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Unfortunately, the used probabilistic method does not allow us to indicate some concrete values of ε for which the limit (3) does not exists or exists but is equal to zero.

The next theorem shows good approximation properties of $L(\lambda, \alpha, s)$ with the arbitrary parameters λ and α.

Theorem 2. *Suppose that the parameters $\lambda, \alpha \in (0, 1]$ are arbitrary, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then there is a closed non-empty set, $\mathfrak{F}_{\lambda,\alpha} \subset \mathcal{H}(\mathfrak{D})$, such that, for every compact set $K \subset D$, $f(s) \in \mathfrak{F}_{\lambda,\alpha}$ and $\varepsilon > 0$, Inequality (2) holds. Moreover, the limit in (3) exists and is positive for all but at most countably many $\varepsilon > 0$.*

The proofs of Theorems 1 and 2 are closely connected to the mean square

$$\int_{T-H}^{T+H} |L(\lambda, \alpha, \sigma + it)|^2 \, \mathrm{d}t$$

for $\sigma > 1/2$.

2. Probabilistic Results

For a topological space, $\mathbb{X}$, we denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of $\mathbb{X}$. We will consider the weak convergence of probability measures on $(\mathcal{H}(\mathfrak{D}), \mathcal{B}(\mathcal{H}(\mathfrak{D})))$. Recall that if

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \, \mathrm{d}P_n = \int_{\mathbb{X}} g \, \mathrm{d}P$$

for every continuous bounded real function g on $\mathbb{X}$, then we say that P_n converges weakly to P as $n \to \infty$; P and P_n, $n \in \mathbb{N}$, are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. The theory of the weak convergence of probability measures is given in the monograph [22].

For $A \in \mathcal{B}(\mathcal{H}(\mathfrak{D}))$, we define

$$P_{T,H}^{\lambda,\alpha}(A) = \frac{1}{H} \mathfrak{m}\{\tau \in [T, T+H] : L(\lambda, \alpha, s + i\tau) \in A\}.$$

We will consider the weak convergence of $P_{T,H}^{\lambda,\alpha}$ as $T \to \infty$ with $H \to \infty$ and $H = o(T)$.

We start investigations of $P_{T,H}^{\lambda,\alpha}$ with the weak convergence of probability measures on a certain group. We put

$$\Omega = \prod_{m \in \mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

Thus, the set Ω consists of all functions $\omega : \mathbb{N}_0 \to \{s \in \mathbb{C} : |s| = 1\}$. On Ω, the operation of pairwise multiplication and the product topology can be defined, and Ω becomes an Abelian topological group. Moreover, according to the well-known Tikhonov theorem, this group is compact. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the invariant Haar measure μ exists.

For $A \in \mathcal{B}(\Omega)$, we set

$$P_{T,H,\alpha}^{\Omega}(A) = \frac{1}{H} \mathfrak{m}\Big\{\tau \in [T, T+H] : \Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big) \in A\Big\}.$$

Lemma 1. *Suppose that $H \to \infty$ as $T \to \infty$. Then, on $(\Omega, \mathcal{B}(\Omega))$, there exists a probability measure, P_α^Ω, such that $P_{T,H,\alpha}^{\Omega}$ converges weakly to P_α^Ω as $T \to \infty$.*

Proof. We use the Fourier transform of $P^{\Omega}_{T,H,\alpha}$. Since Ω is a compact Abelian group, the Fourier transform of $P^{\Omega}_{T,H,\alpha}$ can be defined on the dual group Ω^* (the group of characters) of Ω [23]. It is well known that the group Ω^* is isomorphic to the group

$$\mathcal{G} \overset{\text{def}}{=} \bigoplus_{m\in\mathbb{N}_0} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}$. An element, $\underline{k} = (k_m : m \in \mathbb{N}) \in \mathcal{G}$, where only a finite number of $k_m \in \mathbb{Z}$ are distinct from zero, acts on Ω^* by the formula

$$\omega \mapsto \omega^{\underline{k}} = \prod_{m\in\mathbb{N}_0} \omega^{k_m}(m),$$

where $\omega = (\omega(m) : m \in \mathbb{N}_0)$ are elements of Ω. Therefore, the characters of the group Ω are of the form

$$\prod^*_{m\in\mathbb{N}_0} \omega^{k_m}(m),$$

where the star indicates that only a finite number of integers, k_m, are not zero. Hence, the Fourier transform $\mathcal{F}_{T,H,\alpha}(\underline{k})$ of $P^{\Omega}_{T,H,\alpha}$ is given by

$$\mathcal{F}_{T,H,\alpha}(\underline{k}) = \int_{\Omega} \left(\prod^*_{m\in\mathbb{N}_0} \omega^{k_m}(m) \right) \mathrm{d}P^{\Omega}_{T,H,\alpha}.$$

Thus, by the definition of $P^{\Omega}_{T,H,\alpha}$,

$$\begin{aligned} \mathcal{F}_{T,H,\alpha}(\underline{k}) &= \frac{1}{H} \int_T^{T+H} \left(\prod^*_{m\in\mathbb{N}_0} (m+\alpha)^{-ik_m\tau} \right) \mathrm{d}\tau \\ &= \frac{1}{H} \int_T^{T+H} \exp\left\{ -i\tau \sum^*_{m\in\mathbb{N}_0} k_m \log(m+\alpha) \right\} \mathrm{d}\tau. \end{aligned} \tag{4}$$

If

$$L_\alpha(\underline{k}) \overset{\text{def}}{=} \sum^*_{m\in\mathbb{N}_0} k_m \log(m+\alpha) = 0,$$

then, by (4),

$$\mathcal{F}_{T,H,\alpha}(\underline{k}) = 1. \tag{5}$$

For $\underline{k}$ such that $L_\alpha(\underline{k}) \neq 0$, (4) implies that

$$\mathcal{F}_{T,H,\alpha}(\underline{k}) = \frac{1}{iH} \frac{\exp\{-iTL_\alpha(\underline{k})\} - \exp\{-i(T+H)L_\alpha(\underline{k})\}}{L_\alpha(\underline{k})}.$$

Thus, in this case, since $H \to \infty$ as $T \to \infty$,

$$\lim_{T\to\infty} \mathcal{F}_{T,H,\alpha}(\underline{k}) = 0.$$

From this, (4), and (5) we obtain

$$\lim_{T\to\infty} \mathcal{F}_{T,H,\alpha}(\underline{k}) = \mathcal{F}_\alpha(\underline{k}),$$

where

$$\mathcal{F}_\alpha(\underline{k}) = \begin{cases} 0 & \text{if } L_\alpha(\underline{k}) \neq 0, \\ 1 & \text{if } L_\alpha(\underline{k}) = 0. \end{cases} \tag{6}$$

Since the group Ω is compact, it is the Lévy group; see Theorem 1.4.2 of [23]. Therefore, $P^{\Omega}_{T,H,\alpha}$ converges weakly to the measure P^{Ω}_{α} and the Fourier transform is $\mathcal{F}_\alpha(\underline{k})$. □

Lemma 2. *Suppose that $H \to \infty$ as $T \to \infty$ and that the set $\mathfrak{L}(\alpha)$ is linearly independent over $\mathbb{Q}$. Then $P^{\Omega}_{T,H,\alpha}$ converges weakly to the Haar measure μ as $T \to \infty$.*

Proof. We denote by $\underline{0}$ a collection consisting of zeros. Since the set $\mathfrak{L}(\alpha)$ is linearly independent over $\mathbb{Q}$,

$$L_\alpha(\underline{k}) = 0$$

if and only if $\underline{k} = \underline{0}$. Therefore, by (6),

$$\mathcal{F}_\alpha(\underline{k}) = \begin{cases} 0 & \text{If } \underline{k} \neq \underline{0}, \\ 1 & \text{if } \underline{k} = \underline{0}. \end{cases}$$

Since the latter Fourier transform is the one of the Haar measure, the lemma is proved. □

Lemmas 1 and 2 imply the corresponding limit theorems for probability measures in the space $\mathcal{H}(\mathfrak{D})$. Let $\gamma > 1/2$ be a fixed number, and

$$w_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n}\right)^{\gamma}\right\}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

We introduce the function

$$L_n(\lambda,\alpha,s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} w_n(m,\alpha)}{(m+\alpha)^s}$$

connected to the Lerch zeta-function. Since $m \mapsto w_n(m,\alpha)$ is exponentially decreasing for any n and α, the series defining $L_n(\lambda,\alpha,s)$ is absolutely convergent in any half-plane $\sigma > \sigma_0$ with finite σ_0. Thus, $L_n(\lambda,\alpha,s)$ is an entire function for every fixed n and $\lambda,\alpha \in (0,1]$.

For $A \in \mathcal{B}(\mathcal{H}(\mathfrak{D}))$, we set

$$P^{\lambda,\alpha}_{T,H,n}(A) = \frac{1}{H}\mathfrak{m}\{\tau \in [T,T+H] : L_n(\lambda,\alpha,s+i\tau) \in A\}.$$

Lemma 3. *Suppose that $H \to \infty$ as $T \to \infty$. Then, on $(\mathcal{H}(\mathfrak{D}),\mathcal{B}(\mathcal{H}(\mathfrak{D})))$, there is a probability measure, $P^{\lambda,\alpha}_n$, such that $P^{\lambda,\alpha}_{T,H,n}$ converges weakly to $P^{\lambda,\alpha}_n$ as $T \to \infty$.*

Proof. Consider $v_{\lambda,\alpha,n} : \Omega \to \mathcal{H}(\mathfrak{D})$ given by

$$v_{\lambda,\alpha,n}(\omega) = L_n(\lambda,\alpha,\omega,s),$$

where

$$L_n(\lambda,\alpha,\omega,s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m) w_n(m,\alpha)}{(m+\alpha)^s}.$$

Clearly, the latter series is absolutely convergent in any half-plane $\sigma > \sigma_0$. Hence, the mapping $v_{\lambda,\alpha,n}$ is continuous; therefore, it is $(\mathcal{B}(\Omega),\mathcal{B}(\mathcal{H}(\mathfrak{D})))$-measurable. Thus, each probability measure P on $(\Omega,\mathcal{B}(\Omega))$ defines the unique probability measure $Pv^{-1}_{\lambda,\alpha,n}$ on $(\mathcal{H}(\mathfrak{D}),\mathcal{B}(\mathcal{H}(\mathfrak{D})))$, where

$$Pv^{-1}_{\lambda,\alpha,n}(A) = P(v^{-1}_{\lambda,\alpha,n}A), \quad \forall A \in \mathcal{B}(\mathcal{H}(\mathfrak{D})),$$

and $v^{-1}_{\lambda,\alpha,n}A$ denotes the preimage of A. By the definitions of $P^{\Omega}_{T,H,n}$, $P^{\lambda,\alpha}_{T,H}$ and $v_{\lambda,\alpha,n}$, we have

$$v_{\lambda,\alpha,n}\Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}(m+\alpha)^{-i\tau} w_n(m,\alpha)}{(m+\alpha)^s} = L_n(\lambda,\alpha,s+i\tau)$$

and, for all $A \in \mathcal{B}(\mathcal{H}(\mathfrak{D}))$,

$$P^{\lambda,\alpha}_{T,H,n}(A) = \frac{1}{H}\mathfrak{m}\Big\{\tau \in [T,T+H] : \Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big) \in v^{-1}_{\lambda,\alpha,n}A\Big\} = P^{\Omega}_{T,H,\alpha}\Big(v^{-1}_{\lambda,\alpha,n}A\Big).$$

Therefore, the relation $P^{\lambda,\alpha}_{T,H,n} = P^{\Omega}_{T,H,n}v^{-1}_{\lambda,\alpha,n}$ holds. This, the continuity of $v_{\lambda,\alpha,n}$, Lemma 1, and Theorem 5.1 from [22] on the preservation of weak convergence under continuous mappings show that $P^{\lambda,\alpha}_{T,H,n}$ converges weakly to $P^{\Omega}_{\alpha}v^{-1}_{\lambda,\alpha,n}$ as $T \to \infty$. □

Corollary 1. *Suppose that $H \to \infty$ as $T \to \infty$ and that the set $\mathfrak{L}(\alpha)$ is linearly independent over $\mathbb{Q}$. Then $P^{\lambda,\alpha}_{T,H,n}$ converges weakly to the measure $\mu v^{-1}_{\lambda,\alpha,n}$ as $T \to \infty$.*

Proof. The corollary is an immediate consequence of Lemmas 2 and 3. □

For further investigations, we need some properties of the measure $P^{\lambda,\alpha}_n \stackrel{\text{def}}{=} P^{\Omega}_{\alpha}v^{-1}_{\lambda,\alpha,n}$. Recall that the sequence $\{P_n : n \in \mathbb{N}\}$ of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is tight if, for every $\varepsilon > 0$, there is a compact subset $K = K_\varepsilon \subset \mathbb{X}$ such that $P_n(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$.

Lemma 4. *The sequence $\{P^{\lambda,\alpha}_n, n \in \mathbb{N}\}$ is tight.*

Proof. In [19], the measure

$$P^{\lambda,\alpha}_{T,n}(A) \stackrel{\text{def}}{=} \frac{1}{T}\mathfrak{m}\{\tau \in [0,T] : L_n(\lambda,\alpha,s+i\tau) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}(\mathfrak{D})),$$

was considered, and it was obtained that it weakly converges to the measure $P^{\lambda,\alpha}_n$ as $T \to \infty$ as well. Moreover, it was found that the sequence $\{P^{\lambda,\alpha}_n\}$ is tight; thus, the lemma is true. □

On the probability space $(\Omega, \mathcal{B}(\Omega), \mu)$, we define the $\mathcal{H}(\mathfrak{D})$-valued random element

$$L(\lambda,\alpha,\omega,s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}, \quad \omega \in \Omega,\ s \in \mathfrak{D},$$

and denote by $P_{\lambda,\alpha}$ its distribution. In other words, for $A \in \mathcal{B}(\mathcal{H}(\mathfrak{D}))$,

$$P_{\lambda,\alpha}(A) = \mu\{\omega \in \Omega : L(\lambda,\alpha,\omega,s) \in A\}.$$

Notice that the series for $L(\lambda,\alpha,\omega,s)$, for almost all ω with respect to the Haar measure μ, is uniformly convergent on compact subsets of D; thus, it gives a well-defined $\mathcal{H}(\mathfrak{D})$-valued random element.

Lemma 5. *Suppose that $\lambda \in (0,1]$ and that the set $\mathfrak{L}(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the measure$P^{\lambda,\alpha}_n = \mu v^{-1}_{\lambda,\alpha,n}$ converges weakly to the measure $P_{\lambda,\alpha}$ as $n \to \infty$.*

Proof. The proof is given in Chapter 5 of [15]. To be precise, in [15], the case of transcendental α is discussed; however, in order to conclude it suffices to assume that the set $\mathfrak{L}(\alpha)$ is linearly independent over the field $\mathbb{Q}$ and which follows if α is transcendental. □

For the proof of a limit theorem for $P_{T,H}^{\lambda,\alpha}$ in short intervals, we need an approximation of $L(\lambda,\alpha,s)$ by $L_n(\lambda,\alpha,s)$ in the mean in short intervals. For this, we use a mean square estimate for $L(\lambda,\alpha,s)$ in short intervals.

Recall that the classical notation $a \ll_\theta b$, $a \in \mathbb{C}$, $b > 0$, means that there exists a constant, $c = c(\theta) > 0$, such that $|a| \leqslant cb$.

Lemma 6 (See [24]). *Suppose that $\lambda, \alpha \in (0,1]$ and $\sigma \in (1/2, 7/12]$ are fixed. Then, for $T^{27/82} \leqslant H \leqslant T^\sigma$, the estimate*

$$\int\limits_{T-H}^{T+H} |L(\lambda,\alpha,\sigma+it)|^2 \, \mathrm{d}t \ll_{\lambda,\alpha,\sigma} H$$

uniformly in H is valid.

A proof of the lemma uses the approximate functional equation for the function $L(\lambda,\alpha,s)$, and, for the estimation of the mean squares of Dirichlet polynomials, applies a method of exponential pairs proposed in [25] in the case of the Riemann zeta-function.

Lemma 7. *Let $\lambda \in (0,1]$, $\sigma \geqslant 1/2$ and $|t| \geqslant 2$. Then*

$$L(\lambda,\alpha,\sigma+it) \ll_{\lambda,\alpha,\sigma} |t|^{1/2}.$$

Proof. For $\lambda = 1$ the estimate

$$\zeta(\sigma+it,\alpha) \ll_{\alpha,\sigma} |t|^{1/2}$$

can be found in [26]. For $\lambda \in (0,1)$, by Theorem 3.1.2 from [15],

$$L(\lambda,\alpha,\sigma+it) = \sum_{0 \leqslant m \leqslant |t|} \frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^{\sigma+it}} + O_\lambda(|t|^{-\sigma}) \ll_{\lambda,\alpha,\sigma} |t|^{1/2}.$$

□

Recall the metric in $\mathcal{H}(\mathfrak{D})$ inducing its topology of uniform convergence on compacta. There exists a sequence of compact subsets, $\{K_l : l \in \mathbb{N}\} \subset D$, such that $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$,

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and every compact set $K \subset D$ is contained in some K_l [27]. For $g_1, g_2 \in \mathcal{H}(\mathfrak{D})$, we set

$$\mathfrak{d}(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s)-g_2(s)|}{1+\sup_{s\in K_l} |g_1(s)-g_2(s)|}.$$

Then $\mathfrak{d}$ is the desired metric in $\mathcal{H}(\mathfrak{D})$.

Lemma 8. *Suppose that $\lambda, \alpha \in (0,1]$ and $T^{27/82} \leqslant H \leqslant^{1/2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int\limits_{T}^{T+H} \mathfrak{d}(L_n(\lambda,\alpha,s+i\tau), L(\lambda,\alpha,s+i\tau)) \, \mathrm{d}\tau = 0.$$

Proof. Let

$$\kappa_n(s) = \frac{1}{\gamma} \Gamma\left(\frac{s}{\gamma}\right) n^s.$$

Then, for $s \in \mathfrak{D}$, the integral representation

$$L_n(\lambda,\alpha,s) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} L(\lambda,\alpha,s+z)\kappa_n(z)\,\mathrm{d}z \tag{7}$$

is valid; see Lemma 9 of [19]. By the definition of the metric $\mathfrak{d}$, it suffices to show that, for any compact subset, $K \subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{H}\int\limits_{T}^{T+H}\sup_{s\in K}|L_n(\lambda,\alpha,s+i\tau) - L(\lambda,\alpha,s+i\tau)|\,\mathrm{d}\tau = 0. \tag{8}$$

Let K be a fixed compact set of the strip D. Then there is $\varepsilon > 0$ satisfying $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for all $s = \sigma + it \in K$. We shift the line of integration in (7) to the left. For this, we apply the residue theorem. We take $\gamma_1 = 1/2+\varepsilon-\sigma$ and $\gamma = 1/2+\varepsilon > 1/2$. Then, clearly, $\gamma_1 < -\varepsilon$ and $\gamma_1 \geqslant 2\varepsilon - 1/2$. This shows that the integrand in (7) has, in the strip $\gamma_1 \leqslant \operatorname{Re} z \leqslant \gamma$, a simple pole at the point $z = 0$, and a simple pole at the point $z = 1-s$ if $\lambda = 1$. These observations, together with (7) and the residue theorem, for all $s \in K$, imply, as in [19], that

$$L_n(\lambda,\alpha,s) - L(\lambda,\alpha,s) = \frac{1}{2\pi i}\int\limits_{\gamma_1-i\infty}^{\gamma_1+i\infty} L(\lambda,\alpha,s+z)\kappa_n(z)\,\mathrm{d}z + R_n(s),$$

where

$$R_n(s) = \operatorname*{Res}_{z=1-s} L(\lambda,\alpha,s+z)\kappa_n(z) = \begin{cases} \kappa_n(1-s) & \text{if } \lambda = 1, \\ 0 & \text{if } \lambda \in (0,1), \end{cases}$$

because $\operatorname{Res}_{s=1} L(1,\alpha,s) = \operatorname{Res}_{s=1}\zeta(s,\alpha) = 1$. Hence,

$$\begin{aligned}
&\sup_{s\in K}|L_n(\lambda,\alpha,s+i\tau) - L(\lambda,\alpha,s+i\tau)| \\
&\quad\ll \int\limits_{-\infty}^{\infty}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+iu+i\tau\right)\right|\sup_{s\in K}\left|\kappa_n\left(\frac{1}{2}+\varepsilon-s+iu\right)\right|\mathrm{d}u + \sup_{s\in K}|\kappa_n(1-s-i\tau)| \\
&\quad= \left(\int\limits_{-\infty}^{-\log^2 T} + \int\limits_{-\log^2 T}^{\log^2 T} + \int\limits_{\log^2 T}^{\infty}\right)\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+iu+i\tau\right)\right|\sup_{s\in K}\left|\kappa_n\left(\frac{1}{2}+\varepsilon-s+iu\right)\right|\mathrm{d}u \\
&\qquad+ \sup_{s\in K}|\kappa_n(1-s-i\tau)|.
\end{aligned} \tag{9}$$

For the Gamma-function, the bound that is uniform in $\sigma \in [\sigma_1,\sigma_2]$ with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{10}$$

holds. Hence, for all $s \in K$,

$$\kappa_n\left(\frac{1}{2}+\varepsilon-s+iu\right) \ll_\varepsilon n^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\gamma}|u-t|\right\} \ll_K n^{-\varepsilon}\exp\{-c_1|u|\}, \quad c_1 > 0. \tag{11}$$

Now, by the virtue of Lemma 7, we find

$$\left(\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{\infty}\right)\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+iu+i\tau\right)\right|\sup_{s\in K}\left|\kappa_n\left(\frac{1}{2}+\varepsilon-s+iu\right)\right|\mathrm{d}u$$
$$\ll_{K,\lambda,\alpha} n^{-\varepsilon}\left(\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{\infty}\right)\left(|u|^{1/2}+|\tau|^{1/2}\right)\exp\{-c_1|u|\}\,\mathrm{d}u$$
$$\ll_{K,\lambda,\alpha} n^{-\varepsilon}\left(1+|\tau|^{1/2}\right)\exp\left\{-c_2\log^2 T\right\},\quad c_2>0.$$

From this, we get

$$\begin{aligned}I &\stackrel{\mathrm{def}}{=} \frac{1}{H}\int_T^{T+H}\sup_{s\in K}|L_n(\lambda,\alpha,s+i\tau)-L(\lambda,\alpha,s+i\tau)|\,\mathrm{d}\tau\\
&\ll_{K,\lambda,\alpha}\int_{-\log^2 T}^{\log^2 T}\left(\frac{1}{H}\int_T^{T+H}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+iu+i\tau\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\left|\kappa_n\left(\frac{1}{2}+\varepsilon-s+iu\right)\right|\mathrm{d}u\\
&+\frac{1}{H}\int_T^{T+H}\sup_{s\in K}|\kappa_n(1-s-i\tau)|\,\mathrm{d}\tau+\frac{1}{H}n^{-\varepsilon}\exp\left\{-c_2\log^2 T\right\}\int_T^{T+H}\left(1+|\tau|^{1/2}\right)\mathrm{d}\tau\\
&\stackrel{\mathrm{def}}{=} J_1+J_2+J_3.\end{aligned}\tag{12}$$

The Cauchy–Schwarz inequality gives

$$\frac{1}{H}\int_T^{T+H}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+iu+i\tau\right)\right|\mathrm{d}\tau \ll \left(\frac{1}{H}\int_{T-H-|u|}^{T+H+|u|}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+i\tau\right)\right|^2\mathrm{d}\tau\right)^{1/2}.\tag{13}$$

Now, we will apply Lemma 6. For $T^{27/82}\leqslant H\leqslant T^{1/2}$ and $|u|\leqslant\log^2 T$, as $T\to\infty$, we have $T^{27/82}\leqslant H+|u|\leqslant T^{1/2}+\log^2 T\leqslant T^{1/2+\varepsilon}$. Therefore, Lemma 6 and (13) show that

$$\frac{1}{H}\int_T^{T+H}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+iu+i\tau\right)\right|\mathrm{d}\tau \ll_{K,\lambda,\alpha}\left(\frac{1}{H}(H+|u|)\right)^{1/2}\ll_{K,\lambda,\alpha}(1+|u|)^{1/2}.$$

Hence, in view of (12) and (11),

$$J_1\ll_{K,\lambda,\alpha} n^{-\varepsilon}\int_{-\log^2 T}^{\log^2 T}\sqrt{1+|u|}\exp\{-c_1|u|\}\,\mathrm{d}u\ll_{K,\lambda,\alpha} n^{-\varepsilon}.\tag{14}$$

Moreover, in view (10) again, for $s\in K$,

$$\kappa_n(1-s-i\tau)\ll_\varepsilon n^{1-\sigma}\exp\left\{-\frac{c}{\gamma}|t+\tau|\right\}\ll_K n^{1/2-2\varepsilon}\exp\{-c_2|\tau|\},\quad c_2>0.$$

Thus, by (12),

$$J_2 \ll_K n^{1/2-2\varepsilon} \frac{1}{H} \int_T^{T+H} \exp\{-c_2\tau\} \, d\tau$$
$$\ll_K n^{1/2-2\varepsilon} \frac{1}{H} \left(\exp\left\{-\frac{c_2}{2}T\right\} \int_T^{T+H} \exp\left\{-\frac{c_2}{2}\tau\right\} d\tau \right)$$
$$\ll_K n^{1/2-2\varepsilon} \exp\left\{-\frac{c_2}{2}T\right\}. \tag{15}$$

Clearly,

$$J_3 \ll n^{-\varepsilon} \exp\left\{-c_2 \log^2 T\right\} \sqrt{T}.$$

This, (11), (13), and (15) lead to the estimate

$$I \ll_{K,\lambda,\alpha} n^{-\varepsilon} + n^{1/2-2\varepsilon} \exp\left\{-\frac{c_2}{2}T\right\} + n^{-\varepsilon}\sqrt{T} \exp\left\{-c_2 \log^2 T\right\},$$

and this proves (8). □

To prove a weak convergence for the measure $P^{\lambda,\alpha}_{T,H}$, we use convergence in distribution ($\xrightarrow{\mathcal{D}}$) for random elements, which means the weak convergence of distributions of the corresponding random elements. We will deal with the following general statement.

Lemma 9 (See [22])**.** *Let the metric space $(\mathbb{X}, \rho)$ be separable. Suppose that the $\mathbb{X}$-valued random elements X_{nk} and Y_n, $k \in \mathbb{N}$ and $n \in \mathbb{N}$, are defined on the same probability space $(\widehat{\Omega}, \mathcal{B}, \nu)$;*

$$X_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} X_k, \qquad X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \nu\{\rho(Y_n, X_{nk}) \geqslant \varepsilon\} = 0.$$

Then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$ holds.

Now we state a limit theorem for $P^{\lambda,\alpha}_{T,H}$.

Theorem 3. *Suppose that the parameters $\lambda, \alpha \in (0,1]$ are arbitrary, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then, on $(\mathcal{H}(\mathfrak{D}), \mathcal{B}(\mathcal{H}(\mathfrak{D})))$, there is a probability measure, $P^{\lambda,\alpha}$, that $P^{\lambda,\alpha}_{T,H}$ is weakly convergent to $P^{\lambda,\alpha}$ as $T \to \infty$.*

Proof. On a certain probability space, $(\widehat{\Omega}, \mathcal{B}, \nu)$, we define a random variable, $\theta_{T,H}$, which is uniformly distributed on $[T, T+H]$. Using $\theta_{T,H}$, we introduce the $\mathcal{H}(\mathfrak{D})$-valued random elements

$$L^{\lambda,\alpha}_{T,H,n} = L^{\lambda,\alpha}_{T,H,n}(s) = L_n(\lambda, \alpha, s + i\theta_{T,H})$$

and

$$L^{\lambda,\alpha}_{T,H} = L^{\lambda,\alpha}_{T,H}(s) = L(\lambda, \alpha, s + i\theta_{T,H}),$$

and let $L^{\lambda,\alpha}_n = L^{\lambda,\alpha}_n(s)$ have the distribution $P^{\lambda,\alpha}_n$, where $P^{\lambda,\alpha}_n$ is the limit measure in Lemma 3. Since $P^{\lambda,\alpha}_{T,H,n}$ is the distribution of the random element $L^{\lambda,\alpha,n}_{T,H}$, by Lemma 3, we get

$$L^{\lambda,\alpha}_{T,H,n} \xrightarrow[T\to\infty]{\mathcal{D}} L^{\lambda,\alpha}_n. \tag{16}$$

Lemma 4 asserts that the measure $P_n^{\lambda,\alpha}$ is tight. Therefore, by the Prokhorov theorem (see Theorem 6.1 of [22]), $P_n^{\lambda,\alpha}$ is relatively compact. Hence, there exists a probability measure, $P^{\lambda,\alpha}$, and a sequence, n_r, such that $P_{n_r}^{\lambda,\alpha}$ converges weakly to $P^{\lambda,\alpha}$ as $r \to \infty$. In other words,

$$L_{n_r}^{\lambda,\alpha} \xrightarrow[r\to\infty]{\mathcal{D}} P^{\lambda,\alpha}, \tag{17}$$

which means that $L_{n_r}^{\lambda,\alpha}$ converges in distribution to a random element with the distribution $P^{\lambda,\alpha}$. Note that this mixed notation is convenient and is widely used; see [22]. Moreover, the above definitions and Lemma 8 imply that, for each $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \nu\left\{\mathfrak{d}\left(L_{T,H,n}^{\lambda,\alpha}, L_{T,H}^{\lambda,\alpha}\right) \geqslant \varepsilon\right\} \\ &\quad = \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H}\mathfrak{m}\{\tau \in [T, T+H] : \mathfrak{d}(L_n(\lambda,\alpha,s+i\tau), L(\lambda,\alpha,s+i\tau)) \geqslant \varepsilon\} \\ &\quad \leqslant \frac{1}{H\varepsilon}\int_T^{T+H} \mathfrak{d}(L_n(\lambda,\alpha,s+i\tau), L_n(\lambda,\alpha,s+i\tau))\,\mathrm{d}\tau = 0. \end{aligned} \tag{18}$$

Relations (16) and (17) and the latter equality show that all hypotheses of Lemma 9 are fulfilled. Thus, we have

$$L_{T,H}^{\lambda,\alpha} \xrightarrow[T\to\infty]{\mathcal{D}} P^{\lambda,\alpha},$$

and in other words, $P_{T,H}^{\lambda,\alpha}$ converges weakly to $P^{\lambda,\alpha}$ as $T \to \infty$. □

Theorem 4. *Suppose that $\lambda \in (0,1]$, the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then $P_{T,H}^{\lambda,\alpha}$ converges weakly to the measure $P_{\lambda,\alpha}$ as $T \to \infty$.*

Proof. We repeat the proof of Theorem 3 with one difference: by Lemma 5,

$$L_n^{\lambda,\alpha} \xrightarrow[n\to\infty]{\mathcal{D}} P_{\lambda,\alpha}, \tag{19}$$

where $P_{\lambda,\alpha}$ is the distribution of the random element $L(\lambda,\alpha,\omega,s)$. Therefore, the theorem follows from (16), (18), (19), and Lemma 9. □

3. Proofs of the Main Theorems

The proofs of Theorems 1 and 2 are standard and are based on the Mergelyan theorem [28] and equivalents of weak convergence in Theorems 3 and 4. We start with Theorem 2 because Theorem 1 is a partial case of Theorem 2.

Proof of Theorem 2. Let P be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and $\mathbb{X}$ be a separable space. Recall that the support of P is a closed minimal set, $S \subset \mathbb{X}$, such that $P(S) = 1$. The set S consists of all elements $x \in \mathbb{X}$ such that $P(G(x)) > 0$ for every open neighbourhood, $G(x)$, of x.

Suppose that $\mathfrak{F}_{\lambda,\alpha}$ is the support of the measure $P^{\lambda,\alpha}$ in Theorem 3. Then, by the definition of the support, $\mathfrak{F}_{\lambda,\alpha} \neq \varnothing$ is a closed set. For $f \in \mathfrak{F}_{\lambda,\alpha}$, the compact set $K \subset D$, and any $\varepsilon > 0$, we set

$$G_{\varepsilon,f} = \left\{g \in \mathcal{H}(\mathfrak{D}) : \sup_{s\in K} |f(s) - g(s)| < \varepsilon\right\}.$$

Then $G_{\varepsilon,f}$ is an open neighbourhood of an element of the support. Therefore,

$$P^{\lambda,\alpha}(G_{\varepsilon,f}) > 0, \tag{20}$$

and Theorem 3 in terms of open sets gives

$$\liminf_{T\to\infty} P_{T,H}^{\lambda,\alpha}(G_{\varepsilon,f}) \geqslant P^{\lambda,\alpha}(G_{\varepsilon,f}) > 0.$$

Thus, the definitions of $P_{T,H}^{\lambda,\alpha}$ and $G_{\varepsilon,f}$ imply the first statement of the theorem.

To obtain the second statement of the theorem, we use Theorem 3 in terms of continuity sets. Recall that $A \in \mathcal{B}(\mathbb{X})$ is a continuity set of the measure P if $P(\partial A) = 0$, where ∂A denotes the boundary of A. We observe that $\partial G_{\varepsilon,f}$ lies in the set

$$\left\{ g \in \mathcal{H}(\mathfrak{D}) : \sup_{s\in K} |f(s) - g(s)| = \varepsilon \right\}.$$

Hence, $\partial G_{\varepsilon_1,f} \cap \partial G_{\varepsilon_2,f} = \varnothing$ for $\varepsilon_1 \neq \varepsilon_2$. From this, we have $P^{\lambda,\alpha}(\partial G_{\varepsilon,f}) > 0$ for at most countably many $\varepsilon > 0$. Therefore, the set $G_{\varepsilon,f}$ is a continuity set of the measure $P^{\lambda,\alpha}$ for all but at most countably many $\varepsilon > 0$. Thus, by Theorem 3 in terms of continuity sets and (20),

$$\lim_{T\to\infty} P_{T,H}^{\lambda,\alpha}(G_{\varepsilon,f}) = P^{\lambda,\alpha}(G_{\varepsilon,f}) > 0$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of $P_{T,H}^{\lambda,\alpha}$ and $G_{\varepsilon,f}$ imply the second statement of the theorem. □

Proof of Theorem 1. Differently from Theorem 2, the function $f(s)$ is related to the set K. Therefore, we have to involve the Mergelyan theorem in the approximation of analytic functions by polynomials. By that theorem, for any $\varepsilon > 0$, there is a polynomial, $p_{\varepsilon,f}(s)$, satisfying

$$\sup_{s\in K} |f(s) - p_{\varepsilon,f}(s)| < \frac{\varepsilon}{2}. \tag{21}$$

Put

$$G_{\varepsilon,p_{\varepsilon},f} = \left\{ g \in \mathcal{H}(\mathfrak{D}) : \sup_{s\in K} |g(s) - p_{\varepsilon,f}(s)| < \frac{\varepsilon}{2} \right\}.$$

By Lemma 6.1.7 from [15], it is known that the support of the measure $P_{\lambda,\alpha}$ is the whole space $\mathcal{H}(\mathfrak{D})$. Since $p_{\varepsilon,f}(s) \in \mathcal{H}(\mathfrak{D})$ the set $G_{\varepsilon,p_{\varepsilon},f}$ is an open neighbourhood of an element of the support of $P_{\lambda,\alpha}$. Thus,

$$P_{\lambda,\alpha}(G_{\varepsilon,p_{\varepsilon},f}) > 0. \tag{22}$$

Let $G_{\varepsilon,f}$ be as in the proof of Theorem 2. By the virtue of (21), the inclusion $G_{\varepsilon,p_{\varepsilon},f} \subset G_{\varepsilon,f}$ holds. Therefore, by (22), we have $P_{\lambda,\alpha}(G_{\varepsilon,f}) > 0$, and Theorem 4 yields

$$\liminf_{T\to\infty} P_{T,H}^{\lambda,\alpha}(G_{\varepsilon,f}) \geqslant P_{\lambda,\alpha}(G_{\varepsilon,f}) > 0.$$

The first statement of the theorem is proved.

The second assertion of the theorem follows from the same arguments as the ones given in the proof of Theorem 2 with the measure $P_{\lambda,\alpha}$ in place of $P^{\lambda,\alpha}$ and Theorem 4 in place of Theorem 3.

The theorem is proved. □

Remark 1. *Suppose that the set $\mathfrak{L}(\alpha)$ is linearly independent over $\mathbb{Q}$, $\lambda \in (0,1]$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then, for every compact set $K \subset D$, the analytic function $f(s)$ in D, and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H}\mathfrak{m}\left\{\tau \in [T, T+H] : \sup_{s\in K} |f(s) - L(\lambda, \alpha, s+i\tau)| < \varepsilon\right\} > 0$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{H}\mathfrak{m}\left\{\tau \in [T, T+H] : \sup_{s\in K} |f(s) - L(\lambda, \alpha, s+i\tau)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Proof. We repeat the proof of Theorem 2 with $\mathfrak{F}_{\lambda,\alpha} = \mathcal{H}(\mathfrak{D})$. □

4. Conclusions

Although universality theorems on the approximation of analytic functions by shifts in zeta-functions are not effective in a certain sense, they have a series of theoretical and practical applications. This will stimulate continued research in the field and improve universality results. Usually, the main universality results are stated as theorems on the positivity of the density of approximating shifts in an interval. Clearly, information of such a kind is more useful if the interval is as short as possible. In this paper, we obtained theorems on the approximation of analytic functions by shifts in the Lerch zeta-function $L(\lambda, \alpha, s+i\tau)$ in the interval $[T, T+H]$ with $T^{27/82} \leqslant H \leqslant T^{1/2}$ as $T \to \infty$.

Based on the progress made in this article, the following open problems arise:

1. Improve the lower bound for H. This is closely connected to the mean square estimate

$$\int_{T-H}^{T+H} |L(\lambda, \alpha, \sigma+it)|^2 \, \mathrm{d}t \ll_{\lambda,\alpha} H$$

for $\sigma > 1/2$.

2. Obtain approximation by shifts $L(\lambda, \alpha, s+i\tau_k)$ in short intervals when τ_k runs over a certain discrete set.

3. Extend approximation to the simultaneous kind for a tuple of analytic functions $(f_1(s), \ldots, f_r(s))$ by $(L(\lambda_1, \alpha_1, s+i\tau), \ldots, L(\lambda_r, \alpha_r, s+i\tau))$ in short intervals.

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