DE GRUYTER Open Mathematics 2025; 23: 20250177

**Research Article**

Ramūnas Garunkštis, Tadas Panavas, and Raivydas Šimenas*

# Decompositions of the extended Selberg class functions

https://doi.org/10.1515/math-2025-0177
received October 5, 2023; accepted June 11, 2025

**Abstract:** Let $F(s)$ be a function from the extended Selberg class. We consider decompositions $F(s) = f(h(s))$, where $f$ and $h$ are meromorphic functions. Among other things, we show that $F$ is prime if and only if the greatest common divisor of the orders of all zeros and the pole of $F$ is 1.

**Keywords:** extended Selberg class, prime functions

**MSC 2020:** 11M06

## 1 Introduction

Throughout the article, $s = \sigma + it$ denotes a complex variable. We begin with the definitions of the Selberg and extended Selberg classes. A. Selberg introduced the Selberg class in [1]. Kaczorowski and Perelli introduced the extended Selberg class in [2]. The prototypical example of an element of the class is the Riemann zeta function $\zeta(s)$. For further information about the Selberg class, we refer to survey papers [3–5]. Note that Lekkerkerker [6], Perelli [7], and Matsumoto [8] introduced similar classes.

**Definition.** The *Selberg class* $\mathcal{S}$ consists of functions $F$ satisfying the following axioms:

(i) (ordinary Dirichlet series) $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, absolutely convergent for $\sigma > 1$;

(ii) (analytic continuation) there exists a nonnegative integer $k$ such that $(s-1)^k F(s)$ is an entire function whose growth as $|s| \to \infty$ is

$$|F(s)| \ll e^{|s|^{O(1)}}, \quad |s| \to \infty;$$

(iii) (functional equation) $F(s)$ satisfies a functional equation of type

$$\Lambda_F(s) = \omega \overline{\Lambda_F(1-\bar{s})}. \tag{1}$$

Here, $\Lambda_F(s) \coloneqq F(s)Q^s \prod_{j=1}^{N} \Gamma(\lambda_j s + \mu_j)$ with positive real numbers $Q, \lambda_j$ and complex numbers $\mu_j, \omega$ with $\Re(\mu_j) \geq 0$ and $|\omega| = 1$;

(iv) (Ramanujan hypothesis) $a(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$, where the implicit constant may depend on $\varepsilon$.

(v) (Euler product) $\log F(s) = \sum_{n=1}^{\infty} b_F n^{-s}$, where $b_F(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_F(n) \ll n^{\vartheta}$ for some $\vartheta < 1/2$.

*** Corresponding author: Raivydas Šimenas,** Faculty of Mathematics and Informatics, Institute of Mathematics, Vilnius University, Vilnius, Lithuania, e-mail: raivydas.simenas@mif.vu.lt
**Ramūnas Garunkštis:** Faculty of Mathematics and Informatics, Institute of Mathematics, Vilnius University, Vilnius, Lithuania, e-mail: ramunas.garunkstis@mif.vu.lt
**Tadas Panavas:** Faculty of Mathematics and Informatics, Institute of Mathematics, Vilnius University, Vilnius, Lithuania; Faculty of Business and Technologies, Utenos kolegija Higher Education Institution, Utena, Lithuania, e-mail: panavas.tadas@gmail.com

**Definition.** The *extended Selberg class* $\mathcal{S}^\sharp$ consists of Dirichlet series $F(s)$ that satisfy the first three axioms (i), (ii), and (iii).

The *degree* of $F$ is defined as $d_F \coloneqq 2\sum_{j=1}^{N}\lambda_j$. While for a given $F$, many functional equations of type (1) might exist, the degree of $F$ is an invariant. In addition, note that in the definition of $\Lambda$, the values of $\lambda_j$ and $\mu_j$ need not be pairwise distinct. It is expected that all ratios $\lambda_j/\lambda_m$, $1 \le j, m \le N$, are rational numbers, even more so that every $F \in \mathcal{S}^\sharp$ has the expression $\Lambda_F(s)$ with $\lambda_j = 1/2$ for $1 \le j \le N$, see Perelli [5, Conjectures 4.1 and 4.2].

The zeros of $F(s)$ located at the poles of $\Gamma$ factors appearing in the functional equation are called trivial. They are located at

$$s = -\frac{k + \mu_j}{\lambda_j} \quad \text{with } k \in \mathbb{N}_0 \text{ and } 1 \le j \le N. \tag{2}$$

Hence, the multiplicity of the trivial zero is not greater than $N$. All other zeros are said to be nontrivial.

**Definition.** A function is *transcendental* if it is meromorphic and not rational. This article considers only meromorphic functions defined over the entire complex plane.

**Definition.** (see Gross [9], [10], Chuang and Yang [11, Section 3.2], Urabe [12]). Let $F$ be a meromorphic function. Then an expression

$$F(s) = f(h(s)), \tag{3}$$

where $f$ and $h$ are meromorphic functions, is called a *decomposition* of $F$ with $f$ and $h$ as its left and right components, respectively. $F$ is said to be *prime* in the sense of a decomposition if for every representation of $F$ of the form (3), we have that either $f$ or $h$ is a fractional linear transformation. If every representation of $F$ of the form (3) implies that $f$ or $h$ is rational (respectively: $f$ is a fractional linear transformation whenever $h$ is transcendental, $h$ is a fractional linear transformation whenever $f$ is transcendental), we say that $F$ is *pseudo-prime* (respectively: *left-prime*, *right-prime*). Note that if $F$ is left-prime or right-prime, then $F$ is also pseudo-prime.

Note that in (3), if $f$ is transcendental, then $h$ must be entire. This follows from the fact that the transcendental meromorphic $f$ has an essential singularity or a limit of poles at infinity. If $h$ had a pole at $s_0$, then $f \circ h$ would have an essential singularity or a limit of poles at $s_0$. Consequently, $F$ would not be meromorphic. In particular, if $F$ is right-prime, then in every decomposition (3) with $f$ transcendental, the function $h$ is a polynomial of degree 1.

The first nontrivial example of a prime function is $F(s) = e^s + s$ (see Rosenbloom [13] and Gross [9]). Liao and Yang [14] proved the primeness of the Gamma function and the Riemann zeta function. In [15,16], the primeness of Selberg zeta functions associated with a compact Riemann surface was obtained. The Selberg zeta function does not belong to the extended Selberg class. The concept of decomposition of meromorphic functions can be extended to several complex variables; see Li and Yang [17].

Suppose that $F$ has a decomposition given by $F = f \circ h$. Let $L$ be any fractional linear transformation. Then $F = (f \circ L) \circ (L^{-1} \circ h)$. To obtain around this, we introduce the following definition.

**Definition.** (see Gross [10]) Two decompositions of $F$

$$F = f_1 \circ h_1 \quad \text{and} \quad F = f_2 \circ h_2$$

are said to be *equivalent* if and only if there exists a fractional linear function $L$ such that $f_2 = f_1 \circ L$ and $h_2 = L^{-1} \circ h_1$.

In [9,11,12], the notions *factorization* and *factor* instead of the corresponding notions of *decomposition* and *component* were used. In the (extended) Selberg class case, the notions of *factorization* and *factor* are usually used in the case of multiplication. Both classes $\mathcal{S}$ and $\mathcal{S}^\sharp$ are closed under the multiplication of their elements.

We say that the expression $F(s) = F_1(s)F_2(s)$, where $F, F_1, F_2 \in \mathcal{S}$ (or $F, F_1, F_2 \in \mathcal{S}^\sharp$), is a *factorization* of $F$ in $\mathcal{S}$ (or in $\mathcal{S}^\sharp$) into *factors* $F_1$ and $F_2$. If in every factorization $F(s) = F_1(s)F_2(s)$, either $F_1$ or $F_2$ is a constant, then $F$ is said to be *primitive*. Every $F \in \mathcal{S}$ ($F \in \mathcal{S}^\sharp$) can be factored as a product of primitive $\mathcal{S}$ ($\mathcal{S}^\sharp$) functions. (Theorems 3.5 and 3.11 in Perelli [5].) It is not known whether the factorization into primitive factors is unique. The decomposition into prime functions is not unique (Gross [18, p. 58–59]). However, in particular cases, the decomposition can be unique; see Li [19].

In the present article, the short notation "prime function" always means a prime function in the sense of a decomposition.

**Theorem 1.** *Functions $F \in \mathcal{S}^\sharp$, $d_F \geq 1$, are right-prime.*

The simplest hypothesis would be that every function of the extended Selberg class is either prime or a power of another function from $\mathcal{S}^\sharp$. To this end, we have the following result.

**Theorem 2.** *Suppose $F$ belongs to the extended Selberg class, $d_F \geq 1$, and $F(s) = Q(h(s))$ is a decomposition with $Q$ rational of degree $k \geq 2$ and $h$ meromorphic. Then there is a decomposition $Q_1(h_1(s))$ equivalent to $Q(h(s))$ such that $Q_1(z) = z^k$.*

The following corollary is an immediate consequence of Theorems 1, 2, and the Weierstrass factorization theorem.

**Corollary 3.** *Let $F \in \mathcal{S}^\sharp$, $d_F \geq 1$. Then $F$ is prime if and only if the greatest common divisor of the orders of all zeros and the pole of $F$ is* 1.

If $F = h^k$ for $k \geq 2$, then by our hypothesis, we expect $h \in \mathcal{S}^\sharp$. In the case of the Selberg class $\mathcal{S}$, this was partially confirmed by Molteni [20]. Every $F \in \mathcal{S}$ has a unique Euler product representation, $F(s) = \prod_p F_p(p^{-s})$ for $\sigma > 1$, where $F_p(p^{-s})$ is a holomorphic function on $\sigma > 0$ (see Kaczorowski [3, formula (2.8)]). Let $m_f(\rho)$ denote the order of $\rho$ (with $m_f(\rho) > 0$ if $\rho$ is a zero of $f$ and $m_f(\rho) < 0$ if $\rho$ is a pole of $f$). Then Molteni [20] proved the following result. Let $F \in \mathcal{S}$ and $k \geq 1$. Then $F = X^k$, has a solution $X \in \mathcal{S}$ if and only if $k | m_F(\rho)$ for every $\rho \in \mathbb{C}$ and $k | m_{F_p}(\rho)$ for every $p$, for every $\Re\rho > 0$. As noted in [20], it is expected that $F_p(p^{-s})^{-1}$ is a polynomial in $p^{-s}$ that does not vanish when $\Re s > 0$.

The following two hypotheses about zeros imply that every function of the Selberg class is either prime or a power of another function from $\mathcal{S}$.

*Simple zero conjecture* (Conjecture 7.1 in [4]) Let $F \in \mathcal{S}$ be primitive. Then all but $o(T \log T)$ nontrivial zeros of $F(s)$ up to $T$ are simple.

*Distinct zero conjecture* (Conjecture 7.2 in [4]) Let $F, G \in \mathcal{S}$ be distinct primitive functions. Then all but $o(T \log T)$ nontrivial zeros of $F(s)$ and $G(s)$ are strongly distinct, i.e., placed at different points.

By the simple zero conjecture, Corollary 3, and Lemma 9, a primitive $F \in \mathcal{S}$ is prime. The distinct zero conjecture then implies that $F = F_1^{k_1} F_2^{k_2} \dots F_n^{k_n} \in \mathcal{S}$, where $F_1, F_2 \dots, F_n$ are primitive, is prime if and only if $\mathrm{GCD}(k_1, \dots, k_n) = 1$.

In the next section, we give examples of prime zeta functions from the extended Selberg class. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

## 2 Examples of prime functions from $\mathcal{S}^\#$

Let $d$ be a positive integer. By $\mathcal{S}_d$, we denote the subset of degree $d$ elements of $\mathcal{S}$. Analogically, we will use the notation $\mathcal{S}_d^\sharp$. Typical members of $\mathcal{S}_1$ are Dirichlet $L$-functions with primitive characters. For any character $\chi(n) \bmod \ell$, the Gaussian sum $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{m=1}^{\ell} \chi(m) \exp(2\pi i m/\ell).$$

Note that for a primitive character $\chi$, we have $|\tau(\chi)| = \sqrt{\ell}$. Let $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. Then the functional equation of the Dirichlet $L$-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)$$

with a primitive character $\chi \bmod \ell$ takes the form: if

$$\xi(s,\chi) = \left(\frac{\ell}{\pi}\right)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

then (see Davenport [21, Chapter 9])

$$\xi(s,\chi) = \frac{\tau(\chi)}{i^a q^{1/2}} \xi(1-s,\overline{\chi}).$$

Therefore, any Dirichlet $L$-function for the primitive character is an element of $\mathcal{S}_1$. On the other hand, any element of $\mathcal{S}_1$ is a shifted Dirichlet $L$-function $L(s+i\theta,\chi)$ for some primitive Dirichlet character $\chi$ and real $\theta$ or the Riemann zeta function $\zeta(s)$ (Kaczorowski and Perelli [2, Theorem 3]). All elements of $\mathcal{S}_1$ are prime functions because all their trivial zeros are simple. Moreover, if $\chi$ and $\psi$ are primitive characters with $\chi(-1) = 1$ and $\psi(-1) = -1$, then the Dirichlet $L$-functions $L(s,\chi)$ and $L(s,\psi)$ have simple trivial zeros at $s = -2, -4, -6, \ldots$ and $s = -1, -3, -5, \ldots$, respectively. Hence, $L(s,\chi)L(s,\psi)$ is prime. More examples of prime functions satisfying the assumptions of Theorem 2 can be constructed similarly.

Next, we will show that all degree 1 elements of the extended Selberg class are prime functions. Kaczorowski and Perelli [2] described the extended Selberg class functions of degree $0 \le d \le 1$. To describe degree 1 functions, we need details about degree 0 functions.

For degree 0 elements of the extended Selberg class, there are no $\Gamma$ factors in the functional equation. Let $Q$ and $\omega$ be constants as in the definition of $F \in \mathcal{S}^\sharp$. By $\mathcal{S}_0^\sharp(Q^2,\omega)$ we denote a subclass of $\mathcal{S}_0^\sharp$ with given $Q^2$ and $\omega$. Kaczorowski and Perelli [2, Theorem 1] proved that every $F \in \mathcal{S}_0^\sharp(Q^2,\omega)$ is of the form

$$F(s) = \sum_{n|Q^2} \frac{a(n)}{n^s}.$$

We turn to degree 1 functions. To state their result, we introduce some definitions. First, let us denote $\beta \coloneqq \prod_{j=1}^r \lambda_j^{-2\lambda_j}$ and $q \coloneqq 2\pi Q^2/\beta$. In addition, let $\xi \coloneqq 2\sum_{j=1}^r (\mu_j - 1/2) = \eta + i\theta$ and $\omega^* \coloneqq \omega e^{-i\pi(\eta+1)/2}(Q^2/\beta)^{i\theta}\prod_{j=1}^r \lambda_j^{-2i\Im\mu_j}$. Here, $\omega$, $\mu_j$, $\lambda_j$, and $Q$ come from the functional equation. The triple $(q,\xi,\omega^*)$ is an invariant of $F \in \mathcal{S}_1^\sharp$. The class $\mathcal{S}_1^\sharp$ is the disjoint union over $q \in \mathbb{N}$, $\eta \in \{-1,0\}$, $\theta \in \mathbb{R}$, and $|\omega^*| = 1$ of subclasses $\mathcal{S}_1^\sharp(q,\xi,\omega^*)$ of degree 1 functions (see [2, Theorem 2]).

For a Dirichlet character $\chi$, $f_\chi$ denotes its conductor. Let $\chi^*$ be the primitive Dirichlet character inducing $\chi$. As usual, $\chi_0$ denotes a principal character. By $\omega_{\chi^*}$ denote the $\omega$ factor in the standard functional equation for $L(s,\chi^*)$. Define the following set of Dirichlet characters:

$$\mathcal{X}(q,\xi) \coloneqq \begin{cases} \{\chi(\bmod q) \text{ with } \chi(-1) = 1\} & \text{if } \eta = -1, \\ \{\chi(\bmod q) \text{ with } \chi(-1) = -1\} & \text{if } \eta = 0. \end{cases}$$

From this, we see that for fixed $q$ and $\xi$, the members of $\mathcal{X}(q,\xi)$ are of the same modulus and parity. Theorem 2 of [2] tells that $F \in \mathcal{S}_1^\sharp(q,\xi,\omega^*)$ can be uniquely written as follows:

$$F(s) = \sum_{\chi \in \mathcal{X}(q,\xi)} P_\chi(s+i\theta) L(s+i\theta,\chi^*),$$

where $P_\chi \in \mathcal{S}_0^\sharp(q/f_\chi, \omega^*\overline{\omega}_{\chi^*})$ and $P_{\chi_0} = 0$ if $\theta \neq 0$. This theorem shows that all Dirichlet $L$-functions in the expression for $F$ satisfy the same functional equation. Hence, $F$ will satisfy the same functional equation.

Therefore, we have that if $F \in \mathcal{S}_1^\sharp$, then $F$ satisfies the functional equation $\Lambda_F(s) = \omega\overline{\Lambda_F(1-\bar{s})}$ with

$$\Lambda_F(s) = F(s)Q^s\Gamma\left(\frac{s+a+i\theta}{2}\right),$$

where $a$ is 0 or 1; $|\omega| = 1$ is a complex constant, $\theta$ is real, and $Q > 0$. If $a = 0$, then $F$ has simple trivial zeros at $s = -2 - i\theta, -4 - i\theta, -6 - i\theta, \dots$. If $a = 1$, the trivial simple zeros are at $s = -1 - i\theta, -3 - i\theta, -5 - i\theta, \dots$. Only a finite number of trivial zeros can coincide with nontrivial zeros because nontrivial zeros are located in a vertical strip of a finite width. Therefore, all degree 1 elements of the extended Selberg class are prime functions.

Dedekind zeta functions provide natural examples of prime functions of arbitrary degree. The Dedekind zeta function of a finite extension $K$ of the field of rational numbers of degree $n = r_1 + 2r_2$ is defined for $\sigma > 1$ by

$$L_K(s) = \sum_I \frac{1}{N(I)^s},$$

where $I$ runs over the nonzero ideals of the ring of integers of $K$, $N(I)$ denotes the norm of $I$. This zeta function can be continued analytically to a meromorphic function with a simple pole at $s = 1$. Let $A = 2^{-r_2}\pi^{-n/2}|D_K|^{1/2}$, where $D_K$ is the discrimnant of $K$. Then the function

$$\Phi(s) = A^s\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$$

satisfies the functional equation $\Phi(s) = \Phi(1-s)$ (see Narkiewicz [22, Theorem 7.3]). Thus, any Dedekind zeta function belongs to the Selberg class and is prime because it always has a simple pole at $s = 1$. For any positive integer $n$, there is a Dedekind zeta function of degree $n$.

## 3 Proof of Theorem 1

We will use elements of the Nevanlinna theory.

Let the function $n(r,f)$, $r \geq 0$, denote the number of poles, counting multiplicity, of a meromorphic function $f$ in the disc $|z| \leq r$. The *Nevanlinna counting function* is

$$N(r,f) = \int_0^r (n(t,f) - n(0,f))\frac{\mathrm{d}t}{t} + n(0,f)\log r.$$

Define $\log^+ x = \max\{0, \log x\}$. The *proximity function* is

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\mathrm{d}\theta.$$

The *Nevanlinna characteristic function* then is

$$T(r,f) = N(r,f) + m(r,f).$$

The *order* $0 \leq k \leq \infty$ of a meromorphic function $f$ is defined by

$$k = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}. \tag{4}$$

The following properties of the Nevanlinna characteristic function will be useful. From Hayman [23, Section 1.2], we have

$$T(r,f) = T\left(r, \frac{1}{f}\right) + O(1), \tag{5}$$

$$T\left(r, \sum_{\nu=1}^{p} f_\nu(s)\right) \le \sum_{\nu=1}^{p} T(r, f_\nu(s)) + \log p, \tag{6}$$

$$T\left(r, \prod_{\nu=1}^{p} f_\nu(s)\right) \le \sum_{\nu=1}^{p} T(r, f_\nu(s)) \tag{7}$$

for meromorphic $f, f_\nu$. Further, for a rational function,

$$f(z) = c\frac{z^p + a_1 z^{p-1} + \ldots + a_p}{z^q + b_1 z^{q-1} + \ldots + b_q}, \quad \text{where } c \ne 0,$$

we obtain (see [23, Section 1.3, Example (i)])

$$T(r, f) = \max(p, q) \log r + O(1). \tag{8}$$

Moreover, if $h$ is a meromorphic function, then (Goldberg and Ostrovskij [24, Chapter 1, Theorem 6.1, p. 47])

$$T(r, f \circ h) = \max(p, q) T(r, h) + O(1). \tag{9}$$

If $P(z) = az^p + \ldots$ is a polynomial and $f(z) = \exp(P(z))$, then (see [23, Section 1.3, Example (iii)])

$$T(r, f) = \frac{|a|}{\pi} r^p + o(r^p). \tag{10}$$

Hence, $f(z)$ is of order $p$ if $a \ne 0$.

We proceed with the following notion, which is taken from Chuang et al. [25].

**Definition.** Let $E \subset \mathbb{C}$. Suppose $\theta \in [0, 2\pi]$ is an accumulation point of $S = \{\arg s : s \in E\}$. Then the set $\{s : \arg s = \theta\}$ is an *accumulation line* of $E$.

Liao and Yang [14] used the following proposition to prove the pseudo-primeness of $\zeta$. By his proposition, we shall see that every $F \in \mathcal{S}^\sharp$ with $d_F \ge 1$ is also pseudo-prime. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

**Proposition 4.** *Let $a_1, a_2$ be arbitrary distinct elements of $\overline{\mathbb{C}}$. Let $F$ be a meromorphic function of finite order. Assume that the number of the accumulation lines of $E = \{s : F(s) = a_j \in \overline{\mathbb{C}}, j = 1, 2\}$ is finite. Then $F$ is pseudo-prime.*

Note that here by the solutions of the equation $F(s) = \infty$ we mean the poles of $F$. Proposition 4 is proved in Chuang et al. [25, Theorem 3.12, p. 141]. This reference is not easily accessible, so we provide the proof below. Moreover, our proof differs slightly from the one in [25]. For the proof of the proposition, we will use the following Lemmas 5–7.

**Lemma 5.** *Let $f(z)$ be a meromorphic function not of order zero, and let $h(s)$ be an entire function that is not a polynomial. Then, $f(h(s))$ is of infinite order.*

**Proof.** This is Corollary 1.2 in Edrei and Fuchs [26]. □

**Lemma 6.** *Suppose $f$ is an entire function and there is a sequence $(\omega_n)$ with $\lim \omega_n = \infty$ such that*

$$\bigcup_{n=1}^{\infty} \{s : f(s) = \omega_n\}$$

*has $q$ $(< \infty)$ accumulation lines (except for possibly finitely many $\omega_n$'s). Then $f$ is a polynomial of degree at most $2q$.*

**Proof.** This is Theorem 3.8 from Monakhov [27]. □

**Lemma 7.** *Let $a, b, c, d \in \mathbb{C}$. Let $ad - bc \ne 0$, $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant meromorphic function, and $h(z) = \frac{af(z)+b}{cf(z)+d}$. Then $h(z)$ is transcendental if and only if $f(z)$ is transcendental. Moreover, $h(z)$ and $f(z)$ are of the same order.*

**Proof.** The first statement of the lemma follows from the fact that $f$ is rational if and only if $h$ is rational.

By (5)–(8), we have

$$T(r,h) \le T(r, af + b) + T(r, cf + d) + O(1) \le 2T(r,f) + O(1).$$

Thus, the order of $h$ is not greater than that of $f$. The converse follows in the same way, because the transformation $w \to (aw + b)/(cw + d)$ is invertible. This proves the lemma. □

**Proof of Proposition 4.** First, we show by contradiction that a rational function is pseudo-prime. Suppose $F$ is rational and there is a decomposition $F(s) = f(h(s))$, where $f$ and $h$ are transcendental. Recall that then $h$ is entire. Then by the great Picard theorem, there exists $w \in \mathbb{C}$ such that $w$ is not a pole of $f$ and $h(s) = w$ holds for infinitely many $s$. Hence, $f(h(s)) - f(w)$ has infinitely many roots, which is a contradiction.

If $F$ is not pseudo-prime, then $F(s)$ is transcendental. There is a decomposition $F(s) = f(h(s))$, where $f(z)$ is a transcendental meromorphic function and $h(s)$ is a transcendental entire function. Lemma 5 yields that $f(z)$ is of zero order.

We will show that the equation

$$f(z) = a_j \quad (j = 1, 2) \tag{11}$$

has finitely many roots. Contrary to this, assume that a set $E_1 = \{z : f(z) = a_j \in \overline{\mathbb{C}}, j = 1, 2\}$ is infinite. Hence, there is a sequence $(\omega_n)$ with $\omega_n \in E_1$ and $\lim \omega_n = \infty$. Then

$$E_2 := \bigcup_{n=1}^{\infty} \{s : h(s) = \omega_n\} \subset E.$$

By conditions, $E$ has a finite number of accumulation lines. Hence, the set $E_2$ also has a finite number of accumulation lines. Then Lemma 6 implies that $h$ is a polynomial. This is a contradiction.

Next, we define

$$G(z) := \frac{f(z) - a_1}{f(z) - a_2},$$

if $a_j$, $j = 1, 2$ are finite numbers. In this case, the poles of $f$ in the numerator and denominator cancel out, and by (11), the function $G$ has finitely many zeros and poles. If one of $a_j$, $j = 1, 2$ is infinite, say $a_2 = \infty$, then $G(z) := f(z) - a_1$. By applying (11) with finite $a_1$ and $a_2 = \infty$, we see that $G(z)$ has finitely many zeros and poles also for this case. Then, for each possible pair of $a_1$ and $a_2$, the Weierstrass factorization theorem implies that

$$G(z) = Q(z)e^{L(z)},$$

where $Q(s)$ is a rational function and $L(z)$ is a nonconstant entire function. Lemma 7 yields that $G(s)$ is of zero order since $f(z)$ is such. By the property (7) of the Nevanlinna characteristic sum, we obtain that

$$T(r, G(z)/Q(z)) \le T(r, G(z)) + T(r, 1/Q(z)).$$

Given (8), the rational function $1/Q(z)$ is of zero order. Therefore, $G(z)/Q(z) = e^{L(z)}$ is also a function of zero order. By using Lemma 5 and the fact that the order of the exponential function is 1, we obtain that $L(z)$ is a polynomial, say of degree $n$. Thus, the order of the function $e^{L(z)}$ is $n$ (Hayman [23, Section 1.3, Example (iii)]), which is positive since $L(s)$ is a nonconstant. This contradiction proves the proposition. □

The reasoning in the following proof is similar to the one used in proving the main result in Garunkštis and Steuding [16].

**Proof of Theorem 1.** Given (2), the trivial zeros of $F$ are located in a horizontal strip of bounded height. The Dirichlet series expression and the functional equation imply that the remaining zeros of $F$ are contained in a vertical strip of bounded width. The function $F$ is of finite order by the property (ii) of the definition of the Selberg class and the definition of order (4). Then, by Proposition 4 (applied with $a_1 = 0$ and $a_2 = \infty$), the function $F(s)$ is pseudo-prime.

Next, we show that $F$ is a right-prime function. Assume that

$$F(s) = f(h(s)) \tag{12}$$

is a decomposition. By the definition of a right-prime function, we have to show that if $f$ is transcendental, then $h$ is linear. So, we assume that $f$ is transcendental. Since $F$ is pseudo-prime, $h$ is a polynomial.

Further, we prove that $h$ is of degree 1. For this, we consider the $F(s)$ growth for $\sigma \to +\infty$ and $\sigma \to -\infty$. Functions in $\mathcal{S}^\sharp$ are defined by Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (\sigma > 1).$$

Let $k$ be the smallest index such that $a(k) \neq 0$. Then, for $\sigma \to +\infty$,

$$F(s) = a(k)k^{-s}(1 + o(1)) \tag{13}$$

uniformly in $t$. For a sufficiently small $\varepsilon > 0$, let

$$A = \{s : \pi/2 + \varepsilon \le \arg s \le \pi - \varepsilon \text{ or } \pi + \varepsilon \le \arg s \le 3\pi/2 - \varepsilon\}.$$

We have the formula

$$\Gamma(1 - s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}. \tag{14}$$

Taking the absolute values of (1) and using the identity (14) yields

$$|F(1 - \bar{s})| = \frac{1}{\pi^N}|F(s)||Q^{2s-1}|\prod_{j=1}^{N}|\Gamma(\lambda_j s + \mu_j)||\Gamma(\lambda_j s + 1 - \lambda_j - \overline{\mu_j})||\sin(\pi(\lambda_j(1 - s) + \overline{\mu_j}))|. \tag{15}$$

Note that there exists $\sigma_0$ such that $\sin(\pi(\lambda_j(1 - s) + \overline{\mu_j})) \neq 0$ in the region $A \cap \{s : \sigma < \sigma_0\}$. This together with the Stirling formula (Titchmarsh [28, Section 4.42])

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s\left(1 + O\left(\frac{1}{s}\right)\right), \quad |\arg(s)| < \pi - \varepsilon$$

and (13) implies

$$\lim_{\substack{|s|\to\infty \\ s\in A}} F(s) = \infty. \tag{16}$$

Let $h(s) = a_d s^d + a_{d-1}s^{d-1} + \ldots + a_1 s + a_0$ with $a_d \neq 0$. We consider the preimages $\ell_1, \ldots, \ell_d$ of the half-line

$$\ell = \left\{s : \arg s = \frac{\pi}{2} + \arg a_d\right\}$$

under the action of the polynomial $h$. We number the preimages such that near infinity, the curve $\ell_j$ is close to the half-line

$$L_j := \left\{s : \arg s = \frac{\pi}{2d} + \frac{2j\pi}{d}\right\},$$

where $j \in \{1, \ldots, d\}$. Note that $\arg L_j \neq \pi$, $j \in \{1, \ldots, d\}$. If $d \ge 2$, then there are indices $p, q \in \{1, \ldots, d\}$ such that $L_p$ lies in the half-plane $\sigma > 2$ (except for a finite part of $L_p$) and $L_q$ lies in the set $A$ for sufficiently small $\varepsilon > 0$. Therefore, by formulas (13) and (16), we have

$$\lim_{\substack{|s|\to\infty \\ s\in\ell_p}} |F(s)| \le |a(1)| \quad \text{and} \quad \lim_{\substack{|s|\to\infty \\ s\in\ell_q}} F(s) = \infty. \tag{17}$$

On the other hand, from the facts that $F(s) = f(h(s))$ and $h(\ell_p) = \ell = h(\ell_q)$, we see that $F(\ell_p) = F(\ell_q)$. The last equality contradicts equations (17). Thus $d = 1$. This proves Theorem 1. □

## 4 Proof of Theorem 2

Theorem 2 will be derived from Propositions 8 and 11.

**Proposition 8.** *Suppose $F$ belongs to the extended Selberg class, $d_F \geq 1$, and $F(s) = Q(h(s))$ is a decomposition with $Q$ rational with degree $\geq 2$ and $h$ meromorphic. Then there is a decomposition $Q_1(h_1(s))$, equivalent to $Q(h(s))$, such that $Q_1$ is a polynomial.*

We need the following two lemmas to prove Proposition 8. Let $N_F(\sigma, T)$ count the number of zeros $\rho = \beta + i\gamma$ of $F(s)$ satisfying $\beta \geq \sigma, |\gamma| \leq T$.

**Lemma 9.** *Suppose $F$ belongs to the extended Selberg class, $d_F \geq 1$. Then, for any fixed $\sigma \leq 0$,*

$$N_F(\sigma, T) = \frac{d_F}{\pi} T \log T + O(T), \quad T \to \infty. \tag{18}$$

**Proof.** For $a(1) = 1$, the lemma is a partial case of Theorem 7.7 (with $c = 0$ and $\mathcal{L}(s) = F(s)$) from Steuding [29]. In the proof of Theorem 7.7, the function $\ell(s) = \frac{\mathcal{L}(s)-c}{1-c}$ was used (see the formula (7.3) in [29]). Repeating the proof of Theorem 7.7 with $\ell(s) = \frac{k^s}{a(k)} F(s)$ (see comments at the end of Section 7.2 in [29]), we obtain Lemma 9. □

**Lemma 10.** *For $F$ satisfying the extended Selberg class axioms,*

$$T(r, F) = \frac{d_F}{\pi} r \log r + O(r), \quad r \to \infty.$$

**Proof.** For $a(1) = 1$, this is Theorem 7.9 from Steuding [29]. Next, we consider the general case. For any complex $a(1)$, we have (see formula (7.14) in the proof of Theorem 7.9 in [29])

$$T(r, F) \leq \frac{d_F}{\pi} r \log r + O(r).$$

By (5), we see that

$$T(r, F) = T\left(r, \frac{1}{F}\right) + O(1),$$

and by the definition of the characteristic function,

$$N\left(r, \frac{1}{F}\right) \leq T\left(r, \frac{1}{F}\right).$$

Given (13) and (15), we have that there is $\sigma_1$ such that $F(s)$ has no nontrivial zeros in $\sigma \leq \sigma_1$. Then Lemma 9 together with (2) gives that

$$N\left(r, \frac{1}{F}\right) = \frac{d_F}{\pi} r \log r + O(r).$$

This finishes the proof of Lemma 10. □

**Proof of Proposition 8.** Let

$$F(s) = Q(h(s)),$$

where $Q$ is a rational function with $\deg Q \geq 2$. Then, $h$ is a transcendental meromorphic function because $F$ has an infinite number of zeros. The function $F(s)$ possibly has a pole at $s = 1$. By the last lemma, we see that any $F \in \mathcal{S}^\sharp$ with $d_F > 0$ is an order 1 meromorphic function. Then (9) implies that the order of $h$ equals 1.

We consider cases where $Q(w)$ has no poles, a pole at a single point, poles at two distinct points, and poles at more than two distinct points.

**Case 1** $Q(w)$ has no poles. Then, $Q$ is a polynomial.

**Case 2** $Q(w)$ has a pole at $w_0$ only. Then

$$Q(w) = \frac{P(w)}{(w - w_0)^m},$$

where $P$ is a polynomial, $P(w_0) \neq 0$.

If $m \geq \deg P$, then we define the linear fractional transformation $L(z) = \frac{w_0 z + 1}{z}$ to obtain the equivalent decomposition $F = (Q \circ L) \circ (L^{-1} \circ h)$, where $Q \circ L$ is a polynomial. To make sure of the latter fact, consider

$$Q \circ L(z) = z^m P(w_0 + 1/z).$$

Since $P(w)$ is a polynomial of degree $<m$, multiplying its terms by $z^m$ eliminates all powers from the denominator, and we are left with a polynomial with respect to $z$.

Further, assume that $m < \deg P$. Then, $|Q(w)| \to \infty$ as $|w| \to \infty$, so $h(s)$ cannot have a pole except at $s = 1$, where $F(s)$ may have one. We can derive two sub-cases.

(a) The function $h(s)$ is entire. Then $h(s)$ cannot assume the value $w_0$ except possibly at $s = 1$ and the Hadamard factorization theorem implies that

$$h(s) - w_0 = (s - 1)^k e^{as+b},$$

where $k \geq 0$ is the order of $h$ at $w_0$. From (8) and (10), we have

$$T(r, (s - 1)^k) = k \log r + O(1), \quad T(r, e^{as+b}) = \frac{|a|}{\pi} r + o(r).$$

Then (5)–(7) lead to

$$T(r, F) = O(T(r, h)) = O(r), \quad r \to \infty.$$

This contradicts Lemma 10.

(b) The function $h(s)$ has a pole at $s = 1$. Then it does not assume $w_0$, and we have

$$h(s) - w_0 = \frac{e^{as+b}}{(s - 1)^k},$$

where $k$ is the order of the pole, and we have a contradiction by analogy to case **(a)**.

**Case 3** $Q(w)$ has poles at different $w_1$ and $w_2$. Then $h(s)$ can assume at most one of the values $w_1$ and $w_2$, and if it does assume one of them, it can only be at $s = 1$. Then $h(s)$ either does not assume one of them, for example, $w_1$, and has a $w_2$-point at $s = 1$, or it does not assume both of them. We split our analysis into two cases.

(a) The function $h(s)$ does not assume the value $w_1$ and has a $w_2$-point at $s = 1$. From Hadamard factorization and this, we have that

$$g(s) \coloneqq \frac{h(s) - w_2}{h(s) - w_1} = (s - 1)^k e^{as+b}.$$

for some constant $k$. By Hayman [23, formula (1.5a) and Section 1.3, Exercise (v)], we see that $T(r, h(s)) = T(r, g(s)) + O(1)$. Continuing the discussion, as in Case 2, we obtain a contradiction.

(b) The function $h(s)$ does not assume any of the values $w_1$ and $w_2$. Then we have

$$\frac{h(s) - w_1}{h(s) - w_2} = e^{as+b}.$$

This leads to a contradiction, as in the previous case.

**Case 4** The function $Q(w)$ has poles at different $w_1, w_2, \ldots, w_N$, where $N \geq 3$. Then, $h(s)$ can assume only one of these values, say, $w_N$, since our function $F$ can have a pole only at the point $s = 1$. Therefore, $h(s)$ does not assume at least two values, say, $w_1$ and $w_2$. By Hadamard's factorization, we obtain

$$g(s) := \frac{h(s) - w_1}{h(s) - w_2} = (s-1)^k e^{as+b}.$$

This leads to a contradiction, as in the aforementioned case.

This proves the proposition. □

**Proposition 11.** *Suppose the degree of $F \in S^\sharp$ is greater than or equal to 1. Suppose $F(s) = p(h(s))$ is a decomposition, $h$ is a meromorphic function possibly with pole at $s = 1$, but nowhere else, and $p(z) = a_k z^k + a_{k-1}z^{k-1} + \ldots + a_1 z + a_0$, where $a_k \neq 0$. Then there is a decomposition $P(H(s))$, equivalent to $p(h(s))$ such that $P(z) = a_k z^k$.*

In the proof of Proposition 11, we use trivial zeros (2). Note that these zeros lie in arithmetical progressions.

We first prove two lemmas. In the following lemma, we claim that for a finite number of arithmetical progressions, each progression has an infinite subsequence of elements that coincide or are not very near to the elements of any other sequence.

**Lemma 12.** *Let $\alpha_k > 0$ and $\beta_k \in \mathbb{C}$, $k = 0, 1, \ldots, N$. Then there are $\varepsilon > 0$ and an increasing sequence $n_j$, $j = 1, 2, \ldots$ of positive integers such that, for any $j$ and $1 \leq p \leq N$,*

$$\min_{r\in\mathbb{Z}} |\alpha_0 n_j + \beta_0 - \alpha_p r - \beta_p| \in \{-\infty, -\varepsilon\} \cup \{0\} \cup \{\varepsilon, \infty\}, \quad \varepsilon > 0. \tag{19}$$

**Proof.** In this proof, we use $\{\ldots\}$ to denote the fractional part. Define $\|\alpha\| = \min_{r\in\mathbb{Z}}\{|\alpha - r|\}$. Then we rewrite (19) as follows:

$$\left\| \frac{\alpha_0}{\alpha_p} n_j + \frac{\beta_0 - \beta_p}{\alpha_p} \right\| = 0 \text{ or } > \frac{\varepsilon}{\alpha_p} \quad (p = 1, \ldots, N). \tag{20}$$

Let numbers $1, \alpha_0/\alpha_1, \ldots, \alpha_0/\alpha_M$, where $M \leq N$, be linearly independent over rational numbers. For the remaining coefficients $\alpha_0/\alpha_{M+\ell}$, $\ell = 1, \ldots, N-M$, let there be integers $d_{M+\ell}$ and $d_{M+\ell,r}$, $r = 0, 1, \ldots, N$, such that

$$d_{M+\ell}\frac{\alpha_0}{\alpha_{M+\ell}} = d_{M+\ell,0} + \sum_{p=1}^{M} d_{M+\ell,p}\frac{\alpha_0}{\alpha_p}. \tag{21}$$

Next, we group the coefficients $\alpha_0/\alpha_{M+\ell}$, $\ell = 1, \ldots, N-M$, into rationals and irrationals.

The case of rational coefficients. Suppose, for $\ell = 1, \ldots, L \leq N-M$, we have that all $d_{M+\ell,p} = 0$, $p = 1, \ldots, M \leq N$. Then, for any positive integer $n$ such that $\mathrm{LCM}(d_{M+L+1}, \ldots, d_N)|n$, we have

$$\left\| \frac{\alpha_0}{\alpha_{M+\ell}} n + \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| = \left\| \frac{d_{M+\ell,0}}{d_{M+\ell}} n + \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| = \left\| \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| \quad (1 \leq \ell \leq L). \tag{22}$$

Thus, for $p = M+1, \ldots, N-M$, the statement (19) is true with $n_j = j\mathrm{LCM}(d_{M+L+1}, \ldots, d_N)$.

The case of irrational coefficients. In (21), for any $\ell = L+1, \ldots, N-M$, let there be at least one $i$, $1 \leq i \leq M$, such that $d_{M+\ell,i} \neq 0$, and hence, $\alpha_0/\alpha_{M+\ell}$ are irrational numbers. Assuming that $\mathrm{LCM}(d_{M+L+1}, \ldots, d_N)|n$ from (21), we obtain

$$A(\ell, n) := \left\| \frac{\alpha_0}{\alpha_{M+\ell}} n + \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| = \left\| \sum_{p=1}^{M} \frac{d_{M+\ell,p}}{d_{M+\ell}} \frac{\alpha_0}{\alpha_p} n + \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| \quad (L < \ell \leq N-M). \tag{23}$$

By Kuipers and Niederreiter [30, Example 6.1, p. 48], we have that the multidimensional vector

$$\left\{\frac{\alpha_0}{\alpha_1}n, \ldots, \frac{\alpha_0}{\alpha_M}n\right\},$$

where $n$ runs over all positive multiples of $\mathrm{LCM}(a_{M+L+1}, \ldots, a_N)$, is uniformly distributed modulo 1 in $\mathbb{R}^M$. Thus, by choosing a suitable subsequence, we can control the fractional part of $n\alpha_0/\alpha_{M+\ell}$, $\ell = 1, \ldots, N-M$, but not the integral part. In view of this, we rewrite (23) as follows:

$$A(\ell, n) = \left\| \sum_{p=1}^{M} \frac{d_{M+\ell,p}}{d_{M+\ell}} \left\{\frac{\alpha_0}{\alpha_p}n\right\} + \left\{ \sum_{p=1}^{M} \frac{d_{M+\ell,p}}{d_{M+\ell}} \left[\frac{\alpha_0}{\alpha_p}n\right]\right\} + \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| \quad (L < \ell \leq N-M).$$

If $n$ runs even over all positive integers, then among the numbers

$$\left\| \left\{ \sum_{p=1}^{M} \frac{d_{M+\ell,p}}{d_{M+\ell}} \left[\frac{\alpha_0}{\alpha_p}n\right]\right\} + \frac{\beta_0 - \beta_{M+\ell}}{\alpha_{M+\ell}} \right\| \quad (L < \ell \leq N-M),$$

there are only finitely many different numbers. Let $H$ be the smallest such positive number. If there is no such positive number, then let $H = 1/4$. Next we choose $0 < x_1 < \ldots < x_M$ such that

(i)

$$\max_{L<\ell\leq N-M} \left| \sum_{p=1}^{M} \frac{d_{M+\ell,p}}{d_{M+\ell}} x_p \right| < H/4;$$

(ii) for $k = 1, \ldots, M-1$, if

$$\max_{L<\ell\leq N-M} |d_{M+\ell,k+1}| \neq 0,$$

then

$$\max_{L<\ell\leq N-M} \left| \sum_{p=1}^{k} \frac{d_{M+\ell,p}}{d_{M+\ell}} x_p \right| < \frac{1}{4} \min_{L<\ell\leq N-M} \left( \left| \frac{d_{M+\ell,k+1}}{d_{M+\ell}} x_{k+1} \right| \neq 0 \right);$$

(iii) for $p = 1, \ldots, M$, if $\left\|\frac{\beta_0 - \beta_p}{\alpha_p}\right\| \neq 0$, then we require that

$$x_p < \frac{1}{4} \left\| \frac{\beta_0 - \beta_p}{\alpha_p} \right\|.$$

Hence, there is a small positive number $\delta$, $\delta < x_1$, and the sequence $(n_j)_{j=1}^{\infty}$, such that for all $j$,

(1) $\mathrm{LCM}(d_{M+L+1}, \ldots, d_N) | n_j$;

(2) for $p = 1, \ldots M$,

$$x_p - \delta < \left\{\frac{\alpha_0}{\alpha_p} n_j\right\} < x_p + \delta \tag{24}$$

and

$$x_p + \delta < \frac{1}{2} \left\| \frac{\beta_0 - \beta_p}{\alpha_p} \right\|, \quad \text{if } \left\| \frac{\beta_0 - \beta_p}{\alpha_p} \right\| \neq 0; \tag{25}$$

(3)

$$\max_{L<\ell\leq N-M} \left| \sum_{p=1}^{M} \frac{d_{M+\ell,p}}{d_{M+\ell}} \left\{\frac{\alpha_0}{\alpha_p} n_j\right\} \right| < H/2;$$

(4) for $k = 1, \ldots, M-1$, if $\max_{L<\ell\le N-M}|d_{M+\ell,k+1}| \neq 0$, then

$$\max_{L<\ell\le N-M}\left|\sum_{p=1}^{k}\frac{d_{M+\ell,p}}{d_{M+\ell}}\left\{\frac{\alpha_0}{\alpha_p}n_j\right\}\right| < \frac{1}{2}\min_{L<\ell\le N-M}\left|\frac{d_{M+\ell,k+1}}{d_{M+\ell}}\left\{\frac{\alpha_0}{\alpha_{k+1}}n_j\right\}\right|.$$

This leads to

$$\frac{1}{2}H < \left\|\frac{\alpha_0}{\alpha_{M+\ell}}n_j + \frac{\beta_0-\beta_{M+\ell}}{\alpha_{M+\ell}}\right\| < \frac{3}{2}H \quad (L<\ell\le N-M).$$

By this, (24), (25), (22), and (20), the lemma follows. □

**Lemma 13.** *Suppose the degree of $F \in \mathcal{S}^\sharp$ is greater than or equal to 1. Let $C$ be a small positive real number. Let $s_i^{(M)}$ run only over such trivial zeros of $F$ of some fixed multiplicity $M$ that $\Re s_i^{(M)} \to -\infty$ and that the discs $|s - s_i^{(M)}| \le 2C$ do not intersect and do not contain any additional zeros of $F$. Let $s$ belong to the union of circles $|s - s_i^{(M)}| = C$. Then we have $F(s) \to \infty$ as $\Re s \to -\infty$.*

**Proof.** Given the distribution of trivial zeros (2) and the conditions of the lemma, we have that there is $c > 0$ for which $|\sin(\pi(\lambda_j s + \mu_j))| > c > 0$, $1 \le j \le N$ if $|s - s_0^{(i)}| = C$. Here, $N$ comes from the functional equation (1). Then, the last factor in the equation (15) is strictly positive. In view of (13), the Stirling's formula implies

$$|F(1-\bar{s})| \to \infty \tag{26}$$

for $|s - s_0| = C$, as $\Re s \to \infty$, which gives the proof of the lemma. □

**Proof of Proposition 11.** Let $k_1 < k_2 < \ldots < k_m$ be such multiplicities of the trivial zeros of $F \in \mathcal{S}^\sharp$ that the number of the trivial zeros for each multiplicity $k_i$, $1 \le i \le m$ is infinite. Let $k_{m+1}$ be the multiplicity of the possible pole at $s = 1$. Let the set $\mathcal{M}$ consist of $k_1, k_2, \ldots, k_m$, and possibly $k_{m+1}$. It is expected ([5, Conjecture 4.2]) that for any multiplicity of a trivial zero, there are infinitely many such zeros.

The proof strategy is first to construct for each $M \in \mathcal{M}$, the set of trivial zeros of multiplicity $M$ satisfying the conditions of Lemma 13 and then to apply this lemma together with Rouché's theorem.

From formula (2), the trivial zeros form arithmetic progressions depending on the parameters $\lambda_j$ and $\mu_j$. These progressions may share some members, resulting in zeros of higher multiplicity. The main problem is that a sequence of zeros of multiplicity $M$ could be indefinitely approximated by another sequence of zeros without coinciding.

We now formulate two statements about arithmetic progressions. The first statement is that the intersection of two arithmetical progressions is again an arithmetical progression. The second statement states that if we remove an arithmetical progression from an arithmetical progression and the remaining set has infinitely many elements, then it contains an arithmetical progression.

From the aforementioned statements, we have that if the trivial zeros of a given multiplicity form an infinite set, then this set contains an arithmetical progression. From Lemma 12, it follows that this progression contains a subsequence such that we can draw discs of some small fixed radii satisfying the conditions of Lemma 13 around each of its members. Denote the radii of the discs for each multiplicity in $\mathcal{M}$ by $C_1, \ldots, C_n$. Then, take the minimum of $C_1, \ldots, C_n$ and call it $C$. Denote the set of zeros inside the discs $\mathcal{A}$.

We turn to the decompositions of $F$. There are complex numbers $z_j$, $j = 1, \ldots, k$, such that

$$p(z) = a_k\prod_{j=1}^{k}(z - z_j).$$

Then a decomposition $F = p \circ h$ can be replaced by equivalent decomposition $F = p_j \circ h_j$, $j = 1, \ldots, k$, where

$$h_j(s) = h(s) - z_j \quad \text{and} \quad p_j(z) = p(z + z_j). \tag{27}$$

Then $p_j(0) = 0$ for each $j$. Choose any multiplicity $M$ of a trivial zero belonging to $\mathcal{A}$ and by $(s_i^{(M)})_{i\in\mathbb{N}}$ denote the sequence of such zeros with $i$ increasing as we move to the left of the complex plane. The real parts of this sequence are unbounded as we approach $-\infty$. For each $i$, there is some $j$ such that $h(s_i^{(M)}) = z_j$, so there must be some $j$ that occurs infinitely many times. Fix such a $j$ and the subsequence $\big(s_{i_r}^{(M)}\big)_{r\in\mathbb{N}}$ of $(s_i^{(M)})_{i\in\mathbb{N}}$ such that $h\big(s_{i_r}^{(M)}\big) = z_j$. Let $a_\ell$ denote the coefficients of $p_j$. Thus,

$$p_j(z) = \sum_{\ell=k'}^{k} a_\ell z^\ell, \tag{28}$$

with some $k' \geq 1$ and $a_{k'} \neq 0$. In view of $F = p_j \circ h_j$, we have that

$$M = k'm, \tag{29}$$

where $m$ is the multiplicity of each zero of $h_j$ at $s_{i_r}^{(M)}$.

Next, we consider the function

$$F(s) - a_{k-1}h_j^{k-1}(s) - \ldots - a_{k'}h_j^{k'}(s) = a_k h_j^k(s).$$

If $F(s)$ is large, then $h_j(s)$ is also large and

$$h_j(s) \ll |F(s)|^{\frac{1}{k}}$$

because $F(s)$ is a polynomial of $h_j(s)$. The real parts of the subsequence $(s_{i_r}^{(M)})$ are also unbounded as we approach $-\infty$. This, Lemma 13, and Rouché's theorem give that on the disc $|s - s_{i_r}^{(M)}| \leq C$, for large negative $\Re s_{i_r}^{(M)}$, the functions $F(s)$ and

$$F(s) - a_{k-1}h_j^{k-1}(s) - \ldots - a_{k'}h_j^{k'}(s) = a_k h_j^k(s)$$

have the same number $M$ of zeros. Recall that the multiplicity of each zero of $h_j$ at $s_{i_r}^{(M)}$ is $m$. Hence,

$$M \geq km.$$

This and (29) lead to $k = k'$. Then in view of (28), we conclude Proposition 11. □

**Proof of Theorem 2.** The theorem follows by Propositions 8 and 11. □

**Acknowledgments:** We thank Bao Qin Li for translating the proof of Theorem 3.12 (Proposition 4 in our article) from [25]. We are also grateful to the anonymous reviewers for their valuable comments, which significantly improved the article. We are especially thankful for the suggestion to use the shifts (27), which strengthened Theorem 2.

**Funding information**: This work is funded by the Research Council of Lithuania (LMTLT), agreement No. S-MIP-22-81.

**Author contributions**: All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript. RG and TP prepared the initial version. RŠ joined later.

**Conflict of interest**: The authors state no conflict of interest.

# References


[1] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), Univ. Salerno, Salerno, 1992, pp. 367–385.

[2] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class. I.* $0 \le d \le 1$, Acta Math. **182** (1999), no. 2, 207–241.

[3] J. Kaczorowski, *Axiomatic theory of L-functions: the Selberg class, Analytic number theory*, Lecture Notes in Mathematics, vol. 1891, Springer, Berlin, 2006, pp. 133–209.

[4] A. Perelli, *A survey of the Selberg class of L-functions. II*, Riv. Mat. Univ. Parma **3** (2004), 83–118.

[5] A. Perelli, *A survey of the Selberg class of L-functions. I*, Milan J. Math. **73** (2005), 19–52.

[6] C. G. Lekkerkerker, *On the Zeros of a Class of Dirichlet Series*, Van Gorcum & Co. N.V., Assen, 1955.

[7] A. Perelli, *General L-functions*, Ann. Mat. Pura Appl. **130** (1982), no. 4, 287–306.

[8] K. Matsumoto, *Value-distribution of zeta-functions*, Analytic number theory (Tokyo, 1988), Lecture Notes in Mathematics, vol. 1434, Springer, Berlin, 1990, pp. 178–187.

[9] F. Gross, *On factorization of meromorphic functions*, Trans. Amer. Math. Soc. **131** (1968), 215–222.

[10] F. Gross, *On factorization theory of meromorphic functions*, Comment. Math. Univ. St. Pauli **24** (1975/76), no. 1, 47–60.

[11] C.-T. Chuang and C.-C. Yang, *Fix-points and Factorization of Meromorphic Functions*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1990, Translated from the Chinese.

[12] H. Urabe, *On factorization of certain entire and meromorphic functions*, J. Math. Kyoto Univ. **26** (1986), no. 2, 177–190.

[13] P. C. Rosenbloom, *The fix-points of entire functions*, Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **1952** (1952), 186–192.

[14] L. Liao and C.-C. Yang, *On some new properties of the Gamma function and the Riemann zeta function*, Math. Nachr. **257** (2003), 59–66.

[15] R. Garunkštis, *Selberg zeta-function associated to compact Riemann surface is prime*, Rev. Un. Mat. Argentina **62** (2021), no. 1, 213–218.

[16] R. Garunkštis and J. Steuding, *On primeness of the Selberg zeta-function*, Hokkaido Math. J. **49** (2020), no. 3, 451–462.

[17] B. Q. Li and C.-C. Yang, *Factorization of meromorphic functions in several complex variables*, Several complex variables in China, Contemp. Math., vol. 142, Amer. Math. Soc., Providence, RI, 1993, pp. 61–74.

[18] F. Gross, *Factorization of meromorphic functions and some open problems*, Lecture Notes in Mathematics, vol. 599, Springer, Berlin, 1977, pp. 51–67.

[19] B. Q. Li, *Unique factorizability of the pth power of entire functions*, J. Math. Anal. Appl. **154** (1991), no. 2, 435–445.

[20] G. Molteni, *On the algebraic independence in the Selberg class*, Arch. Math. (Basel) **79** (2002), no. 6, 432–438.

[21] H. Davenport, *Multiplicative number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York-Berlin, 1980, Revised by Hugh L. Montgomery.

[22] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

[23] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[24] A. A. Goldberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs, vol. 236, American Mathematical Society, Providence, RI, 2008.

[25] C. T. Chuang, C. C. Yang, Y. He, G. C. Wen. *Several Topics in Theory of One Complex Variable*, Science Press, Beijing, China, 1995.

[26] A. Edrei and W. H. J. Fuchs, *On the zeros of f(g(z)) where f and g are entire functions*, J. Anal. Math. **12** (1964), 243–255.

[27] V. N. Monakhov, *Boundary value problems with free boundaries for elliptic systems of equations*, Translations of Mathematical Monographs, vol. 57, American Mathematical Society, Providence, RI, 1983, TranslatedRussian by H. H. McFaden.

[28] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, Oxford, 1939.

[29] J. Steuding, *Value-distribution of L-functions*, Lecture Notes in Mathematics, vol. 1877, Springer, Berlin, 2007.

[30] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.