Journal of Complex Networks, 2025, **4**, cnaf008
https://doi.org/10.1093/comnet/cnaf008
Advance access publication 25 July 2025
Article

OXFORD

Two models of sparse and clustered dynamic networks

Mindaugas Bloznelis * and Dominykas Marma

Institute of Computer Science, Faculty of Mathematics and Informatics, Vilnius University, Vilnius, LT-08303, Lithuania

*Corresponding author. Institute of Computer Science, Faculty of Mathematics and Informatics, Vilnius University, Didlaukio 47, Vilnius LT-08303, Lithuania. E-mail: mindaugas.bloznelis@mif.vu.lt.

ABSTRACT

We present two models of sparse dynamic networks that display transitivity—the tendency for nodes sharing a common neighbour to be neighbours of one another. Our first network is a continuous time Markov chain $G = \{G_t = (V, E_t), t \geq 0\}$ whose states are graphs with the common set of nodes $V = \{1, \dots, n\}$. The transitions are defined as follows. Given t, the node pairs $\{i, j\} \subset V$ are assigned independent exponential waiting times A_{ij}. At time $t + \min_{ij} A_{ij}$ the pair $\{i_0, j_0\}$ with $A_{i_0 j_0} = \min_{ij} A_{ij}$ toggles its adjacency status. To mimic clustering patterns of sparse real networks we set intensities a_{ij} of exponential times A_{ij} to be decreasing functions of the degrees of common neighbours of nodes i and j in G_t. Our second network $G' = \{G'_t = (E'_t, V), t \geq 0\}$ is the affiliation network based on a latent Markov chain $H = \{H_t = (V \cup W, E_t), t \geq 0\}$ whose states are bipartite graphs with the bipartition $V \cup W$, where $W = \{1, \dots, m\}$ is an auxiliary set of attributes/affiliations. Nodes $i_1, i_2 \in V$ are adjacent in G'_t whenever i_1 and i_2 have a common neighbour in H_t. We analyse geometric properties of both dynamic networks at stationarity and show that networks possess high clustering. They admit tunable degree distribution and clustering coefficients.

KEYWORDS: dynamic network; stationary network; network Markov chain; clustering coefficient.

1. INTRODUCTION

Many real networks, especially those depicting human interaction, like social networks of friendships, collaboration networks, citation networks, and other show clustering, the propensity of nodes to cluster together by forming relatively small groups with a high density of ties within a group. Clustering is closely related to network transitivity, the tendency for two nodes sharing a common neighbour to be neighbours of one another thus forming a closed triangle of connections. Locally, in a vicinity of a node, this tendency can be quantified by the probability that two randomly selected neighbours of the node are adjacent. The network average of this probability, called the (average) local clustering coefficient, is used to quantify the network transitivity. Another popular measure of network transitivity, the global clustering coefficient, is the (conditional) probability that a randomly selected ordered triple of nodes (u, v, w) forms closed triangle of connections given that u and v are neighbours of w. In many social networks, both clustering coefficients are on the order

Received: 19 November 2024. **Editorial decision:** 16 April 2025. **Accepted:** 29 April 2025

of tens of percent, while the edge density, the probability that two randomly selected nodes are adjacent, is of much smaller order [1]. Often the edge density scales as n^{-1}, where n is the number of nodes in the network. We call networks with such edge densities sparse.

Mathematical modelling of sparse networks displaying clustering/transitivity has attracted considerable attention in the literature, see e.g. [2, 3] and references therein. We briefly review several approaches to modelling of clustered networks. In order to enhance the number of closed triangles in an evolving locally tree-like network Holme and Kim [4] suggested inserting additional edges that close desired fraction of open triangles (paths of length two). Newman [5] generalized the configuration random graph model by prescribing network nodes numbers of closed triangles they participate in. In this way, a predefined number of closed triangles can be introduced into configuration random graph. Bollobás *et al.* [6] built a clustered network by taking a union of randomly located small dense subgraphs of variable sizes. Guillaume and Latapy [7] noted an underlying bipartite structure present in many social networks, where nodes (actors) sharing a common hobby or affiliation are more likely to become friends, and where each hobby/affiliation defines a tightly connected cluster of actors related to it. They suggested modelling a clustered network by first linking actors to affiliations and then connecting actors that share common affiliations. We call such networks affiliation networks. A related class of network models called random intersection graphs represents nodes by finite sets of attributes randomly assigned to nodes; two nodes are adjacent if their attribute sets intersect [8–11], see also [12, 13]. Another source of clustered network models are random geometric graphs, where nodes are randomly selected points of a metric space and where adjacency relation depends on mutual distances between nodes: the closer the nodes the higher the probability of a link joining them [14–16]. We remark that network models mentioned above admit (asymptotic) power law degree distributions.

The present paper is devoted to the modelling of sparse and clustered dynamic networks using Markov chains. By dynamic network, we mean a collection of random graphs $\{G_t = (V, E_t), t \geq 0\}$ sharing the same set of nodes $V = \{1, \dots, n\}$ and having random edge sets E_t, $t \geq 0$. We present two stationary random processes $\{G_t, t \geq 0\}$ with tunable degree distribution and tunable non-vanishing clustering coefficients. Our study is build upon earlier work on dynamic network Markov chains [17–19]. We mention that network Markov chain of [19] is composed of $\binom{n}{2}$ independent Markov chains defining the adjacency status of each node pair $\{i, j\} \subset V$ individually (we refer to Section 2 for details). The network admits tunable edge density and degree distribution, but since the edges are inserted/deleted independently of each other it does not show clustering. Grindrod et al [17] and Užupytė and Wit [18] introduced transitivity into the network Markov chain by relating the birth/death rate of an edge to the number of triangles it closes/opens (cf. [4]). More precisely, they set the birth (death) rate of an edge $i \sim j$ to be an affine function of the number of common neighbours of nodes i and j. Here $i \sim j$ means that i and j are adjacent. A drawback of the models of [17, 18] is that for large n they have a little control over the edge density and clustering strength.

In the present paper, we suggest a remedy to this drawback. Inspired by clustering patterns observed in real networks, where the number of closed triangles incident to a node negatively correlates with the degree of the node ([20–23]) we set the birth rate of an edge $i \sim j$ to be a decreasing function of the degrees of common neighbours of i and j. We show below that such a modification leads to a stationary dynamic network model admiting tunable edge density and clustering coefficients.

Another dynamic clustered network considered in this paper is a stationary affiliation network built upon an underlying bipartite graph valued Markov chain with independent edges. Now the clustering property is caused by the bipartite structure as noted in [7]. We analyse the degree sequence and global clustering coefficient at stationarity using the tools developed for random intersection graphs [8]. We note that earlier work on dynamic affiliation network models ([24–26]) addresses the case where the network size $n = n(t)$ increases with time. Clearly, such networks do not admit stationary distributions.

Finally, we mention the related recent work by Milewska *et al.* [27], where a sparse and clustered dynamic affiliation network is constructed using 2^n independent two state Markow chains: each

subset of $V = \{1, \dots, n\}$ (there are 2^n of them) defines a two state Markow chain (active state and passive state) having exponential holding times with intensities depending on the size of the subset and weights of nodes in it. Furthermore, each active subset defines active clique on the nodes of the subset. The union of cliques that are active at time t defines the instance of the dynamic network at time t. The stationary distribution of this network is given by a random intersection graph.

The rest of the paper is organized as follows. In Section 2, we formally define the network Markov chain and analyse geometric properties of the network analytically and by numerical simulations. In Section 3, we define stationary affiliation network and show the degree distribution and global clustering coefficient. Proofs of the results of Section 3 are given in Supplementary Appendix.

2. NETWORK MARKOV CHAIN

Let $\mathbf{G} = \{G_t = (V, E_t),\ t \geq 0\}$ be a continuous time Markov chain, whose states are graphs on the node set V and transitions are defined as follows. Given G_t (the state occupied at time t), the update takes place at time $t' := t + \min_{ij} A_{ij}$, where $A_{ij} = A_{ij}(G_t)$, $\{i, j\} \in V$ are independent exponential waiting times with intensities $a_{ij} = a_{ij}(G_t)$ defined below. The pair $\{i_0, j_0\}$ with $A_{i_0 j_0} = \min_{ij} A_{ij}$ changes its adjacency status: the edge $i_0 \sim j_0$ is inserted if it is not present at time t; the edge $i_0 \sim j_0$ is removed if it is present at time t. Thus, at time t' the Markov chain jumps to the state $G_{t'} = (V, E_{t'})$, where the edge sets E_t and $E_{t'}$ differ in the single edge $i_0 \sim j_0$.

Let us define the intensities a_{ij} for $\{i, j\} \subset V$. Let $\alpha, \beta, \lambda, \mu \geq 0$ and let λ_i, μ_i, $1 \leq i \leq n$, be positive numbers. Given graph $G = (V, E)$ we assign clustering weights $\nu_{ij}(G, \alpha)$ and $\nu_{ij}(G, \beta)$ to each node pair $\{i, j\} \subset V$, where

$$\nu_{ij}(G, s) = \sum_{v \in N_{ij}} (d_v(G))^{-s}, \qquad s \geq 0. \tag{2.1}$$

For $s = 0$ we have $\nu_{ij}(G, 0) = |N_{ij}|$. Here $N_{ij} = N_{ij}(G)$ stands for the set of common neighbours of nodes i and j in G; $d_v(G)$ denotes the degree of node v in G. Furthermore, each node pair $\{i, j\}$ is assigned intensity

$$a_{ij}(G) = \begin{cases} \lambda_i \lambda_j + \lambda \nu_{ij}(G, \alpha) & \text{for} \quad \{i, j\} \notin E, \\ \left(\mu_i \mu_j - \mu \nu_{ij}(G, \beta)\right)_+ & \text{for} \quad \{i, j\} \in E. \end{cases} \tag{2.2}$$

Here x_+ stands for $\max\{x, 0\}$. A standard argument shows that the chain $\mathbf{G}$ has unique stationary distribution. Chain $\mathbf{G}$ starting with random graph G_0 having such a distribution is called stationary network in what follows.

Let us now explain all elements of the model. (The novel part which is in the ν_{ij} term will be discussed last.) For $\lambda = \mu = 0$ transitions of the chain $\mathbf{G}$ are defined by the transitions of $\binom{n}{2}$ independent Markov chains describing adjacency dynamic of each node pair $\{i, j\} \subset V$ separately. (The Markov chain of the node pair $\{i, j\}$ has two states $i \sim j$ and $i \not\sim j$, where state $i \sim j$ (i and j are adjacent) has exponential holding time with the intensity $\mu_i \mu_j$ and the state $i \not\sim j$ (i and j aren't adjacent) has exponential holding time with the intensity $\lambda_i \lambda_j$.) The stationary network of $\mathbf{G}$ has independent edges and, hence, it lacks the clustering property. Assuming, in addition, that μ_i is the same for each node $i \in V$ ($\mu_i \equiv const$) we obtain a dynamic network considered in [19]. Let us mention that weights λ_i strongly correlate with respective node degrees $d_i(G_t)$, $i \in V$ and are useful in modelling the degree distribution of G_t for large t. In particular, for $\lambda = \mu = 0$ and $\mu_i \equiv c$ the stationary network reproduces Chung-Lu random graph model [28] if we modify the Markow chain to allow multiple parallel edges (we refer to [19] for detailed discussion on the relation with Chung–Lu model [28]). Finally, we remark that large values of λ_i and μ_i (in concert) enhance the variability over time of links incident to node $i \in V$.

Grindrod *et al.* [17] introduced the term $\lambda \nu_{ij}(G, 0)$ to enhance the triadic closure effect. We mention that [17] considers the (discrete) jump chain $\mathbf{G}^* = \{G_k^* = (V, E_k^*), k = 0, 1, 2, \dots\}$

related to **G** defined by (2.2), where $\lambda_i = const_1$, $\mu_i = const_2$ do not depend on i and where $\mu = 0$. More precisely, $\mathbf{G}^*$ represents the list of distinct states visited by the chain **G** arranged in the chronological order. That is, $G_0^* = G_0$, $G_1^* = G_{t_1}$, $G_2^* = G_{t_2}, \ldots$, where $t_1 < t_2 < \ldots$ are the subsequent jump times of continuous chain **G**. Užupytė and Wit [18] complemented the model of [17] by adding the 'triadic protection' term $\mu\nu_{ij}(G,0)$ aimed at reducing the deletion rate of the edges belonging to the closed triangles. They consider the continuous chain **G** defined by (2.2) with $\lambda_i = const_1$, $\mu_i = const_2$.

It has already been mentioned that for large n dynamic networks of [17, 18] permit little control over the edge density, which becomes very sensitive to parameters μ and λ. To overcome such disadvantage we suggest choosing clustering weights $\nu_{ij}(G,s)$ that are decreasing functions of the degrees of common neighbours of i and j. An intuition behind this choice is based on the plausible assumption that for i,j being friends of an individual with a large number of acquaintances makes less impact on the mutual relations between i,j than being friends with a person having just a few contacts. Moreover, [21–23], see also [20], note that in some sparse and clustered real networks the fraction of closed triangles incident to a node scales as a negative power of the degree of that node. Findings of [20–23] motivated our choice of the clustering weights (2.1).

We are most interested in sparse networks, where the number of nodes n is large. In the simplest case, where $\lambda = \mu = 0$ and where $\lambda_i = const_1$ and $\mu_i = const_2$ are the same for each i (we write, for short, $\lambda_i\lambda_j = \lambda_0$ and $\mu_i\mu_j = \mu_0$) each node pair toggles its adjacency status independently and the expected holding time of an edge (respectively, non-edge) is μ_0^{-1} (respectively, λ_0^{-1}). By the law of large numbers the probability that i and j are adjacent in G_t is asymptotically $\mu_0^{-1}/(\mu_0^{-1}+\lambda_0^{-1}) = \lambda_0/(\mu_0+\lambda_0)$ as $t \to \infty$. Hence a snapshot G_t of the stationary network has the distribution of the binomial random graph with the edge density $\lambda_0/(\mu_0+\lambda_0)$. Furthermore, a sparse network is obtained if one chooses $\mu_0 = n$ and $\lambda_0 = c$, where $c > 0$ denotes a number independent of n (think of a sequence of network Markov chains with the number of nodes $n \to \infty$). More generally, for $\lambda = \mu = 0$, $\mu_i\mu_j \equiv n$ and $\sum_{i=1}^n \lambda_i \le cn$ uniformly in n one can obtain a sparse stationary network having independent edges and the degree sequence strongly correlated with the sequence of weights $\{\lambda_i\}$, [19].

The simulation study of Section 2.1 below shows that network Markov chain (2.2) with clustering weights $\nu_{ij}(G,\alpha)$, $\nu_{ij}(G,\beta)$, where $\alpha,\beta > 0$, can produce highly clustered sparse stationary dynamic networks with tunable edge density and clustering coefficients. These empirical findings are supported by a limited analytical study (given in Section 2.2 below) showing upper and lower bounds of the order n^{-1} on the average edge density. In addition, we establish a lower bound of the order n on the average number of closed triangles and in a special case of $\alpha = 2$ we relate the average edge density to the average local clustering coefficient.

Before proceeding further, we introduce some notation. Given a graph $G = (V,E)$ we denote by $\Delta_v(G)$ the number of closed triangles incident to a node $v \in V$. The total number of closed triangles is denoted $N_\Delta(G) = \frac{1}{3}\sum_{v\in V}\Delta_v(G)$ The total number of 2-paths is denoted $N_\Lambda(G) = \sum_{v\in V}\binom{d_v(G)}{2}$. For a node $v \in V$ of degree $d_v(G) \ge 2$ we denote $C_v^{\mathrm{L}}(G) = \Delta_v(G)\binom{d_v(G)}{2}^{-1}$ the local clustering coefficient of v (= probability that two randomly selected neighbours of v are neighbours to each other). In the case where $d_v(G) \le 1$ we put $C_v^{\mathrm{L}}(G) = 0$. The average local clustering coefficient and the global clustering coefficient are denoted

$$\bar{C}^{\mathrm{L}}(G) = \frac{1}{n}\sum_{v\in V} C_v^{\mathrm{L}}(G) \qquad \text{and} \qquad C^{\mathrm{GL}}(G) = \frac{3N_\Delta(G)}{N_\Lambda(G)}.$$

We put $C^{\mathrm{GL}}(G) = 0$ when $N_\Lambda(G) = 0$. The average degree and the average edge density are denoted $\bar{d}(G) = n^{-1}\sum_{v\in V} d_v(G)$ and $e(G) = \binom{n}{2}^{-1}|E|$ respectively. Finally, we denote by $\mathbb{I}_A$ the indicator function of an event (or set) A.

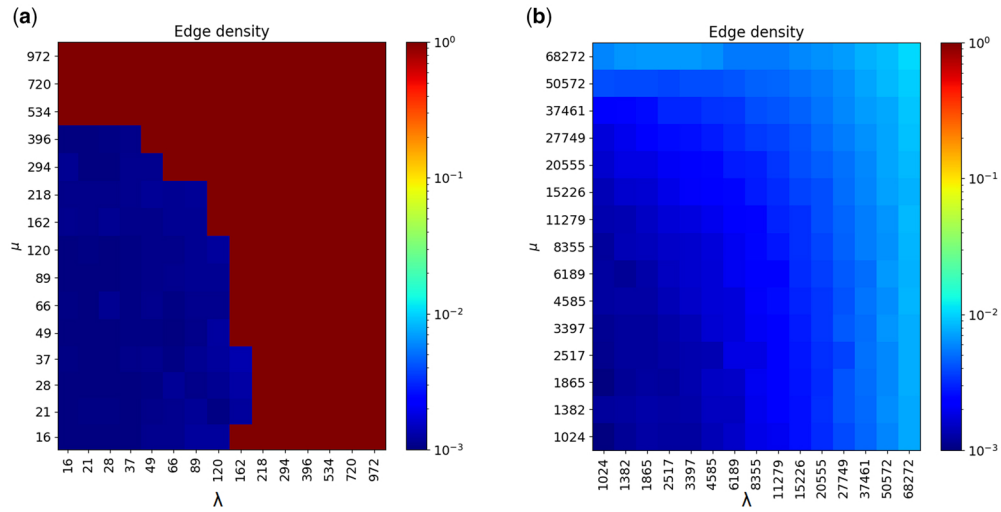

Figure 1. Edge densities in stationary graphs. (a) Heatmap shows edge density of simple triadic model for various values of λ and μ, while the other model parameters are fixed: $\alpha = \beta = 0$, $\lambda_i\lambda_j \equiv 1$, $\mu_i\mu_j \equiv 1000$, $n = 1000$. (b) Heatmap shows edge density of general triadic model for various values of λ and μ, while the other model parameters are fixed: $\alpha = 2.75$, $\beta = 2.5$, $\lambda_i\lambda_j \equiv 1$, $\mu_i\mu_j \equiv 1000$, $n = 1000$.

2.1. Numerical simulations

The aim of the simulation study is twofold: testing the clustering properties of sparse network (2.2) equipped with clustering weights $\nu_{ij}(G, \alpha)$, $\nu_{ij}(G, \beta)$, where $\alpha, \beta > 0$ and comparison of the clustering properties for $\alpha, \beta > 0$ and $\alpha = \beta = 0$ (the case $\alpha = \beta = 0$ corresponds to the setup of [17, 18]).

To address both questions simultaneously we consider a simplified model (2.2), where we assume that $\lambda_i\lambda_j \equiv const_1 := \lambda_0$ and $\mu_i\mu_j \equiv const_2 := \mu_0$, see (2.3) below. Recall that for $\lambda = \mu = 0$, the edges are inserted/deleted independently of each other and the ratio μ_0/λ_0 defines the network edge density $1/(1 + \mu_0/\lambda_0)$ at stationarity. Hence, tuning the ratio μ_0/λ_0 one can achieve the desired edge density. Here we assume that the ratio μ_0/λ_0 is fixed and address the question about tuning parameters λ and μ for achieving desired values of clustering coefficients.

In the simulations we put the number of nodes $n = 1000$, $\mu_0 = n$ and $\lambda_0 = 1$ (for $\lambda = \mu = 0$ such network is sparse at stationarity). We only consider two instances of values of the pair (α, β): the choice of parameters $\alpha = 2.75$ and $\beta = 2.5$ is referred to as 'general triadic model' below; the choice of parameters $\alpha = \beta = 0$ is referred to as 'simple triadic model'. Given (α, β) we generate network Markov chains for different values of (μ, λ) from the range that features variability of the local clustering coefficient (our target parameter). For each choice of (μ, λ) we sample network snaphot G_t out of (approximately) stationary distribution and evaluate the edge density $e(G_t)$ (Fig. 1) and local clustering coefficient $\bar{C}^{\mathrm{L}}(G_t)$ (Fig. 2). To generate an approximately stationary network we run the respective Markov chain starting from an empty graph until $3n^2$ jumps (edge changes) occur. Further simulation steps do not change values of $e(G_t)$ and $\bar{C}^{\mathrm{L}}(G_t)$ beyond the rounding error.

In Figs 1 and 2 values of parameters μ and λ are depicted on the vertical and horizontal axis, respectively. Evenly spaced labels on each axis depict values of geometric sequences with the common ratio 1.35. The colours are put on logarithmic scale and the same scale is applied across different images.

As we can see from Figs 1 and 2, 'general triadic model' admits tunable (average) local clustering coefficient while the edge density remains reasonably small (recall that the ratio μ_0/λ_0 remains

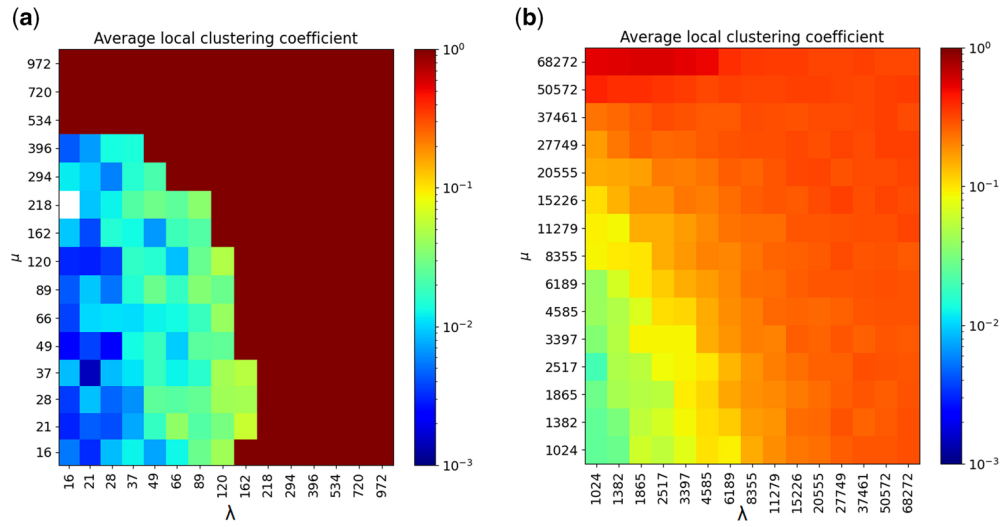

Figure 2. Average local clustering coefficients in stationary graphs. (a) Heatmap shows average clustering coefficient of simple triadic model for various values of λ and μ, while the other model parameters are fixed: $\alpha = \beta = 0$, $\lambda_i\lambda_j \equiv 1$, $\mu_i\mu_j \equiv 1000$, $n = 1000$. (b) Heatmap shows average clustering coefficient of general triadic model for various values of λ and μ, while the other model parameters are fixed: $\alpha = 2.75$, $\beta = 2.5$, $\lambda_i\lambda_j \equiv 1$, $\mu_i\mu_j \equiv 1000$, $n = 1000$.

fixed). On the other hand, 'simple triadic model' shows a swift jump from a sparse graph to the complete graph. Hence while trying to achieve the desired values of the clustering coefficient we are losing control over the edge density.

In Fig. 3a, we examine several clustering characteristics of the stationary network generated by the 'general triadic model' with $\mu = 15000$ and $\lambda = 20000$. Given integer $k \geq 2$, let $g(k)$ denote the number of nodes v of degree $d(v) = k$. Let $f(k) = \frac{1}{g(k)} \sum_{v:\, d(v)=k} C_v^{\mathrm{L}}(G)$ denote the average value of the local clustering coefficient over the set of nodes of degree k. We put $f(k) = 0$ for $g(k) = 0$. We call f the 'local clustering coefficient curve'. The fact that f is mostly decreasing tells us that the local clustering coefficient negatively correlates with node degree, a phenomenon observed in many sparse real networks ([20–23]). The 'general triadic model' reproduces this network property (see [12, 16, 22, 29, 30] for several other models of clustered networks having this property). We also mention that the edge density 0.004 is by two orders less than the average local clustering coefficient. Hence the network is sparse and highly clustered.

Lastly, we touch on the question of the component structure. One may wonder whether the high values of the clustering coefficients are caused by a few (perhaps one) relatively small, but dense subgraphs. Figure 3b shows that this is not the case. The stationary network generated by 'general triadic model' admits a large connected component collecting a fraction of nodes. For simplicity we put $\mu = 0$ (no triad protection). Hence the only remaining parameter to vary is λ. On the horizontal axis, we depict values of $\frac{\lambda}{n}$. We recall that the number of nodes $n = 1000$ remains fixed.

2.2. Analytical results

Let f be a real valued function defined on the set of graphs with the node set V. For example, it can be the number of edges $f(G) = |E|$ of graph $G = (V, E)$, or the number of closed triangles $f(G) = N_\Delta(G)$, etc. For a stationary Markov chain $\mathbf{G}$, the function $t \to \mathbf{E}f(G_t)$ is a constant. Hence $\frac{\partial}{\partial t}\mathbf{E}f(G_t) = 0$. This identity, when applied to properly chosen function f, can give useful information about average characteristics of the network at stationarity. We explore two instances. Choosing $f(G) = |E|$ we show lower and upper bounds for the average edge density $e_t := \mathbf{E}e(G_t)$;

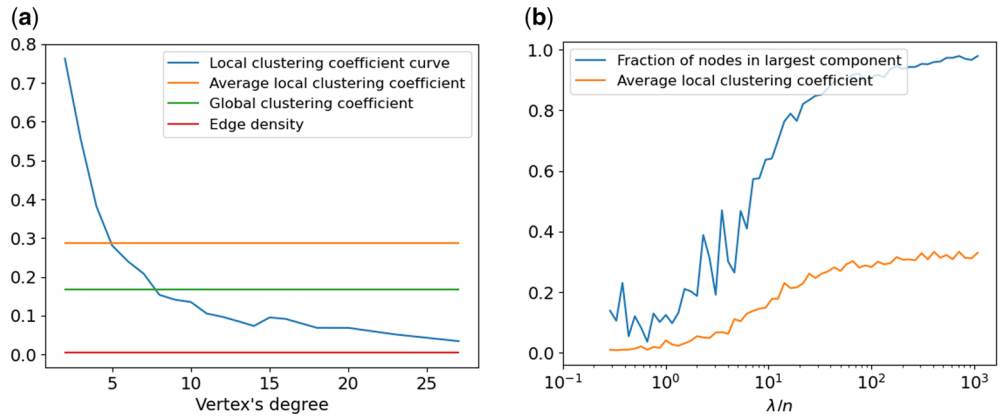

Figure 3. Clustering versus degree and the largest component size. (a) Downward curve shows negative relation between local clustering coefficient and degree in general triadic model with parameters $\alpha = 2.75$, $\beta = 2.5$, $\lambda_i\lambda_j \equiv 1$, $\mu_i\mu_j \equiv 1000$, $\mu = 15000$, $\lambda = 20000$, $n = 1000$. (b) Blue (orange) curve shows relation between the size of the largest component (average clustering coefficient) and the fraction λ/n in general triadic model with parameters $\alpha = 2.75$, $\beta = 2.5$, $\lambda_i\lambda_j \equiv 1$, $\mu_i\mu_j \equiv 1000$, $\mu = 0$, $n = 1000$.

choosing $f(G) = N_\Delta(G)$ we infer about the number of closed triangles. In what follows term triangle is used for the closed triangle exclusively.

Since for stationary $\mathbf{G}$ the average edge density e_t and average clustering coefficient $\mathbf{E}\bar{C}^{\text{L}}(G_t)$ do not depend on t, we sometimes drop the subscript t and write $e = e_t$ and $\bar{C}^{\text{L}} = \mathbf{E}\bar{C}^{\text{L}}(G_t)$. We observe that, by symmetry, the probability distribution of bivariate random variable $(d_v(G_t), \Delta_v(G_t))$ is the same for all $v \in V$. Furthermore, for a stationary network this distribution does not depend on t either. We denote by (d, Δ) a bivariate random variable having the same distribution as $(d_v(G_t), \Delta_v(G_t))$.

To make calculations feasible we assume for the rest of the section that the products $\lambda_i\lambda_j$ and $\mu_i\mu_j$ in (2.2) do not depend on i, j. In this case (2.2) reads as follows

$$a_{ij}(G) = \begin{cases} \lambda_0 + \lambda v_{ij}(G, \alpha) & \text{for} \quad \{i,j\} \notin E, \\ \big(\mu_0 - \mu v_{ij}(G, \beta)\big)_+ & \text{for} \quad \{i,j\} \in E, \end{cases} \tag{2.3}$$

where $\lambda, \lambda_0 > 0$ and $\mu, \mu_0 > 0$.

In Proposition 2.1 below we establish upper and lower bounds on the average edge density of the stationary Markow chain $\mathbf{G}$ defined by (2.3).

Proposition 2.1 We have that

$$e \geq \frac{\lambda_0}{\lambda_0 + \mu_0}. \tag{2.4}$$

For $\alpha, \beta \geq 2$ we have

$$e \leq \frac{\lambda_0 + \frac{1}{n-1}\max\{\lambda, \mu\}}{\lambda_0 + \mu_0}. \tag{2.5}$$

For $\alpha, \beta \geq 1$ and $\lambda_0 + \mu_0 > \max\{\lambda, \mu\}$ we have that

$$e \leq \frac{\lambda_0}{\lambda_0 + \mu_0 - \max\{\lambda, \mu\}}. \tag{2.6}$$

An important conlcusion to draw from inequalities (2.4), (2.5), (2.6) is that for $\frac{\mu_0}{\lambda_0}$ of the order n and $\max\{\lambda, \mu\}$ of the order μ_0 the network G_t is sparse and has average edge density of the order n^{-1} as $n \to +\infty$.

Proof of Proposition 2.1. Equation $\frac{\partial}{\partial t}\mathbf{E}|E_t| = 0$ implies

$$\mathbf{E}\sum_{\{i,j\}\notin E_t} a_{ij}(G_t) = \mathbf{E}\sum_{\{i,j\}\in E_t} a_{ij}(G_t). \tag{2.7}$$

In view of (2.3) we can write the latter identity in the form

$$\mathbf{E}\left(\lambda_0\left(\binom{n}{2} - |E_t|\right) + \nu_t' + \nu_t'' - \mu_0|E_t|\right) = 0, \tag{2.8}$$

where

$$\nu_t' = \lambda \sum_{\{i,j\}\notin E_t}\sum_{v\in N_{ij}} \frac{1}{d_v^{\alpha}} \qquad \text{and} \qquad \nu_t'' = \sum_{\{i,j\}\in E_t} \min\left\{\mu_0,\ \mu\sum_{v\in N_{ij}}\frac{1}{d_v^{\beta}}\right\} \tag{2.9}$$

account for the contribution of the clustering weights $\lambda\nu_{ij}(G_t, \alpha)$ and $\mu\nu_{ij}(G_t, \beta)$. Here we write, for short, $N_{ij} = N_{ij}(G_t)$ and $d_w = d_w(G_t)$. By the linearity of expectation, we obtain from (2.8) that

$$\lambda_0 - (\lambda_0 + \mu_0)e_t + \binom{n}{2}^{-1}\mathbf{E}(\nu_t' + \nu_t'') = 0. \tag{2.10}$$

The inequalities $\nu_t' \geq 0, \nu_t'' \geq 0$ imply $\lambda_0 - (\lambda_0 + \mu_0)e_t \geq 0$. We arrived to lower bound (2.4).

Let us show upper bounds (2.5), (2.6). We estimate

$$\nu_t' \leq \lambda \sum_{\{i,j\}\notin E_t}\sum_{v\in N_{ij}} \frac{1}{d_v^{\min\{\alpha,\beta\}}},$$

$$\nu_t'' \leq \mu \sum_{\{i,j\}\in E_t}\sum_{v\in N_{ij}} \frac{1}{d_v^{\beta}} \leq \mu \sum_{\{i,j\}\in E_t}\sum_{v\in N_{ij}} \frac{1}{d_v^{\min\{\alpha,\beta\}}}.$$

Combining these inequalities we obtain

$$\nu_t' + \nu_t'' \leq \max\{\lambda,\mu\}\sum_{1\leq i<j\leq n}\sum_{v\in N_{ij}} \frac{1}{d_v^{\min\{\alpha,\beta\}}} = \max\{\lambda,\mu\}\sum_{v\in V:\, d_v\geq 2} \frac{1}{d_v^{\min\{\alpha,\beta\}}}\binom{d_v}{2}$$

$$\leq \max\{\lambda,\mu\}\frac{1}{2}\sum_{v\in V} d_v^{2-\min\{\alpha,\beta\}}.$$

For $\min\{\alpha, \beta\} = 2$ we have $\nu_t' + \nu_t'' \leq \frac{n}{2}\max\{\lambda, \mu\}$. Invoking this inequality in (2.10) we obtain (2.5). For $\min\{\alpha, \beta\} = 1$ we have $\nu_t' + \nu_t'' \leq \max\{\lambda, \mu\}|E_t|$. Now (2.10) yields (2.6). *End of the proof.*

The next Proposition 2.2 relates the average edge density to average local clustering coefficient in the special case of $\alpha = 2$ and $\mu = 0$. In this special case, we consider a slightly modified version of (2.3) that includes the 'correction term'

$$\varkappa_{ij}(G) = \frac{1}{n-1}\left(\mathbb{I}_{\{d_i(G)=0\}} + \mathbb{I}_{\{d_j(G)=0\}}\right) + \frac{1}{n-2}\left(\mathbb{I}_{\{d_i(G)=1\}} + \mathbb{I}_{\{d_j(G)=1\}}\right)$$

and replaces $\nu_{ij}(G,2)$ by related quantity $\nu^*_{ij}(G) = \sum_{w\in N_{ij}(G)} \binom{d_w(G)}{2}^{-1}$. We set

$$a_{ij}(G) = \begin{cases} \lambda_0 + \lambda\nu^*_{ij}(G) + \lambda\varkappa_{ij}(G) & \text{for} \quad \{i,j\} \notin E, \\ \mu_0 & \text{for} \quad \{i,j\} \in E. \end{cases} \tag{2.11}$$

The reason for such a modification is that it admits a closed form expression for the average edge density.

Proposition 2.2 For a stationary network Markov chain defined by (2.11) we have that

$$e = \frac{\lambda_0 + \frac{2}{n-1}\lambda(1-\bar{C}^{\mathrm{L}})}{\lambda_0 + \mu_0}. \tag{2.12}$$

Noting that $\bar{C}^{\mathrm{L}} \le 1$ we obtain from (2.12) the upper and lower bounds for the average edge density

$$\frac{\lambda_0}{\lambda_0+\mu_0} \le e \le \frac{\lambda_0 + \frac{2}{n-1}\lambda}{\lambda_0+\mu_0}.$$

Letting $n \to +\infty$ and choosing $\frac{\mu_0}{\lambda_0}$, $\frac{\lambda}{\lambda_0}$ and $\frac{\mu}{\lambda_0}$ of the order n we have that e is of the order n^{-1}. Hence the model produces a sparse dynamic network.

Proof of Proposition 2.2. Equation (2.7) implies

$$\mathbf{E}\sum_{\{i,j\}\notin E_t}\left(\lambda_0 + \lambda\varkappa_{ij} + \lambda\sum_{w\in N_{ij}}\frac{1}{\binom{d_w}{2}}\right) = \mu_0\mathbf{E}|E_t|. \tag{2.13}$$

Here we write, for short, $\varkappa_{ij} = \varkappa_{ij}(G_t)$, $N_{ij} = N_{ij}(G_t)$ and $d_w = d_w(G_t)$. Invoking the identities

$$\sum_{\{i,j\}\notin E_t} 1 = \binom{n}{2} - |E_t|,$$

$$\sum_{\{i,j\}\notin E_t}\varkappa_{ij} = \sum_{w\in V}\mathbb{I}_{\{d_w=0\}} + \sum_{w\in V}\mathbb{I}_{\{d_w=1\}},$$

$$\sum_{\{i,j\}\notin E_t}\sum_{w\in N_{ij}}\frac{1}{\binom{d_w}{2}} = \sum_{w\in V:\, d_w\ge 2}\frac{\binom{d_w}{2} - \Delta_w(G_t)}{\binom{d_w}{2}} = \sum_{w\in V:\, d_w\ge 2}(1 - C^{\mathrm{L}}_w(G_t))$$

and dividing both sides of (2.13) by $\binom{n}{2}$ we have

$$\lambda_0(1-e) + \frac{2\lambda}{n-1}\left(\mathbf{P}\{d\le 1\} + \mathbf{P}\{d\ge 2\} - \bar{C}^{\mathrm{L}}\right) = \mu_0 e,$$

where d denotes the degree of a randomly selected node. We have arrived to (2.12). *End of the proof.*

Our next result establishes a lower bound on the average number of triangles.

Proposition 2.3 Let $0 < \alpha \leq 2$. For a stationary network Markov chain defined by (2.3) we have

$$\mathbf{E}\Delta \geq \frac{\lambda}{4(\lambda_0+\mu_0+\lambda)}\mathbf{P}\{d \geq 2\}. \tag{2.14}$$

An important conlcusion to draw from inequality (2.14) is that choosing $\frac{\mu_0}{\lambda_0}$, $\frac{\lambda}{\lambda_0}$ and $\frac{\mu}{\lambda_0}$ of the order n one can obtain a sparse stationary dynamic network with the property that the average number of triangles incident to a node of degree at least two (formally, the conditional expectation $\mathbf{E}(\Delta|d \geq 2) = \frac{\mathbf{E}\Delta}{\mathbf{P}\{d\geq 2\}}$) is bounded from below by a constant. Note that $d_v(G_t) \leq 1$ implies $\Delta_v(G_t) = 0$. Hence $\Delta_v(G_t) = \Delta_v(G_t)\mathbb{I}_{\{d_v(G_t)\geq 2\}}$ and $\Delta = \Delta\mathbb{I}_{\{d\geq 2\}}$.

Proof of Proposition 2.3. Equation $\frac{\partial}{\partial t}\mathbf{E}N_\Delta(G_t) = 0$ implies

$$\mathbf{E}\sum_{\{i,j\}\notin E_t}|N_{ij}(G_t)|a_{ij}(G_t) = \mathbf{E}\sum_{\{i,j\}\in E_t}|N_{ij}(G_t)|a_{ij}(G_t). \tag{2.15}$$

Here the left sum evaluates the average birth rate of triangles: connecting a pair of non-adjacent nodes i, j by an edge creates $|N_{ij}(G_t)|$ new triangles. The right sum evaluates the average death rate of triangles: deletion of an edge $\{i, j\} \in E_t$ eliminates $|N_{ij}(G_t)|$ triangles from G_t. Furthermore, for $\{i, j\} \in E_t$ we have $a_{ij}(G_t) \leq \mu_0$. Hence the sum on the right of (2.15)

$$\sum_{\{i,j\}\in E_t}|N_{ij}(G_t)|a_{ij}(G_t) \leq \sum_{\{i,j\}\in E_t}|N_{ij}(G_t)|\mu_0 = \sum_{v\in V:\, d_v(G_t)\geq 2}\Delta_v(G_t)\mu_0. \tag{2.16}$$

In the last identity we use the observation that $\Delta_v(G_t)$ counts edges whose both endpoints are adjacent to v. Similarly for the sum on the left of (2.15)

$$\sum_{\{i,j\}\notin E_t}|N_{ij}(G_t)|a_{ij}(G_t) \geq \sum_{v\in V:\, d_v(G_t)\geq 2}\left(\binom{d_v(G_t)}{2} - \Delta_v(G_t)\right)\left(\lambda_0 + \frac{\lambda}{d_v^\alpha(G_t)}\right). \tag{2.17}$$

Here we use the observation that $\binom{d_v(G_t)}{2} - \Delta_v(G_t)$ counts pairs $\{i, j\}$ of neighbours of v that are non-adjacent ($\{i, j\} \notin E_t$). Inequality (2.17) follows from the fact that $a_{ij}(G_t) \geq \lambda_0 + \frac{\lambda}{d_v^\alpha}$ for each $v \in N_{ij}(G_t)$.

Invoking (2.16) and (2.17) in (2.15) we obtain

$$\mathbf{E}\sum_{v\in V:\, d_v(G_t)\geq 2}\left(\binom{d_v(G_t)}{2} - \Delta_v(G_t)\right)\left(\lambda_0 + \frac{\lambda}{d_v^\alpha(G_t)}\right) \leq \mathbf{E}\sum_{v\in V:\, d_v(G_t)\geq 2}\Delta_v(G_t)\mu_0.$$

Recall that the probability distribution of bivariate random variable $(d_v(G_t), \Delta_v(G_t))$ is the same for all $v \in V$. Collecting the terms $\Delta_v(G_t)$ on the right and dividing both sides by n we have

$$\frac{\lambda_0}{2}\mathbf{E}\,(d(d-1)) + \frac{\lambda}{2}\mathbf{E}(d^{1-\alpha}(d-1)\mathbb{I}_{\{d\geq 2\}}) \leq (\lambda_0+\mu_0)\mathbf{E}\Delta + \lambda\mathbf{E}\left(\Delta_v\frac{\mathbb{I}_{\{d_v\geq 2\}}}{d_v^\alpha}\right).$$

Next we upper bound $\lambda\mathbf{E}\left(\Delta_v d_v^{-\alpha}\mathbb{I}_{\{d_v\geq 2\}}\right)\leq\lambda\mathbf{E}\Delta$ and obtain

$$\frac{\lambda}{2}\mathbf{E}\left(d^{1-\alpha}(d-1)\mathbb{I}_{\{d\geq 2\}}\right)\leq(\lambda_0+\mu_0+\lambda)\mathbf{E}\Delta.$$

Furthermore, using inequality $\frac{1}{2}\leq\frac{d-1}{d}\leq\frac{d-1}{d^{\alpha-1}}$, which holds for $0<\alpha\leq 2$ and $d\geq 2$ we lower bound the left side by $\frac{\lambda}{4}\mathbf{E}\mathbb{I}_{\{d_v\geq 2\}}$ and obtain inequality equivalent to (2.14)

$$\frac{\lambda}{4}\mathbf{P}\{d\geq 2\}\leq(\lambda_0+\mu_0+\lambda)\mathbf{E}\Delta.$$

End of the proof.

3. DYNAMIC AFFILIATION NETWORK

Let $\mathbf{H}=\{H_t=(V\cup W,E_t),\ t\geq 0\}$ be a continuous time Markov chain, whose states are bipartite graphs with the bipartition $V\cup W$, where $V=\{1,\dots,n\}$ and $W=\{1,\dots,m\}$. Transitions of $\mathbf{H}$ are defined as follows. Given H_t (the state occupied at time t), the update takes place at time $t':=t+\min_{(i,u)\in V\times W}B_{iu}$ when the pair (i_0,u_0) with $B_{i_0u_0}=\min_{(i,u)\in V\times W}B_{iu}$ changes its adjacency status. Here $B_{iu}=B_{iu}(H_t)$, are independent exponential waiting times with intensities $b_{iu}=b_{iu}(H_t)$ defined below. Thus, at time t' chain $\mathbf{H}$ jumps to the state $H_{t'}=(V\cup W,E_{t'})$, where the edge sets E_t and $E_{t'}$ differ in the single edge $i_0\sim u_0$. Markov chain $\mathbf{H}$ defines dynamic affiliation network $\mathbf{G}'=\{G'_t=(E'_t,V),t\geq 0\}$: for each t any two nodes $i,j\in V$ are adjacent in G'_t whenever i and j have a common neighbour in H_t.

Now we define intensities b_{iu}. We fix $\mu>0$ and assign positive weights y_i and x_u to $i\in V$ and $u\in W$ that model activity of actors and attractiveness of attributes. For a bipartite graph $H=(V\cup W,E)$ we set

$$b_{iu}(H)=\begin{cases}y_ix_u & \text{for}\quad (i,u)\notin E,\\ \mu & \text{for}\quad (i,u)\in E.\end{cases}\tag{3.1}$$

Clearly, $\mathbf{H}$ has a unique stationary distribution defined by the weight sequences $\{y_i\}_{i=1}^n$, $\{x_u\}_{u=1}^m$ and μ. Furthermore, $\mathbf{H}$ comprises of $n\times m$ independent continuous Markov chains describing adjacency dynamic of each node pair $(i,u)\subset V\times W$ separately, where the Markov chain of a pair (i,u) has two states $i\sim u$ and $i\not\sim u$ whose exponential holding times have intensities μ and y_ix_u respectively. Thus, at stationarity, a snaphot H_t represents a random bipartite graph, where edges are inserted independently with probabilities

$$\mathbf{P}\{i\sim u\}=\frac{y_ix_u}{y_ix_u+\mu}=:p_{iu},\tag{3.2}$$

for $(i,u)\in V\times W$. We assume in what follows that dynamic affiliation network $\mathbf{G}'$ is defined by a stationary Markov chain $\mathbf{H}$ satisfying (3.2). In this case, probability distributions of random graphs G'_t and H'_t do not depend on t and with a little abuse of notation we write, for short, $G'=G'_t$ and $H=H_t$.

The random graph G' is closely related to the inhomogeneous random intersection graph introduced in [31] and generalised in [32, 33]. The difference between G' and the random graph of [32, 33] is that in the latter graph node weights $X_1,\dots,X_m$ and attribute weights $Y_1,\dots,Y_n$ are sampled independently at random from given probability distributions P_X and P_Y. Moreover, ratio (3.2) is replaced by $\min\{1,\frac{Y_iX_u}{\sqrt{nm}}\}$. In particular, the distribution of the random graph of [32, 33] is invariant under permutations of the nodes. This is not the case for G', where variable node weights are non-random.

It has been shown in [32, 33] that letting $n, m \to +\infty$ so that $m = m(n)$ is asymptotically linear in n the inhomogeneous random intersection graph admits asymptotic degree distribution (including a power law) and displays clustering. Namely, [33] evaluates the conditional probability

$$\mathbf{P}\{u \sim v | u \sim w, v \sim w\} = \frac{(\mathbf{E}X^3)(\mathbf{E}Y)^3}{(\mathbf{E}X^3)(\mathbf{E}Y)^3 + (\mathbf{E}X^2)^2(\mathbf{E}Y^2)^2(\mathbf{E}Y)^2\sqrt{m/n}} + o(1), \tag{3.3}$$

where $u, v, w \in V$ is arbitrary but given triple of nodes. Here X and Y denote random variables with distributions P_X and P_Y. We remark that the probability in (3.3) is related to the global clustering coefficient. In fact, it is an approximation to the global clustering coefficient, see discussion after Theorem 3.4 below. Now we only remark that the fraction in (3.3) tends to 0 (respectively 1) for $m/n \to +\infty$ (respectively $m/n \to 0$). Hence, in order to get a network with a non-trivial clustering one needs to choose m asymptotically linear in n, say $m/n \to \gamma$, for some $\gamma > 0$.

Let us outline analytical results of this section. For G' being a version of the random graph [32, 33] conditioned on the weights $X_1 = x_1, \dots, X_m = x_m$ and $Y_1 = y_1, \dots, Y_n = y_n$ it is reasonable to ask how the weights of individual nodes affects their degree distributions. We address this question in Theorems 3.1 and 3.3 below. Furthermore, our Theorem 3.4 establishes the first order asymptotic as $n, m \to +\infty$ to the global clustering coefficient $C^{\mathrm{GL}}(G')$.

To proceed further we introduce some notation. By $P_{y,n} = \frac{1}{n}\sum_{i=1}^n \delta_{y_i}$ and $P_{x,m} = \frac{1}{m}\sum_{u=1}^m \delta_{x_u}$ we denote empirical distributions of the sequences $\{y_1, \dots, y_n\}$ and $\{x_1, \dots, x_m\}$. Here δ_t stands for the degenerate distribution that assigns mass 1 to point t. Furthermore, we denote

$$\langle x^s \rangle = \frac{1}{m}\sum_{u\in[m]} x_u^s, \quad \langle y^s \rangle = \frac{1}{n}\sum_{i\in[n]} y_i^s, \quad \beta^2 = \frac{m}{n}, \quad \varkappa = \frac{nm}{\mu^2}.$$

Degrees of G'. Here we show that the expected value $\mathbf{E}d_i$ of the degree $d_i = d_i(G')$ of node i is approximately proportional to its weight y_i. Moreover, d_i has asymptotic compound Poisson distribution as the network size $n \to +\infty$.

THEOREM 3.1 Let $n, m \to +\infty$. Put $\mu = \sqrt{nm}$ and assume that for some $c > 0$ we have $\langle x^3 \rangle \le c$ and $\langle y^2 \rangle \le c$ uniformly in n, m. Then for each i

$$\mathbf{E}d_i = y_i\langle x^2\rangle\langle y\rangle + O\left(\frac{y_i^2}{\sqrt{nm}}\right). \tag{3.4}$$

Theorem 3.1 is an immediate consequence of the following (more general) result.

PROPOSITION 3.2. For each $i \in V$ we have

$$0 \le y_i\varkappa\,\langle x^2\rangle\langle y\rangle - \mathbf{E}d_i \le \frac{\varkappa}{\mu}y_i\left(\langle x^3\rangle\langle y^2\rangle + y_i\langle x^3\rangle\langle y\rangle\right) + \frac{\varkappa^2}{n}y_i^2\langle x^2\rangle^2\langle y^2\rangle + \frac{1}{n}y_i\langle x^2\rangle. \tag{3.5}$$

It follows from Proposition 3.2 that for large n, m random graph G' admits bounded average degrees whenever $\mu = \mu(n, m)$ is of the order $\sqrt{nm}$.

As we are interested in sparse and clustered networks it is reasonable to assume that $\mu = \sqrt{nm}$ and that $m = m(n)$ is asymptotically linear in n as $n \to +\infty$. Theorem 3.1 now implies that $\mathbf{E}d_i$ stays bounded as $n, m \to +\infty$. In this case, the probability distribution of d_i does not concentrate around the expected value $\mathbf{E}d_i$. Theorem 3.3 below shows that d_i has a compound Poisson asymptotic distribution. Recall that compound Poisson distribution is the probability distribution of a randomly stopped sum $\sum_{k=1}^{\Lambda}\xi_k$, where $\xi_1, \xi_2, \dots$ are independent and identically distributed random variables, which are independent of the Poisson random variable Λ. We write $\Lambda \sim \mathscr{P}(\lambda)$,

where $\lambda := \mathbf{E}\Lambda$ denotes the expected value and denote by $\mathscr{CP}(\lambda, P_\xi)$ the (compound Poisson) distribution of $\sum_{k=1}^{\Lambda} \xi_k$. Here P_ξ denotes the (common) probability distribution of ξ_k.

Let $x_1, x_2, \dots$ and $y_1, y_2, \dots$ be positive infinite sequences of weights. In Theorem 3.3, we consider random affiliation networks $G' = G'_{n,m}$, $n, m = 1, 2, \dots$, based on respective bipartite random graphs $H_{n,m}$ whose edges are inserted independently with probabilities (3.2). Note that each $H_{n,m}$ is defined by truncated (finite) sequences $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$.

To formulate our next result we need the following conditions: for $n, m \to +\infty$ we have

- $P_{x,m}$ converges weakly to some probability distribution, say P_X, having a finite first moment $\int sP_X(ds) < \infty$ and $\langle x \rangle$ converges to $\int sP_X(ds)$;
- the family of distributions $\{P_{y,n},\ n = 1, 2, \dots\}$ is uniformly integrable and $\langle y \rangle$ converges to some number $a_y > 0$.

THEOREM 3.3 Let $\mu = \sqrt{nm}$. Let $n \to +\infty$. Assume that $m = m(n)$ is such that m/n converges to some $\gamma > 0$. Assume that (i) and (ii) hold. Denote $a_x = \int sP_X(ds)$ and introduce function $s \to \lambda_s = sa_y\gamma^{-1}$. For each $i = 1, 2, \dots$ the probability distribution of d_i converges weakly to the compound Poisson distribution $\mathscr{CP}(y_i a_x \gamma, Q)$, where the discrete probability distribution Q assigns probabilities

$$Q(\{t\}) = \int \frac{s}{a_x} e^{-\lambda_s} \frac{\lambda_s^t}{t!} P_X(ds). \tag{3.6}$$

to integers $t = 0, 1, 2, \dots$. We note that Q is a mixture of Poisson distributions with a size biased random parameter. To sample from Q one can use the two step procedure: (i) generate a (size biased) random variable $\tilde{X}$ according to the distribution $\mathbf{P}\{\tilde{X} = s\} = \frac{s}{a_x}\mathbf{P}\{X = s\}$, $s = 0, 1, \dots$; and (ii) sample Poisson random variable with rate $\tilde{X}a_y\gamma^{-1}$. The intuition behind the degree distribution formula of Theorem 3.3 is that given node i is linked to (asymptotically) Poissonian number of attributes and each attribute, in addition, is linked to random number of nodes having mixed Poisson distribution.

Clustering in G'. We recall that $N_\Delta(G')$ denotes the number of (closed) triangles in G' and $N_\Lambda(G')$ denotes the number of 2-paths in G'.

THEOREM 3.4 Let $\mu = \sqrt{nm}$. Let $n \to +\infty$. Assume that $m = m(n)$ is such that m/n converges to some $\gamma > 0$. Assume that for some constant $c > 0$ we have $\langle x^5 \rangle < c$ and $\langle y^4 \rangle < c$ for all n. Then

$$N_\Delta(G') = \frac{n}{6\beta}\langle x^3 \rangle \langle y \rangle^3 + o_P(\sqrt{n}), \tag{3.7}$$

$$N_\Lambda(G') = \frac{n}{2\beta}\langle x^3 \rangle \langle y \rangle^3 + \frac{n}{2}\langle x^2 \rangle^2 \langle y^2 \rangle \langle y \rangle^2 + O_P(\sqrt{n}). \tag{3.8}$$

In particular, the global clustering coefficient

$$C^{\mathrm{GL}}(G') = \frac{\langle x^3 \rangle \langle y \rangle^3}{\langle x^3 \rangle \langle y \rangle^3 + \sqrt{\gamma}\langle x^2 \rangle^2 \langle y^2 \rangle \langle y \rangle^2} + o_P(1). \tag{3.9}$$

We remark that conditions $\langle x^5 \rangle < c$ and $\langle y^4 \rangle < c$ of Theorem 3.4 can be relaxed. We expect that the minimal conditions $\langle x^3 \rangle < c$ and $\langle y^2 \rangle < c$ plus the uniform integrability of $t^3 P_{x,m}(dt)$ and $t^2 P_{y,n}(dt)$ would suffice.

We note that the result (3.9) is consistent with the earlier finding (3.3). In particular, the fraction in (3.3) can be obtained from (3.9) if we let $m, n \to +\infty$ and replace averages by respective expectations.

Finally, we comment on the proofs of Theorems 3.3, 3.4, and Proposition 3.2 that are given in the Supplementary Appendix. We mention that the underlying bipartite structure adds a level of complexity. Although the limiting (underlying) random bipartite graph is locally tree-like, the projection graph G' has quite a few short cycles [34]. This makes the analysis of the degree distribution nontrivial. The same comment applies to the global clustering coefficient, where (3.9) is established by showing the concentration of triangle counts and 2-path counts (3.7), (3.8). Here we apply the second moment method (Chebyshev's inequality).

4. CONCLUDING REMARKS

We presented two dynamic network models that generate sparse and clustered stationary networks. Both models seems natural as they mimic dynamics of real network processes. Luckily, for rigorous analysis of dynamic affiliation network we can use techniques developed for random intersection graphs [2, 8, 13, 27, 34, 35]. On the other hand we have only a few rigorous results for stationary Markov chains with clustering like (2.2), (2.3). It would be interesting to learn more about network structure and properties of this model via rigorous analysis.

ACKNOWLEDGEMENTS

We thank anonymous referees for their comments. We thank Information technology research centre of Vilnius University for a high performance computing resources. The diagrams were generated using Matplotlib [36].

SUPPLEMENTARY DATA

Supplementary data are available at *COMNET Journal* online.

FUNDING

None declared.

REFERENCES

1. Newman M. *Networks*, 2nd edn. Oxford: Oxford University Press, 2018.
2. van der Hofstad R. *Random Graphs and Complex Networks. Volume Two*. Volume 54 in Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press, 2024.
3. Kaminski B, Pralat P, Theberge F. *Mining Complex Networks*. New York: Chapman and Hall/CRC, 2021.
4. Holme P, Kim BJ. Growing scale-free networks with tunable clustering. *Phys Rev E* 2002;**65**:026107.
5. Newman MEJ. Random graphs with clustering. *Phys Rev Lett* 2009;**103**:058701.
6. Bollobás B, Janson S, Riordan O. Sparse random graphs with clustering. *Random Struct Alg* 2011;**38**:269–323.
7. Guillaume J-L, Latapy M. Bipartite structure of all complex networks. *Inf Process Lett* 2004;**5**:215–21.
8. Bloznelis M, Godehardt E, Jaworski J et al. Recent progress in complex network analysis. models of random intersection graphs. In: Lausen B, Krolak-Schwerdt S, Böhmer M (eds). *Data Science, Learning by Latent Structures, and Knowledge Discovery. Studies in Classification, Data Analysis, and Knowledge Organization*. Berlin: Springer, 2015, 69–78.
9. Deijfen M, Kets W. Random intersection graphs with tunable degree distribution and clustering. *Probab Eng Inform Sci* 2009;**23**:661–74.
10. Frieze A, Karoński M. *Introduction to Random Graphs*. Cambridge: Cambridge University Press, 2015.
11. Spirakis P, Nikoletseas S, Raptopoulos CH. A guided tour in random intersection graphs. In: Fomin FV, Freivalds R, Kwiatkowska M, Peleg D (eds). Automata, Languages, and Programming. ICALP *Lect. Notes Comput Sci*. Vol. **7966**. Berlin: Springer, 2013, 29–35.
12. Bloznelis M, Leskelä L. Clustering and percolation on superpositions of Bernoulli random graphs. *Random Struct Alg* 2023;**63**:283–342.

13. van der Hofstad R, Komjáthy J, Vadon V. Random intersection graphs with communities. *Adv Appl Probab* 2021;**53**:1061–89.
14. Aiello W, Bonato A, Cooper C *et al.* A spatial web graph model with local influence regions. *Internet Math* 2009;**5**:175–96.
15. Bringmann K, Keusch R, Lengler J. Geometric inhomogeneous random graphs. *Theoret Comput Sci* 2019;**760**:35–54.
16. Fountoulakis N, van der Hoorn P, Müller T *et al.* Clustering in a hyperbolic model of complex networks. *Electron J Probab* 2021;**26**:1–13.
17. Grindrod P, Higham DJ, Parsons MC. Bistability through triadic closure. *Internet Math* 2012;**8**:402–23.
18. Užupytė R, Wit EC. Test for triadic closure and triadic protection in temporal relational event data. *Soc Netw Anal Min* 2020;**10**:20–1.
19. Zhang X, Cristopher Moore C, Newman MEJ. Random graph models for dynamic networks. *Eur Phys J B* 2017;**90**:20.
20. Foudalis I, Jain K, Papadimitriou C *et al.* Modeling social networks through user background and behavior. In: *Algorithms and Models for the Web Graph. Proceedings of the 8th International Workshop, WAW 2011*, Lecture Notes in Computer Science, Vol. 6732. Berlin: Springer, 2011, 85–102.
21. Ravasz E, Somera AL, Mongru DA *et al.* Hierarchical organization of modularity in metabolic networks. *Science* 2002;**297**:1551–5.
22. Ravasz E, Barabási AL. Hierarchical organization in complex networks. *Phys Rev E* 2003;**67**:026112.
23. Vázquez A, Pastor-Satorras R, Vespignani A. Large-scale topological and dynamical properties of Internet. *Phys Rev E* 2002;**65**:066130.
24. Bloznelis M, Götze F. Preferred attachment in affiliation networks. *J Stat Phys* 2014;**156**:800–21.
25. Bloznelis M, Karoński M. Random intersection graph process. *Internet Math* 2015;**11**:385–402.
26. Guillaume J-L, Latapy M. Bipartite graphs as models of complex networks. *Phys A Stat Mech Appl* 2006;**371**:795–813.
27. Milewska M, van der Hofstad R, Zwart B. Dynamic random intersection graph: dynamic local convergence and giant structure. *Random Struct Alg* 2024;**66**:e21264.
28. Chung F, Lu L. The average distances in random graphs with given expected degrees. *Proc Natl Acad Sci USA* 2002;**99**:15879–82.
29. Bloznelis M. Degree and clustering coefficient in sparse random intersection graphs. *Ann Appl Probab* 2013;**23**:1254–1289.
30. Iskhakov L, Kaminski B, Mironov M *et al.* Local clustering coefficient of spatial preferential attachment model. *J Complex Netw* 2020;**8**:cn019.
31. Shang Y. Degree distribution in general random intersection graphs. *Electron J Comb* 2010;**17**:R23.
32. Bloznelis M, Damarackas J. Degree distribution of an inhomogeneous random intersection graph. *Electron J Comb* 2013;**20**:R3.
33. Bloznelis M, Kurauskas V. Clustering function: another view on clustering coefficient. *J Complex Netw* 2016;**4**:61–86.
34. Kurauskas V. On local weak limit and subgraph counts for sparse random graphs. *J Appl Probab* 2022;**59**:755–76.
35. Gröhn T, Karjalainen J, Leskelä L. Clique and cycle frequencies in a sparse random graph model with overlapping communities. *Stoch Models* 2024;**40**:634–58.
36. Hunter JD. Matplotlib: a 2D graphics environment. *Comput Sci Eng* 2007;**9**:90–5.
37. Steele JM. Le Cam's inequality and Poisson approximations. *Am Math Monthly* 1994;**101**:48–54.