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SPECTRAL COVARIANCES AND LIMIT THEOREMS FOR INFINITE-VARIANCE LINEAR PROCESSES AND FIELDS

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Contents

Notation

$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set $\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	the set of integers
$\mathbb{S}^{d-1}$	unit sphere in $\mathbb{R}^d$
A^c	the complement of the set A
$\lfloor x \rfloor$	the largest integer not larger than x (floor)
$\lceil x \rceil$	the smallest integer not smaller that x (ceiling)
$\mathbb{1}_A$	the indicator function of A
$x^{<\alpha>}$	the function $\lvert x\rvert^{\alpha} \operatorname{sign}(x)$
$\xrightarrow{\mathrm{d}}$	convergence in distribution
$\xrightarrow{\mathrm{f.d.d.}}$	convergence in finite dimensional distributions
$\stackrel{\mathrm{d}}{=}$	equality in distribution
$\mathbf{x}$	the vector $(x_1, \ldots, x_d)$
$\mathbf{x} \leqslant \mathbf{y}$	inequalities between two vectors are to be understood component-wise
$\mathbf{nu}$	the vector $(n_1u_1, \ldots, n_du_d)$
$o(f(x))$	$g(x) = o(f(x))$ as $x \to a$ if there exists a function ε such that $\varepsilon(x) \to 0$ as $x \to a$ and $g(x) = f(x)\varepsilon(x)$
$O(f(x))$	$g(x) = O(f(x))$ as $x \to a$ if there exists $M > 0$ such that $\lvert g(x)\rvert \leqslant M \lvert f(x)\rvert$ in the neighbourhood of a
f.d.d.	finite dimensional distributions
i.i.d.	independent identically distributed
s.v.f.	slowly varying function
$S\alpha S$	symmetric α-stable

$\rho(X,Y)$	spectral covariance between X and Y, see (2.7) on page 8
$\tilde{\rho}(X,Y)$	spectral correlation coefficient of (X,Y), see (2.8) on page 8
$[X,Y]_\alpha$	covariation of X on Y, see (2.9) on page 8
$\tau(X,Y)$	codifference between X and Y, see (2.10) on page 8
$I(\theta_1,\theta_2;X,Y)$	generalized codifference between X and Y, see (2.11) on page 8
$\rho_\alpha(X,Y)$	α-spectral covariance between X and Y, see (4.1) on page 37

1 Introduction

1.1 Aims and problems

One of the main aims of this work is to develop theory of spectral covariances. We derive some properties of spectral covariances, investigate asymptotics of the spectral covariance for some infinite-variance linear processes and fields. Namely, we investigate linear process $X_n = \sum_{k=0}^{\infty} c_k \xi_{n-k}$ with asymptotically regularly varying filter $c_k,\ k \in \mathbb{N}$, and i.i.d. α-stable innovations $\xi_k,\ k \in \mathbb{Z}$, we also study the asymptotic dependence between one-step increments of linear fractional stable motion and log-fractional stable motion. In addition, we investigate linear field $X_{k,l} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{(i,j)}(1+i)^{-\beta_1}(1+j)^{-\beta_2}\xi_{k-i,l-j}$, where $\beta_i > 1/\alpha,\ i = 1, 2$, $\xi_{i,j},\ i, j \in \mathbb{Z}$, are i.i.d. α-stable random variables, and coefficients $w_{(i,j)}$ have limits $\lim_{i\to\infty} w_{(i,j)} = w_{(\infty,j)}$, $\lim_{j\to\infty} w_{(i,j)} = w_{(i,\infty)}$ and $\lim_{i,j\to\infty} w_{(i,j)} = 1$. Faced with a complicated picture when investigating the asymptotic behaviour of spectral covariance for this linear field, we introduce another measure of dependence – α-spectral covariance – which we use to investigate the asymptotic dependence structure of d-dimensional linear field $X_{\mathbf{k}} = \sum_{\mathbf{j}\geqslant\mathbf{0}} \left(\prod_{l=1}^{d}(1+j_l)^{-\beta_l}\right)\xi_{\mathbf{k}-\mathbf{j}}$, where $\beta_l > 1/\alpha,\ l = 1, \ldots, d$, and $\xi_{\mathbf{j}},\ \mathbf{j} \in \mathbb{Z}^d$ are i.i.d. α-stable random variables.

The results obtained in [17], relating the asymptotic behaviour of dependence measures to the limit theorems, are generalized to the linear fields.

We answer a question, originally proposed in [52], concerning limit theorems in the case of negative memory. Consider a partial sum process

$S_n(t) = \sum_{k=0}^{\lfloor nt \rfloor} X_k$ of linear processes $X_n = \sum_{i=0}^{\infty} c_i \xi_{n-i}$ with independent identically distributed innovations $\{\xi_i\}$ belonging to the normal domain of attraction of α-stable law, $0 < \alpha \leq 2$. If $|c_k| = k^{-\gamma}$, $k \in \mathbb{N}$, $\gamma > \max(1, 1/\alpha)$, and $\sum_{k=0}^{\infty} c_k = 0$ (the case of negative memory for the stationary sequence $\{X_n\}$), it is known that the normalizing sequence of $S_n(1)$ can grow as $n^{1/\alpha-\gamma+1}$ or remain bounded, if the signs of the coefficients are constant or alternate, respectively. It is of interest to know whether it is possible, given $\lambda \in (0, 1/\alpha - \gamma + 1)$, to change the signs of c_k so that the rate of growth of the normalizing sequence would be n^{λ}. The positive answer is given: we propose a way of choosing the signs and investigate the finite-dimensional convergence of appropriately normalized $S_n(t)$ to linear fractional Lévy motion.

We also generalize (with an additional condition) Theorem 1 in [3] to the case of d-dimensional linear fields. Namely, we investigate convergence in the sense of f.d.d. of appropriately normalized partial sum processes

$$S_{\mathbf{n}}(\mathbf{t}) = \sum_{\mathbf{0} \leqslant \mathbf{k} \leqslant \mathbf{nt}} X_{\mathbf{k}}$$

when $X_{\mathbf{k}} = \sum_{\mathbf{i} \geqslant \mathbf{0}} c_{\mathbf{i}} \xi_{\mathbf{k}-\mathbf{i}}$, coefficients $c_{\mathbf{i}}$ have form $c_{\mathbf{i}} = \prod_{l=1}^{d} a_{i_l}(\gamma_l, l)$ with asymptotically regularly varying with index $-\gamma_l$, $\gamma_l > 1/\alpha$, sequences $a_{i_l}(\gamma_l, l)$, and $\xi_{\mathbf{i}}$ – i.i.d. copies of random variable ξ belonging to the domain of attraction of α-stable law.

1.2 Methods

To prove asymptotic behaviour of dependence measures we use well known results from mathematical analysis. We mostly apply the dominated convergence theorem.

Some known results about slowly varying functions are employed.

To derive Theorems 5.1 and 5.3 we extend the proofs provided in [17] to linear fields. In [17] the proof of Newman's central limit theorem was adapted to the case of infinite variance.

In order to prove the convergence in the sense of f.d.d. of appropriately normalized partial sum processes we use the method of characteristic functions.

1.3 Novelty

The obtained results are new. Most of the results are included in the following publications:

J. Damarackas, V. Paulauskas: Properties of spectral covariance for linear processes with infinite variance, *Lithuanian Mathematical Journal,* 54 (2014) 252–276.

J. Damarackas, V. Paulauskas: On spectral covariance for random fields with infinite variance, *Journal of Multivariate Analysis*, 153 (2017) 156–175.

J. Damarackas: A note on the normalizing sequences for sums of linear processes in the case of negative memory. *Accepted for publication in Lithuanian Mathematical Journal.*

1.4 Acknowledgements

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2 Preliminaries

We begin by recalling some definitions and results about regularly varying functions from [9], which is a standard reference on regularly varying functions. We also prove some lemmas, which will be used in our proofs.

A measurable function $U : (0, \infty) \to (0, \infty)$ is called regularly varying with index η (or η-varying), if for any $\lambda > 0$

$$\lim_{x \to \infty} \frac{U(\lambda x)}{U(x)} = \lambda^{\eta}. \tag{2.1}$$

If $\eta = 0$, the function U is called slowly varying. Every η-varying function U can be written as $U(x) = x^{\eta} L(x)$, where L is a slowly varying function. In order to prove some of the results, we will use the following Potter's theorem (see [9], Theorem 1.5.6, part *iii*))

Theorem 2.1. *If f is a regularly varying function with an index ρ, then for any $A > 1,\ \delta > 0$ there exists $B = B(A, \delta)$ such that for any $x > B,\ y > B$*

$$f(y)(f(x))^{-1} \leqslant A \max\left(\left(yx^{-1}\right)^{\rho+\delta}, \left(yx^{-1}\right)^{\rho-\delta}\right).$$

Suppose L is a s.v.f. It follows from de Bruijn's theorem (see [9], Theorem 1.5.13, for the complete formulation) that there exists a s.v.f. $L^{\#}$, unique up to asymptotic equivalence, such that

$$L^{\#}(x) L(x L^{\#}(x)) \to 1,\ x \to \infty.$$

If this result is applied to the s.v.f. $L(x) = \left(h(x^{1/\alpha})\right)^{-1}$ the existence of a s.v.f. $h_{1/\alpha}$ satisfying

$$h\left(x^{1/\alpha} h_{1/\alpha}^{1/\alpha}(x)\right) \sim h_{1/\alpha}(x),\ x \to \infty, \tag{2.2}$$

is obtained.

The following three lemmas will be useful in our considerations:

Lemma 2.2. *If U is a regularly varying function with index $-\gamma$, $\gamma \geqslant 0$, and $c_k \sim U(k)$, as $k \to \infty$, then for any $\eta > 0$ there exists a constant E such that*

$$|c_k| \leqslant E(1+k)^{\eta-\gamma} \tag{2.3}$$

for all $k \geqslant 0$.

Proof. Since $c_k \sim U(k)$, there exists N_1 such that $|c_k| \leqslant 2U(k)$ for $k \geqslant N_1$. As U is regularly varying with index $-\gamma$, we have $U(k)/k^{-\gamma+\eta} \to 0$, as $k \to \infty$. Also, $k/(k+1) \to 1$, as $k \to \infty$, therefore there exists $N_2 \geqslant N_1$ such that $U(k) \leqslant (1+k)^{-\gamma+\eta}$ for $k \geqslant N_2$. Let us denote

$$E_0 = \max_{k<N_2} \frac{|c_k|}{(1+k)^{\eta-\gamma}}.$$

Setting $E = \max(2, E_0)$ we obtain (2.3). □

Lemma 2.3. *Suppose h is a s.v.f., $q_n \to \infty$, and $f_n : \mathcal{U} \to \mathbb{R}$ is a sequence of functions such that $q_n |f_n(u)| \to \infty$ uniformly for $u \in \mathcal{U}$. For every $\delta > 0$ there exists $N_1 \in \mathbb{N}$ such that*

$$\left| \frac{h(q_n |f_n(u)|)}{h(q_n)} \right| \leqslant 2 \max\{|f_n(u)|^{\delta}, |f_n(u)|^{-\delta}\}, u \in \mathcal{U}, n \geqslant N_1.$$

Proof of Lemma 2.3. This result follows directly from Theorem 2.1. □

Lemma 2.4. *Suppose h is a s.v.f., $f(x) = |x|^{\alpha}$ or $f(x) = x^{<\alpha>}$, $\alpha > 0$, and q_n, y_n are sequences of real numbers such that $q_n \to \infty$, $y_n \to y$. Then*

$$f(y_n) \frac{h(q_n |y_n|^{-1})}{h(q_n)} \to f(y), \ n \to \infty. \tag{2.4}$$

If $y \neq 0$, (2.4) holds with $\alpha \leqslant 0$ as well.

Proof of Lemma 2.4. We begin by assuming $y = 0$. Lemma 2.3 implies that for large n

$$\left| f(y_n) \frac{h(q_n |y_n|^{-1})}{h(q_n)} \right| \leqslant |y_n|^{\alpha} 2 \max\left\{|y_n|^{\frac{\alpha}{2}}, |y_n|^{-\frac{\alpha}{2}}\right\} \leqslant 2 |y_n|^{\frac{\alpha}{2}} \to 0.$$

Now suppose $y \neq 0$. Since $y_n \to y$ holds, there exist $a, b > 0$ and $n_0 \in \mathbb{N}$ such that $a < |y_n|^{-1} < b$ for all $n > n_0$. S.v.f. h has a property that $h(\lambda t)/h(t) \xrightarrow{t\to\infty} 1$ uniformly for $0 < a \leqslant \lambda \leqslant b < \infty$ (see, for instance, [9], Theorem 1.5.2). We obtain $h(q_n |y_n|^{-1})/h(q_n) \to 1$. As f is continuous at y, (2.4) holds. □

Next we move to α-stable random vectors. Let

$$\mathbb{S}^{d-1} = \{\mathbf{s} \in \mathbb{R}^d : \|\mathbf{s}\| = 1\}$$

be the unit sphere in $\mathbb{R}^d$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Letters in bold will be used to denote vectors in $\mathbb{R}^d$. A random vector $\mathbf{X} = (X_1, \dots, X_d)$ is α-stable with parameter $0 < \alpha < 2$, $\alpha \neq 1$, if there exist a finite measure Γ on $\mathbb{S}^{d-1}$ and a vector $\mathbf{b} \in \mathbb{R}^d$ such that the characteristic function (ch.f.) of $\mathbf{X}$ is given by

$$\begin{aligned} &\mathbb{E}\exp\{\mathrm{i}\langle \mathbf{t}, \mathbf{X}\rangle\} \\ &\quad = \exp\left\{-\int_{\mathbb{S}^{d-1}} |\langle \mathbf{t}, \mathbf{s}\rangle|^\alpha \left(1 - \operatorname{sign}\langle \mathbf{t}, \mathbf{s}\rangle \tan\frac{\pi\alpha}{2}\right)\Gamma(\mathrm{d}\mathbf{s}) + \mathrm{i}\langle \mathbf{t}, \mathbf{b}\rangle\right\}. \quad (2.5) \end{aligned}$$

For $\alpha = 1$, we only consider symmetric measures Γ. In such a case, we get the so-called symmetric α-stable ($S\alpha S$) distributions with ch.f. of a very simple form:

$$\mathbb{E}\exp\{\mathrm{i}\langle \mathbf{t}, \mathbf{X}\rangle\} = \exp\left\{-\int_{\mathbb{S}^{d-1}} |\langle \mathbf{t}, \mathbf{s}\rangle|^\alpha \Gamma(\mathrm{d}\mathbf{s})\right\}. \quad (2.6)$$

The measure Γ in (2.5) is called the spectral measure of an α-stable random vector $\mathbf{X}$, and the pair $(\Gamma, \mathbf{b})$ is unique. The Gaussian case $\alpha = 2$ is excluded from this definition since, in the Gaussian case, there is no uniqueness of the spectral measure Γ: different measures Γ may give the same ch.f. Taking $d = 2$ and $\mathbf{b} = 0$ in (2.5), we have an α-stable random vector $\mathbf{X} = (X_1, X_2)$ with spectral measure Γ on $\mathbb{S}^1 = \{\mathbf{s} = (s_1, s_2) \in \mathbb{R}^2 : s_1^2 + s_2^2 = 1\}$.

The spectral covariance of $\mathbf{X}$ (or the spectral covariance between the coordinates X_1 and X_2) is defined as

$$\rho(X_1, X_2) = \int_{\mathbb{S}^1} s_1 s_2 \Gamma(\mathrm{d}\mathbf{s}). \tag{2.7}$$

Also, in analogy with the usual correlation coefficient, the spectral correlation coefficient (s.c.c.) for an α-stable random vector $\mathbf{X}$ is defined as

$$\tilde{\rho}(X_1, X_2) = \int_{\mathbb{S}^1} s_1 s_2 \Gamma(\mathrm{d}\mathbf{s}) \left(\int_{\mathbb{S}^1} s_1^2 \Gamma(\mathrm{d}\mathbf{s}) \int_{\mathbb{S}^1} s_2^2 \Gamma(\mathrm{d}\mathbf{s}) \right)^{-1/2}. \tag{2.8}$$

Suppose X_1 and X_2 are jointly $S\alpha S$ random variables. The covariation of X_1 on X_2 is defined for $1 < \alpha \leqslant 2$ and equals

$$[X_1, X_2]_\alpha = \int_{\mathbb{S}^1} s_1 s_2^{<\alpha-1>} \Gamma(\mathrm{d}\mathbf{s}). \tag{2.9}$$

Another measure of dependence, the codifference, was defined for all $S\alpha S$ random vectors as

$$\tau(X_1, X_2) = \int_{\mathbb{S}^1} \left(|s_1|^\alpha + |s_2|^\alpha - |s_1 - s_2|^\alpha \right) \Gamma(\mathrm{d}\mathbf{s}). \tag{2.10}$$

In the literature one can find two measures of dependence closely related to the codifference. The first is sometimes referred to as the generalized codifference, and is defined as

$$I(\theta_1, \theta_2; X_1, X_2) := \ln \frac{\mathbb{E} \exp\{\mathrm{i}\theta_1 X_1\} \mathbb{E} \exp\{\mathrm{i}\theta_2 X_2\}}{\mathbb{E} \exp\{\mathrm{i}(\theta_1 X_1 + \theta_2 X_2)\}}. \tag{2.11}$$

Another measure is the difference between the joint characteristic function of (X_1, X_2) and the product of their marginal characteristic functions:

$$\begin{aligned} &U(\theta_1, \theta_2; X_1, X_2) \\ &\qquad = \mathbb{E} \exp\{\mathrm{i}(\theta_1 X_1 + \theta_2 X_2)\} - \mathbb{E} \exp\{\mathrm{i}\theta_1 X_1\} \mathbb{E} \exp\{\mathrm{i}\theta_2 X_2\}. \end{aligned} \tag{2.12}$$

The relation between those measures of dependence is as follows

$$\tau(X_1, X_2) = -I(1, -1; X_1, X_2),$$

$$U(\theta_1, \theta_2; X_1, X_2)$$
$$= \mathbb{E}\exp\{\mathrm{i}\theta_1 X_1\}\mathbb{E}\exp\{\mathrm{i}\theta_2 X_2\}\left(\exp(-I(\theta_1, \theta_2; X_1, X_2)) - 1\right).$$

If the process $X(t)$ is stationary and $I(\theta_1, \theta_2; X(0), X(t)) \to 0$, we have

$$U(\theta_1, \theta_2; X_1, X_2) \sim -\mathbb{E}\exp\{\mathrm{i}\theta_1 X(0)\}\mathbb{E}\exp\{\mathrm{i}\theta_2 X(0)\}I(\theta_1, \theta_2; X_1, X_2),$$

i.e., the quantities $U(\theta_1, \theta_2; X_1, X_2)$ and $I(\theta_1, \theta_2; X_1, X_2)$ are asymptotically proportional.

In this work we will investigate linear processes and fields with innovations belonging to the domain of attraction of some α-stable random variable. Random variable ξ belongs to the domain of attraction of α-stable random variable if in the neighbourhood of zero it has characteristic function

$$\mathbb{E}\exp\left(\mathrm{i}x\xi\right) = \begin{cases} \exp\left(-\upsilon_\alpha(x)(1 - \mathrm{i}\beta \mathrm{sign}(x)\tan\frac{\pi\alpha}{2}) + \mathrm{i}x\mu\right) & \text{if } \alpha \neq 1, \\ \exp\left(-\upsilon_\alpha(x)(1 + \mathrm{i}\beta\frac{2}{\pi}\mathrm{sign}(x)\ln(|x|)) + \mathrm{i}x\mu\right) & \text{if } \alpha = 1, \end{cases}$$

where $\upsilon_\alpha(x) = \sigma^\alpha h\left(|x|^{-1}\right)|x|^\alpha(1+o(1))$, h - a s.v.f., $\alpha \in (0, 2], \sigma > 0, |\beta| \leqslant 1, \mu \in \mathbb{R}$. The standard reference on this is [29].

For simplicity let us assume that $\sigma = 1$. We will be working under the assumption that $\beta = 0$ in the case $\alpha = 1$, therefore we can write

$$\mathbb{E}\exp\left(\mathrm{i}x\xi\right) = \exp\left(-h\left(|x|^{-1}\right)(|x|^\alpha - \mathrm{i}\beta x^{\langle\alpha\rangle}\tau_\alpha)(1 + r(x)) + \mathrm{i}x\mu\right), \quad (2.13)$$

where $r(x) \to 0$, as $x \to 0$, $x^{\langle\alpha\rangle} = |x|^\alpha \mathrm{sign}(x)$, $\tau_\alpha = \tan(\pi\alpha/2)$ if $\alpha \neq 1$, and $\tau_1 = 0$.

Now we shall recall the notion of linear stable processes of continuous time. It is well-known what important role α-stable stochastic integrals play in the theory of stable random vectors and processes, that is, integrals of non-random functions with respect to α-stable random measures. The large part of the monograph [58] is devoted to these integrals, therefore we do not provide all definitions of notions, we shall remind the main of them only. We will try to keep the same notation as in [58], referring the reader to this monograph. Let $(E, \mathcal{E}, m)$ be a measurable space with a measure m, and let $\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\}$ and $\beta : E \to [-1, 1]$ be a measurable function. Let us denote by $(\Omega, \mathcal{F}, P)$ the probability space and by $L^0(\Omega)$ the set of all real random variables defined on it.

An independently scattered σ-additive set function $M : \mathcal{E}_0 \to L^0(\Omega)$ is called an α-stable random measure with control measure m and skewness intensity β, if for each $A \in \mathcal{E}_0$ a random variable $M(A)$ is stable with scale, skewness and shift parameters $m(A)^{1/\alpha}$, $m(A)^{-1} \int_A \beta(x)m(\mathrm{d}x)$, and 0, respectively, see Definition 3.3.1 in [58]. Taking $f \in L^\alpha(E, \mathcal{E}, m)$, we get an α-stable random variable

$$X = \int_E f(x)M(\mathrm{d}x),$$

while taking a collection $f_i \in L^\alpha(E, \mathcal{E}, m),\ i = 1, \ldots, k,$ we get an α-stable random vector

$$(X_1, \ldots, X_k), \quad X_i = \int_E f_i(x)M(\mathrm{d}x).$$

Taking a family of functions $\{f_t, t \in T\} \subset L^\alpha(E, \mathcal{E}, m)$ we get an α-stable random process

$$X(t) = \int_E f_t(x)M(\mathrm{d}x),\ t \in T. \tag{2.14}$$

It follows from Property 3.2.1 in [58] that the joint characteristic function of $X(t_1), \ldots, X(t_d)$ is

$$\begin{aligned}\phi_{t_1,\ldots,t_d}&(x_1, \ldots, x_d)\\ &= \exp\left(-\int_E \left|\sum_{j=1}^d x_j f_{t_j}(u)\right|^\alpha \times\right.\\ &\left.\times\left(1 - \mathrm{i}\beta(u)\mathrm{sign}\left(\sum_{j=1}^d x_j f_{t_j}(u)\right)\tan\frac{\pi\alpha}{2}\right) m(\mathrm{d}u)\right)\end{aligned}$$

if $\alpha \neq 1$, and

$$\phi_{t_1,\ldots,t_d}(x_1,\ldots,x_d) = \exp\left(-\int_E \left|\sum_{j=1}^d x_j f_{t_j}(u)\right| \times \times \left(1 + \mathrm{i}\frac{2}{\pi}\beta(u)\mathrm{sign}\left(\sum_{j=1}^d x_j f_{t_j}(u)\right) \ln\left|\sum_{j=1}^d x_j f_{t_j}(u)\right|\right) m(\mathrm{d}u)\right)$$

if $\alpha = 1$.

Important linear stable process of continuous time is linear fractional Lévy motion (LFLM). It is the stochastic process given by

$$Z_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \phi_{\alpha,H}(a,b;t,u) M(\mathrm{d}u), \tag{2.15}$$

with

$$\phi_{\alpha,H}(a,b;t,u) = a\left(((t-u)_+)^{H-\frac{1}{\alpha}} - ((-u)_+)^{H-\frac{1}{\alpha}}\right) + + b\left(((t-u)_-)^{H-\frac{1}{\alpha}} - ((-u)_-)^{H-\frac{1}{\alpha}}\right),$$

M – a stable random measure on $\mathbb{R}$ with Lebesgue control measure, skewness intensity $\beta(u)$ satisfying two additional conditions (we refer the reader to Definition 7.4.1 in [58] for details), $a^2 + b^2 > 0$, $0 < H < 1, 0 < \alpha < 2$ and $H \neq 1/\alpha$.

An extension of LFLM to the case $H = 1/\alpha,\ 1 < \alpha < 2$, is log-fractional stable motion. Suppose M is a stable random measure on $\mathbb{R}$ with Lebesgue control measure and a constant skewness intensity. Log-fractional stable motion is the stochastic process defined as

$$\Lambda_\alpha(t) = \int_{-\infty}^{\infty} (\ln|t-x| - \ln|x|)\, M(\mathrm{d}x),\ t \in \mathbb{R}.$$

The increment process of LFSM is known as linear fractional stable noise, it equals

$$Y_1(t) = Z_{\alpha,H}(a,b;t+1) - Z_{\alpha,H}(a,b;t). \tag{2.16}$$

and forms a stationary sequence. It can be expressed as

$$Y_1(t) = \int_{-\infty}^{\infty} f_t(x) M(\mathrm{d}x), \tag{2.17}$$

where M is a stable random measure on R with Lebesgue control measure, and

$$f_t(x) = a\left((t+1-x)_+^{\eta} - (t-x)_+^{\eta}\right) + b\left((t+1-x)_-^{\eta} - (t-x)_-^{\eta}\right), \tag{2.18}$$

with $\eta = H - 1/\alpha$. Similarly, the log-fractional stable noise equals

$$Y_2(t) = \Lambda_\alpha(t+1) - \Lambda_\alpha(t), \tag{2.19}$$

and can be expressed as (2.17) with

$$f_t(x) = \ln|t+1-x| - \ln|t-x| = \ln\left|\frac{t+1-x}{t-x}\right|. \tag{2.20}$$

Fractional noise processes described above are examples of moving average processes – processes $X(t)$ that can be represented as

$$X(t) = \int_{-\infty}^{\infty} f(t-x) M(\mathrm{d}x), \tag{2.21}$$

where M is a stable random measure with the Lebesgue control measure and $f \in L^\alpha$.

An important $S\alpha S$ moving average process is the Ornstein-Uhlenbeck process, which is defined as

$$X(t) = \int_{-\infty}^{t} \exp(-\lambda(t-x)) M(\mathrm{d}x),$$

where M is $S\alpha S$ stable random measure with Lebesgue control measure.

Next, we recall the notion of association, which has origins in several papers, see [21, 22, 27, 40] and, for more information on association and related notions (positive and negative association, association of measures, etc.), see monograph [12]. Random variables $X_1, \dots, X_n$ are associated if

$$\mathrm{Cov}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0$$

for each pair of functions $f, g : \mathbb{R}^n \to \mathbb{R}$ that are non-decreasing in each coordinate and for which this covariance exists. An infinite collection of random variables is associated if every its finite subset consists of associated random variables. In [39], it was proved that a jointly stable random vector $(X_1, \dots, X_n)$ is associated if and only if its spectral measure Γ_n satisfies the relation

$$\Gamma_n(\mathbb{S}^{n-1} \cap \{[0, \infty)^n \cup (-\infty, 0]^n\}^c) = 0.$$

In the proof of Theorem 5.1, we will use the following multivariate Fekete lemma from [15].

Lemma 2.5 ([15]). *Let $f : \mathbb{Z}_+^d \to [0, \infty)$ satisfy the conditions*

$$f(x_1, \dots, x_j + y_j, \dots, x_d) \le f(x_1, \dots, x_j, \dots, x_d) + f(x_1, \dots, y_j, \dots, x_d) \tag{2.22}$$

for all $x_1, \dots, x_d, y_j \in \mathbb{Z}_+$ and $j \in \{1, \dots, d\}$. Then

$$\lim_{\mathbf{n} \to \infty} \frac{f(n_1, \dots, n_d)}{n_1 \cdots n_d} \tag{2.23}$$

exists and equals

$$\inf_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{f(n_1, \dots, n_d)}{n_1 \cdots n_d}.$$

Remark 2.6. As in the original Fekete lemma ($d = 1$), if instead of subadditivity in each argument (2.22), we have superadditivity in each argument

$$f(x_1, \dots, x_j + y_j, \dots, x_d) \ge f(x_1, \dots, x_j, \dots, x_d) + f(x_1, \dots, y_j, \dots, x_d),$$

then again the limit (2.23) exists, but now it equals

$$\sup_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{f(n_1, \dots, n_d)}{n_1 \cdots n_d}.$$

3 Literature review

3.1 Measures of dependence of processes with infinite variance

Spectral correlation coefficient and spectral covariance

In the paper [53] an attempt was made by Press to introduce a measure of dependence between coordinates of $S\alpha S$ bivariate vector sharing some properties with the usual correlation coefficient. A mistake, later pointed out by Paulauskas in [50], led the author of [53] to believe that all such vectors have ch.f.

$$\begin{aligned}
&\mathbb{E}\exp\left(\mathrm{i}\langle(X_1, X_2), (\theta_1, \theta_2)\rangle\right) \\
&\qquad = \exp\left(-\sum_{i=1}^{m}\left(w_{11}(i)\theta_1^2 + 2w_{12}(i)\theta_1\theta_2 + w_{22}(i)\theta_2^2\right)^{\alpha/2}\right), \quad (3.1)
\end{aligned}$$

where

$$\begin{pmatrix} w_{11}(i) & w_{12}(i) \\ w_{21}(i) & w_{22}(i) \end{pmatrix}, i = 1, \ldots, m,$$

are symmetric positive semi-definite matrices. Press suggested to define the association parameter (a.p.) of (X_1, X_2) as

$$\rho(X_1, X_2) = \frac{\sum_{i=1}^m w_{12}(i)}{\sqrt{\left(\sum_{i=1}^m w_{11}(i)\right)\left(\sum_{i=1}^m w_{22}(i)\right)}}.$$

Having shown that there are $S\alpha S$ distributions with ch.f. that can not be expressed as (3.1), Paulauskas in [50] has suggested another measure of dependence – the spectral correlation coefficient (s.c.c.) defined in (2.8), originally called the generalized association parameter – which could be applied to any bivariate $S\alpha S$ vector. The paper highlighted some good features of this dependence measure, namely, it was shown that the s.c.c. $\tilde{\rho}$ has the following properties:

Proposition 3.1.

1. *$|\tilde{\rho}| \le 1$, and if the coordinates of $\mathbf{X}$ are independent then $\tilde{\rho} = 0$;*

2. *if $|\tilde{\rho}| = 1$, then the distribution of $\mathbf{X}$ is concentrated on a line, i.e., coordinates X_1 and X_2 are linearly dependent;*

3. *if $\alpha = 2$, $\tilde{\rho}$ coincides with a correlation coefficient of a Gaussian random vector with characteristic function* (2.6)*;*

4. *$\tilde{\rho}$ is independent of α and depends only on the spectral measure Γ of $\mathbf{X}$.*

5. *if a random vector $\mathbf{X}$ is sub-Gaussian with ch.f.*

$$\exp\left\{-(\sigma_1^2 t_1^2 + 2r\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2)^{\alpha/2}\right\},$$

where σ_1^2, σ_2^2 are variances and r is the correlation coefficient of underlying Gaussian vector, then the spectral correlation coefficient equals r.

For a long time, except for a brief mention in [58], there was almost no literature dealing with this measure of dependence.

The interest in the spectral correlation coefficient was revived in [24], where it was compared to a newly introduced measure of dependence.

An unpublished paper [51] by Paulauskas followed, first part of which can be considered as a program for developing theory of spectral covariances. In the paper Paulauskas introduced another measure of dependence – the spectral covariance – an analogue of the usual covariance defined by (2.7). The spectral covariance and s.c.c. in [51] were referred to as α-covariance and α-correlation coefficient. It was shown that the notion of spectral covariance and s.c.c. can be extended to general bivariate α-stable random vectors. Also, it was shown how those measures of dependence could be used to measure dependence between coordinates of a vector belonging to the normal domain of attraction of α-stable random vector. Later, in [18], it was noticed that the same approach can be used to vectors belonging to the domain of attraction of α-stable random vector: let $\xi = (\xi_1, \xi_2)$ be a random vector satisfying the following condition: there exist a number $0 < \alpha < 2$, a s.v.f. L, and a finite measure Γ on $\mathbb{S}^1$ such that

$$\lim_{x\to\infty} \frac{x^\alpha}{L(x)} \mathbb{P}\left(\|\xi\| > x,\ \xi\|\xi\|^{-1} \in A\right) = \Gamma(A) \tag{3.2}$$

for any Borel set A on $\mathbb{S}^1$ with $\Gamma(\partial A) = 0$. It is well known (see [56]) that this condition is necessary and sufficient for (ξ_1, ξ_2) to belong to the domain of attraction of an α-stable random vector $\mathbf{X} = (X_1, X_2)$ with exponent α and spectral measure Γ. The spectral covariance and s.c.c. of (ξ_1, ξ_2) were defined by means of the measure Γ in the same way as these quantities are defined for an α-stable random vector $\mathbf{X} = (X_1, X_2)$:

$$\rho(\xi_1, \xi_2) = \rho(X_1, X_2) = \int_{\mathbb{S}^1} s_1 s_2 \Gamma(\mathrm{d}\mathbf{s}),$$

and similarly for $\tilde{\rho}(\xi_1, \xi_2)$; see [18] for motivation of such a definition.

Also, it was demonstrated in [51] that the notion of spectral covariance can be naturally extended to α-stable random vectors with values in $\mathbb{R}^d$ or even in separable Banach space.

The spectral covariance and s.c.c. for stochastic integrals (and linear processes, which can be considered as a particular case of stochastic integrals) does not depend on the skewness intensity $\beta(x)$. To see this, let (X_1, X_2) be a bivariate $S\alpha S$ random vector, defined by means of stochastic integrals, i.e.,

$$(X_1, X_2) \stackrel{\mathrm{d}}{=} \left(\int_E f_1(x) M(\mathrm{d}x),\ \int_E f_2(x) M(\mathrm{d}x)\right),$$

where $\stackrel{\mathrm{d}}{=}$ stands for equality in distribution. In Chapter 3.2 in [58] one can find the following expression of the spectral measure Γ of the random vector (X_1, X_2) via control measure m, skewness intensity β and functions f_1, f_2:

$$\Gamma(A) = \int_{g^{-1}(A)} \frac{1+\beta(x)}{2} m_1(\mathrm{d}x) + \int_{g^{-1}(-A)} \frac{1-\beta(x)}{2} m_1(\mathrm{d}x),$$

where A is a Borel set in $\mathbb{S}^1$,

$$g^{-1}(A) = \{x \in E_+ : (g_1(x), g_2(x)) \in A\},$$

$$E_+ = \{x \in E : f_1^2(x) + f_2^2(x) > 0\},$$

$$m_1(\mathrm{d}x) = \left(f_1^2(x) + f_2^2(x)\right)^{\alpha/2} m(\mathrm{d}x),$$

and

$$g_i(x) = \frac{f_i(x)}{(f_1^2(x) + f_2^2(x))^{1/2}}, \quad i = 1, 2.$$

From this expression it is easy to see that in the calculation of spectral covariance the expression $s_1 s_2 \Gamma(\mathrm{d}s) + (-s_1)(-s_2)\Gamma(-\mathrm{d}s)$ is equal to $s_1 s_2 m_1(\mathrm{d}x) \mid_{(g_1(x), g_2(x)) = \pm(s_1, s_2)}$, where the last expression means that the differential is calculated at points where vector (g_1, g_2) is equal to (s_1, s_2) or $-(s_1, s_2)$. Therefore, we get the following expressions of spectral covariance and spectral correlation coefficient:

$$\rho(X_1, X_2) = \int_E \frac{f_1(x) f_2(x)}{(f_1^2(x) + f_2^2(x))^{\frac{2-\alpha}{2}}} m(\mathrm{d}x), \tag{3.3}$$

$$\tilde{\rho}(X_1, X_2) = \frac{\int_E \frac{f_1(x) f_2(x)}{(f_1^2(x) + f_2^2(x))^{\frac{2-\alpha}{2}}} m(\mathrm{d}x)}{\left(\int_E \frac{f_1(x)^2}{(f_1^2(x) + f_2^2(x))^{\frac{2-\alpha}{2}}} m(\mathrm{d}x) \int_E \frac{f_2(x)^2}{(f_1^2(x) + f_2^2(x))^{\frac{2-\alpha}{2}}} m(\mathrm{d}x)\right)^{1/2}}.$$

Formally in the above written formulae one should integrate over $E_+ = \{x \in E : f_1^2(x) + f_2^2(x) > 0\}$, but for convenience of writing we agree that the integrand is equal to zero if $f_1^2(x) + f_2^2(x) = 0$.

Another interesting topic, especially for practitioners, is the estimation of spectral covariances, for which we must have estimates of the spectral measure Γ and (in the case of estimation of other measures of dependence) of the parameter α. Whereas for univariate heavy-tailed distributions, estimation of the tail index is developed quite well, this cannot be said about the multivariate case. Here we restrict ourselves to giving references [16, 20, 49, 55], where estimation of the parameters of multivariate heavy-tailed distributions is considered. In [33], the first estimate of the spectral covariance (where as in [50], the term "generalized association parameter" is used) is constructed.

Other measures of dependence

In this section we present an overview of research involving measures of dependence for random variables with infinite variance. This overview is in no way complete, but is sufficient to illustrate the field of research and to survey some recent results.

The covariation was introduced by Miller in [46] and since then was widely investigated. Together with its generalization to pth order random variables, it naturally appears in many settings. We refer the reader to the paper [14], or [13], where a connection between the covariation, conditional moments, and James orthogonality was established.

In [2, 4], the quantity $U(\theta_1, \theta_2; X_1, X_2)$, defined in (2.12), as a measure of dependence between the coordinates of an α-stable vector (X_1, X_2), was considered. Later on (see [36, 37]), it was noted that, instead of this difference, it is more convenient to consider the generalized codifference $I(\theta_1, \theta_2; X_1, X_2)$, defined in (2.11).

Many properties of codifference and covariation are presented in [58]. For example,

Theorem 3.2. *For a $S\alpha S$, $0 < \alpha \leqslant 2$, stationary moving average process*

X_t,

$$\lim_{t\to\infty} \tau(X_t, X_0) = 0.$$

For the proof see Theorem 4.7.3 in [58]. The exact asymptotic rate of decay of dependence measures is of interest – in some papers it is used to classify long memory, long-range dependence. There are many papers dealing with asymptotic behaviour of dependence measures.

In [36] ARMA time series with $S\alpha S$ innovations were investigated and the codifference was used as a substitute for the usual covariance. It was shown that $\tau(X_n, X_0)$ is bounded above by exponentially decaying function and exact asymptotics of $\tau(X_n, X_0)$ were evaluated in some cases.

In [37] the asymptotic dependence structure of time series $X(n)$ satisfying FARIMA equation was studied. To be precise, the codifference and covariation were used to investigate dependence between $X(0)$ and $X(n)$ as $n \to \infty$. $X(n)$ is the unique solution of

$$\Phi(B)X(n) = \Theta(B)(1 - B)^{-d}\epsilon_n,$$

where Φ and Θ are polynomials with real coefficients and no common roots, Θ has no roots in the closed unit disk, B is the backshift operator, and ϵ_n is a sequence of i.i.d. $S\alpha S$ random variables.

Solutions of FARIMA equation investigated in [37] are a particular case of processes

$$X(n) = \sum_{j=0}^{\infty} c_j \epsilon_{n-j} \tag{3.4}$$

where ϵ_j are i.i.d. $S\alpha S$ random variables with ch.f. $\exp(-|t|^\alpha)$, and c_j are asymptotically equivalent to some regularly varying with index $\rho < -1/\alpha$ function $U(j)$. In [38] such more general linear processes were investigated and the following theorems were proved:

Theorem 3.3 ([38]). *Suppose* $0 < \alpha \leqslant 2$, $\rho < -1/\alpha$, *and consider the moving average process* (3.4).

1. *If $\alpha > 1$, $\rho(\alpha - 1) < -1$ and $c_j \sim U(j)$ for some regularly varying with index ρ function U, then*

$$\lim_{n\to\infty} \frac{\tau(X_n, X_0)}{U(n)} = \alpha \sum_{j=0}^{\infty} c_j^{<\alpha-1>}.$$

2. *If $\alpha \geqslant 1$, $\rho(\alpha - 1) > -1$ and*

$$\left(\frac{c_j}{U(j)} - 1\right) = O(j^{-1}) \tag{3.5}$$

for some non-increasing regularly varying with index ρ function U, then

$$\lim_{n\to\infty} \frac{\tau(X_n, X_0)}{nU^{\alpha}(n)} = \int_0^{\infty} y^{\rho\alpha} + (y+1)^{\rho\alpha} - (y^{\rho} - (y+1)^{\rho})^{\alpha} \, \mathrm{d}y. \tag{3.6}$$

3. *If $\alpha \leqslant 1$, then* (3.6) *holds, provided* (3.5) *holds for some non-increasing convex regularly varying with index ρ function U whose derivative U' satisfies*

$$\left|1 - \frac{U'(x+n)}{U'(x)}\right| = O(x^{-1}) \text{ a.e., as } x \to \infty. \tag{3.7}$$

Theorem 3.4 ([38])**.** *Suppose $1 < \alpha \leqslant 2$, $\rho < -1/\alpha$, and consider the moving average process* (3.4)*.*

1. *If $\rho(\alpha - 1) < -1$ and $c_j \sim U(j)$ for some regularly varying with index ρ function U, then*

$$\lim_{n\to\infty} \frac{[X_n, X_0]_{\alpha}}{U(n)} = \sum_{j=0}^{\infty} c_j^{<\alpha-1>}.$$

2. *If $\rho(\alpha - 1) > -1$ and* (3.5) *holds for some non-increasing regularly varying with index ρ function U, then*

$$\lim_{n\to\infty} \frac{[X_n, X_0]_{\alpha}}{nU^{\alpha}(n)} = \int_0^{\infty} (y+1)^{\rho} y^{\rho(\alpha-1)} \mathrm{d}y. \tag{3.8}$$

In [4] the asymptotic dependence structure of linear fractional stable noise and log-fractional stable noise with a constant skewness function $\beta(x) \equiv \beta$ was investigated. We state the results for the symmetric processes, as it was done in [58].

Theorem 3.5. *Consider linear fractional stable noise* $Y_1(t)$ *defined by* (2.16) *and suppose it is symmetric.*

If $0 < \alpha < 1,\ 0 < H < 1$ *or* $1 < \alpha < 2,\ 1 - 1/(\alpha(\alpha-1)) < H < 1$, $H \neq 1/\alpha$, *then*

$$I(\theta_1, \theta_2; Y_1(t), Y_1(0)) \sim B(\theta_1, \theta_2) t^{\alpha H - \alpha}, \tag{3.9}$$

as $t \to \infty$.

If $\alpha = 1,\ 0 < H < 1$, *then* (3.9) *holds if either* $\operatorname{sign}(ab) = 1$ *or* $\operatorname{sign}(\theta_1\theta_2) = -1$. *If* $\alpha = 1$, $\operatorname{sign}(ab) \neq 1$ *and* $\operatorname{sign}(\theta_1\theta_2) \neq -1$ *then* $B(\theta_1, \theta_2) = 0$ *and*

$$I(\theta_1, \theta_2; Y_1(t), Y_1(0)) \sim -2(1-H)(|a\theta_1| + |b\theta_2|) t^{H-2}$$

as $t \to \infty$.

If $1 < \alpha < 2,\ 0 < H < 1 - 1/(\alpha(\alpha-1))$, *then*

$$I(\theta_1, \theta_2; Y_1(t), Y_1(0)) \sim F(\theta_1, \theta_2) t^{H - 1/\alpha - 1}$$

as $t \to \infty$.

The constants $B(\theta_1, \theta_2)$ *and* $F(\theta_1, \theta_2)$ *are as follows*

$$\begin{aligned} B(\theta_1, \theta_2) = \left|H - \frac{1}{\alpha}\right|^\alpha \Bigg(& |a|^\alpha \int_{-\infty}^0 \left|\theta_1(1-x)^{H-1/\alpha-1} + \theta_2(-x)^{H-1/\alpha-1}\right|^\alpha - \\ & - \left|\theta_1(1-x)^{H-1/\alpha-1}\right|^\alpha - \left|\theta_2(-x)^{H-1/\alpha-1}\right|^\alpha \mathrm{d}x + \\ & + \int_0^1 \left|a\theta_1(1-x)^{H-1/\alpha-1} - b\theta_2 x^{H-1/\alpha-1}\right|^\alpha - \\ & - \left|a\theta_1(1-x)^{H-1/\alpha-1}\right|^\alpha - \left|b\theta_2 x^{H-1/\alpha-1}\right|^\alpha \mathrm{d}x + \\ & + |b|^\alpha \int_0^\infty \left|\theta_2(1+x)^{H-1/\alpha-1} + \theta_1 x^{H-1/\alpha-1}\right|^\alpha - \\ & - \left|\theta_2(1+x)^{H-1/\alpha-1}\right|^\alpha - \left|\theta_1 x^{H-1/\alpha-1}\right|^\alpha \mathrm{d}x \Bigg), \end{aligned}$$

$$
\begin{aligned}
F(\theta_1, \theta_2) = (H\alpha - 1)\times& \\
\times \Bigg(a\theta_1 \Big(\int_{-\infty}^{0} & \left(a\theta_2 \left((1-x)^{H-1/\alpha} - (-x)^{H-1/\alpha} \right) \right)^{<\alpha-1>} \mathrm{d}x + \\
&+ \int_0^1 \left(\theta_2 \left(a(1-x)^{H-1/\alpha} - bx^{H-1/\alpha} \right) \right)^{<\alpha-1>} \mathrm{d}x + \\
&+ \int_1^{\infty} \left(b\theta_2 \left((x-1)^{H-1/\alpha} - x^{H-1/\alpha} \right) \right)^{<\alpha-1>} \mathrm{d}x \Big) + \\
&+ b\theta_2 \Big(\int_{-\infty}^{0} \left(b\theta_1 \left((1-x)^{H-1/\alpha} - (-x)^{H-1/\alpha} \right) \right)^{<\alpha-1>} + \\
&+ \int_0^1 \left(\theta_1 \left(b(1-x)^{H-1/\alpha} - ax^{H-1/\alpha} \right) \right)^{<\alpha->} \mathrm{d}x + \\
&\qquad + \int_1^{\infty} \left(a\theta_1 \left((x-1)^{H-1/\alpha} - x^{H-1/\alpha} \right) \right)^{<\alpha-1>} \mathrm{d}x \Big) \Bigg).
\end{aligned}
$$

Theorem 3.6. *Consider log-fractional stable noise $Y_2(t)$ defined by* (2.19) *and suppose it is symmetric. Then*

$$
I(\theta_1, \theta_2; Y_2(t), Y_2(0)) \sim G(\theta_1, \theta_2) t^{1-\alpha},
$$

where

$$
\begin{aligned}
G(\theta_1, \theta_2) = \int_{-\infty}^{1} & \left(\left| \frac{\theta_1}{1+x} - \frac{\theta_2}{x} \right|^{\alpha} - \left| \frac{\theta_1}{1+x} \right|^{\alpha} - \left| \frac{\theta_2}{x} \right|^{\alpha} \right) \mathrm{d}x + \\
&+ \int_0^{\infty} \left(\left| \frac{\theta_2}{1+x} - \frac{\theta_1}{x} \right|^{\alpha} - \left| \frac{\theta_2}{1+x} \right|^{\alpha} - \left| \frac{\theta_1}{x} \right|^{\alpha} \right) \mathrm{d}x.
\end{aligned}
$$

Remark 3.7. For comparison, without the assumption of symmetry the rate of decay in Theorem 3.6 would be the same, but the expression of $G(\theta_1, \theta_2)$ would be different. It follows from Theorem 2.4 in [4] that

$$
\begin{aligned}
G(\theta_1, \theta_2) = \int_{-\infty}^{1} & \left(\xi \left(\frac{\theta_1}{1-x} + \frac{\theta_2}{-x} \right) - \xi \left(\frac{\theta_1}{1-x} \right) - \xi \left(\frac{\theta_2}{-x} \right) \right) \mathrm{d}x + \\
&+ \int_0^{\infty} \left(\overline{\xi} \left(\frac{\theta_2}{1+x} + \frac{\theta_1}{x} \right) - \overline{\xi} \left(\frac{\theta_2}{1+x} \right) - \overline{\xi} \left(\frac{\theta_1}{x} \right) \right) \mathrm{d}x.
\end{aligned}
$$

where

$$
\xi(u) = |u|^{\alpha} \left(1 - \mathrm{i}\beta \mathrm{sign}(u) \tan \frac{\pi\alpha}{2} \right),
$$

and $\overline{\xi}(u)$ denotes the complex conjugate of $\xi(u)$.

Another paper that deals with codifference for non-symmetric processes is [54], where the asymptotic behaviour of codifference was investigated for α-stable random process, expressed as a sum of stochastic integrals with respect to non-symmetric α-stable random measure.

In the paper [41] the decay rates of the codifference and covariation for increments of infinite-variance renewal-reward process were examined.

In [42] the codifference and covariation were used to investigate the dependence structure of linear log-fractional stable motion – a process defined for $1 < \alpha \leqslant 2$ as

$$\begin{aligned} &Y(t) \\ &= \int_{-\infty}^{\infty} a\left(\ln_0(t-x)_+ - \ln_0(-x)_+\right) + b\left(\ln_0(t-x)_- - \ln_0(-x)_-\right) M(\mathrm{d}x), \end{aligned}$$

where $|a| + |b| > 0$, M is a $S\alpha S$ random measure with Lebesgue control measure, and

$$\ln_0 x = \begin{cases} \ln x \text{ if } x > 0, \\ 0 \text{ otherwise.} \end{cases}$$

It was shown that codifference of this process decays faster than covariation if $a \neq b$ and $ab \neq 0$.

In [43] linear fractional stable noise in the previously uninvestigated boundary case $1 < \alpha < 2,\ H = 1 - 1/(\alpha(\alpha - 1))$ was examined. In the paper the asymptotic behaviour of (2.12) was studied.

The paper [44] deals with symmetric log-fractional stable noise. It was shown that as $t \to \infty$, the codifference satisfies $\tau(Y_2(t), Y_2(0)) \sim C_1 t^{1-\alpha}$ and covariation $[Y_2(t), Y_2(0)]_\alpha \sim C_2 t^{1-\alpha}$, where C_1 and C_2 are positive constants.

A symmetrized and normalized version of the covariation for $S\alpha S$ random variables was introduced in [23]. Later, in [24] a new measure of dependence, called the signed symmetric covariation coefficient, was introduced. A modified version of this measure was defined in [33]. For details and properties of these measures of dependence we refer the reader to the original papers.

It seems that there were almost no attempts to investigate the asymptotic dependence structure of linear fields with infinite variance. As an exception we can mention the papers [34, 35]. In the first paper Chentsov type random fields, introduced by Takenaka in [62], are considered, and in the second – linear fields

$$X(\mathbf{t}) = \int_{\mathbb{R}^n} \left(p(\mathbf{x}-\mathbf{t})^{H-n/\alpha} - p(\mathbf{x})^{H-n/\alpha}\right) M(\mathrm{d}\mathbf{x}),\ \mathbf{t} \in \mathbb{R}^n,$$

where M is $S\alpha S$ random measure with Lebesgue control measure, p is arbitrary norm on $\mathbb{R}^n$, and $H \in (0,1)$, $0 < \alpha \leqslant 2$. Both papers investigate the asymptotic dependence structure as $u \to \infty$ of one-step increment of projection processes

$$X_{\mathbf{e}}(u) = X((u+1)\mathbf{e}) - X(u\mathbf{e}), u \in \mathbb{R},$$

where $\mathbf{e} \in \mathbb{R}^n$.

We do not discuss the relation of the measures of dependence, based on the spectral measure of α-stable or regularly varying random vectors, with measures of dependence of different nature, which can be defined for random vectors with infinite variance, such as Spearman's ρ, Kendal's τ, or the distance covariance. The last mentioned measure has some similarity with the codifference since, as a measure of dependence between the coordinates of a vector (X_1, X_2), the weighted (with some specific weight) L_2-norm of the difference $\mathbb{E}\exp\{\mathrm{i}(sX_1+tX_2)\} - \mathbb{E}\exp\{\mathrm{i}sX_1\}\mathbb{E}\exp\{\mathrm{i}tX_2\}$ is taken; see [60, 61]. Recently, another distance based measure of dependence for stable random variables was introduced, its properties were studied and compared with codifference and covariation, see [1].

3.2 Limit theorems and memory

Relation between the spectral covariance and limit theorems

The classical central limit theorem is well known – if we have a sequence of i.i.d. random variables $X_k,\ k \in \mathbb{N}$, with finite variance σ^2 and mean μ, then

$$\frac{\sum_{k=1}^{n}(X_k - \mu)}{\sqrt{n}} \xrightarrow{\text{d}} \mathcal{N}(0, \sigma^2), \tag{3.10}$$

where $\mathcal{N}(0, \sigma^2)$ is the normal distribution.

It turned out that (3.10) also holds for sequences of dependent random variables, provided the dependence is not too strong. The well known central limit theorem of Newman (see [47], where this theorem was proved for fields on $\mathbb{Z}^d$, or [48], where functional CLT was proved in the case $d = 1$) states that if for a stationary and associated sequence $X_1, X_2, \ldots$ with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$, the series of covariances converges, that is, if

$$\sum_{k=2}^{\infty} \mathbb{E}X_1 X_k < \infty, \tag{3.11}$$

then

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}} \xrightarrow{\text{d}} \mathcal{N}(0, \sigma^2), \tag{3.12}$$

where $\sigma^2 = \mathbb{E}X_1^2 + 2\sum_{k=2}^{\infty} \mathbb{E}X_1 X_k$.

The condition (3.11) is equivalent to convergence of the sequence

$$K_n = \sum_{k=1}^{n} \mathbb{E}X_1 X_k,$$

and is optimal in a sense that it can not be weakened – an example of stationary associated sequence was provided in [28] with $K_n \sim \ln n$ for which $\sum_{k=1}^{n} X_k / \sqrt{n K_n}$ does not have any non-degenerate limit distribution. In [59] a strictly stationary associated random sequence is constructed which does not satisfy the central limit theorem and such that K_n is an arbitrary s.v.f.

In earlier papers devoted to limit theorems with stable law limits (see, e.g., [30–32]), some conditions on weak dependence or mixing were used; for the latest results in limit theorems with stable limits, we refer the reader to [7], where a large list of references can be found. A different approach is used in [17], where the spectral covariance (of course, without using this name) was used in limit theorems for associated sequences with infinite variance.

Before stating the results from [17], we recall some notions. A sequence $X_1, X_2, \dots$ is jointly α-stable if for any $n \in \mathbb{N}$ there exist a spectral measure Γ_n on $\mathbb{S}^{n-1}$, and a vector $\mathbf{b}_n \in \mathbb{R}^n$ such that the vector $(X_1, \dots, X_n)$ has ch.f. (2.5) with $\mathbf{b} = \mathbf{b}_n$ and $\Gamma = \Gamma_n$. If, additionally, the sequence is stationary, then $\mathbf{b}_n = (b, \dots, b)$ for some $b \in \mathbb{R}$. A stable vector $(X_1, \dots, X_n)$ is strictly α-stable if either $\beta_n = 0$ in the case $\alpha \neq 1$ or $\int_{\mathbb{S}^{n-1}} s_i \Gamma_n(d\mathbf{s}) = 0,\ i = 1, \dots, n$, in the case $\alpha = 1$,

Let us denote $S_n = \sum_{k=1}^n X_k$ and by $\rho(k),\ k \geq 2$, the spectral covariance of a two-dimensional stable vector (X_1, X_k). If this vector is associated, then $\rho(k) \geq 0$. We are now ready to state the following result from [17]:

Theorem 3.8 ([17]). *Let $X_1, X_2, \dots$ be a stationary, associated, and jointly α-stable sequence.*

If $0 < \alpha < 1$, then

$$\frac{S_n}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mu, \tag{3.13}$$

where μ is a strictly α-stable distribution.

If $\alpha = 1$, then there exist constants A_n such that $S_n/n - A_n \stackrel{\mathrm{d}}{=} X_1$.

If $1 < \alpha < 2$ and

$$\sum_{k=2}^{\infty} \rho(k) < \infty, \tag{3.14}$$

then

$$\frac{S_n - \mathbb{E}S_n}{n^{1/\alpha}} = \frac{S_n - nb}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mu, \tag{3.15}$$

where μ is a non-degenerate strictly α-stable distribution.

It was noted in [17] that the limit law μ in (3.13) may be degenerate, which can be seen from the following example given in [17]. Let X be a strictly α-stable random variable, $0 < \alpha < 2$, and define $X_i \equiv X$ for all $i \in \mathbb{N}$. Then this sequence is stationary, associated, and jointly α-stable, and

$$n^{-1/\alpha} S_n = n^{1-1/\alpha} X.$$

This equality shows that, for such a sequence and $0 < \alpha < 1$, the limit μ is degenerate. The same example shows that, without condition (3.14), relation (3.15) may fail since, in the case $1 < \alpha < 2$, the sequence $n^{-1/\alpha} S_n$ diverges. Also, condition (3.14) suggests that, when considering random variables with infinite variance, the spectral covariance is a natural candidate to substitute the usual covariance.

If we were able in Theorem 3.8 to change stationary, associated, and jointly α-stable sequence $X_1, X_2, \ldots$ by a stationary and associated sequence that belongs to the domain of attraction of $X_1, X_2, \ldots$, then we would get a complete generalization of Newman's theorem for associated sequences with infinite variance. Unfortunately, in attempt to do this in [17], a condition stronger than (3.14) is assumed. In order to state this result, we need more notation from [17]. Let $\{X_i,\ i \in \mathbb{N}\}$ be an arbitrary stationary sequence, and let $\{Y_i,\ i \in \mathbb{N}\}$ be a stationary and jointly strictly α-stable sequence. We say that $\{X_i,\ i \in \mathbb{N}\}$ belongs to the domain of strict normal attraction of $\{Y_i,\ i \in \mathbb{N}\}$ and write $\{X_i\} \in \mathcal{D}_{\text{sn}}(\{Y_i\})$ if, for each $m \in \mathbb{N}$, the distribution of the vector $\xi_m = (X_1, \ldots, X_m)$ belongs to the domain of strict normal attraction of the α-stable random vector $\eta_m = (Y_1, \ldots, Y_m)$. This means that, for each $m \geq 1$,

$$n^{-1/\alpha} \sum_{j=1}^{n} \xi_{m,j} \xrightarrow{\text{d}} \eta_m \quad \text{as } n \to \infty,$$

where $\xi_{m,j},\ j \geq 1$, are independent copies of ξ_m. The subscript "sn" stands for "strict normal" and points out that the convergence is required without centering and that $n^{-1/\alpha}$ is used for normalization, whereas in the general definition of the domain of attraction, centering and general regularly varying functions are allowed for normalization.

For an associated and stationary sequence $\{X_i,\ i \in \mathbb{N}\}$ for fixed $A > 0$ and $0 < \alpha < 2$, the following quantity is introduced

$$I_\alpha^A(X_i, X_j) = \sup_{b \geq A} b^{\alpha-2} \int_{-b}^{b} \int_{-b}^{b} H_{(X_i,X_j)}(x,y)\mathrm{d}x\mathrm{d}y, \tag{3.16}$$

where

$$H_{(X_i,X_j)}(x,y) = \mathbb{P}(X_i \leq x, X_j \leq y) - \mathbb{P}(X_i \leq x)\mathbb{P}(X_j \leq y).$$

Let, as before, $S_n = \sum_{k=1}^n X_k$, and let $Z_n = \sum_{k=1}^n Y_k$.

Theorem 3.9 ([17]). *Let $\{X_i,\ i \in \mathbb{N}\}$ be a stationary associated sequence such that $\{X_i\} \in \mathcal{D}_{\mathrm{sn}}(\{Y_i\})$, where $\{Y_i,\ i \in \mathbb{N}\}$ is a stationary and jointly strictly α-stable sequence, $0 < \alpha < 2$, and Γ_n is symmetric for all n, if $\alpha = 1$. If*

$$\sum_{k=2}^{\infty} I_\alpha^A(X_1, X_k) < \infty \tag{3.17}$$

for some $A > 0$, then there exists a strictly α-stable distribution μ such that

$$\frac{S_n}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mu$$

and

$$\frac{Z_n}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mu. \tag{3.18}$$

Remark 3.10. It is appropriate to mention the paper [45], where the result similar to Theorem 3.9 is proved under conditions allowing the divergence of the series in (3.17).

Memory

The notion of (long) memory is of interest both in theory and practice, there are many papers that mention this concept. However, different authors can understand this notion differently – there is no one universally accepted definition. For example, in the literature one can encounter the following definitions of memory for processes with finite variance:

Definition 1. A stationary process $(X_t,\ t \in \mathbb{Z})$ is called a long memory process in the covariance sense, if $\sum_{j=0}^{\infty} |\mathrm{Cov}(X_j, X_0)| = \infty$.

Definition 2. A stationary process $(X_t,\ t \in \mathbb{Z})$ with a spectral density function f_X is called a long memory process in the spectral density sense if

$$\frac{\sup f_X(\lambda)}{\inf f_X(\lambda) = \infty}.$$

Definition 3. A stationary process $(X_t,\ t \in \mathbb{Z})$ is called a long memory process in the covariance sense with a speed of convergence of order $2d,\ 0 < d < 1/2$, if there exists a constant C (dependent on d) such that

$$\mathrm{Cov}(X_t, X_0) \sim Ct^{2d-1},$$

as $t \to \infty$.

Definition 4. A stationary process $(X_t,\ t \in \mathbb{Z})$ is said to have Allen variance long memory if

$$\frac{\mathrm{Var}\left(\sum_{k=1}^{n} X_k\right)}{n} \to \infty,$$

as $n \to \infty$.

We refer the reader to the papers [26] and [57] for a extensive overview.

The definitions of memory above use concepts that require the process to have finite variance. It is not immediately clear how to extend the notion of long memory to the infinite-variance processes. For example, in [38] long memory was based on decay rate of codifference.

In a recent paper [52] it was suggested to classify memory (with respect to summation operation) of general stationary sequences by means of the growth of normalizing sequences for partial sums as follows:

Definition 5. Say $\{X_n, n \in \mathbb{Z}\}$ is a stationary sequence that is not subordinated, has finite variance and zero mean or is jointly regularly varying with index $0 < \alpha \leqslant 2$, $\mathbb{E}X_0 = 0$ if $\alpha > 1$, and X_0 is symmetric if $\alpha = 1$. Also, suppose there exists a normalizing sequence $A_n = n^{1/\alpha+\delta}L(n)$, where $-1/\alpha < \delta \leqslant 1 - 1/\alpha$ and L is a s.v.f., and a constant $b \in \mathbb{R}$ such that $A_n^{-1}(\sum_{i=1}^{\lfloor nt \rfloor} X_i - \lfloor nt \rfloor b)$ converges in the sense of finite dimensional distributions to some stochastically continuous process that is not identically zero. The sequence $\{X_n, n \in \mathbb{Z}\}$ has zero memory if $\delta = 0$, positive memory if $\delta > 0$, and negative memory if $\delta < 0$. The sequence has strongly negative memory if for any sequence $B_n \to \infty$, the sequence $B_n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} X_i$ weakly converges to zero.

Suppose $X_n = \sum_{i=0}^{\infty} c_i \xi_{n-i}$, where ξ_i are i.i.d. $S\alpha S$ random variables, $\sum_{i=0}^{\infty} |c_i|^{\alpha} < \infty$. In [52] the following conjecture is stated, which, if true, would relate the notion of spectral covariance and negative memory:

Conjecture 3.11. In the case $1 < \alpha < 2, \sum_{j=0}^{\infty} c_j = 0$ and

$$c_j = j^{-\beta}(1 + O(j^{-h(\alpha)})),\ j \geqslant 1,$$

with some function h, we should get that $\rho(X_n, X_0)$ is of order $C(\alpha, \beta)n^{1-\beta\alpha}$.

In [52] the following definition was proposed for stationary random field $\mathcal{X} = \{X_{\mathbf{k}},\ \mathbf{k} \in \mathbb{Z}^d\}$ that is not subordinated and is with finite variance or is jointly regularly varying with index $0 < \alpha \leqslant 2$, $\mathbb{E}X_{\mathbf{0}} = 0$ if $\alpha > 1$, and $X_{\mathbf{0}}$ is symmetric if $\alpha = 1$.

Definition 6. Suppose a stationary random field $\mathcal{X}$ is as described above and $\bar{\delta} = (\delta_1, \ldots, \delta_d)$. $\mathcal{X}$ has directional $\bar{\delta}$-memory is there exist slowly varying functions $L_i,\ i = 1, \ldots, d$, such that

$$A_{\mathbf{n}}^{-1} \sum_{\mathbf{1} \leqslant \mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}$$

converges in distribution to a nondegenerate law, where

$$A_{\mathbf{n}} = \prod_{i=1}^{d} n_i^{1/\alpha+\delta_i} L_i(n_i),\ -1/\alpha < \delta_i < 1 - 1/\alpha.$$

The field has isotropic memory if $\delta_i = \delta$ for $i = 1, \ldots, d$ and this isotropic memory can be positive, zero, or negative, depending on the sign of δ.

Limit theorems for linear processes and fields

Linear processes

$$X_n = \sum_{i \in \mathbb{Z}} c_i \xi_{n-i}, \tag{3.19}$$

with innovations $\xi_i, i \in \mathbb{Z}$, being independent identically distributed(i.i.d.) random variables and coefficients $\{c_i\}$ such that series (3.19) converges a.s., are widely investigated as they allow to model various types of dependence and memory. A question often studied is the convergence of appropriately normalized and centered partial sum process $S_n(t) := \sum_{k=0}^{\lfloor nt \rfloor} X_k, t > 0,\ S_n(0) = 0$.

Suppose that ξ_1 belongs to the domain of attraction of α-stable law, i.e., $a_n^{-1} \sum_{i=1}^{n} \xi_i - b_n$ converges in distribution to α-stable random variable, $\alpha \in (0, 2]$. It is well-known that then $a_n = n^{1/\alpha} L(n)$, where L is a slowly varying function (s.v.f.). Centering sequence b_n is simple: if $\alpha > 1$ one has $b_n = a_n^{-1} n E\xi_1$, if $\alpha < 1$, centering is not needed at all, and only in the case $\alpha = 1$ centering is a little bit more complicated. In order to avoid these complications in the case $\alpha = 1$ it is often assumed that the innovations are symmetric. The case $\alpha = 2$ (i.e., when ξ_1 belongs to the domain of attraction of a Gaussian law) is investigated deeply, many results concerning the convergence of $S_n(t)$, or more general partial sum processes, formed by stationary sequences with finite variance are documented in series of monographs, starting from classical monographs [10,11] and ending with a recent one [25]. The case $0 < \alpha < 2$ is less investigated, but the interest in the convergence of partial sum processes, formed by stationary sequences with infinite variance to stable limits during the last two decades had increased greatly. This can be explained by the fact that many processes in practice can be modelled using heavy-tailed distributions. We mention only several papers, which are, in our opinion, close to the present work. Namely, we refer the reader to papers [3, 5, 8, 19, 30, 31], also one can find more references in the paper [6].

Important case of processes satisfying $\sum_{i\in\mathbb{Z}}|c_i| < \infty$ is investigated in [5] where necessary and sufficient conditions were provided for convergence in finite-dimensional distributions $A_n^{-1}S_n(t) \xrightarrow{\text{f.d.d.}} \sum_{i\in\mathbb{Z}} c_i Z(t)$, where $Z(t)$ is α-stable Lévy motion and $A_n = n^{1/\alpha}L(n)$. However, in the case $\sum_{i\in\mathbb{Z}} c_i = 0$ the limit is 0 and different normalizing sequence is needed to get a non-degenerate limit (we say that A_n is a normalizing sequence of $S_n(1)$ if $A_n^{-1}S_n(1)$ converges in distribution to some non-degenerate random variable. In what follows A_n denotes the normalizing sequence of $S_n(1)$). Using the terminology from [52] one can say that in [5] the case of zero memory was considered. Processes with negative memory were little investigated in the literature. It seems that only in [3] all three cases of memory are considered, but even there the proofs are given only in the case of zero and positive memories, the proof in the case of negative memory is left for readers.

To formulate Theorem 1 from [3], we need some notation.

Let $X_k = \sum_j a(k-j)\xi_j,\ k \in \mathbb{N}$, where ξ_j are i.i.d. random variables such that

$$P(\xi_1 < -t) = (C_1 + o(1))t^{-\alpha}h(t),\ P(\xi_1 > t) = (C_2 + o(1))t^{-\alpha}h(t),$$

as $t \to \infty$, where $C_1 \geqslant 0$, $C_2 \geqslant 0$, $C_1 + C_2 > 0$, and h is a s.v.f.. If $\alpha = 1$ it is assumed that $C_1 = C_2$. This is equivalent to ξ_1 having characteristic function

$$\mathbb{E}\exp(\mathrm{i}t\xi_1) = \exp\left(-(C + o(1))|t|^\alpha H(|t|^{-1})(1 - \mathrm{i}D\operatorname{sign}(t))\right),$$

as $t \to 0$, where ,

$$C = \begin{cases} (C_1 + C_2)\Gamma(|1-\alpha|)\cos(\alpha\pi/2) \text{ if } \alpha \neq 1, \\ (C_1 + C_2)\pi/2 \text{ if } \alpha = 1, \end{cases}$$

$$D = \begin{cases} \tan(\alpha\pi/2)(C_1 - C_2)/(C_1 + C_2) \text{ if } \alpha \neq 1, \\ 0 \text{ if } \alpha = 1, \end{cases}$$

and H is a s.v.f. related to h (we refer the reader to the original paper for details). Let us denote

$$Y^{(\beta)}(t) = \begin{cases} \int_{-\infty}^{\infty} \left((t-x)_+^{1-\beta} - (-x)_+^{1-\beta} \right) M(\mathrm{d}x) \text{ if } \beta \neq 1, \\ \int_{-\infty}^{\infty} \mathbb{1}_{(0,t)}(x) M(\mathrm{d}x) \text{ if } \beta = 1, \end{cases}$$

where M is an α-stable measure with Lebesgue control measure and constant skewness intensity D.

Theorem 3.12 ([3]). *Suppose X_k is as described above.*

i) If $\sum_j |a(j)| < \infty$ and $\sum_j a(j) \neq 0$, then the process

$$Y_n(t) = \frac{\sum_{k=1}^{\lfloor nt \rfloor} X_k}{C^{1/\alpha} \left|\sum_j a(j)\right| n^{1/\alpha} H_\alpha^{1/\alpha}(n)}, \quad t \geqslant 0,$$

as $n \to \infty$ converges in finite dimensional distributions to the process $Y^{(1)}$ defined above.

ii) If $\alpha > 1,\ 1/\alpha < \beta < 1$ and

$$a(j) = \begin{cases} j^{-\beta} L(j), \ j \in \mathbb{N}, \\ 0 \textit{ otherwise}, \end{cases}$$

where L is a s.v.f., then

$$Y_n(t) = \frac{\sum_{k=1}^{\lfloor nt \rfloor} X_k}{|1-\beta|^{-1} C^{1/\alpha} n^{1/\alpha+1-\beta} L(n) H_\alpha^{1/\alpha}(n)}, \tag{3.20}$$

as $n \to \infty$ converges in finite dimensional distributions to the process $Y^{(\beta)}$.

iii) If $0 < \alpha \leqslant 2,\ \max(1, 1/\alpha) < \beta < 1/\alpha + 1$,

$$\sum_{k=0}^{n} a(k) = (\beta - 1)^{-1} n^{1-\beta} L(n),$$

where L is a Zygmund s.v.f., and $a(k) = 0$ for $k \leqslant 0$, then the process (3.20) *converges in finite dimensional distributions to the process $Y^{(\beta)}$.*

Remark 3.13. Paper [3] deals with slowly varying functions, however, there is a gap in the proof of Theorem 3.12, part (ii). The relation (27) in [3] claims that

$$\sup_j \left| \frac{h\left(A_N \left|\sum_{r=1}^n u_r a_{\lfloor Nt_r \rfloor}(j)\right|^{-1}\right)}{h\left(A_N \left|a_N(j)\right|^{-1}\right)} - 1 \right| \to 0,\ N \to \infty, \tag{3.21}$$

here h is a s.v.f., $a_N(j) = \sum_{k=1}^N a(k-j)$ and $a(k) = k^{-\beta} L(k) \mathbb{1}_{[1,\infty)}(k)$ with a s.v.f. L. Let us take, for example, $L(x) = 1, h(x) = \ln(x)$, $n = 1, u_1 = 1, t_1 = 2$. Now (3.21) simplifies to

$$\sup_j \left| \frac{\ln\left(A_N \left|a_{2N}(j)\right|^{-1}\right)}{\ln\left(A_N \left|a_N(j)\right|^{-1}\right)} - 1 \right| \to 0,\ N \to \infty, \tag{3.22}$$

with the normalizing sequence $A_N = KN^{1/\alpha+1-\beta} \ln^{1/\alpha}(N)$, $K > 0$. For any j we have

$$\sup_j \left| \frac{\ln\left(A_N \left|a_{2N}(j)\right|^{-1}\right)}{h\left(A_N \left|a_N(j)\right|^{-1}\right)} - 1 \right| \geqslant \left| \frac{\ln\left(A_N \left|a_{2N}(j)\right|^{-1}\right)}{\ln\left(A_N \left|a_N(j)\right|^{-1}\right)} - 1 \right|.$$

If $j_N = N - 1$,

$$a_N(j_N) = 1,\ a_{2N}(j_N) \sim N^{1-\beta} \int_0^1 u^{-\beta} \mathrm{d}u,$$

which implies

$$\frac{\ln\left(A_N \left|a_{2N}(j_N)\right|^{-1}\right)}{\ln\left(A_N \left|a_N(j_N)\right|^{-1}\right)} \to \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} + 1 - \beta} \neq 1,$$

proving that (3.22) does not hold.

4 Spectral covariances

4.1 The problem and results

As was mentioned in the literature overview, the spectral covariance did not attract the attention of fellow scientists in the field. In this chapter we investigate asymptotic behaviour of the spectral covariance for some processes and compare them to known results about other measures of dependence. We also study the asymptotic dependence structure of a certain linear field.

We begin by introducing another measure of dependence. For an α-stable random vector $\mathbf{X} = (X_1, X_2)$ with spectral measure Γ, we introduce the quantity

$$\rho_\alpha(X_1, X_2) = \int_{\mathbb{S}^1} s_1^{\langle\alpha/2\rangle} s_2^{\langle\alpha/2\rangle} \Gamma(\mathrm{d}\mathbf{s}), \tag{4.1}$$

which we call the α-spectral covariance, emphasizing that this new measure of dependence depends not only on the spectral measure Γ, but also on α. Just like the spectral covariance, this notion can be extended to random vectors satisfying (3.2) with the same spectral measure Γ. Motivation for the introduction of this dependence measure will be given later.

Let us compare the measures of dependence by looking at a simple example.

Example 4.1. Let Z_1, Z_2 be two i.i.d. $S\alpha S$ random variables with ch.f. $\exp(-|t|^\alpha)$ and let $X_c = Z_1 + cZ_2,\ Y = Z_2$. Then it is easy to calculate

$$[X_c, Y]_\alpha = c,\ [Y, X_c]_\alpha = c^{<\alpha-1>},$$

$$\rho(c) := \rho(X_c, Y) = \frac{c}{(1+c^2)^{\frac{2-\alpha}{2}}},\ \tilde{\rho}(c) := \tilde{\rho}(X_c, Y) = \frac{c}{\sqrt{(1+c^2)^{1-\frac{\alpha}{2}} + c^2}},$$

$$\rho_\alpha(c) := \rho_\alpha(X_c, Y) = c^{\frac{\alpha}{2}},$$

and

$$\tau(c) := \tau(X_c, Y) = |c|^\alpha + 1 - |c-1|^\alpha .$$

We see that $\rho(c) \sim c$ and $\tilde{\rho}(c) \sim c$ as $c \to 0$, independently of α (which is very natural), while for the codifference we have

$$\tau(c) \sim \begin{cases} \alpha c, & \text{for } \alpha > 1, \\ |c|^\alpha, & \text{for } \alpha < 1. \end{cases}$$

The codifference in the case $\alpha = 1$ looks strange

$$\tau(c) = |c| + 1 - |c-1| = \begin{cases} 0, & \text{for } c < 0, \\ 2c, & \text{for } 0 \le c < 1, \\ 2, & \text{for } c \ge 1. \end{cases}$$

This expression means that in the case $\alpha = 1$ codifference does not show dependence between $Z_1 - Z_2$ and Z_2, but shows it between $Z_1 + Z_2$ and Z_2. Also the behaviour of spectral correlation coefficient $\tilde{\rho}(c)$ as $|c| \to \infty$ is natural: $\lim_{c\to\pm\infty} \tilde{\rho}(c) = \pm 1$, while the behaviour of $\tau(c)$ is not so natural, for example, if $\alpha = 1$, then $\lim_{c\to-\infty} \tau(c) = 0$ and $\lim_{c\to-\infty} \tau(c) = 2$.

Assuming that the spectral measure Γ is the main parameter "responsible" for the dependence between the coordinates of the α-stable vector $\mathbf{X}$, we can define other such measures of dependence. Let $g : \mathbb{S}^1 \to \mathbb{R}$ be a function such that the integral $\int_{\mathbb{S}^1} g(s_1, s_2)\Gamma(\mathrm{d}\mathbf{s})$ is correctly defined and the following conditions are satisfied:

$$g(s_1, s_2) = 0 \quad \text{if} \quad s_1 s_2 = 0, \tag{4.2}$$

$$g(s_1, s_2) = g(s_2, s_1), \tag{4.3}$$

$$g(s_1, s_2) = g(-s_1, -s_2). \tag{4.4}$$

Then we can define

$$\rho(g; X_1, X_2) = \int_{\mathbb{S}^1} g(s_1, s_2)\Gamma(\mathrm{d}\mathbf{s}). \tag{4.5}$$

Condition (4.2) is the principal one; for a measure Γ concentrated on the axes (which means that the coordinates X_1 and X_2 are independent), it ensures that $\rho(g; X_1, X_2) = 0$. Condition (4.3) gives us the symmetry $\rho(g; X_1, X_2) = \rho(g; X_2, X_1)$, whereas condition (4.4) means that, for any non-symmetric measure Γ, $\int_{\mathbb{S}^1} g(g_1, s_2)\Gamma(\mathrm{d}\mathbf{s}) = \int_{\mathbb{S}^1} g(s_1, s_2)\Gamma_1(\mathrm{d}\mathbf{s})$, where Γ_1 is the symmetrized spectral measure, that is, $\Gamma_1(\mathrm{d}\mathbf{s}) = (\Gamma(\mathrm{d}\mathbf{s}) + \Gamma(-\mathrm{d}\mathbf{s}))/2$. To define an analogue of the spectral correlation coefficient using a general function g, we should also define g on $\{(s_1, s_2) : s_1 = s_2, |s_1| \le 1\}$ (not only on $\mathbb{S}^1$) and require the following inequality to hold:

$$\left| \frac{\int_{\mathbb{S}^1} g(s_1, s_2)\Gamma(\mathrm{d}\mathbf{s})}{\left(\int_{\mathbb{S}^1} g(s_1, s_1)\Gamma(\mathrm{d}\mathbf{s}) \int_{\mathbb{S}^1} g(s_2, s_2)\Gamma(\mathrm{d}\mathbf{s})\right)^{1/2}} \right| \le 1. \tag{4.6}$$

This notion can also be extended to random vectors belonging to the domain of attraction of α-stable vector $\mathbf{X}$ (i.e., satisfying (3.2)), like it was done with the spectral covariance and α-spectral covariance.

The following result shows that we can define a quite large class of measures of dependence, containing as particular cases the spectral covariance, α-spectral covariance, and codifference.

Proposition 4.2. *Two families of functions*

$$f_{1,\beta}(s_1, s_2) = s_1^{\langle \beta/2 \rangle} s_2^{\langle \beta/2 \rangle},$$

$$f_{2,\beta}(s_1, s_2) = |s_1|^\beta + |s_2|^\beta - |s_1 - s_2|^\beta, \quad 0 < \beta \le 2,$$

satisfy conditions (4.2)–(4.4) *and* (4.6) *and, via formula* (4.5), *generate families of measures of dependence for bivariate vectors regularly varying with index* $0 < \alpha \le 2$ *and spectral measure* Γ. *The spectral covariance is obtained by taking* $g = f_{1,2}$ *or* $g = f_{2,2}/2$ *in* (4.5), *whereas the* α*-spectral covariance and codifference are obtained by taking* $g = f_{1,\alpha}$ *and* $g = f_{2,\alpha}$*, respectively.*

In the case of the spectral covariance (2.7), we have a measure of dependence independent of the value of α, which means that all α-stable random vectors $\mathbf{X} = (X_1, X_2)$ with the same spectral measure Γ have the same spectral covariance for all $0 < \alpha \leq 2$. The spectral covariance becomes a member of two families of measures of dependence, both depending on the parameter $\beta > 0$:

$$\rho_\beta(X_1, X_2) = \int_{\mathbb{S}^1} s_1^{\langle \beta/2 \rangle} s_2^{\langle \beta/2 \rangle} \Gamma(\mathrm{d}\mathbf{s})$$

and

$$\tau_\beta(X_1, X_2) = \int_{\mathbb{S}^1} \left(|s_1|^\beta + |s_2|^\beta - |s_1 - s_2|^\beta \right) \Gamma(\mathrm{d}\mathbf{s}). \tag{4.7}$$

It would be natural to call these measures as the β-spectral covariance and β-codifference, respectively, leaving the traditional name codifference for the case $\beta = \alpha$. In what follows, unless stated otherwise, when referring to codifference we mean the measure of dependence (4.7) with $\beta = \alpha$.

The idea of codifference was based on the logarithm of the ratio of characteristic functions (see (2.11)), and from this formula expression (2.10) is obtained only for $S\alpha S$ random vectors. For non-symmetric α-stable random vectors, the ratio of the characteristic functions can be complex-valued. Definitely, it is not easy to give a meaning to a complex-valued measure of dependence. In our approach, we have the same expression (2.10) (or even a more general measure, the β-codifference) for any α-stable random vectors. Moreover, the β-codifference can be applied to vectors belonging to the domain of attraction of an α-stable random vector in the same way as it is done for the spectral and α-spectral covariances.

In general, it is impossible to compare the α-spectral covariance with the spectral covariance, or more generally, to compare $\rho_\beta(X_1, X_2)$ with two different $0 < \beta_1, \beta_2 \leq 2$. However, in some cases, for example, when a stable vector (X_1, X_2) is associated, it is possible to make a comparison.

Proposition 4.3. *If a stable vector (X_1, X_2) is associated, then, for any $0 < \alpha < 2$,*

$$\rho_{\beta_1}(X_1, X_2) \leq \rho_{\beta_2}(X_1, X_2) \text{ if } \beta_1 > \beta_2$$

and, in particular,

$$\rho(X_1, X_2) \leq \rho_\alpha(X_1, X_2). \tag{4.8}$$

If $1 < \alpha < 2$*, then*

$$\rho(X_1, X_2) \leq [X_1, X_2]_\alpha. \tag{4.9}$$

If $1 \leq \alpha \leq 2$*, then there exist constants* $c_1, c_2 > 0$ *such that*

$$c_1\rho(X_1, X_2) \leq \tau(X_1, X_2) \leq c_2\rho(X_1, X_2). \tag{4.10}$$

Remark 4.4. We can make the following conclusions from this proposition. Suppose X_n is an associated sequence of stable random variables. It follows from (4.10) that the series $\sum_{i=2}^{\infty} \rho(X_1, X_i)$ converges if and only if the series $\sum_{i=2}^{\infty} \tau(X_1, X_i)$ converges, therefore, in Theorem 3.8 the condition (3.14) can be substituted by $\sum_{i=2}^{\infty} \tau(X_1, X_i) < \infty$ to obtain an equivalent statement. Inequalities (4.8) and (4.9) imply that $\sum_{i=2}^{\infty} \rho(X_1, X_i) \leqslant \sum_{i=2}^{\infty} \rho_\alpha(X_1, X_i)$ and $\sum_{i=2}^{\infty} \rho(X_1, X_i) \leqslant \sum_{i=2}^{\infty} [X_i, X_1]_\alpha$, thus, condition $\sum_{i=2}^{\infty} \rho_\alpha(X_1, X_i) < \infty$ or $\sum_{i=2}^{\infty} [X_i, X_1]_\alpha < \infty$ can be used instead of (3.14), however, then the statement becomes weaker.

Relation between β-spectral covariance and β-codifference, and, in particular, between the spectral covariance, α-spectral covariance, and codifference can be demonstrated taking the Ornstein–Uhlenbeck process as an example.

Example 4.5. Consider the Ornstein–Uhlenbeck process

$$X(t) = \int_{-\infty}^{t} \exp\{-\lambda(t - x)\} M(\mathrm{d}x), \ t \in \mathbb{R}, \tag{4.11}$$

with general α-stable random measure M. Let us denote the β-spectral covariance function for this process by $\rho_\beta(t) = \rho_\beta(X(0), X(t))$ $(0 < \beta \leq 2)$, the normalized β-spectral covariance function by $\bar{\rho}_\beta(t) = \rho_\beta(t)(\rho_\beta(0))^{-1}$, and similar notations for the β-codifference: $\tau_\beta(t) = \tau_\beta(X(0), X(t))$ and

$\bar{\tau}_\beta(t) = \tau_\beta(X(0), X(t))(\tau_\beta(0))^{-1}$. Due to the symmetry, it suffices to consider only $t \geq 0$. It is not difficult to get the following expressions:

$$\rho_\beta(t) = \frac{1}{\lambda\alpha} \mathrm{e}^{-\lambda t\beta/2} \left(1 + \mathrm{e}^{-2\lambda t}\right)^{(\alpha-\beta)/2},$$

$$\bar{\rho}_\beta(t) = 2^{-(\alpha-\beta)/2} \mathrm{e}^{-\lambda t\beta/2} \left(1 + \mathrm{e}^{-2\lambda t}\right)^{(\alpha-\beta)/2},$$

$$\tau_\beta(t) = \frac{1}{\alpha\lambda} \left(1 + \mathrm{e}^{-\lambda t\beta} - \left|1 - \mathrm{e}^{-\lambda t}\right|^\beta\right) \left(\frac{1 + \mathrm{e}^{-2\lambda t}}{2}\right)^{(\alpha-\beta)/2},$$

$$\bar{\tau}_\beta(t) = \frac{1}{2} \left(1 + \mathrm{e}^{-\lambda t\beta} - \left|1 - \mathrm{e}^{-\lambda t}\right|^\beta\right) \left(1 + \mathrm{e}^{-2\lambda t}\right)^{(\alpha-\beta)/2}.$$

From these expressions we see that exponential decay of all these functions depends only on β, whereas the constants depend on both parameters α and β. Taking $\beta = 2$, $\beta = \alpha$ in $\bar{\rho}_\beta(t)$ and $\beta = \alpha$ in $\bar{\tau}_\beta(t)$, we get the following asymptotic relations for the normalized spectral covariance, α-spectral covariance, and codifference, respectively:

$$\bar{\rho}(t) \sim 2^{(2-\alpha)/2} \mathrm{e}^{-\lambda t}, \tag{4.12}$$

$$\bar{\rho}_\alpha(t) = \mathrm{e}^{-\alpha\lambda t/2}, \tag{4.13}$$

$$\bar{\tau}(t) \sim \begin{cases} \frac{\alpha}{2} \mathrm{e}^{-\lambda t} & \text{if } 1 < \alpha < 2, \\ \mathrm{e}^{-\lambda t} & \text{if } \alpha = 1, \\ \frac{1}{2} \mathrm{e}^{-\alpha\lambda t} & \text{if } 0 < \alpha < 1. \end{cases} \tag{4.14}$$

From three relations (4.12)–(4.14) the most complicated is relation (4.14) for the codifference, but the exponential rate of decay is the slowest in (4.13) for the α-spectral covariance due to the exponent $\alpha/2 < 1$. It seems that preference must be given to the spectral covariance, which gives the exponential decay independent of α and coinciding with the decay of the covariance function of Gaussian Ornstein–Uhlenbeck process ($\alpha = 2$), has simple and continuous with respect to α expression of the constant in the asymptotic relation (4.12). Here it is appropriate to note that the constants in the asymptotic relation (4.14) differ from the corresponding constants in the same relation in [58] since we use the normalization of

$\tau_\beta(t)$ by the value $\tau_\beta(0)$, whereas in [58], the scale parameter of $X(0)$ is used for the normalization.

It is convenient to introduce function

$$V_\alpha(x,y) = \frac{xy}{(x^2+y^2)^{\frac{2-\alpha}{2}}}.$$

It is easy to see that V_α is continuous and for $c>0$ we have $V_\alpha(cx,cy) = c^\alpha V_\alpha(x,y)$. Also we have the following:

Lemma 4.6. *Suppose* $1 \leqslant \alpha \leqslant 2$ *and* $x, y \geqslant 0$. *If* $x_1 \geqslant x$ *and* $y_1 \geqslant y$, *then*

$$V_\alpha(x,y) \leqslant V_\alpha(x_1,y_1).$$

Proof. If $\alpha = 2$ or $xy = 0$ the claim is trivial, hence we assume $\alpha < 2$ and $x, y > 0$. Let us denote $\gamma = (2-\alpha)/2$, then

$$V_\alpha(x,y) = xy(x^2+y^2)^{-\gamma} = \left(x^{2-\frac{1}{\gamma}}y^{-\frac{1}{\gamma}} + x^{-\frac{1}{\gamma}}y^{2-\frac{1}{\gamma}}\right)^{-\gamma}.$$

Since $\alpha \geqslant 1$, we have $2 - 1/\gamma \leqslant 0$, thus $x^{2-\frac{1}{\gamma}} \geqslant x_1^{2-\frac{1}{\gamma}}$ and $y^{2-\frac{1}{\gamma}} \geqslant y_1^{2-\frac{1}{\gamma}}$. We have

$$x^{2-\frac{1}{\gamma}}y^{-\frac{1}{\gamma}} + x^{-\frac{1}{\gamma}}y^{2-\frac{1}{\gamma}} \geqslant x_1^{2-\frac{1}{\gamma}}y_1^{-\frac{1}{\gamma}} + x_1^{-\frac{1}{\gamma}}y_1^{2-\frac{1}{\gamma}},$$

implying

$$\begin{aligned} V_\alpha(x,y) = \left(x^{2-\frac{1}{\gamma}}y^{-\frac{1}{\gamma}} + x^{-\frac{1}{\gamma}}y^{2-\frac{1}{\gamma}}\right)^{-\gamma} \\ \leqslant \left(x_1^{2-\frac{1}{\gamma}}y_1^{-\frac{1}{\gamma}} + x_1^{-\frac{1}{\gamma}}y_1^{2-\frac{1}{\gamma}}\right)^{-\gamma} = V_\alpha(x_1,y_1). \end{aligned}$$

□

We begin our study of asymptotic behaviour of the spectral covariance by investigating linear processes

$$X(k) = \sum_{j=0}^{\infty} c_j \epsilon_{k-j}, \ k \in \mathbb{Z}, \tag{4.15}$$

where $\epsilon_i,\ i \in \mathbb{Z}$ are i.i.d. α-stable random variables with characteristic function

$$\begin{cases} \exp\left(-|t|^\alpha \left(1 - \mathrm{i}\beta \mathrm{sign}(t) \tan\frac{\pi\alpha}{2}\right)\right) \text{ if } \alpha \neq 1, \\ \exp\left(-|t|\right) \text{ if } \alpha = 1, \end{cases}$$

and a filter c_j satisfying

$$\sum_{j=0}^{\infty} |c_j|^\alpha < \infty.$$

This condition ensures the a.s. convergence of the series (4.15). Without any additional assumptions it is difficult to say anything about the decay rate of $\rho(n) := \rho(X(0), X(n))$. We assume that

$$c_i \sim U(i), \text{ as } i \to \infty, \tag{4.16}$$

where U is regularly varying with index $\eta = -\kappa$ and $\kappa > 1/\alpha$. Using the properties of U it is not difficult to show that $\sum_i |U(i)|^\alpha < \infty$ and the stable process (4.15) is defined correctly. Motivation to investigate such processes comes from [37].

The asymptotic dependence structure of the process above is described in the following theorem.

Theorem 4.7. *Suppose that condition* (4.16) *holds. For a linear process* $X(n)$ *defined in* (4.15) *we have:*

If $\alpha \leq 1$ *and* $\kappa > 1/\alpha$ *or* $1 < \alpha \leqslant 2$ *and* $1/\alpha < \kappa < 1/(\alpha - 1)$, *then*

$$\rho(n) \sim nU^\alpha(n) \int_0^\infty \frac{\left(t^\kappa (1+t)^\kappa\right)^{1-\alpha}}{\left(t^{2\kappa} + (1+t)^{2\kappa}\right)^{(2-\alpha)/2}} \mathrm{d}t. \tag{4.17}$$

If $\alpha > 1$ *and* $\kappa > 1/(\alpha - 1)$, *then*

$$\lim_{n\to\infty} \frac{\rho(n)}{U(n)} = \sum_{i=0}^{\infty} c_i^{\langle \alpha - 1 \rangle}. \tag{4.18}$$

In the case $\alpha > 1$ *and* $\kappa = 1/(\alpha - 1)$ *we assume stronger condition than* (4.16), *namely, we assume that* $c_i \sim i^{-\kappa}$, *then*

$$\rho(n) \sim n^{-\kappa} \ln n. \tag{4.19}$$

We make the following observations: First of all, comparing Theorem 4.7 to Theorem 3.3 and Theorem 3.4, we see that the decay rate of the spectral covariance coincides with that of codifference and covariation. In the case $\alpha > 1$ and $\kappa > 1/(\alpha - 1)$ the constant $\sum_{i=0}^{\infty} c_i^{<\alpha-1>}$ appears in all three theorems.

Condition $\sum_{j=0}^{\infty} c_j = 0$ has no influence on the limits in Theorem 4.7 in the case $\alpha < 2$, therefore, we can disprove Conjecture 3.11. The same can be said about codifference and covariation. Consequently, the notion of negative memory is not as closely related to these measures of dependence as it is in the case $\alpha = 2$.

In order to obtain the same order of decay some additional and quite strong conditions on coefficients or the function U are imposed in theorems 3.3 and 3.4 (see (3.5) and (3.7)). The only assumption we make is that the coefficients are asymptotically regularly varying (except in the case $\alpha > 1,\ \kappa = 1/(\alpha - 1)$, which was excluded from the formulation in [37,38], only mentioning that there is a "phase transition").

Suppose that $1 < \alpha < 2$ and $c_j \sim j^{-\kappa}$, $\kappa > 1/\alpha$. If $c_j \geqslant 0$, the process $X(n)$ is associated. It is easy to determine from Theorem 4.7 that $\sum_{k=2}^{\infty} \rho(k) < \infty$ if and only if $\kappa > 2/\alpha$. For $\kappa > 2/\alpha$ we can apply Theorem 3.8 and conclude that

$$\sum_{k=1}^{n} X(k)/n^{1/\alpha} \xrightarrow{\mathrm{d}} \mu, \tag{4.20}$$

where μ is a non-degenerate strictly α-stable distribution. Theorem 3.12 reveals that (4.20) holds for $\kappa > 1$. Therefore, for $1 < \kappa < 2/\alpha$ the condition $\sum_{k=2}^{\infty} \rho(k) < \infty$ is not satisfied, but (4.20) holds.

We could investigate a more general linear process $Z(k) = \sum_{j=0}^{\infty} c_j \eta_{k-j}$, $k \in \mathbb{Z}$, where $\eta_j, j \in \mathbb{Z}$, are i.i.d. random variables belonging to the normal domain of attraction of an α-stable random variable, and coefficients c_j satisfy (4.16). It is easy to show that the finite-dimensional distributions of the process $Z(k),\ k \in \mathbb{Z}$, belong to the normal domain of attraction of the corresponding distributions of the process $X(k)$ defined by (4.15). Therefore, by definition, $\rho(Z(n), Z(0)) = \rho(X(k), X(0))$ and Theorem (4.7) holds for the process $Z(k),\ k \in \mathbb{Z}$, as well.

Next we formulate a general fact on spectral covariances of a stationary α-stable moving average process (2.21). This is an analogue of Theorem 3.2:

Theorem 4.8. *Suppose $\rho(g; \cdot, \cdot)$ is a general measure of dependence defined by* (4.5) *and* $|g(s_1, s_2)| \leqslant C\,|s_1 s_2|^{\alpha/2}$. *If X_t is an α-stable,* $0 < \alpha \leqslant 2$, *moving average process, then*

$$\lim_{t\to\infty} \rho(g; X_t, X_0) = 0.$$

Corollary 4.9. *For an α-stable,* $0 < \alpha \leqslant 2$, *moving average process X_t,*

$$\lim_{t\to\infty} \rho(X_t, X_0) = 0, \quad \lim_{t\to\infty} \rho_\alpha(X_t, X_0) = 0.$$

Let us determine the rate of decay of the spectral covariance for log-fractional and linear fractional stable noises.

Theorem 4.10. *Let $Y_2(t)$ be a log-fractional stable noise defined by* (2.19). *Then*

$$\rho(Y_2(0), Y_2(t)) \sim Ct^{1-\alpha}, \quad t \to \infty, \tag{4.21}$$

where

$$C = 2\int_0^\infty \frac{\big(y(1+y)\big)^{1-\alpha}}{\big(y^2+(1+y)^2\big)^{\frac{2-\alpha}{2}}}\mathrm{d}y - \int_0^1 \frac{\big(y(1-y)\big)^{1-\alpha}}{\big(y^2+(1-y)^2\big)^{\frac{2-\alpha}{2}}}\mathrm{d}y. \tag{4.22}$$

Remark 4.11. Comparing this result to Theorem 3.6 we see that the rate of decay of codifference is the same. Relation (4.21) holds independently of skewness intensity β, it is not so with codifference, as was observed in Remark 3.7.

In the next theorem we consider a linear fractional stable noise process, which is more complicated comparing with the log-fractional stable noise, since the integrand $f_t(x)$ from (2.18) depends on more parameters. We show that the asymptotic behaviour is different for different values of the parameters, present in (2.18). Let us introduce two regions $S = S_1 \cup S_2$ and U, where

$$S_1 = \{(H, \alpha) : 0 < \alpha \leqslant 1, 0 < H < 1\},$$

$$S_2 = \{(H, \alpha) : 1 < \alpha < 2,\ 1 - 1/(\alpha(\alpha - 1)) < H < 1,\ H \neq 1/\alpha\},$$

$$U = \{(H, \alpha) : 1 < \alpha < 2,\ 0 < H < 1 - 1/(\alpha(\alpha - 1))\}.$$

Theorem 4.12. *Let $Y_1(t)$ be linear fractional stable noise defined by* (2.16)*. Then for $(\alpha, H) \in S$*

$$\rho(Y_1(0), Y_1(t)) \sim C_1(a, b, \alpha, H) t^{\alpha H - \alpha}, \quad t \to \infty, \tag{4.23}$$

while for $(\alpha, H) \in U$

$$\rho(Y_1(0), Y_1(t)) \sim C_2(a, b, \alpha, H) t^{H - 1 - 1/\alpha}, \quad t \to \infty, \tag{4.24}$$

where $C_1(a, b, \alpha, H)$ and $C_2(a, b, \alpha, H)$ are defined in (4.60) *and* (4.69)*, respectively.*

Remark 4.13. The same remark as the Remark 4.11 can be made about Theorem 4.12.

Let us move on to investigating linear fields

$$X_{\mathbf{k}} = \sum_{\mathbf{i} \geqslant \mathbf{0}} c_{\mathbf{i}} \epsilon_{\mathbf{k} - \mathbf{i}},\ \mathbf{k} \in \mathbb{Z}^d, \tag{4.25}$$

where $\epsilon_{\mathbf{i}},\ \mathbf{i} \in \mathbb{Z}^d$, are i.i.d. random variables with $S\alpha S$ distribution (this assumption is made only for the simplicity of writing), and $c_{\mathbf{i}},\ \mathbf{i} \geqslant \mathbf{0}$, are real numbers satisfying the condition

$$\sum_{\mathbf{i} \geqslant \mathbf{0}} |c_{\mathbf{i}}|^\alpha < \infty.$$

We are interested in the expression of the quantities $\rho(X_{\mathbf{0}}, X_{(s_1k_1,\ldots,s_dk_d)})$ and $\rho_\alpha(X_{\mathbf{0}}, X_{(s_1k_1,\ldots,s_dk_d)})$ via the filter $\{c_{\mathbf{i}}\}$ and in their asymptotic behaviour as $\min_{1\le i\le d} k_i \to \infty$ for a fixed collection of signs $s_1,\ldots,s_d \in \{-1,1\}$.

For any $\mathbf{a} = (a_1,\ldots,a_d) \in \mathbb{Z}^d$, we denote

$$Q_{\mathbf{a}} = \left\{\mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{Z}^d : x_i \ge -a_i, i = 1,\ldots,d\right\},$$

$\mathbf{a}_+ = ((a_1)_+,\ldots,(a_d)_+)$ and $\mathbf{a}_- = ((a_1)_-,\ldots,(a_d)_-)$, where $(\cdot)_+ = \max(\cdot,0)$ and $(\cdot)_- = -\min(\cdot,0)$. We have the following expressions for the spectral covariances of a linear field in the general case $d \ge 2$.

Proposition 4.14. *For a linear field defined in* (4.25)*, for any* $\mathbf{k} \in \mathbb{Z}^d$*, it holds that*

$$\rho(X_{\mathbf{0}}, X_{\mathbf{k}}) = \rho(X_{\mathbf{k}_-}, X_{\mathbf{k}_+}) = \sum_{\mathbf{j}\in Q_{\mathbf{0}}} \frac{c_{\mathbf{j}+\mathbf{k}_-} c_{\mathbf{j}+\mathbf{k}_+}}{\left(c^2_{\mathbf{j}+\mathbf{k}_-} + c^2_{\mathbf{j}+\mathbf{k}_+}\right)^{(2-\alpha)/2}} \tag{4.26}$$

and

$$\rho_\alpha(X_{\mathbf{0}}, X_{\mathbf{k}}) = \rho_\alpha(X_{\mathbf{k}_-}, X_{\mathbf{k}_+}) = \sum_{\mathbf{j}\in Q_{\mathbf{0}}} c^{\langle\alpha/2\rangle}_{\mathbf{j}+\mathbf{k}_-} c^{\langle\alpha/2\rangle}_{\mathbf{j}+\mathbf{k}_+}. \tag{4.27}$$

For linear random processes ($d = 1$, $X_k = \sum_{i\ge 0} c_i \epsilon_{k-i}$, $k \in \mathbb{Z}$), we easily obtain the asymptotic decay of $\rho(X_0, X_n)$ (see Theorem 4.7). It turned out that the generalization to the case $d > 1$ is not so trivial. Since the main difficulties in passing from the case $d = 1$ to the case $d > 1$ can be seen in the case $d = 2$, we consider mainly this case and use the notation without letters in bold. Let us denote

$$X_{k,l} = \sum_{i,j=0}^{\infty} c_{i,j}\epsilon_{k-i,l-j}, \ (k,l) \in \mathbb{Z}^2, \tag{4.28}$$

where $\epsilon_{k,l}$, $(k,l) \in \mathbb{Z}^2$, are i.i.d. $S\alpha S$ random variables, and $c_{i,j}$, $i,j \ge 0$, are real numbers satisfying the condition

$$\sum_{i,j=0}^{\infty} |c_{i,j}|^\alpha < \infty. \tag{4.29}$$

We are interested in the asymptotic behaviour of $\rho(k,l) := \rho(X_{0,0}, X_{k,l})$ under some assumptions on the regular behaviour of the filter $\{c_{i,j}\}$. We can consider two types of such behaviour:

$$c_{i,j} \sim \frac{1}{(i^{\beta_1} + j^{\beta_2})^{\beta_3}} \quad \text{and, in particular,} \quad c_{i,j} \sim \frac{1}{\|(i,j)\|^{\beta_3}}$$

with some positive $\beta_i,\ i = 1, 2, 3$, such that (4.29) is satisfied, or

$$c_{i,j} \sim i^{-\beta_1} j^{-\beta_2} \tag{4.30}$$

with some $\beta_k > 1/\alpha,\ k = 1, 2$. Here $c_{i,j} \sim a_{i,j}$ means that

$$\lim_{i,j\to\infty} \frac{c_{i,j}}{a_{i,j}} = 1.$$

We investigate linear fields with filters satisfying condition (4.30), and this choice is motivated by two reasons: first, the behaviour of $\rho(k,l)$ for the field satisfying (4.30) is easier to investigate, and, second, linear fields with such a filter motivated the definition of directional memory (see Definition 6). For linear random processes the relation $c_i \sim i^{-\beta}$ was sufficient to obtain the asymptotics of the spectral covariance. For linear fields, it is not sufficient to have (4.30) since we also must control partial limits of the filter over rows and columns (see conditions (A2) and (A3) in Theorem 4.17). The expression of $\rho(k,l)$ is different for the cases $k > 0, l > 0$ and $k > 0, l < 0$: for $n, m \in \mathbb{N}$ we have

$$\rho(n,m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{c_{i,j}c_{i+n,j+m}}{\left(c_{i,j}^2 + c_{i+n,j+m}^2\right)^{\frac{2-\alpha}{2}}},$$

$$\rho(n,-m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{c_{i+n,j}c_{i,j+m}}{\left(c_{i+n,j}^2 + c_{i,j+m}^2\right)^{\frac{2-\alpha}{2}}}.$$

Due to the stationarity, the remaining two cases ($k < 0,\ l > 0$ and $k < 0,\ l < 0$) can be transformed into the first ones. Investigation of such expressions is quite difficult, and the main cause of these difficulties is the exponent $(2-\alpha)/2$ at the norm of points $(c_{i+n,j}, c_{i,j+m})$ in the denominator, which disappears only in the case $\alpha = 2$. This observation motivated us to introduce α-spectral covariance. It turned out that investigation of the asymptotics of $\rho_\alpha(X_{0,0}, X_{k,l})$ is much simpler and with a possible extension to the case $d \geq 3$.

Let us assume that the coefficients $c_{\mathbf{i}}$ in (4.28) have the form

$$c_{i,j} = w_{(i,j)}(1+i)^{-\beta_1}(1+j)^{-\beta_2}, \quad i, j \geq 0, \tag{4.31}$$

where $\beta_k > 1/\alpha$, $k = 1, 2$, and the coefficients $w_{(i,j)}$ satisfy the following conditions:

(A1) there exists $\lim_{i,j\to\infty} w_{(i,j)} = 1$,

(A2) for every $i \geq 0$, there exists $\lim_{j\to\infty} w_{(i,j)} = w_{(i,\cdot)} > 0$,

(A3) for every $j \geq 0$, there exists $\lim_{i\to\infty} w_{(i,j)} = w_{(\cdot,j)} > 0$.

The following two theorems describe the asymptotic dependence structure of the linear field described above. Theorem 4.15 deals with $\rho(k, l)$ as $k \to \infty,\ l \to \infty$, and Theorem 4.16 treats the case $k \to \infty,\ l \to -\infty$.

Theorem 4.15. *Suppose that a linear field* (4.28) *with coefficients* $c_{i,j}$ *having form* (4.31), *satisfies conditions* (A1)–(A3). *Then the asymptotic behaviour of spectral covariance* $\rho(n, m)$ *is as follows.*

1. *If* $1 < \alpha \leqslant 2$ *and* $\beta_i > \frac{1}{\alpha-1}$, $i = 1, 2$,

$$\lim_{n,m\to\infty} \frac{\rho(n, m)}{n^{-\beta_1} m^{-\beta_2}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{(i,j)}^{\langle\alpha-1\rangle} (1+i)^{-\beta_1(\alpha-1)}(1+j)^{-\beta_2(\alpha-1)}. \tag{4.32}$$

2. *If* $1 < \alpha \leqslant 2$ *and* $\frac{1}{\alpha} < \beta_i < \frac{1}{\alpha-1}$, $i = 1, 2$ *or* $0 < \alpha \leqslant 1$ *and* $\beta_i > \frac{1}{\alpha}$, $i = 1, 2$,

$$\lim_{n,m\to\infty} \frac{\rho(n, m)}{n^{1-\alpha\beta_1} m^{1-\alpha\beta_2}} = \int_0^\infty \int_0^\infty V_\alpha(t^{-\beta_1} s^{-\beta_2},\ (t+1)^{-\beta_1}(s+1)^{-\beta_2}) \mathrm{d}t\mathrm{d}s. \tag{4.33}$$

3. *If* $1 < \alpha \leqslant 2$ *and* $\beta_1 > \frac{1}{\alpha-1}$, $\frac{1}{\alpha} < \beta_2 < \frac{1}{\alpha-1}$,

$$\lim_{n,m\to\infty} \frac{\rho(n,m)}{n^{-\beta_1} m^{1-\beta_2\alpha}} = \sum_{i=0}^{\infty} w_{(i,\infty)}^{\alpha-1} (1+i)^{-\beta_1(\alpha-1)} \int_0^{\infty} u^{-\beta_2(\alpha-1)} (1+u)^{-\beta_2} \mathrm{d}u.$$

4. *If* $1 < \alpha \leqslant 2$ *and* $\beta_1 = \frac{1}{\alpha-1}$, $\frac{1}{\alpha} < \beta_2 < \frac{1}{\alpha-1}$,

$$\lim_{n,m\to\infty} \frac{\rho(n,m)}{n^{-\beta_1} m^{1-\beta_2\alpha} \ln(n)} = \int_0^{\infty} v^{-\beta_2(\alpha-1)} (1+v)^{-\beta_2} \,\mathrm{d}v.$$

5. *If* $1 < \alpha \leqslant 2$ *and* $\beta_1 = \frac{1}{\alpha-1}$, $\beta_2 > \frac{1}{\alpha-1}$,

$$\lim_{n,m\to\infty} \frac{\rho(n,m)}{n^{-\beta_1} m^{-\beta_2} \ln(n)} \sum_{j=0}^{\infty} w_{(\infty,j)}^{\alpha-1} (1+j)^{-\beta_2(\alpha-1)}.$$

6. *If* $1 < \alpha \leqslant 2$ *and* $\beta_1 = \frac{1}{\alpha-1}$, $\beta_2 = \frac{1}{\alpha-1}$,

$$\lim_{n,m\to\infty} \frac{\rho(n,m)}{n^{-\beta_1} m^{-\beta_2} \ln(n) \ln(m)} = 1.$$

Theorem 4.16. *Suppose that a linear field* (4.28) *with coefficients* $c_{i,j}$ *having form* (4.31)*, satisfies conditions* (A1)–(A3)*. Then the asymptotic behaviour of spectral covariance* $\rho(n,-m)$ *is as follows.*

1. *If* $1 < \alpha \le 2$, $\frac{1}{\beta_1} + \frac{1}{\beta_2} > \alpha$ *and* $\frac{1}{\alpha} < \beta_i < \frac{1}{\alpha-1}$, $i = 1, 2$, *or* $0 < \alpha \le 1$ *and* $\frac{1}{\beta_1} + \frac{1}{\beta_2} > \alpha$*:*

$$\lim_{n,m\to\infty} \frac{\rho(n,-m)}{n^{1-\beta_1\alpha} m^{1-\beta_2\alpha}} = \int_0^{\infty}\int_0^{\infty} V_\alpha((1+u)^{-\beta_1} v^{-\beta_2},\ u^{-\beta_1}(1+v)^{-\beta_2}) \mathrm{d}u\mathrm{d}v.$$

2. *If* $1 < \alpha \le 2$, $\frac{1}{\alpha-1} < \beta_1$ *and* $\frac{1}{\alpha} < \beta_2 < 1$*:*

$$\lim_{n,m\to\infty} \frac{\rho(n,-m)}{n^{-\beta_1} m^{1-\beta_2\alpha}} = \sum_{i=0}^{\infty} w_{(i,\infty)}^{\alpha-1} (1+i)^{-\beta_1(\alpha-1)} \int_0^{\infty} v^{-\beta_2} (1+v)^{-\beta_2(\alpha-1)} \mathrm{d}v.$$

3. *If* $1 < \alpha \leq 2$, $\beta_1 = \frac{1}{\alpha-1}$ *and* $\frac{1}{\alpha} < \beta_2 < 1$:

$$\lim_{n,m\to\infty} \frac{\rho(n,-m)}{n^{1-\beta_1\alpha} m^{1-\beta_2\alpha} \ln(n)} = \int_0^\infty v^{-\beta_2}(1+v)^{-\beta_2(\alpha-1)} \mathrm{d}v\mathrm{d}s.$$

4. *If* $1 < \alpha \leq 2$ *and* $\frac{1}{\alpha-1} < \beta_i$, $i = 1, 2$:

a) *If* m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to 0$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1(\alpha-1)} m_n^{-\beta_2}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{(\infty,j)}^{\alpha-1} w_{(i,\infty)} (1+j)^{-\beta_2(\alpha-1)} (1+i)^{-\beta_1}.$$

b) *If* m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to c \in (0;\infty)$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1(\alpha-1)} m_n^{-\beta_2}} = \frac{1}{c}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(w_{(\infty,j)}(1+j)^{-\beta_2},\ c w_{(i,\infty)}(1+i)^{-\beta_1}).$$

c) *If* $m_n \to \infty$ *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to \infty$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{(\infty,j)} w_{(i,\infty)}^{\alpha-1} (1+j)^{-\beta_2} (1+i)^{-\beta_1(\alpha-1)}.$$

5. *If* $1 < \alpha < 2$, $\frac{1}{\alpha-1} < \beta_1$ *and* $1 < \beta_2 < \frac{1}{\alpha-1}$:

a) *If* m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to 0$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\frac{\beta_1}{\beta_2}} m_n^{1-\beta_2\alpha}} = \sum_{i=0}^{\infty}\int_0^\infty V_\alpha(v^{-\beta_2}, w_{(i,\infty)}(1+i)^{-\beta_1})\mathrm{d}v.$$

b) *If* m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to c \in (0;\infty)$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(w_{(\infty,j)} c^{-1}(1+j)^{-\beta_2}, w_{(i,\infty)}(1+i)^{-\beta_1}).$$

c) If $m_n \to \infty$ *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to \infty$*:*

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{(\infty,j)} w_{(i,\infty)}^{\alpha-1} (1+j)^{-\beta_2} (1+i)^{-\beta_1(\alpha-1)}.$$

6. *If* $1 < \alpha < 2,\ 1 < \beta_i < \frac{1}{\alpha-1},\ i = 1, 2$ *and* $\frac{1}{\beta_1} + \frac{1}{\beta_2} < \alpha$ *or* $0 < \alpha \leq 1$ *and* $\frac{1}{\beta_1} + \frac{1}{\beta_2} < \alpha$*:*

   a) If m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to 0$*:*

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{-\frac{\beta_1}{\beta_2}} m_n^{1-\beta_2\alpha}} = \sum_{i=0}^{\infty} \int_0^{\infty} V_\alpha(v^{-\beta_2},\ w_{(i,\infty)}(1+i)^{-\beta_1}) \mathrm{d}v.$$

   b) If m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to c \in (0; \infty)$*:*

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{-\beta_1(\alpha-1)} m_n^{-\beta_2}} = \frac{1}{c} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} V_\alpha(w_{(\infty,j)}(1+j)^{-\beta_2},\ w_{(i,\infty)} c^2 (1+i)^{-\beta_1}).$$

   c) If $m_n \to \infty$ *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to \infty$*:*

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{1-\beta_1\alpha} m_n^{-\frac{\beta_2}{\beta_1}}} = \sum_{j=0}^{\infty} \int_0^{\infty} V_\alpha(u^{-\beta_1},\ w_{(\infty,j)}(1+j)^{-\beta_2}) \mathrm{d}u.$$

7. *If* $1 < \alpha < 2,\ \frac{1}{\alpha-1} < \beta_1$ *and* $\beta_2 = \frac{1}{\alpha-1}$*:*

   a) If m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to 0$*:*

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{-\frac{\beta_1}{\beta_2}} m_n^{1-\beta_2\alpha} \ln\left(n^{-\frac{\beta_1}{\beta_2}} m_n\right)} = \sum_{i=0}^{\infty} w_{(i,\infty)}(1+i)^{-\beta_1}.$$

   b) If m_n *is a sequence such that* $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to c \in (0; \infty)$*:*

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = c \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} V_\alpha(w_{(\infty,j)} c^{-1} (1+j)^{-\beta_2},\ w_{(i,\infty)}(1+i)^{-\beta_1}).$$

c) If $m_n \to \infty$ is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to \infty$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{(\infty,j)} w_{(i,\infty)}^{\alpha-1} (1+j)^{-\beta_2}(1+i)^{-\beta_1(\alpha-1)}.$$

8. If $1<\alpha<2,\ \beta_1 = \frac{1}{\alpha-1}$ and $1<\beta_2<\frac{1}{\alpha-1}$

a) If m_n is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to 0$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\frac{\beta_1}{\beta_2}} m_n^{1-\beta_2\alpha}} = \sum_{i=0}^{\infty} \int_0^{\infty} V_\alpha(w_{(i,\infty)}\,(1+i)^{-\beta_1}\,,\ s^{-\beta_2})\mathrm{d}s.$$

b) If m_n is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to c \in (0;\infty)$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(w_{(\infty,j)} c^{-1}(1+j)^{-\beta_2},\ w_{(i,\infty)}(1+i)^{-\beta_1}).$$

c) If $m_n \to \infty$ is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to \infty$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{1-\beta_1\alpha} m_n^{-\frac{\beta_2}{\beta_1}} \ln\left(n m_n^{-\frac{\beta_2}{\beta_1}}\right)} = \sum_{j=0}^{\infty} w_{(\infty,j)}\,(1+j)^{-\beta_2}\,.$$

9. If $1<\alpha<2$ and $\beta_i = \frac{1}{\alpha-1},\ i=1,2$:

a) If m_n is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to 0$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1/\beta_2} m_n^{1-\beta_2\alpha} \ln\left(n^{-\beta_1/\beta_2} m_n\right)} = \sum_{i=0}^{\infty} w_{(i,\infty)}(1+i)^{-\beta_1}.$$

b) If m_n is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to c \in (0;\infty)$:

$$\lim_{n\to\infty} \frac{\rho(n,-m_n)}{n^{-\beta_1} m_n^{-\beta_2(\alpha-1)}} = c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(w_{(\infty,j)} c^{-1}(1+j)^{-\beta_2},\ w_{(i,\infty)}(1+i)^{-\beta_1}).$$

c) If $m_n \to \infty$ is a sequence such that $\frac{m_n^{-\beta_2}}{n^{-\beta_1}} \to \infty$:

$$\lim_{n\to\infty} \frac{\rho(n, -m_n)}{n^{1-\beta_1\alpha} m_n^{-\beta_2/\beta_1} \ln\left(n m_n^{-\beta_2/\beta_1}\right)} = \sum_{j=0}^{\infty} w_{(\infty,j)}(1+j)^{-\beta_2}.$$

The cases

- $\alpha \leqslant 1$ and $1/\beta_1 + 1/\beta_2 = \alpha$;
- $1 < \alpha < 2$, $\beta_1 > 1/(\alpha - 1)$ and $\beta_2 = 1$;
- $1 < \alpha < 2$, $\beta_1 = 1/(\alpha - 1)$ and $\beta_2 = 1$;
- $1 < \alpha < 2$, $1 < \beta_i < 1/(\alpha - 1)$ and $1/\beta_1 + 1/\beta_2 = \alpha$;

were not considered in Theorem 4.16. We feel that the cases examined are sufficient to reveal the complex dependence structure of this linear field.

A remark on the codifference for linear random fields is appropriate here. According to Proposition 4.3, the codifference in the case of associated random variables (e.g., when the coefficients of the filter are non-negative) for $1 \leq \alpha < 2$ is equivalent to the spectral covariance. Taking into account the complexity of the asymptotic behaviour of the spectral covariance for linear fields, we can expect that asymptotic behaviour of the codifference for the linear random field is also complicated.

As the following theorem shows, asymptotic behaviour of α-spectral covariance is simpler. For the formulation, it is convenient to denote $\gamma_k = \beta_k\alpha/2,\ k = 1, 2$, and

$$K(a) = \int_0^\infty v^{-a}(1+v)^{-a}\,\mathrm{d}v := B(1-a, 2a-1), \quad 1/2 < a < 1,$$

where $B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t = \int_0^\infty t^{z-1}(1+t)^{-(z+w)}\mathrm{d}t$ is the beta function.

Let $n, m \in \mathbb{N}$, $s \in \{-1, 1\}$, and $\rho_\alpha(k_1, k_2) := \rho_\alpha(X_{0,0}, X_{k_1,k_2})$. From (4.27) and (4.31) we have

$$\rho_\alpha(n, sm) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_{i,j,n,m,s}(1+i)^{-\gamma_1}(1+j)^{-\gamma_2}(1+i+n)^{-\gamma_1}(1+j+m)^{-\gamma_2}, \quad (4.34)$$

where

$$q_{i,j,n,m,s} = \begin{cases} w_{(i,j)}^{\langle\alpha/2\rangle} w_{(i+n,j+m)}^{\langle\alpha/2\rangle}, & s = 1, \\ w_{(i,j+m)}^{\langle\alpha/2\rangle} w_{(i+n,j)}^{\langle\alpha/2\rangle}, & s = -1. \end{cases}$$

There are six main sets of the parameters β_1, β_2 giving different asymptotic behaviours of $\rho_\alpha(n, sm)$. The rest of the sets give symmetric results.

Theorem 4.17. *Suppose that a linear field* (4.28) *with coefficients* $c_{i,j}$ *of the form* (4.31) *satisfies conditions* (A1)–(A3). *Then the asymptotic behaviour of the* α*-spectral covariance* (4.34) *is as follows:*

1. *If* $1/2 < \gamma_i < 1, i = 1, 2$*:*

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{1-2\gamma_1} m^{1-2\gamma_2}} = K(\gamma_1)K(\gamma_2). \quad (4.35)$$

2. *If* $\gamma_1 > 1$ *and* $1/2 < \gamma_2 < 1$*:*

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{-\gamma_1} m^{1-2\gamma_2}} = \sum_{i=0}^{\infty} w_{(i,\cdot)}^{\alpha/2} (1+i)^{-\gamma_1} K(\gamma_2). \quad (4.36)$$

3. $\gamma_i > 1, i = 1, 2$:

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{-\gamma_1} m^{-\gamma_2}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j,s)^{\langle\alpha/2\rangle} (1+i)^{-\gamma_1} (1+j)^{-\gamma_2}, \quad (4.37)$$

where $W(i,j,1) = w_{(i,j)}$ *and* $W(i,j,-1) = w_{(i,\cdot)} w_{(\cdot,j)}$.

4. *If* $1/2 < \gamma_1 < 1, \gamma_2 = 1$*:*

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{1-2\gamma_1} m^{1-2\gamma_2} \ln m} = K(\gamma_1). \quad (4.38)$$

5. *If* $\gamma_1 > 1$ *and* $\gamma_2 = 1$*:*

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{1-2\gamma_1} m^{1-2\gamma_2} \ln m} = \sum_{i=0}^{\infty} w_{(i,\cdot)}^{\alpha/2} (1+i)^{-\gamma_1}. \tag{4.39}$$

6. *If* $\gamma_1 = 1, \gamma_2 = 1$*:*

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{1-2\gamma_1} (\ln n) m^{1-2\gamma_2} (\ln m)} = 1. \tag{4.40}$$

As it was mentioned before the statement of the theorem, we do not state results in symmetric cases. For example, if $1/2 < \gamma_1 < 1, \gamma_2 > 1$, then, instead of (4.36), we have

$$\lim_{n,m\to\infty} \frac{\rho_\alpha(n, sm)}{n^{1-2\gamma_1} m^{-\gamma_2}} = \sum_{j=0}^{\infty} w_{(\cdot,j)}^{\alpha/2} (1+j)^{-\gamma_2} K(\gamma_1).$$

Assuming conditions slightly stronger than (A2) and (A3), we can get simpler expressions of the constants in the asymptotic relations of Theorem 4.17.

Corollary 4.18. *If* $w_{(i,\cdot)} \equiv 1$, $w_{(\cdot,j)} \equiv 1$, *then the right-hand sides of relations* (4.36) *and* (4.39) *are* $\zeta(\gamma_1)K(\gamma_2)$ *and* $\zeta(\gamma_1)$, *respectively, and the right-hand side of relation* (4.37) *in the case of* $s = -1$ *is* $\zeta(\gamma_1)\zeta(\gamma_2)$, *where*

$$\zeta(t) := \sum_{i=0}^{\infty} (1+i)^{-t} = \frac{1}{\Gamma(t)} \int_0^\infty \frac{s^{t-1} \mathrm{e}^{-s}}{1 - \mathrm{e}^{-s}} \mathrm{d}s, \quad t > 1.$$

Remark 4.19. If the sum on the right hand side of (4.32) or (4.37) equals 0, a different normalization is required to obtain a non-zero limit. Under more general conditions (e.g., not requiring the limits to be positive in (A2) and (A3)) the limits in some other cases might equal 0. We do not investigate the asymptotic behaviour of dependence measures in such cases here, these questions are left for our future research.

Now we discuss a generalization of the results of Theorem 4.17 to the case $d \geq 3$. In order to get the same generality as in Theorem 4.17, we should consider a random linear field defined in (4.25) with coefficients of the form

$$c_{\mathbf{i}} = w_{\mathbf{i}} \prod_{k=1}^{d} (1 + i_k)^{-\beta_k} \tag{4.41}$$

with $\beta_k > 1/\alpha,\ k = 1, \ldots, d$, and a function w (of the argument $\mathbf{i} = (i_1, \ldots, i_d)$) having properties similar to those stated in (A1)–(A3). Although there are no principal difficulties to investigate the α-spectral covariance in this general case, the formulations and proofs become complicated since now, instead of conditions (A1)–(A3), there will be much more various partial limits of the function w. Therefore, we consider the simple case where $w_{\mathbf{i}} \equiv 1$, that is, we consider a random linear field (4.25) with coefficients

$$c_{\mathbf{i}} = \prod_{k=1}^{d} (1 + i_k)^{-\beta_k}, \tag{4.42}$$

where $\beta_k > 1/\alpha,\ k = 1, \ldots, d$. Let us denote $\gamma_l = \alpha\beta_l/2$, $l = 1, \ldots, d$.

Proposition 4.20. *Suppose that a random linear field* (4.25) *with coefficients* (4.42) *satisfies the following condition: there are integer numbers* $u, v \geqslant 0,\ 0 \leq u+v \leq d$, *such that* $1/2 < \gamma_{l_i} < 1$ *for* $i = 1, \ldots, u, \quad \gamma_{l_i} > 1$ *for* $i = u+1, \ldots, u+v$, *and* $\gamma_{l_i} = 1$ *for* $i = u+v+1, \ldots, d$. *Then, adopting the convention that* $\prod_{\varnothing} = 1$, *we have*

$$\frac{\rho_\alpha(X_{\mathbf{0}}, X_{\mathbf{k}})}{\left(\prod_{i=1}^{u} |k_{l_i}|^{1-2\gamma_{l_i}}\right)\left(\prod_{i=u+1}^{u+v} |k_{l_i}|^{-\gamma_{l_i}}\right)\left(\prod_{i=u+v+1}^{d} |k_{l_i}|^{-1} \ln(|k_{l_i}|)\right)} \to \left(\prod_{i=1}^{u} K(\gamma_{l_i})\right)\left(\prod_{i=u+1}^{u+v} \zeta(\gamma_{l_i})\right) \tag{4.43}$$

as $\min_{1 \leqslant l \leqslant d} |k_l| \to \infty$.

It is worth noting that in the general case of coefficients from (4.41), under appropriate conditions on w, we will have more complicated constants on the right-hand side of the relation (4.43), but the same rate of decay (provided the constants are not zero).

4.2 Proofs

Proof of Proposition 4.2. Clearly, functions $f_{1,\beta}$ and $f_{2,\beta}$ satisfy conditions (4.2)–(4.4), and the function $f_{1,\beta}$ satisfies (4.6). Thus, it remains to prove that $f_{2,\beta}$ also satisfies this inequality. Using the inequality $|1-|x|^p| \leq |1-x|^p\,,\ x \in \mathbb{R},\ 0 < p \leq 1$, taking into account that $0 < \beta/2 \leq 1$, and assuming that $s_1 \neq 0$, we have

$$\left|1 - \left|\frac{s_2}{s_1}\right|^{\beta/2}\right| \leq \left|1 - \frac{s_2}{s_1}\right|^{\beta/2}.$$

Multiplying both sides of this inequality by $|s_1|^{\beta/2}$ and then taking squares of both sides, we easily get the following inequality (which also holds for $s_1 = 0$):

$$|s_1|^\beta + |s_2|^\beta - |s_1 - s_2|^\beta \leq 2|s_1 s_2|^{\beta/2}. \tag{4.44}$$

From the inequality $|s_1+s_2|^\beta \leqslant |s_1|^\beta + 2^{\beta/2}|s_1 s_2|^{\beta/2} + |s_2|^\beta$ it follows that

$$|s_1 + s_2|^\beta - |s_1|^\beta - |s_2|^\beta \leqslant 2|s_1 s_2|^{\beta/2}. \tag{4.45}$$

From (4.44) and (4.45) we get $\left||s_1+s_2|^\beta - |s_1|^\beta - |s_2|^\beta\right| \leqslant 2|s_1 s_2|^{\beta/2}$. Therefore,

$$\begin{aligned}
&\left|\int_{\mathbb{S}^1} \left(|s_1|^\beta + |s_2|^\beta - |s_1 - s_2|^\beta\right) \Gamma(\mathrm{d}\mathbf{s})\right| \\
&\qquad\qquad \leqslant 2\int_{\mathbb{S}^1} |s_1 s_2|^{\beta/2}\Gamma(\mathrm{d}\mathbf{s}) \\
&\qquad\qquad\qquad \leqslant 2\left(\int_{\mathbb{S}^1} |s_1|^\beta\, \Gamma(\mathrm{d}\mathbf{s}) \int_{\mathbb{S}^1} |s_2|^\beta\, \Gamma(\mathrm{d}\mathbf{s})\right)^{1/2}.
\end{aligned}$$

It is easy to see that the last inequality coincides with (4.6) for $f_{2,\beta}$. □

Proof of Proposition 4.3. The vector (X_1, X_2) is associated, so its spectral measure is concentrated in the first and third quadrants. Inequality (4.8) follows from the inequality $|s_1 s_2|^{\beta_1} \leqslant |s_1 s_2|^{\beta_2}$, and inequality (4.9) follows from $|s_1 s_2| \leqslant |s_1|\,|s_2|^{\alpha-1}$.

Denoting $s_1 = \sin\phi,\ s_2 = \cos\phi$, it is easy to see by symmetry that in order to prove (4.10), it is sufficient to show that there exist constants

c_1, c_2 such that

$$c_1 \cos\phi \sin\phi \leqslant |\cos\phi|^\alpha + |\sin\phi|^\alpha - |\cos\phi - \sin\phi|^\alpha \leqslant c_2 \cos\phi \sin\phi$$

for $\phi \in [0, \pi/4]$. Applying the inequalities

$$\alpha x - \frac{\alpha}{2} x^2 \leqslant 1 - (1-x)^\alpha \leqslant \alpha x, \; x \in [0,1],$$

we obtain

$$\begin{aligned}
&|\sin\phi|^\alpha + |\cos\phi|^\alpha - |\cos\phi - \sin\phi|^\alpha \\
&\qquad \leqslant |\sin\phi|^\alpha + |\cos\phi|^\alpha \, \alpha \tan\phi \\
&= |\sin\phi|^\alpha + \alpha \sin\phi \, |\cos\phi|^{\alpha-1} = \sin\phi\cos\phi \left((\sin\phi)^{\alpha-1} (\cos\phi)^{-1} + \alpha (\cos\phi)^{\alpha-2} \right) \\
&\qquad \leqslant \sin\phi \cos\phi \left((2^{-1/2})^{\alpha-2} + \alpha (2^{-1/2})^{\alpha-2} \right)
\end{aligned}$$

and

$$\begin{aligned}
&|\sin\phi|^\alpha + |\cos\phi|^\alpha - |\sin\phi - \cos\phi|^\alpha \\
&\qquad \geqslant |\cos\phi|^\alpha \left(|\tan\phi|^\alpha + \alpha \tan\phi - \frac{\alpha}{2} (\tan\phi)^2 \right) \\
&\qquad \geqslant |\cos\phi|^\alpha \left(\alpha \tan\phi + (\tan\phi)^\alpha - (\tan\phi)^2 \right) \\
&\qquad \geqslant \alpha \, |\cos\phi|^\alpha \tan\phi = \alpha \, |\cos\phi|^{\alpha-1} \sin\phi \geqslant \alpha \cos\phi \sin\phi.
\end{aligned}$$

□

Proof of Proposition 4.14. Due to the stationarity of the field, the vectors $(X_\mathbf{0}, X_\mathbf{k})$ and $(X_{\mathbf{0}+\mathbf{k}_-}, X_{\mathbf{k}+\mathbf{k}_-}) = (X_{\mathbf{k}_-}, X_{\mathbf{k}_+})$ have the same distribution. If $\mathbf{a} \in Q_\mathbf{0}$, then we have $Q_\mathbf{0} \subset Q_\mathbf{a}$ and

$$X_\mathbf{a} = \sum_{\mathbf{i} \in Q_\mathbf{0}} c_\mathbf{i} \varepsilon_{\mathbf{a}-\mathbf{i}} = \sum_{\mathbf{j} \in Q_\mathbf{a}} c_{\mathbf{j}+\mathbf{a}} \varepsilon_{-\mathbf{j}} = \sum_{\mathbf{j} \in Q_\mathbf{0}} c_{\mathbf{j}+\mathbf{a}} \varepsilon_{-\mathbf{j}} + \sum_{\mathbf{j} \in Q_\mathbf{a} \setminus Q_\mathbf{0}} c_{\mathbf{j}+\mathbf{a}} \varepsilon_{-\mathbf{j}}.$$

Notice that the sets $Q_{\mathbf{k}_-} \setminus Q_\mathbf{0}$ and $Q_{\mathbf{k}_+} \setminus Q_\mathbf{0}$ have no elements in common, therefore

$$\theta_1 X_{\mathbf{k}_-} + \theta_2 X_{\mathbf{k}_+}$$
$$= \sum_{\mathbf{j}\in Q_{\mathbf{0}}} \theta_1 c_{\mathbf{j}+\mathbf{k}_-}\varepsilon_{-\mathbf{j}} + \sum_{\mathbf{j}\in Q_{\mathbf{k}_-}\setminus Q_{\mathbf{0}}} \theta_1 c_{\mathbf{j}+\mathbf{k}_-}\varepsilon_{-\mathbf{j}} +$$
$$+ \sum_{\mathbf{j}\in Q_{\mathbf{0}}} \theta_2 c_{\mathbf{j}+\mathbf{k}_+}\varepsilon_{-\mathbf{j}} + \sum_{\mathbf{j}\in Q_{\mathbf{k}_+}\setminus Q_{\mathbf{0}}} \theta_2 c_{\mathbf{j}+\mathbf{k}_+}\varepsilon_{-\mathbf{j}}.$$

Now we easily obtain the ch.f. of the vector $(X_{\mathbf{k}_-}, X_{\mathbf{k}_+})$:

$$\mathbb{E}\exp\{\mathrm{i}(\theta_1 X_{\mathbf{k}_-} + \theta_2 X_{\mathbf{k}_+})\}$$
$$= \exp\Bigg\{-|\theta_1|^\alpha \sum_{\mathbf{j}\in Q_{\mathbf{k}_-}\setminus Q_{\mathbf{0}}} |c_{\mathbf{j}+\mathbf{k}_-}|^\alpha - \sum_{\mathbf{j}\in Q_{\mathbf{0}}} |\theta_1 c_{\mathbf{j}+\mathbf{k}_-} + \theta_2 c_{\mathbf{j}+\mathbf{k}_+}|^\alpha -$$
$$- |\theta_2|^\alpha \sum_{\mathbf{j}\in Q_{\mathbf{k}_+}\setminus Q_{\mathbf{0}}} |c_{\mathbf{j}+\mathbf{k}_+}|^\alpha \Bigg\}.$$

The obtained ch.f. allows us to see the structure of the spectral measure Γ, and we obtain expressions (4.26) and (4.27). □

Proof of Theorem 4.7. We need to investigate the asymptotic behaviour of

$$\rho(n) = \sum_{i=0}^{\infty} V_\alpha(c_i, c_{i+n}) = \sum_{i=0}^{\infty} \frac{c_i c_{i+n}}{(c_i^2 + c_{i+n}^2)^{\frac{2-\alpha}{2}}}$$

as $n \to \infty$. For simplicity of writing we assume that $c_i \neq 0,\ i \in \mathbb{N}_0$.

Function U is regularly varying with index $-\kappa$, therefore, we can write $U(i) = i^{-\kappa}h(i)$, where h is a s.v.f.

We begin with the case $\alpha > 1$ and $\kappa > 1/(\alpha-1)$. Lemma 2.2 applied with $\gamma = \kappa$ and $\eta = (\kappa - 1/(\alpha-1))/2$ implies

$$|c_i| \leqslant E(1+i)^{-\kappa+\eta},\ i \geqslant 0.$$

This yields

$$\sum_{i=0}^{\infty} |c_i|^{\alpha-1} \leqslant E^{\alpha-1} \sum_{i=0}^{\infty} (1+i)^{(-\kappa+\eta)(\alpha-1)} < \infty,$$

since $(-\kappa+\eta)(\alpha-1) > -1$.

Let us investigate the convergence of $V_\alpha(c_i, c_{i+n})$ for a fixed $i \in \mathbb{N}_0$. Lemma 2.2 implies $|c_{i+n}| \leqslant E(i+n)^{-\kappa/2} \to 0$, as $n \to \infty$. Also, for fixed i

$$\frac{c_{i+n}}{U(n)} = \frac{c_{i+n}}{U(i+n)} \frac{U(i+n)}{U(n)} = \frac{c_{i+n}}{U(i+n)} \frac{(i+n)^{-\kappa}}{n^{-\kappa}} \frac{h(i+n)}{h(n)} \to 1,$$

since, by Lemma 2.4 applied with $q_n = n$, $y_n = (1+i/n)^{-1}$, and $f \equiv 1$, we have

$$\frac{h(i+n)}{h(n)} = \frac{h(n(1+\frac{i}{n}))}{h(n)} \to 1, \text{ as } n \to \infty.$$

Therefore

$$\frac{V_\alpha(c_i, c_{i+n})}{U(n)} = \frac{c_i \frac{c_{i+n}}{U(n)}}{(c_i^2 + c_{i+n}^2)^{\frac{2-\alpha}{2}}} \to \frac{c_i}{(c_i^2)^{\frac{2-\alpha}{2}}} = c_i^{<\alpha-1>}.$$

We shall show that for large n, the sequence $V_\alpha(c_i, c_{i+n})/U(n), i \in \mathbb{N}_0$, is dominated by a summable sequence. By using the inequality $c_i^2 + c_{i+n}^2 \geqslant c_i^2$ we obtain

$$\left| \frac{V_\alpha(c_i, c_{i+n})}{U(n)} \right| \leqslant |c_i|^{\alpha-1} \frac{|c_{i+n}|}{U(n)}.$$

We proceed to show that for large n

$$|c_{i+n}|/U(n) \leqslant 4, \quad \text{for all } i \in \mathbb{N}_0.$$

For large n we have $|c_{i+n}| \leqslant 2U(i+n)$. Theorem 2.1 applied to $f = U$, $A = 2$ and any $\delta \in (0, \kappa)$ implies the existence of $B > 0$ such that

$$\frac{U(n+i)}{U(n)} \leqslant 2\left(1 + \frac{i}{n}\right)^{-\kappa+\delta} \leqslant 2, \text{ for } n \geqslant B.$$

Hence

$$\left| \frac{V_\alpha(c_i, c_{i+n})}{U(n)} \right| \leqslant 4\,|c_i|^{\alpha-1}.$$

Applying the dominated convergence theorem we obtain

$$\frac{\rho(n)}{U(n)} = \sum_{i=0}^{\infty} \frac{V_\alpha(c_i, c_{i+n})}{U(n)} \to \sum_{i=0}^{\infty} |c_i|^{\alpha-1}, \text{ as } n \to \infty.$$

Now let us consider the case $\alpha > 1$ and $1/\alpha < \kappa < 1/(\alpha-1)$ or $\alpha \leqslant 1$ and $\kappa > 1/\alpha$. Let us fix $\epsilon = (1+\kappa(1-\alpha))/(2(\kappa+2))$ and $\eta = \epsilon/2$.

Function U is regularly varying with index $-\kappa$, therefore, there exists N such that

$$n^{-\kappa+\eta} \leqslant U(n) \leqslant n^{-\kappa+\eta}, \; n \geqslant N.$$

Suppose n is large enough so that $n^\epsilon > N+1$. We split $\rho(n)$ into three sums

$$\begin{aligned}\rho(n) &= \sum_{i=0}^{\infty} V_\alpha(c_i, c_{i+n}) = S_{1,n} + S_{2,n} + S_{3,n} \\ &:= \sum_{i=0}^{N-1} V_\alpha(c_i, c_{i+n}) + \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} V_\alpha(c_i, c_{i+n}) + \sum_{i=\lfloor n^\epsilon \rfloor}^{\infty} V_\alpha(c_i, c_{i+n}).\end{aligned}$$

Inequality $c_i^2 + c_{i+n}^2 \geqslant c_i^2$ implies

$$|V_\alpha(c_i, c_{i+n})| \leqslant |c_i|^{\alpha-1} |c_{i+n}|,$$

hence

$$\begin{aligned}|S_{1,n}| &\leqslant \sum_{i=0}^{N-1} |V_\alpha(c_i, c_{i+n})| \leqslant \sum_{i=0}^{N-1} |c_i|^{\alpha-1} |c_{i+n}| \\ &\leqslant \max_{0 \leqslant j < N} |c_j|^{\alpha-1} \sum_{i=0}^{N-1} |c_{i+n}| \leqslant \max_{0 \leqslant j < N} |c_j|^{\alpha-1} \sum_{i=0}^{N-1} E(i+n)^{-\kappa+\eta} \\ &\leqslant E \max_{0 \leqslant j < N} |c_j|^{\alpha-1} N n^{-\kappa+\eta},\end{aligned}$$

and

$$\begin{aligned}|S_{2,n}| &\leqslant \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} |V_\alpha(c_i, c_{i+n})| \leqslant \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} |c_i|^{\alpha-1} |c_{i+n}| \\ &\leqslant E n^{-\kappa+\eta} \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} |c_i|^{\alpha-1} = E n^{-\kappa+\eta} \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} |c_i|^{\alpha} |c_i|^{-1} \\ &\leqslant E n^{-\kappa+\eta} \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} |c_i|^{\alpha} i^{\kappa+\eta} \leqslant E n^{-\kappa+\eta} \sum_{i=N}^{\lfloor n^\epsilon \rfloor - 1} |c_i|^{\alpha} n^{\epsilon(\kappa+\eta)} \\ &\leqslant E n^{-\kappa+\eta+\epsilon(\kappa+\eta)} \sum_{i=0}^{\infty} |c_i|^{\alpha} \leqslant E n^{-\kappa+\epsilon(\kappa+2)} \sum_{i=0}^{\infty} |c_i|^{\alpha}.\end{aligned}$$

We have thus obtained $S_{1,n} = O(n^{-\kappa+\eta})$ and $S_{2,n} = O(n^{-\kappa+\epsilon(\kappa+2)})$, therefore $S_{1,n} + S_{2,n} = O(n^{-\kappa+\epsilon(\kappa+2)})$. Since $\epsilon(\kappa+2) < \kappa + 1 - \alpha\kappa$, we obtain

$$\frac{n^{-\kappa+\epsilon(\kappa+2)}}{nU^\alpha(n)} = n^{-\kappa+\epsilon(\kappa+2)-1+\alpha\kappa} h^{-\alpha}(n) \to 0,$$

as $n \to \infty$, and, consequently,

$$\frac{S_{1,n} + S_{2,n}}{nU^{\alpha}(n)} \to 0. \tag{4.46}$$

It remains to deal with $S_{3,n}$. We have

$$\begin{aligned} S_{3,n} = \sum_{i=\lfloor n^{\epsilon} \rfloor}^{\infty} V_{\alpha}(c_i, c_{i+n}) &= \int_{\lfloor n^{\epsilon} \rfloor}^{\infty} V_{\alpha}(c_{\lfloor v \rfloor}, c_{\lfloor v \rfloor + n}) \mathrm{d}v \\ &= \int_{\lfloor n^{\epsilon} \rfloor / n}^{\infty} V_{\alpha}(c_{\lfloor nv \rfloor}, c_{\lfloor nv \rfloor + n}) \mathrm{d}nv \\ &= nU^{\alpha}(n) \int_{\lfloor n^{\epsilon} \rfloor / n}^{\infty} V_{\alpha}(U^{-1}(n) c_{\lfloor nv \rfloor}, U^{-1}(n) c_{\lfloor nv \rfloor + n}) \mathrm{d}v. \end{aligned}$$

Let us show that $V_{\alpha}(U^{-1}(n) c_{\lfloor nv \rfloor}, U^{-1}(n) c_{\lfloor nv \rfloor + n})$ converges point-wise and is dominated by an integrable function.

For $v > 0$, due to Lemma 2.4, we have

$$\frac{c_{\lfloor nv \rfloor}}{U(n)} = \frac{c_{\lfloor nv \rfloor}}{U(\lfloor nv \rfloor)} \frac{U(\lfloor nv \rfloor)}{U(n)} = \frac{c_{\lfloor nv \rfloor}}{U(\lfloor nv \rfloor)} \frac{\lfloor nv \rfloor^{-\kappa}}{n^{-\kappa}} \frac{h(n \frac{\lfloor nv \rfloor}{n})}{h(n)} \to v^{-\kappa},$$

and, similarly,

$$\frac{c_{\lfloor nv \rfloor + n}}{U(n)} \to (1+v)^{-\kappa},$$

as $n \to \infty$. As the function V_{α} is continuous, we get

$$V_{\alpha}\left(\frac{c_{\lfloor nv \rfloor}}{U(n)}, \frac{c_{\lfloor nv \rfloor + n}}{U(n)} \right) \to V_{\alpha}(v^{-\kappa}, (1+v)^{-\kappa}).$$

Using inequality $|V_{\alpha}(x, y)| \leqslant |x|^{\alpha - 1} |y|$ we obtain

$$\left| V_{\alpha}\left(\frac{c_{\lfloor nv \rfloor}}{U(n)}, \frac{c_{\lfloor nv \rfloor + n}}{U(n)} \right) \right| \leqslant \left| \frac{c_{\lfloor nv \rfloor}}{U(n)} \right|^{\alpha - 1} \left| \frac{c_{\lfloor nv \rfloor + n}}{U(n)} \right| = \left| \frac{c_{\lfloor nv \rfloor}}{U(n)} \right|^{\alpha} \left| \frac{c_{\lfloor nv \rfloor + n}}{c_{\lfloor nv \rfloor}} \right|.$$

Since $c_i \sim U(i)$, there exists N such that for $i \geqslant N$ the following inequality holds

$$\frac{1}{2} U(i) \leqslant c_i \leqslant 2U(i),$$

therefore, if $\lfloor nv \rfloor \geqslant N$,

$$\begin{aligned} &\left| V_{\alpha}\left(\frac{\lfloor nv \rfloor}{U(n)}, \frac{c_{\lfloor nv \rfloor + n}}{U(n)} \right) \right| \\ &\qquad \leqslant \left| \frac{2U(\lfloor nv \rfloor)}{U(n)} \right|^{\alpha} \left| \frac{2U(\lfloor nv \rfloor + n)}{\frac{1}{2} U(\lfloor nv \rfloor)} \right| = 2^{\alpha + 2} \left(\frac{U(\lfloor nv \rfloor)}{U(n)} \right)^{\alpha} \frac{U(\lfloor nv \rfloor + n)}{U(\lfloor nv \rfloor)}. \end{aligned}$$

We apply Theorem 2.1 to $f = U$, $A = 2$ and $\delta = \min(\kappa - 1/\alpha, 1 - \kappa(\alpha - 1))/4$ and obtain the existence of $B > 0$ such that for $x, y > B$

$$\frac{U(y)}{U(x)} \leqslant 2\max\left(\left(\frac{y}{x}\right)^{-\kappa+\delta}, \left(\frac{y}{x}\right)^{-\kappa-\delta}\right) \leqslant 2\left(\frac{y}{x}\right)^{-\kappa}\left(\left(\frac{y}{x}\right)^{\delta} + \left(\frac{y}{x}\right)^{-\delta}\right).$$

Consequently, if $n^\epsilon > \max(N, B) + 1$, we have

$$\begin{aligned}
&\left|V_\alpha\left(\frac{\lfloor nv\rfloor}{U(n)}, \frac{c_{\lfloor nv\rfloor+n}}{U(n)}\right)\right| \mathbb{1}_{\{\lfloor nv\rfloor > \lfloor n^\epsilon\rfloor\}} \\
&\qquad \leqslant 2^{\alpha+2}\mathbb{1}_{\{\lfloor nv\rfloor > \lfloor n^\epsilon\rfloor\}}\left(2\left(\left(\frac{\lfloor nv\rfloor}{n}\right)^{-\kappa+\delta} + \left(\frac{\lfloor nv\rfloor}{n}\right)^{-\kappa-\delta}\right)\right)^{\alpha} \times \\
&\qquad\qquad \times 2\left(\left(\frac{\lfloor nv\rfloor + n}{\lfloor nv\rfloor}\right)^{-\kappa+\delta} + \left(\frac{\lfloor nv\rfloor + n}{\lfloor nv\rfloor}\right)^{-\kappa-\delta}\right) \\
&\leqslant 2^{2\alpha+3}\left(\left(\left(\frac{v}{2}\right)^{-\kappa+\delta} + \left(\frac{v}{2}\right)^{-\kappa-\delta}\right)\right)^{\alpha}\left(\left(\frac{v+2}{2v}\right)^{-\kappa+\delta} + \left(\frac{v+2}{2v}\right)^{-\kappa-\delta}\right),
\end{aligned}$$

since for $\lfloor nv\rfloor \geqslant 2$ we have $\lfloor nv\rfloor \geqslant nv/2$. Let us denote the dominating function as $G(v)$. Function G is continuous on $(0, \infty)$, as $v \to \infty$ we have $G(v) = O(v^{(-\kappa+\delta)\alpha})$, and, as $v \downarrow 0$, we have $G(v) = O(v^{(-\kappa-\delta)\alpha+\kappa-\delta})$. We conclude that G is integrable, since $(-\kappa+\delta)\alpha < -1$ and $(-\kappa-\delta)\alpha+\kappa-\delta > -1$ by the choice of δ.

The dominated convergence theorem implies

$$\frac{S_{3,n}}{nU^\alpha(n)} \to \int_0^\infty V_\alpha(v^{-\kappa}, (1+v)^{-\kappa})\mathrm{d}v,$$

as $n \to \infty$. Relation $\rho(n) = S_{1,n} + S_{2,n} + S_{3,n}$ and (4.46) yields (4.17).

It remains to consider the case $1 < \alpha \leq 2$, $\kappa = 1/(\alpha - 1)$. We split $\rho(n)$ into three parts:

$$\rho(n) = V_\alpha(c_0, c_n) + S_{1,n} + S_{2,n},$$

where $S_{1,n} = \sum_{i=1}^{n-1} V_\alpha(c_i, c_{i+n})$ and $S_{2,n} = \sum_{i=n}^{\infty} V_\alpha(c_i, c_{i+n})$.

We have

$$|V_\alpha(c_0, c_n)| \leqslant |c_0|^{\alpha-1}|c_n| = O(n^{-\kappa}),$$

therefore

$$\frac{V_\alpha(c_0, c_n)}{n^{-\kappa}\ln(n)} \to 0,$$

as $n \to \infty$.

Let us now show that $S_{1,n}/(n^{-\kappa}\ln(n)) \to 1$. We have

$$\begin{aligned}
S_{1,n} &= \sum_{i=1}^{n} V_\alpha(c_i, c_{i+n}) = \int_1^n V_\alpha(c_{\lfloor v \rfloor}, c_{\lfloor v \rfloor + n}) \mathrm{d}v \\
&= \int_0^1 V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) \mathrm{d}n^v = \int_0^1 V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) n^v \ln(n) \mathrm{d}v \\
&= n^{-\kappa} \ln(n) \int_0^1 n^\kappa V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) n^v \mathrm{d}v.
\end{aligned}$$

We continue by investigating the point-wise convergence of $n^\kappa V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) n^v$ for $v \in (0,1)$. Since $c_i \sim i^{-\kappa}$, we have

$$\begin{aligned}
n^\kappa V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) n^v &= \frac{n^\kappa c_{\lfloor n^v \rfloor} c_{\lfloor n^v \rfloor + n} n^v}{\left(c_{\lfloor n^v \rfloor}^2 + c_{\lfloor n^v \rfloor + n}^2\right)^{\frac{2-\alpha}{2}}} \\
&\sim \frac{n^\kappa \lfloor n^v \rfloor^{-\kappa} n^{-\kappa} n^v}{\left(\lfloor n^v \rfloor^{-2\kappa}\right)^{\frac{2-\alpha}{2}}} = \lfloor n^v \rfloor^{-\kappa(\alpha-1)} n^v = \lfloor n^v \rfloor^{-1} n^v \to 1,
\end{aligned}$$

as $n \to \infty$.

Let us show that $n^\kappa V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) n^v$ is dominated by an integrable function on $(0,1)$. As before,

$$\begin{aligned}
n^\kappa \left| V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) \right| n^v &\leqslant n^{\kappa+v} \left| c_{\lfloor n^v \rfloor} \right|^{\alpha-1} \left| c_{\lfloor n^v \rfloor + n} \right| \\
&\leqslant n^{\kappa+v} E^\alpha \lfloor n^v \rfloor^{-\kappa(\alpha-1)} (\lfloor n^v \rfloor + n)^{-\kappa} \leqslant n^v E^\alpha \lfloor n^v \rfloor^{-\kappa(\alpha-1)} \\
&= E^\alpha \frac{n^v}{\lfloor n^v \rfloor} \leqslant 2E^\alpha.
\end{aligned}$$

Constant function is integrable on a finite interval, therefore, the dominated convergence theorem implies

$$\frac{S_{1,n}}{n^{-\kappa}\ln(n)} = \int_0^1 n^\kappa V_\alpha(c_{\lfloor n^v \rfloor}, c_{\lfloor n^v \rfloor + n}) n^v \mathrm{d}v \to 1,$$

as $n \to \infty$.

We continue by investigating $S_{2,n}$. We have

$$\begin{aligned}|S_{2,n}| \leqslant \sum_{i=n}^{\infty} |V_\alpha(c_i, c_{i+n})| \leqslant \sum_{i=n}^{\infty} |c_i|^{\alpha-1} |c_{i+n}| \\ \leqslant E^\alpha \sum_{i=n}^{\infty} i^{-\kappa(\alpha-1)}(i+n)^{-\kappa} \\ = E^\alpha \int_n^\infty \lfloor v \rfloor^{-1} (\lfloor v \rfloor + n)^{-\kappa} \mathrm{d}v \\ \leqslant E^\alpha \int_n^\infty \left(\frac{v}{2}\right)^{-1} \left(\frac{v}{2} + n\right)^{-\kappa} \mathrm{d}v \\ = E^\alpha 2^{1+\kappa} \int_1^\infty (nv)^{-1} (nv + 2n)^{-\kappa} \mathrm{d}nv \\ = E^\alpha 2^{1+\kappa} n^{-\kappa} \int_1^\infty v^{-1}(v+2)^{-\kappa} \mathrm{d}v.\end{aligned}$$

The integral is finite, therefore

$$\frac{S_{2,n}}{n^{-\kappa}\ln(n)} \to 0,$$

as $n \to \infty$.

In conclusion we see that

$$\frac{\rho(n)}{n^{-\kappa}\ln(n)} = \frac{V_\alpha(c_0, c_n) + S_{1,n} + S_{2,n}}{n^{-\kappa}\ln(n)} \to 1,$$

as $n \to \infty$. The proof is complete.

□

Proof of Theorem 4.8. Inequality $|g(s_1, s_2)| \leqslant C |s_1 s_2|^{\alpha/2}$ implies that

$$|\rho(g; X_1, X_2)| \leqslant C \int_E |s_1 s_2|^{\alpha/2} \Gamma(\mathrm{d}\mathbf{s}) = C \int_{-\infty}^{+\infty} |f(-x)f(t-x)|^{\frac{\alpha}{2}} \mathrm{d}x. \quad (4.47)$$

We need to show that

$$r(t) = \int_{-\infty}^{+\infty} g(x)g(x-t)\mathrm{d}x \to 0, \quad t \to \infty,$$

where $g(x) = |f(-x)|^{\alpha/2}$. Notice that $g \in L_2$, since $f \in L_\alpha$. Let us define an operator $B_t : L_2 \to L_2$ by $B_t g(x) = g(x-t)$. Now $r(t)$ can be written as

$$r(t) = \langle g, B_t g \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in L_2.

Fix $0 < \varepsilon < 1$. Let us denote $M := \max(1, \|g\|_{L_2})$. It is well known that the family of step functions S is dense in L_2. Therefore we can find a step function $s \in S$ such that $\|g - s\|_{L_2} \leqslant \frac{\varepsilon}{3M}$. With the help of Cauchy-Schwarz inequality it is easy to show that the inequality

$$0 \leqslant |\langle g, B_t g \rangle| \leqslant \varepsilon + |\langle s, B_t s \rangle|$$

holds. Since s is a step function, there exists $K > 0$ such that $s(x) = 0$ for $|x| > K$. Thus $\langle s, B_t s \rangle = 0$ and $\langle g, B_t g \rangle \leqslant \varepsilon$ provided that $t > 2K$. This means that $\langle g, B_t g \rangle \to 0$ as $t \to \infty$. This and inequality (4.47) imply that $\rho(g; X_1, X_2) \to 0, t \to \infty$. The theorem is proved. □

Proof of Theorem 4.10. We must investigate the asymptotic behaviour of the integral

$$\rho(t) = \int_{-\infty}^{\infty} V_\alpha(f_0(x), f_t(x)) \mathrm{d}x.$$

Denoting $a_0 = -\infty,\ a_1 = 0,\ a_2 = 1,\ a_3 = t,\ a_4 = t+1,\ a_5 = \infty$, we have

$$\rho(t) = \sum_{i=1}^{5} I_i(t), \text{ where } I_i(t) = \int_{a_{i-1}}^{a_i} V_\alpha(f_0(x), f_t(x)) \mathrm{d}x. \tag{4.48}$$

Simple change of variables allows to verify that $I_5(t) = I_1(t), \quad I_4(t) = I_2(t)$, therefore it is sufficient to consider $I_1(t),\ I_2(t),\ I_3(t)$ only. Using the change of variables again and denoting

$$g_t(y) := tV_\alpha \left(\left(\ln \left(1 + (ty)^{-1}\right), \ln\left(1 + (t(1+y))^{-1}\right) \right), \ y \in (0; \infty), \right.$$

we have

$$I_1(t) = \int_0^\infty g_t(y) \mathrm{d}y.$$

For a fixed $y \in (0; \infty)$

$$t \ln \left(1 + (ty)^{-1}\right) \to y^{-1}, \quad t \ln \left(1 + (t(1+y))^{-1}\right) \to (1+y)^{-1}, \ t \to \infty,$$

therefore one can easily see that for a fixed $y \in (0; \infty)$

$$g_t(y) t^{\alpha - 1} \to g(y) := (y(1+y))^{-1} \left(y^{-2} + (1+y)^{-2}\right)^{\frac{\alpha-2}{2}}, \ t \to \infty.$$

Applying the elementary inequality $a^2 + b^2 \geqslant 2ab$ we have

$$g(y) \le g_0(y) := 2^{\frac{\alpha-2}{2}}(y(1+y))^{-\alpha/2},$$

therefore, taking into account that $\alpha \in (1;2)$, function g is integrable over interval $(0;\infty)$. Moreover, one can easily verify that for all $y \in (0;\infty)$ and $t \ge 1$

$$t^{\alpha-1} g_t(y) \le g_0(y),$$

thus, applying the dominated convergence theorem, we get

$$\lim_{t\to\infty} \frac{I_1(t)}{t^{1-\alpha}} = \lim_{t\to\infty} \int_0^\infty \frac{g_t(y)}{t^{1-\alpha}} \mathrm{d}y = \int_0^\infty \lim_{t\to\infty} \frac{g_t(y)}{t^{1-\alpha}} \mathrm{d}y = \int_0^\infty g(y)\mathrm{d}y. \quad (4.49)$$

Now we show that the term $I_2(t)$ (and $I_4(t)$, as well) is of the smaller order than $I_1(t)$. Namely, it is not difficult to get the estimate

$$|I_2(t)| \leqslant \int_0^1 \left|\ln \frac{1-x}{x}\right|^{\frac{\alpha}{2}} \left|\ln \frac{t+1-x}{t-x}\right|^{\frac{\alpha}{2}} \mathrm{d}x \leqslant z(t),$$

where $z(t) = (t-1)^{-\alpha/2} \int_0^1 \left|\ln(1-x)x^{-1}\right|^{\frac{\alpha}{2}} \mathrm{d}x$. Hence,

$$0 \leqslant \frac{I_2(t)}{t^{1-\alpha}} \leqslant z(t)t^{\alpha-1} \leqslant C_1 t^{\frac{\alpha}{2}-1} \to 0, \text{ as } t \to \infty, \text{ and } I_2(t) = o(t^{1-\alpha}). \quad (4.50)$$

It remains to investigate the term $I_3(t)$ and its investigation is very similar to that of $I_1(t)$, therefore we shall provide only the main steps. As above, it is possible to write

$$I_3(t) = \int_0^1 h_{t-1}(y)\mathrm{d}y,$$

where

$$h_t(y) = -tV_\alpha\left(\ln\left(1+(ty)^{-1}\right), \ln\left(1+(t(1-y))^{-1}\right)\right).$$

Then we prove that, for all $y \in (0;1)$, $h_t(y)t^{1-\alpha} \to h(y)$, as $t \to \infty$, where $h(y) = -V_\alpha(y^{-1},(1-y)^{-1})$. This limit function is integrable over interval $(0;1)$, and $\left|h_t(y)t^{\alpha-1}\right| \leqslant (y(1-y))^{-\alpha/2}$. Therefore, applying the dominated convergence theorem we get

$$\lim_{t\to\infty}\frac{I_3(t)}{t^{1-\alpha}} = \lim_{t\to\infty}\left(\frac{t-1}{t}\right)^{1-\alpha}\int_0^1 \frac{h_{t-1}(y)}{(t-1)^{1-\alpha}}\mathrm{d}y$$
$$= \int_0^1 \lim_{t\to\infty}\frac{h_t(y)}{t^{1-\alpha}}\mathrm{d}y = \int_0^1 h(y)\mathrm{d}y. \quad (4.51)$$

Collecting the relations (4.48), (4.49), (4.50), and (4.51), we get the assertion (4.21) with (4.22). The theorem is proved. □

Proof of Theorem 4.12. The proof of this theorem is similar to that one of the previous theorem, only it is more complicated due to the fact that the function $f_t(x)$ is more complicated. Again, we must investigate the asymptotic behaviour of the integral

$$\rho(t) = \int_{-\infty}^{\infty} V_\alpha\left(f_0(x), f_t(x)\right)\mathrm{d}x. \quad (4.52)$$

It follows from (2.18) that

$$f_t(x) = \begin{cases} a\left((t+1-x)^\eta - (t-x)^\eta\right) & \text{, for } x \leqslant t, \\ a(t+1-x)^\eta - b(x-t)^\eta & \text{, for } t < x \leqslant t+1, \\ b\left((x-t-1)^\eta - (x-t)^\eta\right) & \text{, for } t+1 < x. \end{cases}$$

Without loss of generality we assume that $t \geq 2$. As in the proof of the previous theorem we shall use the subdivision of the integral in (4.52) into five integrals, as it is done in (4.48). Using some changes of variables we can show the following two equalities

$$I_5^{(a,b)}(t) = I_1^{(b,a)}(t), \quad I_4^{(a,b)}(t) = I_2^{(b,a)}(t), \quad (4.53)$$

therefore it remains to investigate the asymptotic behaviour of $I_1^{(a,b)}(t)$, $I_2^{(a,b)}(t)$, and $I_3^{(a,b)}(t)$. As it was mentioned, this behaviour is different for different values of possible main parameters α and H. We start with asymptotic behaviour of the terms $I_i^{(a,b)}(t)$, $i = 1,2,3$ in the region S. Elementary calculations show that

$$I_1^{(a,b)}(t) = t^{\alpha H - \alpha}\left|a\right|^\alpha \int_0^\infty h_t(x)\mathrm{d}x,$$

where $h_t(x) := V_\alpha\left(g_t(x), g_t(x+1)\right), \quad g_t(x) = t\left((t^{-1}+x)^\eta - x^\eta\right)$. It is evident that $g_t(x) \to \eta x^{\eta-1}$, as $t\to\infty$, therefore

$$h_t(x) \xrightarrow{t\to\infty} |\eta|^\alpha h(x) := |\eta|^\alpha V_\alpha\left(x^{\eta-1}, (x+1)^{\eta-1}\right).$$

Since

$$h(x) \sim \begin{cases} C_3 x^{(\eta-1)(\alpha-1)} & \text{, for } x\to 0, \\ C_4 x^{(\eta-1)\alpha} & \text{, for } x\to\infty, \end{cases}$$

then, in order for h to be integrable over $(0,\infty)$, the following conditions must hold:

$$(\eta-1)(\alpha-1) > -1, \quad (\eta-1)\alpha < -1. \tag{4.54}$$

It is not difficult to verify that for $(\alpha, H)\in S$ both inequalities are valid: to see that the second one holds it is sufficient to note that $(\eta-1)\alpha = (H-1)\alpha - 1 < -1$, while for the first inequality one must separately consider the cases $0<\alpha\leqslant 1$ and $1<\alpha<2$.

In order to apply the dominated convergence theorem, we must bound h_t by integrable function. By the mean value theorem one can write $g_t(x) = (\eta)(x+c)^{\eta-1}$, here $c\in(0;\frac{1}{t})$ is intermediate value, therefore, we have

$$|\eta|(x+1)^{\eta-1} \leqslant |g_t(x)| \leqslant |\eta| x^{\eta-1}. \tag{4.55}$$

Now we can estimate $|h_t(x)|$:

$$|h_t(x)| \leqslant |g_t(x) g_t(x+1)| \left(g_t^2(x)\right)^{(\alpha-2)/2} = |g_t(x)|^{\alpha-1} |g_t(x+1)|,$$

and using (4.55) we get $|h_t(x)| \le h_0(x)$, where

$$h_0(x) \le \begin{cases} |\eta|^\alpha (x+1)^{(\eta-1)\alpha} & \text{, for } 0<\alpha\le 1, \\ |\eta|^\alpha x^{(\eta-1)(\alpha-1)}(x+1)^{\eta-1} & \text{, for } 1<\alpha<2, \end{cases}$$

The function h_0 is integrable over over $(0,\infty)$ due to inequalities (4.54), which, as we had seen, are valid in the domain S. Thus, we can apply the dominated convergence theorem, and we get

$$\frac{I_1^{(a,b)}(t)}{|a|^\alpha t^{\alpha H-\alpha}} = \int_0^\infty h_t(x)\mathrm{d}x \xrightarrow{t\to\infty} \int_0^\infty h(x)\mathrm{d}x.$$

Therefore, taking into account (4.53) we get

$$I_1^{(a,b)}(t) \sim |a|^{\alpha}\, t^{\alpha H-\alpha} \int_0^{\infty} h(x)\mathrm{d}x, \quad t \to \infty, \tag{4.56}$$

$$I_5^{(a,b)}(t) \sim |b|^{\alpha}\, t^{\alpha H-\alpha} \int_0^{\infty} h(x)\mathrm{d}x, \quad t \to \infty. \tag{4.57}$$

The next step is to show that

$$I_j^{(a,b)}(t) = o(t^{\alpha H-\alpha}), \quad t \to \infty, j = 2, 4, \tag{4.58}$$

and this is achieved by estimating the term $I_2^{(a,b)}(t)$ as follows

$$|I_2^{(a,b)}(t)| \leqslant (t-1)^{\eta-1}\, |a|\, |\eta| \int_0^1 |(a(1-x)^{\eta} - bx^{\eta})|^{\alpha-1}\, \mathrm{d}x.$$

Note that for parameters $(\alpha, H) \in S$ the integral in the last inequality is finite (for this it is sufficient to consider the behaviour of the integrand at the endpoints of the interval $(0;1)$ in the area S_2 and at the point where $a(1-x)^{\eta} - bx^{\eta} = 0$ in S_1) and $\eta - 1 - \alpha H + \alpha < 0$. It remains to investigate $I_3^{(a,b)}(t)$, and for the convenience we consider $I_3^{(a,b)}(t+1)$. Simple considerations lead to the equality

$$I_3^{(a,b)}(t+1) = -t^{\alpha H-\alpha} \int_0^1 k_t(x)\mathrm{d}x,$$

where $k_t(x) := V_{\alpha}\left(ag_t(1-x), bg_t(x)\right)$, and the function g_t was defined when considering $I_1^{(a,b)}(t)$. Now it is easy to see that

$$k_t(x) \xrightarrow{t\to\infty} k(x) := |\eta|^{\alpha}\, V_{\alpha}\left(a(1-x)^{\eta-1}, bx^{\eta-1}\right),$$

and the function on the left-hand side of the last relation is integrable (this can be verified as above). A little bit more complicated is majorizing of $k_t(x)$ by integrable function. We choose different majorizing functions for $x \in (0, 1/2)$ and for $x \in (1/2, 1)$. Namely, for $x \in (0, 1/2)$ we use the estimate

$$\begin{aligned}|k_t(x)| &\leqslant |ag_t(1-x)bg_t(x)| \left(b^2 g_t^2(x)\right)^{(\alpha-2)/2} \\ &= |a|\, |b|^{\alpha-1}\, |g_t(1-x)|\, |g_t(x)|^{\alpha-1}.\end{aligned}$$

For $1 < \alpha < 2$ we use the estimate

$$|a|\,|b|^{\alpha-1}\,|g_t(1-x)|\,|g_t(x)|^{\alpha-1} \leqslant |a|\,|b|^{\alpha-1}\,|\eta|^{\alpha}\,(1-x)^{\eta-1}x^{(\eta-1)(\alpha-1)},$$

while for $0 < \alpha \leqslant 1$ we estimate

$$|a|\,|b|^{\alpha-1}\,|g_t(1-x)|\,|g_t(x)|^{\alpha-1} \leqslant |a|\,|b|^{\alpha-1}\,|\eta|^{\alpha}\,(1-x)^{\eta-1}(1+x)^{(\eta-1)(\alpha-1)}.$$

Again, taking into account (4.54) we can verify that both majorizing functions are integrable over $(0,1/2)$. For $x \in (1/2,1)$ we use the estimate

$$|k_t(x)| \leqslant \frac{|ag_t(1-x)bg_t(x)|}{\left(a^2g_t^2(1-x)\right)^{\frac{2-\alpha}{2}}} = |b|\,|a|^{\alpha-1}\,|g_t(1-x)|^{\alpha-1}\,|g_t(x)|\,,$$

and then in a similar manner we construct the majorizing function, which we do not provide here. Finally, applying the dominated convergence theorem, we get

$$I_3^{(a,b)}(t+1) \sim -\,|\eta|^{\alpha}\left(t^{\alpha H-\alpha}\right)\int_0^1 k(x)\mathrm{d}x. \tag{4.59}$$

Collecting relations (4.56)-(4.59) we get (4.23) and the constant C_1 has the following expression

$$C_1(a,b,\alpha,H) = |\eta|\left((|a|^{\alpha}+|b|^{\alpha})\int_0^{\infty} h(x)\mathrm{d}x - \int_0^{\infty} k(x)\mathrm{d}x\right). \tag{4.60}$$

Now it remains to investigate the asymptotic behaviour of $\rho(t)$ in the region U, and since the scheme of investigation is the same, we provide only the main steps. We use the same division of the integral (4.48) and again it is sufficient to investigate three terms $I_j^{(a,b)},\ j=1,2,3$. The first term we can express as follows:

$$I_1^{(a,b)}(t) = |a|^{\alpha}\int_0^{\infty} u_t(x)\mathrm{d}x,$$

where $u_t(x) = V_{\alpha}\left(v(x), t^{\eta-1}g_{1,t}(x)\right)$, $v(x) = (1+x)^{\eta} - x^{\eta}$, $g_{1,t}(x) = t\left((1+\frac{1+x}{t})^{\eta} - (1+\frac{x}{t})^{\eta}\right)$. It is easy to see that $g_{1,t}(x) \xrightarrow{t\to\infty} \eta$, therefore

$$u_t(x)t^{1-\eta} \xrightarrow{t\to\infty} u(x),$$

where $u(x) = -\eta(-v(x))^{\alpha-1} = -\eta\left(x^{\eta} - (1+x)^{\eta}\right)^{\alpha-1}$. It is not difficult to make sure that for $(\alpha, H) \in U$ the function u is integrable over $(0, \infty)$. It remains to show that the family of functions $\left\{u_t(\cdot)t^{1-\eta}, t \geqslant 2\right\}$ is bounded by integrable function. To this aim we use the inequality $|\eta^{-1}u_t(x)| \leq |v(x)|^{\alpha-1}|g_{1,t}(x)|$ and, noting that $|g_{1,t}(x)| \leq |\eta|$, we get

$$|\eta^{-1}u_t(x)| \leq |\eta||v(x)|^{\alpha-1}.$$

We got integrable majorizing function, therefore, applying the dominated convergence theorem, we get

$$I_1^{(a,b)}(t) \sim t^{\eta-1}\,|a|^{\alpha}\left(\frac{1}{\alpha} - H\right)\int_0^{\infty}\left(x^{\eta} - (1+x)^{\eta}\right)^{\alpha-1}\mathrm{d}x, \tag{4.61}$$

$$I_5^{(a,b)}(t) \sim t^{\eta-1}\,|b|^{\alpha}\left(\frac{1}{\alpha} - H\right)\int_0^{\infty}\left(x^{\eta} - (1+x)^{\eta}\right)^{\alpha-1}\mathrm{d}x. \tag{4.62}$$

In contrast to the region S, in U the term $I_2^{(a,b)}(t)$ has impact to the asymptotic behaviour of $\rho(t)$. We can write

$$I_2^{(a,b)}(t) = \int_0^1 z_t(x)\mathrm{d}x,$$

where $z_t(x) = V_{\alpha}\left(v(x,a,b), a g_{2,t}(x)\right)$, $v(x,a,b) = a(1-x)^{\eta} - bx^{\eta}$, $g_{2,t}(x) = (t+1-x)^{\eta} - (t-x)^{\eta}$. Then we perform the same steps as above: first we note that

$$z_t(x)t^{1-\eta} \xrightarrow{t\to\infty} \eta a\left(a(1-x)^{\eta} - bx^{\eta}\right)^{<\alpha-1>},$$

and the limit function is integrable over $(0,1)$. This follows from the fact that, due to $\alpha - 1 > 0$, the only points where function $|a(1-x)^{\eta} - bx^{\eta}|^{\alpha-1}$ is unbounded are the endpoints of the interval, and the integrability of this function follows from the inequality $\eta(\alpha-1) > -1$, which holds in the region U. Taking into account that for $x \in (0,1)$, $t \geqslant 2$

$$\left|\frac{g_{2,t}(x)}{t^{\eta-1}}\right| = \frac{|\eta|}{t^{-1}}\int_0^{\frac{1}{t}}\left(1 - \frac{x}{t} + y\right)^{\eta-1}\mathrm{d}y \leqslant |\eta|\left(\frac{1}{2}\right)^{\eta-1},$$

we get the majorant

$$\left|z_t(x)t^{1-\eta}\right| \leqslant 2^{1-\eta}\,|a|\,|\eta|\,|a(1-x)^{\eta} - bx^{\eta}|^{\alpha-1}.$$

Again, applying the dominated convergence theorem, we get

$$I_2^{(a,b)}(t) \sim t^{\eta-1}\eta a \int_0^1 \left(a(1-x)^\eta - bx^\eta\right)^{<\alpha-1>} \mathrm{d}x, \tag{4.63}$$

$$I_4^{(a,b)}(t) \sim t^{\eta-1}\eta b \int_0^1 \left(b(1-x)^\eta - ax^\eta\right)^{<\alpha-1>} \mathrm{d}x. \tag{4.64}$$

It remains to investigate the third term, and as above, we shall consider $I_3^{(a,b)}(t+1)$, which can be written as

$$I_3^{(a,b)}(t+1) = \int_1^{t+1} V_\alpha\left(ag_{2,t+1}(x), bv_1(x)\right) \mathrm{d}x,$$

where $v_1(x) = (x-1)^\eta - x^\eta$.

Now it is convenient to divide interval $(1, t+1)$ into two intervals $(1, (t+2)/2), ((t+2)/2, t+1)$. Then the above written integral we divide into two integrals, and we get

$$I_3^{(a,b)}(t+1) = I_{3,1}^{(a,b)}(t+1) + I_{3,2}^{(a,b)}(t+1). \tag{4.65}$$

Simple change of variables $x = t+2-y$ shows that

$$I_{3,2}^{(a,b)}(t+1) = I_{3,1}^{(b,a)}(t+1), \tag{4.66}$$

therefore it is sufficient to consider only $I_{3,1}^{(a,b)}(t+1)$. Denoting

$$w_t(x) = \frac{g_{2,t+1}(x)v_1(x)\mathbb{1}_{(1;\frac{t+2}{2})}(x)}{\left(a^2 g_{2,t+1}^2(x) + b^2 v_1^2(x)\right)^{\frac{2-\alpha}{2}}},$$

we can write

$$I_{3,1}^{(a,b)}(t+1) = -ab\int_1^\infty w_t(x)\mathrm{d}x. \tag{4.67}$$

The investigation of the last integral goes along the same lines as for previous integrals: we have

$$w_t(x)t^{1-\eta} \xrightarrow{t\to\infty} |b|^{\alpha-2}(-\eta)\left((x-1)^\eta - x^\eta\right)^{\alpha-1},$$

and we construct the integrable majorizing function. Denoting $g_3(y) := g_{3,t,x}(y) = (1 + (1-x)/t + y)^\eta$ we have

$$\left|w_t(x)t^{1-\eta}\right| \leqslant |b|^{\alpha-2}\, t(g_3(0) - g_3(1/t))v_1^{\alpha-1}(x).$$

Estimating for $x \in (1; \frac{t+2}{2})$ (we recall that outside of this interval function w_t vanishes)

$$|g_3(0) - g_3(1/t)| = \left| \int_0^{1/t} g_3'(y) \mathrm{d}y \right| \leqslant |\eta| t^{-1} \left(1 + \frac{1-x}{t} \right)^{\eta - 1} \leqslant 2^{1-\eta} |\eta| t^{-1},$$

we get the integrable majorizing function:

$$\left| w_t(x) t^{1-\eta} \right| \leqslant |b|^{\alpha-2} |\eta| 2^{1-\eta} \left((x-1)^\eta - x^\eta \right)^{\alpha-1}.$$

Once more applying the dominated convergence theorem and taking into account (4.65)-(4.67), we get

$$\begin{aligned} & I_3^{(a,b)}(t+1) \\ & \qquad \sim \left(ab^{<\alpha-1>} + ba^{<\alpha-1>} \right) \eta t^{\eta-1} \int_1^\infty \left((x-1)^\eta - x^\eta \right)^{\alpha-1} \mathrm{d}x. \quad (4.68) \end{aligned}$$

Collecting relations (4.61)-(4.64) and (4.68) we get (4.24) with

$$C_2(a, b, \alpha, H) = -\eta \sum_{i=1}^4 d_i J_i, \qquad (4.69)$$

where $d_1 = |a|^\alpha + |b|^\alpha,\ d_2 = -a,\ d_3 = -b,\ d_4 = -ab^{<\alpha-1>} - ba^{<\alpha-1>}$,

$$J_1 = \int_0^\infty \left(x^\eta - (1+x)^\eta \right)^{\alpha-1} \mathrm{d}x, \quad J_2 = \int_0^1 \left(a(1-x)^\eta - bx^\eta \right)^{<\alpha-1>} \mathrm{d}x,$$

$$J_3 = \int_0^1 \left(b(1-x)^\eta - ax^\eta \right)^{<\alpha-1>} \mathrm{d}x, \quad J_4 = \int_1^\infty \left((x-1)^{H-\frac{1}{\alpha}} - x^{H-\frac{1}{\alpha}} \right)^{\alpha-1} \mathrm{d}x.$$

The theorem is proved. □

Proof of Theorem 4.15. We will not provide all of the proofs as this would make the thesis very long. Here we will investigate the following sets of parameters:

1. $1 < \alpha \leqslant 2,\ \beta_i > \frac{1}{\alpha-1},\ i = 1, 2;$
2. $1 < \alpha \leqslant 2,\ \frac{1}{\alpha} < \beta_i < \frac{1}{\alpha-1},\ i = 1, 2;$
3. $1 < \alpha \leqslant 2,\ \beta_1 > \frac{1}{\alpha-1},\ \frac{1}{\alpha} < \beta_2 < \frac{1}{\alpha-1};$
4. $1 < \alpha \leqslant 2,\ \beta_1 = \frac{1}{\alpha-1},\ \beta_2 = \frac{1}{\alpha-1};$

We feel that the provided proofs illustrate the ideas used, the other sets of parameters are dealt with in a similar way.

For convenience of writing let us assume that $w_{(i,j)} \neq 0$ for all $i, j \geqslant 0$. It then follows from assumptions (A1)-(A3) that there exist constants $d, e > 0$ such that

$$d < \left|w_{(i,j)}\right| < e, \ i, j \geqslant 0.$$

We need to investigate the asymptotic behaviour of

$$\rho(n,m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c_{i,j}, c_{i+n,j+m}) \tag{4.70}$$

as $\min(n,m) \to \infty$.

We begin with the case $1 < \alpha \leqslant 2, \ \beta_i > \frac{1}{\alpha-1}, \ i = 1, 2$. We have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |c_{i,j}|^{\alpha-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |w_{(i,j)}(1+i)^{-\beta_1}(1+j)^{-\beta_2}|^{\alpha-1}$$

$$\leqslant e^{\alpha-1}\sum_{i=0}^{\infty}(1+i)^{-\beta_1(\alpha-1)}\sum_{j=0}^{\infty}(1+j)^{-\beta_2(\alpha-1)} < \infty.$$

Let us show that

$$\frac{\rho(n,m)}{n^{-\beta_1}m^{-\beta_2}} \to \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}^{<\alpha-1>}, \tag{4.71}$$

as $n, m \to \infty$.

We have

$$\frac{c_{i+n,j+m}}{n^{-\beta_1}m^{-\beta_2}} = \frac{w_{(i+n,j+m)}(1+i+n)^{-\beta_1}(1+j+m)^{-\beta_2}}{n^{-\beta_1}m^{-\beta_2}} \to 1,$$

as $n, m \to \infty$, therefore

$$\frac{V_\alpha(c_{i,j}, c_{i+n,j+m})}{n^{-\beta_1}m^{-\beta_2}} = \frac{c_{i,j}\frac{c_{i+n,j+m}}{n^{-\beta_1}m^{-\beta_2}}}{\left(c_{i,j}^2 + c_{i+n,j+m}^2\right)^{\frac{2-\alpha}{2}}} \to c_{i,j}^{<\alpha-1>}.$$

Using the inequality $|V_\alpha(x,y)| \leqslant |x|^{\alpha-1}|y|$ we obtain

$$\frac{|V_\alpha(c_{i,j}, c_{i+n,j+m})|}{n^{-\beta_1}m^{-\beta_2}} \leqslant |c_{i,j}|^{\alpha-1}\frac{|c_{i+n,j+m}|}{n^{-\beta_1}m^{-\beta_2}}$$
$$< |c_{i,j}|^{\alpha-1}\frac{e(1+i+n)^{-\beta_1}(1+j+m)^{-\beta_2}}{n^{-\beta_1}m^{-\beta_2}} < e\,|c_{i,j}|^{\alpha-1}.$$

Since $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|c_{i,j}|^{\alpha-1}<\infty$, the dominated convergence theorem applies yielding (4.71).

We continue with the case $1<\alpha\leqslant 2,\ \frac{1}{\alpha}<\beta_i<\frac{1}{\alpha-1},\ i=1,2.$ We will show that

$$\frac{\rho(n,m)}{n^{1-\alpha\beta_1}m^{1-\alpha\beta_2}} \to \int_0^\infty\int_0^\infty \frac{t^{-\beta_1}(t+1)^{-\beta_1}s^{-\beta_2}(s+1)^{-\beta_2}}{\left(t^{-2\beta_1}s^{-2\beta_2}+(t+1)^{-2\beta_1}(s+1)^{-2\beta_2}\right)^{\frac{2-\alpha}{2}}}\mathrm{d}t\mathrm{d}s. \quad (4.72)$$

We have

$$\begin{aligned}\rho(n,m) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c_{i,j},c_{i+n,j+m})\\ &= \int_0^\infty\int_0^\infty V_\alpha(c_{\lfloor t\rfloor,\lfloor s\rfloor},c_{\lfloor t\rfloor+n,\lfloor s\rfloor+m})\mathrm{d}t\mathrm{d}s\\ &= \int_0^\infty\int_0^\infty V_\alpha(c_{\lfloor nt\rfloor,\lfloor ms\rfloor},c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m})\mathrm{d}nt\mathrm{d}ms\\ &= n^{1-\alpha\beta_1}m^{1-\alpha\beta_2}\int_0^\infty\int_0^\infty V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor},n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m})\mathrm{d}t\mathrm{d}s.\end{aligned}$$

For fixed values of $t,s>0$ we have

$$n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor} = \frac{w_{(\lfloor nt\rfloor,\lfloor ms\rfloor)}(1+\lfloor nt\rfloor)^{-\beta_1}(1+\lfloor ms\rfloor)^{-\beta_2}}{n^{-\beta_1}m^{-\beta_2}} \to t^{-\beta_1}s^{-\beta_2},$$

as $n,m\to\infty$. Similarly

$$n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m} \to (t+1)^{-\beta_1}(s+1)^{-\beta_2}.$$

As the function V_α is continuous, we obtain

$$V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor},n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m}) \to V_\alpha(t^{-\beta_1}s^{-\beta_2},(t+1)^{-\beta_1}(s+1)^{-\beta_2}),$$

as $n,m\to\infty$.

We see that

$$n^{\beta_1}m^{\beta_2}\left|c_{\lfloor nt\rfloor,\lfloor ms\rfloor}\right| = \frac{\left|w_{(\lfloor nt\rfloor,\lfloor ms\rfloor)}\right|(1+\lfloor nt\rfloor)^{-\beta_1}(1+\lfloor ms\rfloor)^{-\beta_2}}{n^{-\beta_1}m^{-\beta_2}} < et^{-\beta_1}s^{-\beta_2},$$

and, similarly,

$$n^{\beta_1}m^{\beta_2}\left|c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m}\right| < e(t+1)^{-\beta_1}(s+1)^{-\beta_2},$$

thus

$$\begin{aligned}&\left|V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor},n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m})\right|\\&\qquad\leqslant\left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor}\right|^{\alpha-1}\left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor+m})\right|\\&\qquad\qquad< e^\alpha t^{-\beta_1(\alpha-1)}s^{-\beta_2(\alpha-1)}(t+1)^{-\beta_1}(s+1)^{-\beta_2}.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^\alpha t^{-\beta_1(\alpha-1)}s^{-\beta_2(\alpha-1)}(t+1)^{-\beta_1}(s+1)^{-\beta_2}\mathrm{d}t\mathrm{d}s\\&\qquad=e^\alpha\int_0^\infty t^{-\beta_1(\alpha-1)}(t+1)^{-\beta_1}\mathrm{d}t\int_0^\infty s^{-\beta_2(\alpha-1)}(s+1)^{-\beta_2}\mathrm{d}s<\infty,\end{aligned}$$

the dominated convergence theorem yields (4.72).

Next we investigate the case $1<\alpha\leqslant 2,\ \beta_1>\frac{1}{\alpha-1},\ \frac{1}{\alpha}<\beta_2<\frac{1}{\alpha-1}$. We have

$$\begin{aligned}\rho(n,m)&=\sum_{i=0}^\infty\sum_{j=0}^\infty V_\alpha(c_{i,j},c_{i+n,j+m})\\&=\sum_{i=0}^\infty\int_0^\infty V_\alpha(c_{i,\lfloor s\rfloor},c_{i+n,\lfloor s\rfloor+m})\mathrm{d}s\\&=\sum_{i=0}^\infty\int_0^\infty V_\alpha(c_{i,\lfloor ms\rfloor},c_{i+n,\lfloor ms\rfloor+m})\mathrm{d}ms\\&=n^{-\beta_1}m^{1-\alpha\beta_2}\sum_{i=0}^\infty\int_0^\infty n^{\beta_1}V_\alpha(m^{\beta_2}c_{i,\lfloor ms\rfloor},m^{\beta_2}c_{i+n,\lfloor ms\rfloor+m})\mathrm{d}s.\end{aligned}$$

The function under the integral equals

$$\begin{aligned}&n^{\beta_1}V_\alpha(m^{\beta_2}c_{i,\lfloor ms\rfloor},m^{\beta_2}c_{i+n,\lfloor ms\rfloor+m})\\&\qquad=\frac{m^{\beta_2}c_{i,\lfloor ms\rfloor}n^{\beta_1}m^{\beta_2}c_{i+n,\lfloor ms\rfloor+m}}{\left(m^{2\beta_2}c_{i,\lfloor ms\rfloor}^2+m^{2\beta_2}c_{i+n,\lfloor ms\rfloor+m}^2\right)^{\frac{2-\alpha}{2}}}.\end{aligned}\tag{4.73}$$

Recall that

$$m^{\beta_2}c_{i,\lfloor ms\rfloor}=m^{\beta_2}w_{(i,\lfloor ms\rfloor)}(1+i)^{-\beta_1}(1+\lfloor ms\rfloor)^{-\beta_2}\tag{4.74}$$

and

$$\begin{aligned}&n^{\beta_1}m^{\beta_2}c_{i+n,\lfloor ms\rfloor+m}\\&\qquad=n^{\beta_1}m^{\beta_2}w_{(i+n,\lfloor ms\rfloor+m)}(1+i+n)^{-\beta_1}(1+\lfloor ms\rfloor+m)^{-\beta_2}\end{aligned}\tag{4.75}$$

therefore, for fixed values of $i \in \mathbb{N}_0$ and $s > 0$

$$m^{\beta_2} c_{i,\lfloor ms \rfloor} \to w_{(i,\infty)} (1+i)^{-\beta_1} s^{-\beta_2},$$

$$n^{\beta_1} m^{\beta_2} c_{i+n,\lfloor ms \rfloor + m} \to (s+1)^{-\beta_2}.$$

Consequently, $m^{\beta_2} c_{i+n,\lfloor ms \rfloor + m} \to 0$, and from (4.73) we obtain

$$\begin{aligned} n^{\beta_1} V_\alpha(m^{\beta_2} c_{i,\lfloor ms \rfloor}, m^{\beta_2} c_{i+n,\lfloor ms \rfloor + m}) \\ \to \frac{w_{(i,\infty)}(1+i)^{-\beta_1} s^{-\beta_2} (s+1)^{-\beta_2}}{\left(w_{(i,\infty)} (1+i)^{-\beta_1} s^{-\beta_2} \right)^{2-\alpha}} \\ = w_{(i,\infty)}^{\alpha-1} (1+i)^{-\beta_1(\alpha-1)} s^{-\beta_2(\alpha-1)} (s+1)^{-\beta_2}. \end{aligned}$$

Let us show that $n^{\beta_1} \left| V_\alpha(m^{\beta_2} c_{i,\lfloor ms \rfloor}, m^{\beta_2} c_{i+n,\lfloor ms \rfloor + m}) \right|$ is dominated by a good function. From (4.74) and (4.75) we see that

$$m^{\beta_2} \left| c_{i,\lfloor ms \rfloor} \right| \leqslant e(1+i)^{-\beta_1} s^{-\beta_2}$$

and

$$n^{\beta_1} m^{\beta_2} \left| c_{i+n,\lfloor ms \rfloor + m} \right| \leqslant e(s+1)^{-\beta_2}.$$

We can now estimate

$$\begin{aligned} n^{\beta_1} \left| V_\alpha(m^{\beta_2} c_{i,\lfloor ms \rfloor}, m^{\beta_2} c_{i+n,\lfloor ms \rfloor + m}) \right| \\ \leqslant n^{\beta_1} \left(m^{\beta_2} \left| c_{i,\lfloor ms \rfloor} \right| \right)^{\alpha-1} m^{\beta_2} \left| c_{i+n,\lfloor ms \rfloor + m} \right| \\ \leqslant e^\alpha (1+i)^{-\beta_1(\alpha-1)} s^{-\beta_2(\alpha-1)} (1+s)^{-\beta_2}, \end{aligned}$$

and, since

$$\begin{aligned} \sum_{i=0}^{\infty} \int_0^\infty (1+i)^{-\beta_1(\alpha-1)} s^{-\beta_2(\alpha-1)} (1+s)^{-\beta_2} \mathrm{d}s \\ = \sum_{i=0}^{\infty} (1+i)^{-\beta_1(\alpha-1)} \int_0^\infty s^{-\beta_2(\alpha-1)} (1+s)^{-\beta_2} \mathrm{d}s < \infty, \end{aligned}$$

the dominated convergence theorem implies

$$\frac{\rho(n,m)}{n^{-\beta_1} m^{1-\beta_2\alpha}} \to \sum_{i=0}^{\infty} w_{(i,\infty)}^{\alpha-1} (1+i)^{-\beta_1(\alpha-1)} \int_0^\infty s^{-\beta_2(\alpha-1)} (1+s)^{-\beta_2} \mathrm{d}s,$$

as $n, m \to \infty$.

In the case $1 < \alpha \leqslant 2,\ \beta_1 = \frac{1}{\alpha-1},\ \beta_2 = \frac{1}{\alpha-1}$ we split $\rho(n,m)$ as follows:

$$\rho(n,m) = \sum_{k=0}^{4} S_{k,n,m},$$

where

$$S_{0,n,m} = \sum_{i=0 \text{ or } j=0} V_\alpha(c_{i,j}, c_{i+n,j+m}),$$

$$S_{1,n,m} = \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} V_\alpha(c_{i,j}, c_{i+n,j+m}),$$

$$S_{2,n,m} = \sum_{i=1}^{n-1} \sum_{j=m}^{\infty} V_\alpha(c_{i,j}, c_{i+n,j+m}),$$

$$S_{3,n,m} = \sum_{i=n}^{\infty} \sum_{j=m}^{\infty} V_\alpha(c_{i,j}, c_{i+n,j+m}),$$

$$S_{4,n,m} = \sum_{i=n}^{\infty} \sum_{j=1}^{m-1} V_\alpha(c_{i,j}, c_{i+n,j+m}).$$

Let us begin with $S_{1,n,m}$. We have

$$\begin{aligned} S_{1,n,m} &= \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} V_\alpha(c_{i,j}, c_{i+n,j+m}) \\ &= \int_1^n \int_1^m V_\alpha(c_{\lfloor t \rfloor, \lfloor s \rfloor}, c_{\lfloor t \rfloor + n, \lfloor s \rfloor + m}) \mathrm{d}s\mathrm{d}t \\ &= \int_0^1 \int_0^1 V_\alpha(c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor}, c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) \mathrm{d}m^s \mathrm{d}n^t \\ &= \ln(n)\ln(m) \int_0^1 \int_0^1 V_\alpha(c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor}, c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) m^s n^t \mathrm{d}s\mathrm{d}t \\ &= \ln(n)\ln(m) n^{-\beta_1} m^{-\beta_2} \times \\ &\times \int_0^1 \int_0^1 n^{\beta_1} m^{\beta_2} V_\alpha(c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor}, c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) m^s n^t \mathrm{d}s\mathrm{d}t. \end{aligned}$$

Let us now investigate the point-wise convergence of the function under the integral. We have

$$\begin{aligned} & n^{\beta_1} m^{\beta_2} V_\alpha(c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor}, c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) m^s n^t \\ &= \frac{m^s n^t c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor} n^{\beta_1} m^{\beta_2} c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}}{\left(c^2_{\lfloor n^t \rfloor, \lfloor m^s \rfloor} + c^2_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}\right)^{\frac{2-\alpha}{2}}}. \quad (4.76) \end{aligned}$$

Since $c_{k,l} \sim k^{-\beta_1} l^{-\beta_2}$, $\lfloor n^t \rfloor \sim n^t$, $\lfloor m^s \rfloor \sim m^s$, and for $t, s \in (0,1)$ also $\lfloor n^t \rfloor + n \sim n$ and $\lfloor m^s \rfloor + m \sim m$, we obtain

$$
\begin{aligned}
n^{\beta_1} m^{\beta_2} V_\alpha(c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor}, c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) m^s n^t \\
\sim \frac{m^s n^t n^{-\beta_1 t} m^{-\beta_2 s} n^{\beta_1} m^{\beta_2} n^{-\beta_1} m^{-\beta_2}}{\left(n^{-2\beta_1 t} m^{-2\beta_2 s}\right)^{\frac{2-\alpha}{2}}} \\
= m^s n^t n^{-\beta_1(\alpha-1)t} m^{-\beta_2(\alpha-1)s} = m^s n^t n^{-t} m^{-s} = 1.
\end{aligned}
$$

Let us show that absolute value of (4.76) is bounded by an integrable function. We have

$$
\begin{aligned}
n^{\beta_1} m^{\beta_2} \left| V_\alpha(c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor}, c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) \right| m^s n^t \\
\leqslant n^{\beta_1} m^{\beta_2} \left| c_{\lfloor n^t \rfloor, \lfloor m^s \rfloor} \right|^{\alpha-1} \left| c_{\lfloor n^t \rfloor + n, \lfloor m^s \rfloor + m}) \right| m^s n^t \\
< e^\alpha n^{\beta_1} m^{\beta_2} m^s n^t (1 + \left\lfloor n^t \right\rfloor)^{-\beta_1(\alpha-1)} (1 + \lfloor m^s \rfloor)^{-\beta_2(\alpha-1)} \times \\
\times (1 + \left\lfloor n^t \right\rfloor + n)^{-\beta_1} (1 + \lfloor m^s \rfloor + m)^{-\beta_2} \\
\leqslant e^\alpha n^{\beta_1} m^{\beta_2} m^s n^t n^{-t} m^{-s} n^{-\beta_1} m^{-\beta_2} = e^\alpha.
\end{aligned}
$$

A constant function is integrable on $(0,1)^2$, thus, the dominated convergence theorem applies yielding

$$
\frac{\rho(n,m)}{n^{-\beta_1} m^{-\beta_2} \ln(n) \ln(m)} \to \int_0^1 \int_0^1 1 \mathrm{d}s \mathrm{d}t = 1.
$$

Let us now examine the quantities

$$
Z_1(n,\beta) = \sum_{i=0}^{n-1} (1+i)^{-1} (1+i+n)^{-\beta},
$$

$$
Z_2(n,\beta) = \sum_{i=n}^{\infty} (1+i)^{-1} (1+i+n)^{-\beta}.
$$

We have

$$
\begin{aligned}
Z_1(n,\beta) \leqslant \sum_{i=0}^{n-1} (1+i)^{-1} n^{-\beta} = n^{-\beta} \left(1 + \sum_{i=1}^{n-1} (1+i)^{-1}\right) \\
= n^{-\beta} \left(1 + \int_1^n (1 + \lfloor v \rfloor)^{-1} \mathrm{d}v\right) \leqslant n^{-\beta} \left(1 + \int_1^n v^{-1} \mathrm{d}v\right) \\
= n^{-\beta} \left(1 + \ln(n)\right),
\end{aligned}
$$

$$Z_2(n,\beta) \leqslant \sum_{i=n}^{\infty}(1+i)^{-1-\beta} = \int_n^{\infty}(1+\lfloor v\rfloor)^{-1-\beta}\mathrm{d}v$$
$$\leqslant \int_n^{\infty} v^{-1-\beta}\mathrm{d}v = \frac{n^{-\beta}}{\beta},$$

therefore $Z_1(n,\beta) = O(n^{-\beta}\ln(n))$ and $Z_2(n,\beta) = O(n^{-\beta})$. We will employ these results to deal with $S_{i,n,m},\ i = 0,2,3,4$. We will apply inequality

$$|V_\alpha(c_{i,j}, c_{i+n,j+m})| \leqslant |c_{i,j}|^{\alpha-1}|c_{i+n,j+m}|$$
$$\leqslant e^{\alpha}(1+i)^{-1}(1+j)^{-1}(1+i+n)^{-\beta_1}(1+j+m)^{-\beta_2}$$

Let us begin with $S_{0,n,m}$:

$$|S_{0,n,m}| \leqslant \sum_{i=0}^{\infty}|V_\alpha(c_{i,0}, c_{i+n,m})| + \sum_{j=0}^{\infty}|V_\alpha(c_{0,j}, c_{n,j+m})|$$
$$\leqslant e^{\alpha}\sum_{i=0}^{\infty}(1+i)^{-1}(1+i+n)^{-\beta_1}(1+m)^{-\beta_2} +$$
$$+ e^{\alpha}\sum_{j=0}^{\infty}(1+j)^{-1}(1+n)^{-\beta_1}(1+j+m)^{-\beta_2}$$
$$\leqslant e^{\alpha}m^{-\beta_2}(Z_1(n,\beta_1) + Z_2(n,\beta_1)) + e^{\alpha}n^{-\beta_1}(Z_1(m,\beta_2) + Z_2(m,\beta_2)),$$

hence

$$\frac{S_{0,n,m}}{n^{-\beta_1}m^{-\beta_2}\ln(n)\ln(m)} \to 0.$$

We continue with $S_{2,n,m}$:

$$|S_{2,n,m}| \leqslant e^{\alpha}\sum_{i=1}^{n-1}\sum_{j=m}^{\infty}(1+i)^{-1}(1+j)^{-1}(1+i+n)^{-\beta_1}(1+j+m)^{-\beta_2}$$
$$\leqslant e^{\alpha}\sum_{i=0}^{n-1}(1+i)^{-1}(1+i+n)^{-\beta_1}\sum_{j=m}^{\infty}(1+j)^{-1}(1+j+m)^{-\beta_2}$$
$$= e^{\alpha}Z_1(n,\beta_1)Z_2(m,\beta_2),$$

therefore

$$\frac{S_{2,n,m}}{n^{-\beta_1}m^{-\beta_2}\ln(n)\ln(m)} \to 0.$$

Similarly we show that

$$\frac{S_{3,n,m} + S_{4,n,m}}{n^{-\beta_1} m^{-\beta_2} \ln(n) \ln(m)} \to 0.$$

Consequently

$$\frac{\rho(n,m)}{n^{-\beta_1} m^{-\beta_2} \ln(n) \ln(m)} \to 1,$$

as $n, m \to \infty$.

This finishes our proof. □

Proof of Theorem 4.16. For convenience of writing we assume that $w_{(i,j)} \neq 0$ for all $i, j \geqslant 0$. It then follows from assumptions (A1)-(A3) that there exist constants $d, e > 0$ such that

$$d < \left| w_{(i,j)} \right| < e, \ i, j \geqslant 0.$$

We need to investigate the asymptotic behaviour of

$$\rho(n, -m) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} V_\alpha(c_{i+n,j}, c_{i,j+m})$$

as $\min(n, m) \to \infty$.

Similarly to the proof of Theorem 4.15, we consider only part of the sets of parameters, since the other cases are investigated in a similar way. Here we will provide the proofs for the following cases:

1. $1 < \alpha \leq 2, \ \frac{1}{\alpha} < \beta_i < \frac{1}{\alpha - 1}, \ i = 1, 2$ and $\frac{1}{\beta_1} + \frac{1}{\beta_2} > \alpha$;
2. $1 < \alpha \leq 2, \ \frac{1}{\alpha - 1} < \beta_1$ and $\frac{1}{\alpha} < \beta_2 < 1$;
3. $1 < \alpha < 2, \ \beta_1 = \frac{1}{\alpha - 1}$ and $1 < \beta_2 < \frac{1}{\alpha - 1}$;

We begin with the first case. The spectral covariance equals

$$\begin{aligned}
\rho(n, -m) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} V_\alpha(c_{i+n,j}, c_{i,j+m}) \\
&= \int_0^\infty \int_0^\infty V_\alpha(c_{\lfloor t \rfloor + n, \lfloor s \rfloor}, c_{\lfloor t \rfloor, \lfloor s \rfloor + m}) \mathrm{d}s \mathrm{d}t \\
&= \int_0^\infty \int_0^\infty V_\alpha(c_{\lfloor nt \rfloor + n, \lfloor ms \rfloor}, c_{\lfloor nt \rfloor, \lfloor ms \rfloor + m}) \mathrm{d}ms \mathrm{d}nt \\
&= n^{1-\alpha\beta_1} m^{1-\alpha\beta_2} \int_0^\infty \int_0^\infty V_\alpha(n^{\beta_1} m^{\beta_2} c_{\lfloor nt \rfloor + n, \lfloor ms \rfloor}, n^{\beta_1} m^{\beta_2} c_{\lfloor nt \rfloor, \lfloor ms \rfloor + m}) \mathrm{d}s \mathrm{d}t.
\end{aligned}$$

Suppose $t, s > 0$, we have

$$
\begin{aligned}
&n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor} \\
&\qquad = n^{\beta_1}m^{\beta_2}w_{(\lfloor nt\rfloor+n,\lfloor ms\rfloor)}(1+\lfloor nt\rfloor+n)^{-\beta_1}(1+\lfloor ms\rfloor)^{-\beta_2} \\
&\qquad\qquad \to (1+t)^{-\beta_1}s^{-\beta_2},
\end{aligned}
$$

similarly

$$
n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m} \to t^{-\beta_1}(1+s)^{-\beta_2}.
$$

Due to the continuity of the function V_α we obtain

$$
\begin{aligned}
&V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}, n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m}) \\
&\qquad \to V_\alpha((1+t)^{-\beta_1}s^{-\beta_2}, (1+t)^{-\beta_1}s^{-\beta_2}),
\end{aligned}
$$

as $n, m \to \infty$.

Let us show that on $(0,\infty)^2$ the function

$$
\left|V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}, n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m})\right| \tag{4.77}
$$

is bounded above by an integrable function.

If $t > 0, s \geqslant 1$ we have

$$
\begin{aligned}
&\left|V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}, n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m})\right| \\
&\qquad \leqslant \left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}\right| \left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m}\right|^{\alpha-1} \\
&\qquad \leqslant e^\alpha \left(\frac{1+\lfloor nt\rfloor+n}{n}\right)^{-\beta_1}\left(\frac{1+\lfloor ms\rfloor}{m}\right)^{-\beta_2}\times \\
&\qquad \times \left(\frac{1+\lfloor nt\rfloor}{n}\right)^{-\beta_1(\alpha-1)}\left(\frac{1+\lfloor ms\rfloor+m}{m}\right)^{-\beta_2(\alpha-1)} \\
&\qquad\qquad \leqslant e^\alpha(t+1)^{-\beta_1}s^{-\beta_2}t^{-\beta_1(\alpha-1)}(1+s)^{-\beta_2(\alpha-1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty\int_1^\infty (t+1)^{-\beta_1}s^{-\beta_2}t^{-\beta_1(\alpha-1)}(1+s)^{-\beta_2(\alpha-1)}\mathrm{d}s\mathrm{d}t \\
&\qquad = \int_0^\infty t^{-\beta_1(\alpha-1)}(t+1)^{-\beta_1}\mathrm{d}t\int_1^\infty s^{-\beta_2}(1+s)^{-\beta_2(\alpha-1)}\mathrm{d}s < \infty.
\end{aligned}
$$

If $t \geqslant 1, 0 < s <$ we can bound

$$\begin{aligned}
&\left|V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}, n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m})\right| \\
&\qquad \leqslant \left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}\right|^{\alpha-1}\left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m}\right| \\
&\qquad\qquad \leqslant e^\alpha(t+1)^{-\beta_1(\alpha-1)}s^{-\beta_2(\alpha-1)}t^{-\beta_1}(1+s)^{-\beta_2},
\end{aligned}$$

and the dominating function is integrable:

$$\begin{aligned}
&\int_0^\infty\int_1^\infty(t+1)^{-\beta_1}s^{-\beta_2}t^{-\beta_1(\alpha-1)}(1+s)^{-\beta_2(\alpha-1)}\mathrm{d}s\mathrm{d}t \\
&\qquad = \int_1^\infty t^{-\beta_1}(t+1)^{-\beta_1(\alpha-1)}\mathrm{d}t\int_0^1 s^{-\beta_2(\alpha-1)}(1+s)^{-\beta_2}\mathrm{d}s < \infty.
\end{aligned}$$

It remains to bound (4.77) on $(0,1)^2$. We notice that

$$\left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}\right| \leqslant e(1+t)^{-\beta_1}s^{-\beta_2} \leqslant es^{-\beta_2},$$

$$\left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m}\right| \leqslant et^{-\beta_1}(1+s)^{-\beta_2} \leqslant et^{-\beta_1}.$$

Due to Lemma 4.6 we obtain

$$\begin{aligned}
&\left|V_\alpha(n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}, n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m})\right| \\
&\qquad = V_\alpha(\left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor+n,\lfloor ms\rfloor}\right|, \left|n^{\beta_1}m^{\beta_2}c_{\lfloor nt\rfloor,\lfloor ms\rfloor+m}\right|) \\
&\qquad\qquad \leqslant V_\alpha(es^{-\beta_2}, et^{-\beta_1}),
\end{aligned}$$

and proceed to show that

$$I := \int_0^1\int_0^1 V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s\mathrm{d}t < \infty. \tag{4.78}$$

We split I into two integrals: $I = I_1 + I_2$, where

$$I_1 = \int_0^1\int_0^{t^{\beta_1/\beta_2}} V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s\mathrm{d}t,$$

$$I_2 = \int_0^1\int_{t^{\beta_1/\beta_2}}^1 V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s\mathrm{d}t,$$

and proceed to show that both integrals are finite.

Since $-\beta_2(\alpha-1) > -1$, we have

$$\begin{aligned}\int_0^{t^{\beta_1/\beta_2}} & V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s \\ &\leqslant \int_0^{t^{\beta_1/\beta_2}} s^{-\beta_2(\alpha-1)} t^{-\beta_1}\mathrm{d}s = t^{-\beta_1}\int_0^{t^{\beta_1/\beta_2}} s^{-\beta_2(\alpha-1)}\mathrm{d}s \\ &= t^{-\beta_1}\frac{(t^{\beta_1/\beta_2})^{1-\beta_2(\alpha-1)}}{1-\beta_2(\alpha-1)} = \frac{t^{-\beta_1+\beta_1/\beta_2-\beta_1(\alpha-1)}}{1-\beta_2(\alpha-1)} \\ &= \frac{t^{\beta_1/\beta_2-\beta_1\alpha}}{1-\beta_2(\alpha-1)},\end{aligned}$$

therefore, since $\beta_1/\beta_2 - \beta_1\alpha > -1$, we obtain

$$I_1 \leqslant \int_0^1 \frac{t^{\beta_1/\beta_2-\beta_1\alpha}}{1-\beta_2(\alpha-1)}\mathrm{d}t < \infty.$$

We continue with I_2. Now

$$\begin{aligned}\int_{t^{\beta_1/\beta_2}}^1 & V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s \\ &\leqslant \int_{t^{\beta_1/\beta_2}}^1 s^{-\beta_2} t^{-\beta_1(\alpha-1)}\mathrm{d}s \\ &= t^{-\beta_1(\alpha-1)}\int_{t^{\beta_1/\beta_2}}^1 s^{-\beta_2}\mathrm{d}s \\ &= t^{-\beta_1(\alpha-1)}\frac{1-t^{\beta_1(1-\beta_2)/\beta_2}}{1-\beta_2},\end{aligned}$$

therefore,

$$\int_{t^{\beta_1/\beta_2}}^1 V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s = O(t^{-\beta_1\max(\alpha-1,\alpha-1/\beta_2)})$$

as $t \to 0$ and

$$\int_{t^{\beta_1/\beta_2}}^1 V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s \to 0$$

as $t \to 1$. Condition $\beta_1\max(\alpha-1, \alpha-1/\beta_2) < 1$ is satisfied, hence $I_2 < \infty$ and (4.78) holds.

The dominated convergence theorem applies yielding

$$\frac{\rho(n,-m)}{n^{1-\alpha\beta_1}m^{1-\alpha\beta_2}} \to \int_0^\infty\int_0^\infty V_\alpha((1+t)^{-\beta_1}s^{-\beta_2}, (1+t)^{-\beta_1}s^{-\beta_2})\mathrm{d}s\mathrm{d}t.$$

We continue by investigating the case $1 < \alpha \leq 2$, $\frac{1}{\alpha-1} < \beta_1$ and $\frac{1}{\alpha} < \beta_2 < 1$. We express the spectral covariance as

$$\rho(n,-m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c_{i+n,j}, c_{i,j+m})$$
$$= \sum_{i=0}^{\infty}\int_0^\infty V_\alpha(c_{i+n,\lfloor s\rfloor}, c_{i,\lfloor s\rfloor+m})\mathrm{d}s$$
$$= \sum_{i=0}^{\infty}\int_0^\infty V_\alpha(c_{i+n,\lfloor ms\rfloor}, c_{i,\lfloor ms\rfloor+m})\mathrm{d}ms$$
$$= n^{-\beta_1}m^{1-\beta_2\alpha}\sum_{i=0}^{\infty}\int_0^\infty n^{\beta_1}V_\alpha(m^{\beta_2}c_{i+n,\lfloor ms\rfloor}, m^{\beta_2}c_{i,\lfloor ms\rfloor+m})\mathrm{d}s.$$

Since

$$n^{\beta_1}m^{\beta_2}c_{i+n,\lfloor ms\rfloor} \to s^{-\beta_2}$$

and

$$m^{\beta_2}c_{i,\lfloor ms\rfloor+m} \to w_{(i,\infty)}(1+i)^{-\beta_1}(1+s)^{-\beta_2},$$

we obtain

$$n^{\beta_1}V_\alpha(m^{\beta_2}c_{i+n,\lfloor ms\rfloor}, m^{\beta_2}c_{i,\lfloor ms\rfloor+m})$$
$$= \frac{n^{\beta_1}m^{\beta_2}c_{i+n,\lfloor ms\rfloor}m^{\beta_2}c_{i,\lfloor ms\rfloor+m}}{\left(m^{2\beta_2}c^2_{i+n,\lfloor ms\rfloor} + m^{2\beta_2}c^2_{i,\lfloor ms\rfloor+m}\right)^{\frac{2-\alpha}{2}}}$$
$$\to w^{\alpha-1}_{(i,\infty)}(1+i)^{-\beta_1(\alpha-1)}s^{-\beta_2}(1+s)^{-\beta_2(\alpha-1)},$$

as $n,m\to\infty$.

Let us now find an upper bound for $n^{\beta_1}\left|V_\alpha(m^{\beta_2}c_{i+n,\lfloor ms\rfloor}, m^{\beta_2}c_{i,\lfloor ms\rfloor+m})\right|$. We have

$$n^{\beta_1}\left|V_\alpha(m^{\beta_2}c_{i+n,\lfloor ms\rfloor}, m^{\beta_2}c_{i,\lfloor ms\rfloor+m})\right|$$
$$\leqslant n^{\beta_1}\left|m^{\beta_2}c_{i+n,\lfloor ms\rfloor}\right|\left|m^{\beta_2}c_{i,\lfloor ms\rfloor+m}\right|^{\alpha-1}$$
$$< e^\alpha(1+i)^{-\beta_1(\alpha-1)}s^{-\beta_2}(1+s)^{-\beta_2(\alpha-1)}.$$

Since

$$\sum_{i=0}^{\infty}\int_0^\infty(1+i)^{-\beta_1(\alpha-1)}s^{-\beta_2}(1+s)^{-\beta_2(\alpha-1)}\mathrm{d}s$$
$$= \sum_{i=0}^{\infty}(1+i)^{-\beta_1(\alpha-1)}\int_0^\infty s^{-\beta_2}(1+s)^{-\beta_2(\alpha-1)}\mathrm{d}s < \infty,$$

we can apply the dominated convergence theorem to obtain

$$\frac{\rho(n,-m)}{n^{-\beta_1}m^{1-\beta_2\alpha}} \to \sum_{i=0}^{\infty}\int_0^{\infty} w_{(i,\infty)}^{\alpha-1}(1+i)^{-\beta_1(\alpha-1)}s^{-\beta_2}(1+s)^{-\beta_2(\alpha-1)}\mathrm{d}s.$$

Let us now investigate the case $1<\alpha<2$, $\beta_1=\frac{1}{\alpha-1}$ and $1<\beta_2<\frac{1}{\alpha-1}$. We begin by assuming that $h_n = m_n^{-\beta_2}/n^{-\beta_1} \to c \in (0;\infty)$. In what follows, in order not to overload the notation, instead of m_n we write m.

The spectral covariance equals

$$\rho(n,-m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c_{i+n,j}, c_{i,j+m})$$
$$= m^{-\beta_2\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(h_n^{-1}n^{\beta_1}c_{i+n,j}, m^{\beta_2}c_{i,j+m}).$$

Since

$$h_n^{-1}n^{\beta_1}c_{i+n,j} \to c^{-1}w_{(\infty,j)}(1+j)^{-\beta_2}$$

and

$$m^{\beta_2}c_{i,j+m} \to w_{(i,\infty)}(1+i)^{-\beta_1},$$

continuity of V_α implies

$$V_\alpha(m^{\beta_2}n^{-\beta_1}n^{\beta_1}c_{i+n,j}, m^{\beta_2}c_{i,j+m})$$
$$\to V_\alpha(c^{-1}w_{(\infty,j)}(1+j)^{-\beta_2}, w_{(i,\infty)}(1+i)^{-\beta_1}),$$

as $n\to\infty$.

We have

$$n^{\beta_1}|c_{i+n,j}| \leqslant e\left(\frac{1+i+n}{n}\right)^{-\beta_1}(1+j)^{-\beta_2} \leqslant e(1+j)^{-\beta_2}$$

and, similarly,

$$m^{\beta_2}c_{i,j+m} \leqslant e(1+i)^{-\beta_1}.$$

Since $h_n^{-1}\to c^{-1}>0$, for large n we have

$$h_n^{-1} < 2c^{-1},$$

therefore, by Lemma 4.6,

$$\begin{aligned}
&\left|V_\alpha(h_n^{-1}n^{\beta_1}c_{i+n,j}, m^{\beta_2}c_{i,j+m})\right| \\
&\qquad\qquad\leqslant V_\alpha(e2c^{-1}(1+j)^{-\beta_2}, e(1+i)^{-\beta_1}) \\
&\qquad\qquad\qquad\qquad\leqslant E^\alpha V_\alpha((1+j)^{-\beta_2}, (1+i)^{-\beta_1}),
\end{aligned}$$

where $E = e\max(1, 2c^{-1})$. In order to apply the dominated convergence theorem we need to show that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha((1+j)^{-\beta_2}, (1+i)^{-\beta_1}) < \infty. \tag{4.79}$$

Using Lemma 4.6, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha((1+j)^{-\beta_2}, (1+i)^{-\beta_1}) \\
&\qquad = \int_0^\infty\int_0^\infty V_\alpha((1+\lfloor s\rfloor)^{-\beta_2}, (1+\lfloor t\rfloor)^{-\beta_1})\mathrm{d}s\mathrm{d}t \\
&\qquad\qquad = \int_1^\infty\int_1^\infty V_\alpha(\lfloor s\rfloor^{-\beta_2}, \lfloor t\rfloor^{-\beta_1})\mathrm{d}s\mathrm{d}t \\
&\qquad\qquad\qquad \leqslant \int_1^\infty\int_1^\infty V_\alpha(2s^{-\beta_2}, 2t^{-\beta_1})\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

since $\lfloor x\rfloor \geqslant x/2$ for $x \geqslant 1$. Let us show that

$$I := \int_1^\infty\int_1^\infty V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s\mathrm{d}t < \infty.$$

We split this integral into two

$$I = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \int_1^\infty\int_1^{t^{\beta_1/\beta_2}} V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s\mathrm{d}t, \\
I_2 &= \int_1^\infty\int_{t^{\beta_1/\beta_2}}^\infty V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

and examine them separately.

We shall show that I_1 is finite. We have

$$\int_1^{t^{\beta_1/\beta_2}} V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s$$

$$\begin{aligned} &\leqslant \int_1^{t^{\beta_1/\beta_2}} s^{-\beta_2(\alpha-1)} t^{-\beta_1} \mathrm{d}s \\ &= t^{-\beta_1} \int_1^{t^{\beta_1/\beta_2}} s^{-\beta_2(\alpha-1)} \mathrm{d}s \\ &= t^{-\beta_1} \frac{t^{\beta_1/\beta_2(1-\beta_2(\alpha-1))} - 1}{1-\beta_2(\alpha-1)} \\ &\leqslant t^{-\beta_1} \frac{t^{\beta_1/\beta_2(1-\beta_2(\alpha-1))}}{1-\beta_2(\alpha-1)} \\ &= \frac{t^{\beta_1/\beta_2-\beta_1\alpha}}{1-\beta_2(\alpha-1)}. \end{aligned}$$

Since $\beta_1/\beta_2 - \beta_1\alpha < -1$, the integral I_1 is finite. We continue by examining I_2. Since $\beta_2 > 1$ we have

$$\int_{t^{\beta_1/\beta_2}}^{\infty} V_\alpha(s^{-\beta_2}, t^{-\beta_1})\mathrm{d}s$$

$$\begin{aligned} &\leqslant \int_{t^{\beta_1/\beta_2}}^{\infty} s^{-\beta_2} t^{-\beta_1(\alpha-1)} \mathrm{d}s \\ &= t^{-\beta_1(\alpha-1)} \int_{t^{\beta_1/\beta_2}}^{\infty} s^{-\beta_2} \mathrm{d}s \\ &= t^{-\beta_1(\alpha-1)} \frac{t^{\beta_1/\beta_2(1-\beta_2)}}{\beta_2-1} = \frac{t^{\beta_1/\beta_2-\beta_1\alpha}}{\beta_2-1}. \end{aligned}$$

In the case under investigation we have $\beta_1/\beta_2 - \beta_1\alpha < -1$, therefore $I_2 < \infty$. Consequently $I < \infty$ and (4.79) holds.

The dominated convergence theorem implies

$$\frac{\rho(n,m)}{m^{-\beta_2\alpha}} \to \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c^{-1}w_{(\infty,j)}(1+j)^{-\beta_2}, w_{(i,\infty)}(1+i)^{-\beta_1}),$$

as $n \to \infty$.

Let us now assume that $m = m_n$ is a sequence such that

$$h_n = m^{-\beta_2}/n^{-\beta_1} \to 0,$$

and introduce notation $g_n = mn^{-\beta_1/\beta_2}$. In order not to overload the notation in what follows instead of g_n we simply write g. Since $h_n \to 0$, we have $g \to \infty$, as $n \to \infty$.

The spectral covariance equals

$$\rho(n,-m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c_{i+n,j}, c_{i,j+m})$$
$$= \sum_{i=0}^{\infty}\int_0^{\infty} V_\alpha(c_{i+n,\lfloor s\rfloor}, c_{i,\lfloor s\rfloor+m})\mathrm{d}s$$
$$= \sum_{i=0}^{\infty}\int_0^{\infty} V_\alpha(c_{i+n,\lfloor gs\rfloor}, c_{i,\lfloor gs\rfloor+m})\mathrm{d}gs$$
$$= gm^{-\beta_2\alpha}\sum_{i=0}^{\infty}\int_0^{\infty} V_\alpha(m^{\beta_2}c_{i+n,\lfloor gs\rfloor}, m^{\beta_2}c_{i,\lfloor gs\rfloor+m})\mathrm{d}s.$$

As $n \to \infty$, we have

$$m^{\beta_2}c_{i+n,\lfloor gs\rfloor} = m^{\beta_2}w_{(i+n,\lfloor gs\rfloor)}(1+i+n)^{-\beta_1}(1+\lfloor gs\rfloor)^{-\beta_2}$$
$$= m^{\beta_2}g^{-\beta_2}n^{-\beta_1}w_{(i+n,\lfloor gs\rfloor)}\left(\frac{1+i+n}{n}\right)^{-\beta_1}\left(\frac{1+\lfloor gs\rfloor}{g}\right)^{-\beta_2}$$
$$= w_{(i+n,\lfloor gs\rfloor)}\left(\frac{1+i+n}{n}\right)^{-\beta_1}\left(\frac{1+\lfloor gs\rfloor}{g}\right)^{-\beta_2},$$

thus, $m^{\beta_2}c_{i+n,\lfloor gs\rfloor} \to s^{-\beta_2}$, as $n \to \infty$, and $m^{\beta_2}\left|c_{i+n,\lfloor gs\rfloor}\right| \leqslant es^{-\beta_2}$. Also

$$m^{\beta_2}c_{i,\lfloor gs\rfloor+m} = m^{\beta_2}w_{(i,\lfloor gs\rfloor+m)}(1+i)^{-\beta_1}\left(1+\lfloor gs\rfloor+m\right)^{-\beta_2}$$
$$= w_{(i,\lfloor gs\rfloor+m)}(1+i)^{-\beta_1}\left(\frac{1+\lfloor gs\rfloor+m}{m}\right)^{-\beta_2}.$$

From the equality above we see that $m^{\beta_2}c_{i,\lfloor gs\rfloor+m} \to w_{(i,\infty)}(1+i)^{-\beta_1}$, as $n \to \infty$, and $\left|m^{\beta_2}c_{i,\lfloor gs\rfloor+m}\right| \leqslant e(1+i)^{-\beta_1}$.

As V_α is continuous, the obtained relations imply

$$V_\alpha(m^{\beta_2}c_{i+n,\lfloor gs\rfloor}, m^{\beta_2}c_{i,\lfloor gs\rfloor+m}) \to V_\alpha(s^{-\beta_2}, w_{(i,\infty)}(1+i)^{-\beta_1}).$$

Due to Lemma 4.6 we obtain

$$\left|V_\alpha(m^{\beta_2}c_{i+n,\lfloor gs\rfloor}, m^{\beta_2}c_{i,\lfloor gs\rfloor+m})\right|$$
$$\leqslant V_\alpha(es^{-\beta_2}, e(1+i)^{-\beta_1}) = e^\alpha V_\alpha(s^{-\beta_2}, (1+i)^{-\beta_1}).$$

If we show that

$$\sum_{i=0}^{\infty}\int_0^{\infty} V_\alpha(s^{-\beta_2}, (1+i)^{-\beta_1})\mathrm{d}s < \infty, \tag{4.80}$$

the dominated convergence theorem will imply

$$\frac{\rho(n,-m)}{gm^{-\beta_2\alpha}} \to \sum_{i=0}^{\infty} V_\alpha(s^{-\beta_2}, w_{(i,\infty)}(1+i)^{-\beta_1})\mathrm{d}s. \tag{4.81}$$

A simple change of variables gives us

$$\begin{aligned}
&\int_0^\infty V_\alpha(s^{-\beta_2},(1+i)^{-\beta_1})\mathrm{d}s \\
&\qquad = \int_0^\infty V_\alpha\left(\left((1+i)^{\beta_1/\beta_2}s\right)^{-\beta_2},(1+i)^{-\beta_1}\right)\mathrm{d}(1+i)^{\beta_1/\beta_2}s \\
&\qquad = (1+i)^{\beta_1/\beta_2}\int_0^\infty V_\alpha\left((1+i)^{-\beta_1}s^{-\beta_2},(1+i)^{-\beta_1}\right)\mathrm{d}s \\
&\qquad\qquad = (1+i)^{\beta_1/\beta_2-\beta_1\alpha}\int_0^\infty V_\alpha\left(s^{-\beta_2},1\right)\mathrm{d}s,
\end{aligned}$$

therefore

$$\sum_{i=0}^{\infty}\int_0^\infty V_\alpha(s^{-\beta_2},(1+i)^{-\beta_1})\mathrm{d}s = \sum_{i=0}^{\infty}(1+i)^{\beta_1/\beta_2-\beta_1\alpha}\int_0^\infty V_\alpha\left(s^{-\beta_2},1\right)\mathrm{d}s.$$

The sum is finite, since in the case under consideration we have $\beta_1/\beta_2 - \beta_1\alpha < -1$. The integral is finite since $1 < \beta_2 < 1/(\alpha-1)$, as $t \to 0$ we have $V_\alpha\left(s^{-\beta_2},1\right) \sim s^{-\beta_2(\alpha-1)}$, and $V_\alpha\left(s^{-\beta_2},1\right) \sim s^{-\beta_2}$ as $t \to \infty$. Therefore (4.80) holds yielding (4.81).

It remains to investigate the behaviour of $\rho(n,-m)$ under the assumption that $m = m_n$ is a sequence such that $m_n \to \infty$ and $h_n \to \infty$. Let us denote $f_n = nm^{-\beta_2/\beta_1}$. In order to have simpler notation, in what follows we will write f instead of f_n. Since $h_n \to \infty$, we have $f \to \infty$.

We express the spectral covariance as follows

$$\rho(n,-m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_\alpha(c_{i+n,j}, c_{i,j+m}) = \sum_{j=0}^{\infty}\int_0^\infty V_\alpha(c_{\lfloor t\rfloor+n,j}, c_{\lfloor t\rfloor,j+m})\mathrm{d}t,$$

and split it up as

$$\rho(n,-m) = S_0 + S_1 + S_2, \tag{4.82}$$

where

$$S_0 = \sum_{j=0}^{\infty} V_\alpha(c_{n,j}, c_{0,j+m}),$$

$$S_1 = \sum_{j=0}^{\infty}\int_1^f V_\alpha(c_{\lfloor t\rfloor+n,j}, c_{\lfloor t\rfloor,j+m})\mathrm{d}t,$$

$$S_2 = \sum_{j=0}^{\infty} \int_f^{\infty} V_\alpha(c_{\lfloor t \rfloor + n, j}, c_{\lfloor t \rfloor, j+m}) \mathrm{d}t,$$

Let us begin by investigating S_1. We have

$$\begin{aligned} S_1 = \sum_{j=0}^{\infty} \int_1^{f} V_\alpha(c_{\lfloor t \rfloor + n, j}, c_{\lfloor t \rfloor, j+m}) \mathrm{d}t \\ = \sum_{j=0}^{\infty} \int_0^{1} V_\alpha(c_{\lfloor f^t \rfloor + n, j}, c_{\lfloor f^t \rfloor, j+m}) \mathrm{d}f^t \\ = \sum_{j=0}^{\infty} \int_0^{1} V_\alpha(c_{\lfloor f^t \rfloor + n, j}, c_{\lfloor f^t \rfloor, j+m}) f^t \ln(f) \mathrm{d}t \\ = n^{-\beta_1 \alpha} f \ln(f) \sum_{j=0}^{\infty} \int_0^{1} f^{-1} V_\alpha(n^{\beta_1} c_{\lfloor f^t \rfloor + n, j}, n^{\beta_1} c_{\lfloor f^t \rfloor, j+m}) f^t \mathrm{d}t. \end{aligned}$$

Let us investigate $f^{-1} V_\alpha(n^{\beta_1} c_{\lfloor f^t \rfloor + n, j}, n^{\beta_1} c_{\lfloor f^t \rfloor, j+m}) f^t$. We have

$$f^{-1} V_\alpha(n^{\beta_1} c_{\lfloor f^t \rfloor + n, j}, n^{\beta_1} c_{\lfloor f^t \rfloor, j+m}) f^t = \frac{n^{\beta_1} c_{\lfloor f^t \rfloor + n, j} f^{-1} f^t n^{\beta_1} c_{\lfloor f^t \rfloor, j+m}}{\left(n^{2\beta_1} c^2_{\lfloor f^t \rfloor + n, j} + n^{2\beta_1} c^2_{\lfloor f^t \rfloor, j+m}\right)^{\frac{2-\alpha}{2}}}. \quad (4.83)$$

Relations $n^{\beta_1} c_{\lfloor f^t \rfloor + n, j} \to w_{(\infty, j)}(1+j)^{-\beta_2}$ and $n^{\beta_1} c_{\lfloor f^t \rfloor, j+m} \to \infty$ imply

$$\left(n^{2\beta_1} c^2_{\lfloor f^t \rfloor + n, j} + n^{2\beta_1} c^2_{\lfloor f^t \rfloor, j+m}\right)^{\frac{2-\alpha}{2}} \sim \left(n^{\beta_1} c_{\lfloor f^t \rfloor, j+m}\right)^{2-\alpha},$$

hence

$$\begin{aligned} f^{-1} V_\alpha(n^{\beta_1} c_{\lfloor f^t \rfloor + n, j}, n^{\beta_1} c_{\lfloor f^t \rfloor, j+m}) f^t \\ \sim n^{\beta_1} c_{\lfloor f^t \rfloor + n, j} f^{-1} f^t n^{\beta_1(\alpha-1)} c^{\alpha-1}_{\lfloor f^t \rfloor, j+m} \\ \sim w_{(\infty, j)}(1+j)^{-\beta_2} f^{-1} f^t n^{\beta_1(\alpha-1)} f^{-\beta_1(\alpha-1)t} m^{-\beta_2(\alpha-1)} \\ = w_{(\infty, j)}(1+j)^{-\beta_2} f^{-1} f^t n^1 f^{-t} m^{-\beta_2/\beta_1} \\ = w_{(\infty, j)}(1+j)^{-\beta_2}. \end{aligned}$$

We can also bound the absolute value of (4.83) as follows

$$
\begin{aligned}
&\left|f^{-1}V_\alpha(n^{\beta_1}c_{\lfloor f^t\rfloor+n,j}, n^{\beta_1}c_{\lfloor f^t\rfloor,j+m})f^t\right| \\
&\qquad\leqslant f^{-1}f^t\left|n^{\beta_1}c_{\lfloor f^t\rfloor+n,j}\right|\left|n^{\beta_1}c_{\lfloor f^t\rfloor,j+m}\right|^{\alpha-1} \\
&\qquad\leqslant f^{-1}f^t e^\alpha n^{\beta_1}(1+\left\lfloor f^t\right\rfloor+n)^{-\beta_1}(1+j)^{-\beta_2}\times \\
&\qquad\times n^{\beta_1(\alpha-1)}(1+\left\lfloor f^t\right\rfloor)^{-\beta_1(\alpha-1)}(1+j+m)^{-\beta_2(\alpha-1)} \\
&\qquad\leqslant f^{-1}f^t e^\alpha n^{\beta_1}n^{-\beta_1}(1+j)^{-\beta_2}nf^{-t}m^{-\beta_2(\alpha-1)} \\
&\qquad= e^\alpha(1+j)^{-\beta_2}f^{-1}nm^{-\beta_2/\beta_1} = e^\alpha(1+j)^{-\beta_2}.
\end{aligned}
$$

Since

$$\sum_{j=0}^{\infty}\int_0^1(1+j)^{-\beta_2}\mathrm{d}t = \sum_{j=0}^{\infty}(1+j)^{-\beta_2} < \infty,$$

the dominated convergence theorem implies

$$\frac{S_1}{n^{-\beta_1\alpha}f\ln(f)} \to \sum_{j=0}^{\infty}\int_0^1 w_{(\infty,j)}(1+j)^{-\beta_2}\mathrm{d}t = \sum_{j=0}^{\infty}w_{(\infty,j)}(1+j)^{-\beta_2}. \quad (4.84)$$

Next we will show that $S_0 = o(n^{-\beta_1\alpha}f\ln(f))$ and $S_2 = o(n^{-\beta_1\alpha}f\ln(f))$. We begin with S_0.

$$
\begin{aligned}
|S_0| &\leqslant \sum_{j=0}^{\infty}|V_\alpha(c_{n,j}, c_{0,j+m})| \leqslant \sum_{j=0}^{\infty}|c_{n,j}|\,|c_{0,j+m}|^{\alpha-1} \\
&\leqslant \sum_{j=0}^{\infty}e^\alpha n^{-\beta_1}(1+j)^{-\beta_2}m^{-\beta_2(\alpha-1)} = e^\alpha\sum_{j=0}^{\infty}(1+j)^{-\beta_2}n^{-\beta_1-1}f \\
&= e^\alpha\sum_{j=0}^{\infty}(1+j)^{-\beta_2}n^{-\beta_1\alpha}f,
\end{aligned}
$$

thus we have obtained

$$S_0 = o(n^{-\beta_1\alpha}f\ln(f)). \quad (4.85)$$

Let us deal with S_2. Since

$$\left|c_{\lfloor t\rfloor+n,j}\right| \leqslant e(1+\lfloor t\rfloor+n)^{-\beta_1}(1+j)^{-\beta_2} \leqslant en^{-\beta_1}(1+j)^{-\beta_2},$$

and

$$\left|c_{\lfloor t\rfloor,j+m}\right| \leqslant e(1+\lfloor t\rfloor)^{-\beta_1}(1+j+m)^{-\beta_2} \leqslant et^{-\beta_1}m^{-\beta_2},$$

employing Lemma 4.6 we obtain

$$
\begin{aligned}
|S_2| &\leqslant \sum_{j=0}^{\infty} \int_f^{\infty} \left| V_\alpha(c_{\lfloor t \rfloor + n, j}, c_{\lfloor t \rfloor, j+m}) \right| \mathrm{d}t \\
&\leqslant \sum_{j=0}^{\infty} \int_f^{\infty} V_\alpha(en^{-\beta_1}(1+j)^{-\beta_2}, et^{-\beta_1} m^{-\beta_2}) \mathrm{d}t \\
&= e^{\alpha} n^{-\beta_1 \alpha} \sum_{j=0}^{\infty} (1+j)^{-\beta_2 \alpha} \int_f^{\infty} V_\alpha(1, n^{\beta_1}(1+j)^{\beta_2} t^{-\beta_1} m^{-\beta_2}) \mathrm{d}t. \qquad (4.86)
\end{aligned}
$$

Let us change the variable in integral

$$
\begin{aligned}
&\int_f^{\infty} V_\alpha(1, n^{\beta_1}(1+j)^{\beta_2} t^{-\beta_1} m^{-\beta_2}) \mathrm{d}t \\
&= \int_{(1+j)^{-\beta_2/\beta_1}}^{\infty} V_\alpha\left(1, n^{\beta_1}(1+j)^{\beta_2} (ft(1+j)^{\beta_2/\beta_1})^{-\beta_1} m^{-\beta_2}\right) \mathrm{d}ft(1+j)^{\beta_2/\beta_1} \\
&= f(1+j)^{\beta_2/\beta_1} \int_{(1+j)^{-\beta_2/\beta_1}}^{\infty} V_\alpha\left(1, t^{-\beta_1}\right) \mathrm{d}t. \qquad (4.87)
\end{aligned}
$$

We shall find an upper bound for the integral $\int_{(1+j)^{-\beta_2/\beta_1}}^{\infty} V_\alpha\left(1, t^{-\beta_1}\right) \mathrm{d}t$. We have

$$
\begin{aligned}
&\int_{(1+j)^{-\beta_2/\beta_1}}^{\infty} V_\alpha\left(1, t^{-\beta_1}\right) \mathrm{d}t \\
&= \int_{(1+j)^{-\beta_2/\beta_1}}^{1} V_\alpha\left(1, t^{-\beta_1}\right) \mathrm{d}t + \int_1^{\infty} V_\alpha\left(1, t^{-\beta_1}\right) \mathrm{d}t \\
&\leqslant \int_{(1+j)^{-\beta_2/\beta_1}}^{1} t^{-\beta_1(\alpha-1)} \mathrm{d}t + \int_1^{\infty} t^{-\beta_1} \mathrm{d}t \\
&= \frac{\beta_2}{\beta_1} \ln(1+j) + \frac{1}{\beta_1 - 1}.
\end{aligned}
$$

Therefore (4.87) is bounded above by

$$
f(1+j)^{\beta_2/\beta_1} \left(\frac{\beta_2}{\beta_1} \ln(1+j) + \frac{1}{\beta_1 - 1} \right),
$$

and now (4.86) implies

$$
\begin{aligned}
|S_2| &\leqslant e^{\alpha} n^{-\beta_1 \alpha} \sum_{j=0}^{\infty} (1+j)^{-\beta_2 \alpha} f(1+j)^{\beta_2/\beta_1} \left(\frac{\beta_2}{\beta_1} \ln(1+j) + \frac{1}{\beta_1 - 1} \right) \\
&= e^{\alpha} n^{-\beta_1 \alpha} f \sum_{j=0}^{\infty} (1+j)^{\beta_2/\beta_1 - \beta_2 \alpha} \left(\frac{\beta_2}{\beta_1} \ln(1+j) + \frac{1}{\beta_1 - 1} \right).
\end{aligned}
$$

The sum is finite, because $\beta_2/\beta_1 - \beta_2\alpha < -1$, therefore $S_2 = O(n^{-\beta_1 \alpha} f)$, implying

$$
S_2 = o(n^{-\beta_1 \alpha} f \ln(f)). \qquad (4.88)
$$

Recalling (4.82) and collecting (4.84),(4.85) and (4.88), we obtain

$$\frac{\rho(n,-m)}{n^{-\beta_1\alpha} f \ln(f)} \to \sum_{j=0}^{\infty} w_{(\infty,j)}(1+j)^{-\beta_2}.$$

This completes our proof. □

Proof of Theorem 4.17. Let us note that assumptions (A1)–(A3) imply the existence of a positive number K such that $\left|w_{(i,j)}\right| \leq K$ for all i, j.

In the case $\gamma_i > 1,\ i = 1, 2$, we write (4.34) as

$$\rho_\alpha(n, sm) = n^{-\gamma_1} m^{-\gamma_2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_1(i, j, n, m, s),$$

where

$$\begin{aligned} & f_1(i,j,n,m,s) \\ & \qquad = q_{i,j,n,m,s}(1+i)^{-\gamma_1}(1+j)^{-\gamma_2}\left(1+\frac{i+1}{n}\right)^{-\gamma_1}\left(1+\frac{j+1}{m}\right)^{-\gamma_2}. \end{aligned}$$

For fixed i, j, we have

$$f_1(i,j,n,m,s) \to W(i,j,s)^{\langle\alpha/2\rangle}(1+i)^{-\gamma_1}(1+j)^{-\gamma_2},\ (n,m)\to\infty,$$

where $W(i, j, 1) = w_{(i,j)}$ and $W(i, j, -1) = w_{(i,\cdot)} w_{(\cdot,j)}$. Since

$$|f_1(i,j,n,m,s)| \leq K^\alpha (1+i)^{-\gamma_1}(1+j)^{-\gamma_2}$$

and $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1+i)^{-\gamma_1}(1+j)^{-\gamma_2} < \infty$, the dominated convergence theorem implies (4.37).

Although the assumptions (A2) and (A3) hold point-wise, it is possible to show that, due to the assumption (A1), the convergence in both these conditions is uniform. Therefore, for an arbitrary $\epsilon > 0$, there exists $N := N(\epsilon) \in \mathbb{N}$ such that, for any $i \geq 0$ (assuming that $n > N$),

$$(1-\epsilon)^\alpha w_{(i,\cdot)}^{\alpha/2} \leq q_{i,j,n,m,s} \leq (1+\epsilon)^\alpha w_{(i,\cdot)}^{\alpha/2},\ j \geq N. \tag{4.89}$$

At this point, it is convenient to introduce the notation

$$h_{i,n,a} := (1+i)^{-a}\,(1+i+n)^{-a}.$$

Let us split $\rho_\alpha(n, sm)$ into two sums

$$\rho_\alpha(n, sm) = \Sigma_1 + \Sigma_2$$
$$:= \sum_{i=0}^{\infty}\sum_{j=0}^{N-1} q_{i,j,n,m,s} h_{i,n,\gamma_1} h_{j,m,\gamma_2} + \sum_{i=0}^{\infty}\sum_{j=N}^{\infty} q_{i,j,n,m,s} h_{i,n,\gamma_1} h_{j,m,\gamma_2}.$$

Using (4.89), we estimate Σ_2 as

$$(1-\epsilon)^\alpha G_n B_{m,N} \leq \Sigma_2 \leq (1+\epsilon)^\alpha G_n B_{m,N}, \tag{4.90}$$

where $G_n = \sum_{i=0}^{\infty} w_{(i,\cdot)}^{\alpha/2} h_{i,n,\gamma_1}$ and $B_{m,N} = \sum_{j=N}^{\infty} h_{j,m,\gamma_2}$.

If $\gamma_1 > 1$, then we can write

$$G_n = n^{-\gamma_1} \sum_{i=0}^{\infty} w_{(i,\cdot)}^{\alpha/2} (1+i)^{-\gamma_1} \left(1 + \frac{i+1}{n}\right)^{-\gamma_1},$$

and the monotone convergence theorem implies

$$\frac{G_n}{n^{-\gamma_1}} \to \sum_{i=0}^{\infty} w_{(i,\cdot)}^{\alpha/2} (1+i)^{-\gamma_1}, \ n \to \infty.$$

Next, we investigate the case $1/2 < \gamma_1 < 1$. We can write

$$G_n = \int_0^\infty w_{(\lfloor t \rfloor,\cdot)}^{\alpha/2} h_{\lfloor t \rfloor,n,\gamma_1} \mathrm{d}t = \int_0^\infty w_{(\lfloor nt \rfloor,\cdot)}^{\alpha/2} h_{\lfloor nt \rfloor,n,\gamma_1} \mathrm{d}nt$$
$$= n^{1-2\gamma_1} \int_0^\infty w_{(\lfloor nt \rfloor,\cdot)}^{\alpha/2} \frac{h_{\lfloor nt \rfloor,n,\gamma_1}}{n^{-2\gamma_1}} \mathrm{d}t.$$

For fixed $t \in (0, \infty)$, we have

$$w_{(\lfloor nt \rfloor,\cdot)}^{\alpha/2} \frac{h_{\lfloor nt \rfloor,n,\gamma_1}}{n^{-2\gamma_1}} \to t^{-\gamma_1}(1+t)^{-\gamma_1}$$

and

$$\left| w_{(\lfloor nt \rfloor,\cdot)}^{\alpha/2} \frac{h_{\lfloor nt \rfloor,n,\gamma_1}}{n^{-2\gamma_1}} \right| \leqslant K^{\alpha/2} t^{-\gamma_1}(1+t)^{-\gamma_1}.$$

Since the dominating function is integrable, the dominated convergence theorem implies

$$\frac{G_n}{n^{1-2\gamma_1}} \to \int_0^\infty t^{-\gamma_1}(1+t)^{-\gamma_1} \mathrm{d}t = K(\gamma_1), \ n \to \infty.$$

Next, we deal with G_n in the case $\gamma_1 = 1$. Let us assume that $n \geqslant 2$. We have

$$G_n = \int_0^\infty w^{\alpha/2}_{(\lfloor t \rfloor, \cdot)} h_{\lfloor t \rfloor, n, 1} \mathrm{d}t = \int_0^\infty w^{\alpha/2}_{(\lfloor t \rfloor, \cdot)} (1 + \lfloor t \rfloor)^{-1} (1 + \lfloor t \rfloor + n)^{-1} \mathrm{d}t.$$

The change of variables $t = n^y$ gives

$$\begin{aligned} G_n = \int_{-\infty}^\infty w^{\alpha/2}_{(\lfloor n^y \rfloor, \cdot)} (1 + \lfloor n^y \rfloor)^{-1} (1 + \lfloor n^y \rfloor + n)^{-1} n^y \ln(n) \mathrm{d}y \\ = n^{-1} \ln(n) \int_{-\infty}^\infty f_2(n, y) \mathrm{d}y, \end{aligned}$$

where

$$f_2(n, y) = w^{\alpha/2}_{(\lfloor n^y \rfloor, \cdot)} \left(1 + \frac{\lfloor n^y \rfloor + 1}{n}\right)^{-1} \frac{n^y}{1 + \lfloor n^y \rfloor}.$$

For a fixed $y \in \mathbb{R} \backslash \{0, 1\}$, we have

$$f_2(n, y) \to \begin{cases} 0 \text{ if } y < 0 \text{ or } y > 1, \\ 1 \text{ if } 0 < y < 1. \end{cases}$$

Convergence in the set $\{0, 1\}$ does not matter since it is a zero-measure set. Also,

$$f_2(n, y) \leqslant F(y) := \begin{cases} K^{\alpha/2} 2^y \text{ if } y \leqslant 0, \\ K^{\alpha/2} \text{ if } 0 < y < 1, \\ K^{\alpha/2} 2^{1-y} \text{ if } y \geqslant 1. \end{cases}$$

The dominating function F is integrable in $\mathbb{R}$. Therefore, the dominated convergence theorem implies

$$\frac{G_n}{n^{-1} \ln(n)} \to \int_0^1 1 \mathrm{d}t = 1 \text{ as } n \to \infty.$$

We denote

$$s_{n,\gamma} = \begin{cases} n^{1-2\gamma} \text{ if } 1/2 < \gamma < 1, \\ n^{-1} \ln(n) \text{ if } \gamma = 1, \\ n^{-\gamma} \text{ if } \gamma > 1; \end{cases} \qquad g_\gamma = \begin{cases} K(\gamma) \text{ if } 1/2 < \gamma < 1, \\ 1 \text{ if } \gamma = 1, \\ \sum_{i=0}^\infty w^{\alpha/2}_{(i, \cdot)} (1 + i)^{-\gamma} \text{ if } \gamma > 1. \end{cases}$$

In conclusion, for $\gamma_1 > 1/2$, we have

$$\frac{G_n}{s_{n,\gamma_1}} \to g_{\gamma_1} \text{ as } n \to \infty.$$

Similarly, if $1/2 < \gamma_2 \leqslant 1$, then we can show that

$$\frac{B_{m,N}}{s_{m,\gamma_2}} \to g_{\gamma_2} \text{ as } m \to \infty.$$

Inequality (4.90) implies

$$(1-\varepsilon)^\alpha g_{\gamma_1} g_{\gamma_2} \leqslant \liminf_{(n,m)\to\infty} \frac{\Sigma_2}{s_{n,\gamma_1} s_{m,\gamma_2}},$$

$$\limsup_{(n,m)\to\infty} \frac{\Sigma_2}{s_{n,\gamma_1} s_{m,\gamma_2}} \leqslant (1+\varepsilon)^\alpha g_{\gamma_1} g_{\gamma_2}.$$

The sum Σ_1 can be bounded above as follows:

$$\begin{aligned}
|\Sigma_1| &\leqslant K^\alpha N m^{-\gamma_2} \sum_{i=0}^{\infty} h_{i,n,\gamma_1} \\
&\leqslant K^\alpha N m^{-\gamma_2} \sup_{k\geqslant 0} w_{(k,\cdot)}^{-\alpha/2} \sum_{i=0}^{\infty} w_{(i,\cdot)}^{\alpha/2} h_{i,n,\gamma_1} \\
&= K^\alpha N m^{-\gamma_2} \sup_{k\geqslant 0} w_{(k,\cdot)}^{-\alpha/2} G_n.
\end{aligned}$$

In the case $1/2 < \gamma_2 \leqslant 1$, we have that $m^{-\gamma_2} = o(s_{m,\gamma_2})$ as $m \to \infty$; thus, $\Sigma_1 = o(\Sigma_2)$ as $(n,m) \to \infty$. Therefore, if $\gamma_1 > 1/2$ and $1/2 < \gamma_2 \leqslant 1$, then we have

$$(1-\varepsilon)^\alpha g_{\gamma_1} g_{\gamma_2} \leqslant \liminf_{(n,m)\to\infty} \frac{\rho_\alpha(n, sm)}{s_{n,\gamma_1} s_{m,\gamma_2}},$$

$$\limsup_{(n,m)\to\infty} \frac{\rho_\alpha(n, sm)}{s_{n,\gamma_1} s_{m,\gamma_2}} \leqslant (1+\varepsilon)^\alpha g_{\gamma_1} g_{\gamma_2}.$$

Arbitrariness of $\varepsilon > 0$ implies

$$\lim_{(n,m)\to\infty} \frac{\rho_\alpha(n, sm)}{s_{n,\gamma_1} s_{m,\gamma_2}} = g_{\gamma_1} g_{\gamma_2}.$$

□

Proof of Proposition 4.20. Substituting $c_{\mathbf{i}}$ from (4.42) into (4.27), we obtain the following expression for the α-spectral covariance:

$$\rho_\alpha(X_{\mathbf{0}}, X_{\mathbf{k}}) = \rho_\alpha(X_{\mathbf{k}_-}, X_{\mathbf{k}_+}) = \prod_{l=1}^{d} r_{\gamma_l}(|k_l|), \tag{4.91}$$

where $r_\gamma(n) := \sum_{i=0}^{\infty}(1+i)^{-\gamma}(1+i+n)^{-\gamma}$. Obviously, it is sufficient to investigate the asymptotic behaviour of $r_\gamma(n)$ as $n \to \infty$. Using the same steps as in the proof of Theorem 4.17 and mainly applying the monotone or dominated convergence theorem, in the cases $\gamma > 1$, $1/2 < \gamma < 1$, and $\gamma = 1$, we can prove the following three relations, respectively:

$$\frac{r_\gamma(n)}{n^{-\gamma}} \to \zeta(\gamma), \ \frac{r_\gamma(n)}{n^{1-2\gamma}} \to K(\gamma), \quad \text{and} \ \frac{r_\gamma(n)}{n^{-1}\ln(n)} \to 1 \quad \text{as } n \to \infty.$$

Now from (4.91) and from the last three relations we easily get (4.43). $\square$

5 Limit theorems

5.1 The problem and results

Limit theorems

Now we will formulate two limit theorems for stationary and associated random fields, which generalize the results from [17]. Note that limit theorems for stationary associated random fields with finite variance are deeply investigated; a lot of information on the CLT in this case can be found in [12]. On the other hand, limit theorems for stationary random fields with infinite variance are less investigated; therefore, it seems interesting to investigate limit theorems for associated stationary random fields with infinite variance.

We consider the case $d = 2$, although there are no principal difficulties to consider the general case $d \geq 2$, apart from a more complicated notation. Let $\mathbb{X} = \{X_{i,j},\ (i,j) \in \mathbb{Z}^2\}$ be a stationary random field (by stationarity we mean strict stationarity with respect to translation operation). We say that a random field $\mathbb{X}$ is associated if, for any finite set $A \subset \mathbb{Z}^2$, the collection of random variables $X_{i,j},\ (i,j) \in A$, is associated and is jointly (strictly) α-stable if the collection $X_{i,j},\ (i,j) \in A$, is jointly (strictly) α-stable. First, we state an analogue of Theorem 3.8 for a stationary associated jointly α-stable random field $\mathbb{X} = \{X_{i,j},\ (i,j) \in \mathbb{Z}^2\}$. We denote $\rho(i,j) = \rho(X_{0,0}, X_{i,j})$ and

$$S_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j}, \quad \bar{S}_{n,m} = \frac{S_{n,m}}{n^{1/\alpha} m^{1/\alpha}}. \tag{5.1}$$

Theorem 5.1. *Let $\mathbb{X}$ be a stationary associated jointly α-stable field.*

If $0 < \alpha < 1$, *then*

$$\bar{S}_{n,m} \xrightarrow{\mathrm{d}} \mu \quad \text{as } n, m \to \infty, \tag{5.2}$$

where μ *is a strictly* α*-stable distribution.*

If $\alpha = 1$, *then there exist constants* $A_{n,m}$ *such that*

$$\bar{S}_{n,m} - A_{n,m} \stackrel{\mathrm{d}}{=} X_1.$$

If $1 < \alpha < 2$ *and*

$$\sum_{(i,j)\in\mathbb{Z}^2} \rho(i,j) < \infty, \tag{5.3}$$

then

$$\frac{S_{n,m} - \mathbb{E}S_{n,m}}{n^{1/\alpha}m^{1/\alpha}} = \frac{S_{n,m} - nmb}{n^{1/\alpha}m^{1/\alpha}} \xrightarrow{\mathrm{d}} \mu \text{ as } n, m \to \infty, \tag{5.4}$$

where μ *is a non-degenerate strictly* α*-stable distribution.*

An observation similar to Remark 4.4 can be made here: the quantity $\rho(i,j)$ in (5.3) can be substituted by the codifference $\tau(i,j) := \tau(X_{0,0}, X_{i,j})$ to obtain an equivalent statement. Changing the spectral covariance by the α-spectral covariance (or covariation) in (5.3) makes the statement of Theorem 5.1 weaker, however, as can be seen from Theorems 4.15, 4.16 and 4.17, the asymptotic behaviour of α-spectral covariance is simpler.

Theorem 4.17 and Proposition 4.3 allow us to verify condition (5.3) for linear random fields.

Corollary 5.2. *Suppose that a linear field* (4.28) *with coefficients* $c_{i,j}$ *of the form* (4.31) *with* $w_{(i,j)} \geqslant 0$ *satisfies conditions (A1)–(A3). If* $1 < \alpha < 2$ *and* $\beta_i > 2/\alpha$, $i = 1, 2$, *then*

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} X_{i,j}}{n^{1/\alpha}m^{1/\alpha}} \xrightarrow{\mathrm{d}} \mu \quad \text{as } n, m \to \infty,$$

where μ *is a non-degenerate strictly* α*-stable distribution.*

Now we shall state a generalization of Theorem 3.9. Let $\{X_{i,j},\ i, j \in \mathbb{Z}\}$ and $\{Y_{i,j},\ i, j \in \mathbb{Z}\}$ be stationary and associated random fields, and, additionally, let $\{Y_{i,j},\ i,j \in \mathbb{Z}\}$ be jointly strictly α-stable, $0 < \alpha < 2$. We say that $\{X_{i,j},\ i,j \in \mathbb{Z},\}$ belongs to the domain of strict normal attraction of $\{Y_{i,j},\ i,j \in \mathbb{Z},\}$ and write $\{X_{i,j}\} \in \mathscr{D}_{\mathrm{sn}}(\{Y_{i,j}\})$ if, for each $(n,m) \in \mathbb{Z}_+^2$, the distribution of the mn-dimensional vector $\mathbf{X}_{n,m} = (X_{1,1}, X_{1,2}, \ldots, X_{1,m}, X_{2,1}, \ldots, X_{2,m}, \ldots, X_{n,1}, \ldots, X_{n,m})$ belongs to the domain of strict normal attraction of the mn-dimensional α-stable random vector $\mathbf{Y}_{n,m} = (Y_{1,1}, Y_{1,2}, \ldots, Y_{1,m}, Y_{2,1}, \ldots, Y_{2,m}, \ldots, Y_{n,1}, \ldots, Y_{n,m})$. The spectral measure of $\mathbf{Y}_{n,m}$ is a measure on $\mathbb{S}^{mn-1}$, we denote it by $\Gamma_{n,m}$.

We shall use the notation (5.1), but now assuming that the field $\{X_{i,j},\ i,j \in \mathbb{Z}\}$ is in the domain of attraction of an α-stable field $\{Y_{i,j},\ i,j \in \mathbb{Z}\}$. Using the function $I_\alpha^A(X_i, X_j)$ defined in (3.16), let us denote (using bold letters for two-dimensional indices)

$$I_\alpha^A(\mathbf{i},\mathbf{k}) := I_\alpha^A(X_{\mathbf{i}}, X_{\mathbf{k}}), \quad I_\alpha^A(\mathbf{k}) := I_\alpha^A(\mathbf{0},\mathbf{k}),$$

$$\bar{Z}_{n,m} = n^{-1/\alpha} m^{-1/\alpha} \sum_{i=1}^{n} \sum_{j=1}^{m} Y_{i,j}.$$

Theorem 5.3. *Let $\{X_{i,j},\ i,j \in \mathbb{Z}\}$ be a stationary associated field such that $\{X_{i,j}\} \in \mathscr{D}_{sn}(\{Y_{i,j}\})$, where $\{Y_{i,j},\ i,j \in \mathbb{Z}\}$ is a stationary and jointly strictly α-stable field, $0 < \alpha < 2$, and $\Gamma_{n,m}$ is symmetric for all n,m if $\alpha = 1$.*

If

$$\sum_{\mathbf{j} \in \mathbb{Z}^2} I_\alpha^A(\mathbf{j}) < \infty \tag{5.5}$$

for some $A > 0$, then there exists a strictly α-stable distribution μ such that

$$\bar{S}_{n,m} \xrightarrow{\mathrm{d}} \mu \tag{5.6}$$

and

$$\bar{Z}_{n,m} \xrightarrow{\mathrm{d}} \mu \quad \text{as } n,m \to \infty. \tag{5.7}$$

Linear fields

In this section we are going to investigate the convergence of finite dimensional distributions of appropriately normalized partial sum process

$$S_{\mathbf{n}}(\mathbf{t}) = \sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}} X_{\mathbf{k}}, \tag{5.8}$$

as $\min(n_1,\ldots,n_d) \to \infty$. Here $X_{\mathbf{k}} = \sum_{\mathbf{j}\geqslant\mathbf{0}} c_{\mathbf{j}}\xi_{\mathbf{k}-\mathbf{j}}$, $\mathbf{k} \in \mathbb{Z}_+^d$, is a d-dimensional linear random field, $\xi_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^d$ are independent copies of random variable ξ, belonging to the domain of attraction of α-stable random variable, see (2.13), (if $\alpha = 1$ we assume that the random variable is symmetric). We assume that coefficients $c_{\mathbf{j}}$ are of the form

$$c_{\mathbf{j}} = c_{(j_1,\ldots,j_d)} = \prod_{l=1}^{d} a_{j_l}(\gamma_l, l),\ \mathbf{j} \geqslant \mathbf{0},$$

where

$$a_j(\gamma_l, l) \sim (1+j)^{-\gamma_l} L_l(j),\ \text{as } j \to \infty, \tag{5.9}$$

with $\gamma_l > 1/\alpha$ and some s.v.f. L_l, $l = 1,\ldots,d$. If $\gamma_l = 1$ we make a simplifying assumption $L_l \equiv 1$.

Remark 5.4. In what follows we assume that $\gamma_l > 1/\alpha$ holds without explicitly mentioning it. For example, sometimes instead of writing $\gamma_l > 1$ and $\gamma_l > 1/\alpha$ we will just write $\gamma_l > 1$.

Let us denote

$$s_{n,\gamma_l,l} = \begin{cases} 1 \text{ if } \gamma_l > 1 \text{ and } \sum_{j=0}^{\infty} a_j(\gamma_l,l) \neq 0, \\ n^{1-\gamma_l} L_l(n) \text{ if } 1 < \gamma_l < 1 + 1/\alpha \text{ and } \sum_{j=0}^{\infty} a_j(\gamma_l,l) = 0, \\ \ln n \text{ if } \gamma_l = 1, \\ n^{1-\gamma_l} L_l(n) \text{ if } 1/\alpha < \gamma_l < 1, \end{cases} \tag{5.10}$$

and

$$A_{\mathbf{n}} = h_{1/\alpha}^{1/\alpha} \left(\prod_{j=1}^{d} n_j\right) \prod_{j=1}^{d} \left(n_j^{\frac{1}{\alpha}} s_{n_j,\gamma_j,j}\right), \tag{5.11}$$

here $h_{1/\alpha}$ is a s.v.f. corresponding to h from (2.13) and satisfying (2.2). We also define

$$H_{\gamma_l}(u,t,l) = \begin{cases} \sum_{k=0}^{\infty} a_k(\gamma_l, l)\mathbb{1}_{[0,t)}(u) \text{ if } \gamma_l > 1 \text{ and } \sum_{j=0}^{\infty} a_j(\gamma_l, l) \neq 0, \\ \frac{(t-u)_+^{1-\gamma_l} - (-u)_+^{1-\gamma_l}}{1-\gamma_l} \text{ if } 1 < \gamma_l < 1 + 1/\alpha \text{ and } \sum_{j=0}^{\infty} a_j(\gamma_l, l) = 0, \\ \mathbb{1}_{[0,t)}(u) \text{ if } \gamma_l = 1, \\ \frac{(t-u)_+^{1-\gamma_l} - (-u)_+^{1-\gamma_l}}{1-\gamma_l} \text{ if } 1/\alpha < \gamma_l < 1, \end{cases}$$

and

$$\mathcal{H}(\mathbf{u},\mathbf{t}) = \prod_{l=1}^{d} H_{\gamma_l}(u_l, t_l, l).$$

We are now ready to formulate our result.

Theorem 5.5. *For the process* $S_{\mathbf{n}}(\mathbf{t})$ *defined by* (5.8) *and normalizing sequence* (5.11) *we have*

$$A_{\mathbf{n}}^{-1} S_{\mathbf{n}}(\mathbf{t}) \xrightarrow{\text{f.d.d.}} I(\mathbf{t}), \qquad (5.12)$$

where $I(\mathbf{t})$ *is* α*-stable stochastic integral defined by*

$$I(\mathbf{t}) = \int_{\mathbb{R}^d} \mathcal{H}(\mathbf{u},\mathbf{t}) M(\mathrm{d}\mathbf{u}),$$

with Lebesgue control measure and skewness intensity $\beta(x) \equiv \beta$.

Remark 5.6. Normalizing sequence in (5.12) contains factor $h_{1/\alpha}^{1/\alpha}\left(\prod_{j=1}^{d} n_j\right)$ which is absent in the Definition 6. Since this factor can not be expressed as a product of slowly varying functions $\prod_{j=1}^{d} l_j(n_j)$, Theorem 5.5 reveals that the definition of memory for stationary fields in [52] needs revision – it does not classify linear fields with general innovations belonging to the domain of attraction of α-stable random variable (it applies only to linear fields with innovations belonging to the normal domain of attraction).

Negative memory

We consider a partial sum process $S_n(t) = \sum_{k=0}^{\lfloor nt \rfloor} X_k$ of linear processes $X_n = \sum_{i=0}^{\infty} c_i \xi_{n-i}$ with independent identically distributed innovations $\{\xi_i\}$

belonging to the domain of attraction of α-stable law, $0 < \alpha \leq 2$. If

$$|c_k| = k^{-\gamma}, k \in \mathbb{N},\ \gamma > \max(1/\alpha, 1),\ \text{and}\ \sum_{k=0}^{\infty} c_k = 0 \tag{5.13}$$

(the case of negative memory for the stationary sequence $\{X_n\}$), it is known that the normalizing sequence of $S_n(1)$ can grow as $n^{1/\alpha-\gamma+1}$ or remain bounded, if the signs of the coefficients of $c_k,\ k \in \mathbb{N}$, are constant or alternate, respectively. It is of interest to know whether it is possible, given $\lambda \in (0, 1/\alpha-\gamma+1)$, to change the signs of c_k so that the rate of growth of the normalizing sequence would be n^{λ}. The following theorem gives a positive answer: we propose a way of choosing the signs and investigate the finite-dimensional convergence of appropriately normalized $S_n(t)$ to linear fractional Lévy motion.

For $\theta \geqslant 1$ let us denote $T = T_\theta = \{k : k = 2\left\lfloor l^\theta \right\rfloor - 1$ for some $l \in \mathbb{N},\ l \geqslant 3\}$.

Theorem 5.7. *Suppose that* $\max(1/\alpha, 1) < \gamma < 1 + 1/\alpha$, $1 \leqslant \theta < \alpha/(\alpha\gamma - 1)$ *and for* $n \geqslant 0$ *let us define*

$$s_n = \begin{cases} 1 \ \textit{if}\ n \in T, \\ (-1)^n \ \textit{otherwise.} \end{cases}$$

Consider a linear process $X_n = \sum_{i=0}^{\infty} b_i \xi_{n-i}$ *where* $b_k = s_k k^{-\gamma},\ k \in \mathbb{N}$, $b_0 = -\sum_{k=1}^{\infty} b_k$ *and* $\xi_i,\ i \in \mathbb{N}$, *is a sequence of i.i.d. random variables having ch. f.* (2.13).

The sequence of linear processes $\bar{S}_n(t) = A_n^{-1} \sum_{k=0}^{\lfloor nt \rfloor} X_k$ *converges in finite-dimensional distributions to the LFLM*

$$Z_{\alpha, 1/\alpha+1/\theta-\gamma}\left(-2^{1-1/\theta}/(\gamma\theta - 1), 0; t\right)$$

with skewness intensity $\beta(u) \equiv \beta$. *Here* $A_n = n^{1/\alpha+1/\theta-\gamma} h_{1/\alpha}^{1/\alpha}(n)$ *with* $h_{1/\alpha}$ *satisfying* (2.2).

Assumption $|b_k| = k^{-\gamma}$ could be changed by a more general $|b_k| = k^{-\gamma}(a + o(1)),\ k \to \infty$, requiring only minor changes to the proof. We do not do this in order to keep the proof technically simpler.

Theorem 5.7 answers the problem proposed by Paulauskas in [52]:

Corollary 5.8. *Suppose ξ_i belong to the normal domain of attraction of α-stable law. Given $\gamma \in (\max(1/\alpha, 1), 1 + 1/\alpha)$ and any $\lambda \in (0, 1/\alpha - \gamma + 1)$, it is possible to choose the signs of the coefficients $c_k, k \in \mathbb{N}$, satisfying* (5.13) *so that A_n would grow as n^λ.*

5.2 Proofs

Proof of Theorem 5.1. Let Y be a stable random variable with spectral measure Γ_Y and shift parameter b. Since the unit sphere on the line is two points, the parameters $b,\ \sigma_+(Y) = \Gamma_Y(\{1\})$, and $\sigma_-(Y) = \Gamma_Y(\{-1\})$ completely determine the distribution of Y, which we denote by

$$\gamma_Y(b, \sigma_+(Y), \sigma_-(Y)).$$

Recall that

$$\mathbf{X}_{n,m} = (X_{1,1}, X_{1,2}, \dots, X_{1,m}, X_{2,1}, \dots, X_{2,m}, \dots, X_{n,1}, \dots, X_{n,m}). \quad (5.14)$$

Since $X_{i,j},\ 1 \le i \le n,\ 1 \le j \le m$, are associated and jointly α-stable, the random variable $S_{n,m}$, as a linear combination from the vector (5.14), is the stable distribution $\gamma_{S_{n,m}}(bnm, \sigma_+(S_{n,m}), \sigma_-(S_{n,m}))$, where b is the shift parameter of $X_{1,1}$, and

$$\begin{aligned}
\sigma_+(S_{n,m}) &= c^+_{n,m} = \int_{\mathbb{S}^{nm-1}} \left(\sum_{i=1}^{n} \sum_{j=1}^{m} s_{i,j} \right)^{\alpha} \mathbb{1}_{\{s_{i,j} \geqslant 0,\ 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}} \Gamma_{n,m}(\mathrm{d}\mathbf{s}), \\
\sigma_-(S_{n,m}) &= c^-_{n,m} = \int_{\mathbb{S}^{nm-1}} \left(-\sum_{i=1}^{n} \sum_{j=1}^{m} s_{i,j} \right)^{\alpha} \mathbb{1}_{\{s_{i,j} \leqslant 0,\ 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}} \Gamma_{n,m}(\mathrm{d}\mathbf{s}).
\end{aligned}$$

Here $\Gamma_{n,m}$ denotes the spectral measure of the stable vector (5.14) on the unit sphere $\mathbb{S}^{nm-1}$. As in [17], we must show that the parameters of the stable distribution $\gamma_{\bar{S}_{n,m}}$ are converging, and to this end, we shall use Lemma 2.5 in the case $d = 2$. We will show that $c^+_{n,m}$ and $c^-_{n,m}$, as functions $\mathbb{Z}^2_+ \to [0,\infty)$, are subadditive and superadditive in the cases $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively, and additive in the case $\alpha = 1$. We start with the case $0 < \alpha < 1$. It is convenient to denote vectors in $\mathbb{R}^{nm}$ by $(s_{1,1},\ldots,s_{1,m},s_{2,1},\ldots,s_{2,m},\ldots,s_{n,1},\ldots,s_{n,m})$. Applying the inequality $(x+y)^\alpha \leqslant x^\alpha + y^\alpha$, $x, y \geqslant 0, 0 \leqslant \alpha \leqslant 1$, we can write

$$\begin{aligned}
c^+_{n+k,m} &= \sigma_+(S_{n+k,m}) \\
&= \int_{\mathbb{S}^{(n+k)m-1}} \left(\sum_{i=1}^{n+k}\sum_{j=1}^{m} s_{i,j}\right)^\alpha \mathbb{1}_{\{s_{i,j}\geqslant 0,\ 1\leqslant i\leqslant n+k, 1\leqslant j\leqslant m\}}\Gamma_{n+k,m}(\mathrm{d}\mathbf{s}) \\
&\leqslant \int_{\mathbb{S}^{(n+k)m-1}} \left(\left(\sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j}\right)^\alpha + \left(\sum_{i=n+1}^{n+k}\sum_{j=1}^{m} s_{i,j}\right)^\alpha\right) \times \\
&\times \mathbb{1}_{\{s_{i,j}\geqslant 0,\ 1\leqslant i\leqslant n+k, 1\leqslant j\leqslant m\}}\Gamma_{n+k,m}(\mathrm{d}\mathbf{s}) \\
&= \sigma_+(S_{n,m}) + \sigma_+(S_{n+k,m} - S_{n,m}) \\
&= \sigma_+(S_{n,m}) + \sigma_+(S_{k,m}) = c^+_{n,m} + c^+_{k,m}.
\end{aligned}$$

Exactly in the same way, we can show that $c^+_{n,m+l} \leqslant c^+_{n,m} + c^+_{n,l}$, and we have that $c^+_{n,m}$ is subadditive, the same can be shown for $c^-_{n,m}$. Therefore, by Lemma 2.5 we have that there exist the limits

$$\lim_{n,m\to\infty} \frac{c^+_{n,m}}{nm}, \quad \lim_{n,m\to\infty} \frac{c^-_{n,m}}{nm}.$$

Since the random variable $\bar{S}_{n,m}$ is stable with parameters $b(nm)^{1-1/\alpha}$, $c^+_{n,m}/(nm)$, and $c^-_{n,m}/(nm)$, which are convergent as $n,m \to \infty$, we get (5.2). In the case $1 < \alpha < 2$, we use the inequality $(x+y)^\alpha \geqslant x^\alpha + y^\alpha$, $x, y \geqslant 0$, and now we prove that $c^+_{n,m}$ and $c^-_{n,m}$ are superadditive. From Remark 2.6 we get that the limits

$$\lim_{n,m\to\infty} \frac{c^+_{n,m}}{nm}, \quad \lim_{n,m\to\infty} \frac{c^-_{n,m}}{nm}$$

exist, but now these limits are equal to the corresponding quantities, defined by changing $\lim_{n,m\to\infty}$ into $\sup_{(n,m)\in\mathbb{Z}^2_+}$. Using condition (5.3), we will show that these limits are finite. Let us denote $V_{n,m} = \{(i,j) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$. Using the association property, we can write

$$\begin{aligned} c^+_{n,m} + c^-_{n,m} &= \int_{\mathbb{S}^{nm-1}} \left| \sum_{\mathbf{i}\in V_{n,m}} s_{\mathbf{i}} \right|^\alpha \Gamma_{n,m}(\mathrm{d}\mathbf{s}) \qquad (5.15) \\ &= \int_{\mathbb{S}^{nm-1}} \left(\sum_{\mathbf{i}\in V_{n,m}} |s_{\mathbf{i}}| \right)^\alpha \Gamma_{n,m}(\mathrm{d}\mathbf{s}) \\ &\leq \int_{\mathbb{S}^{nm-1}} \left(\sum_{\mathbf{i}\in V_{n,m}} |s_{\mathbf{i}}| \right)^2 \Gamma_{n,m}(\mathrm{d}\mathbf{s}) \\ &= \sum_{\mathbf{i},\mathbf{j}\in V_{n,m}} \int_{\mathbb{S}^{nm-1}} s_{\mathbf{i}} s_{\mathbf{j}} \Gamma_{n,m}(\mathrm{d}\mathbf{s}) = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 = \sum_{\mathbf{i}\in V_{n,m}} \int_{\mathbb{S}^{nm-1}} s_{\mathbf{i}}^2 \Gamma_{n,m}(\mathrm{d}\mathbf{s}) \\ \leqslant \sum_{\mathbf{i}\in V_{n,m}} \int_{\mathbb{S}^{nm-1}} |s_{\mathbf{i}}|^\alpha \Gamma_{n,m}(\mathrm{d}\mathbf{s}) = nm(c^+_{1,1} + c^-_{1,1}) \qquad (5.16) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{\mathbf{i}\neq\mathbf{j}\in V_{n,m}} \int_{\mathbb{S}^{nm-1}} s_{\mathbf{i}} s_{\mathbf{j}} \Gamma_{n,m}(\mathrm{d}\mathbf{s}) \qquad (5.17) \\ &\leqslant \sum_{\mathbf{i}\neq\mathbf{j}\in V_{n,m}} \int_{\mathbb{S}^{nm-1}} \frac{s_{\mathbf{i}}}{\sqrt{s_{\mathbf{i}} + s_{\mathbf{j}}^2}} \frac{s_{\mathbf{j}}}{\sqrt{s_{\mathbf{i}}^2 + s_{\mathbf{j}}^2}} \left(s_{\mathbf{i}}^2 + s_{\mathbf{j}}^2\right)^{\alpha/2} \mathbb{1}_{\{s_{\mathbf{i}}^2 + s_{\mathbf{j}}^2 > 0\}} \Gamma_{n,m}(\mathrm{d}\mathbf{s}) \\ &\leqslant \sum_{\mathbf{i}\neq\mathbf{j}\in V_{n,m}} \rho(X_{\mathbf{i}}, X_{\mathbf{j}}) = \sum_{\mathbf{i}\neq\mathbf{j}\in V_{n,m}} \rho(X_{\mathbf{0}}, X_{\mathbf{i}-\mathbf{j}}) \\ &= nm \sum_{\mathbf{k}\in D_{n-1,m-1}} \rho(X_{\mathbf{0}}, X_{\mathbf{k}}) \leqslant nm \sum_{\mathbf{k}\in\mathbb{Z}^2} \rho(X_{\mathbf{0}}, X_{\mathbf{k}}), \end{aligned}$$

where $D_{n-1,m-1} = \{(i,j) \in \mathbb{Z}^2 : |i| \leqslant n-1, |j| \leqslant m-1, |i| + |j| > 0\}$. From (5.15)–(5.17) and condition (5.3) it follows that

$$\sup_{(n,m)\in\mathbb{Z}^2_+} \frac{c^+_{n,m} + c^-_{n,m}}{nm} < \infty.$$

It remains to note that, in this case, centering is needed since the shift parameter for $\bar{S}_{n,m}$ is $b(nm)^{1-1/\alpha}$, and (5.4) is proved.

The case $\alpha = 1$ is easy. As in [17], we can show that $c^+_{n,m}$ and $c^-_{n,m}$ are additive, and therefore, $c^+_{n,m} = nmc^+_{1,1}$ and $c^-_{n,m} = nmc^-_{1,1}$. In the case of symmetric spectral measure $\Gamma_{X_{1,1}}$, the parameters of $\bar{S}_{n,m}$ are $(b, c^+_{1,1}, c^-_{1,1})$, whereas in the general case centering may be needed. $\square$

Proof of Corollary 5.2. Since the coefficients c_k are non-negative, the investigated linear field is associated. We will show that

$$\sum_{(i,j)\in\mathbb{Z}^2} \rho_\alpha(i,j) < \infty.$$

This, together with Proposition 4.3, implies (5.3), and Theorem 5.1 gives the result stated in the corollary. Since the field is stationary, it suffices to show that

$$\sum_{i\geqslant 0}\sum_{j\in\mathbb{Z}} \rho_\alpha(i,j) < \infty. \tag{5.18}$$

According to Theorem 4.17, we have the asymptotic relation (4.37), which implies that $\sum_{i\geqslant N}\sum_{j\geqslant N}\rho_\alpha(i,sj) < 1$ for N large enough and $s\in\{-1,1\}$. Similarly as in the proof of Theorem 4.17, for fixed m, we have

$$\rho_\alpha(n,sm) \sim n^{-\gamma_1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lim_{n\to\infty} q_{i,j,n,m,s}(1+i)^{-\gamma_1}(1+j)^{-\gamma_2}(1+j+m)^{-\gamma_2},$$

as $n\to\infty$, and, for fixed n, we have

$$\rho_\alpha(n,sm) \sim m^{-\gamma_2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lim_{m\to\infty} q_{i,j,n,m,s}(1+i)^{-\gamma_1}(1+j)^{-\gamma_2}(1+i+n)^{-\gamma_1},$$

as $m\to\infty$. These relations imply that the row series and the column series are finite. Thus, (5.18) holds. $\square$

Proof of Theorem 5.3. The proof goes along the same lines as in Newman's CLT (see [47]), adapted to the case of infinite variance in [17]. First, we outline the proof. The main step is showing the following relation (recall that $\bar{S}_{n,m} = S_{n,m}(nm)^{-1/\alpha}$):

$$\lim_{m_1,m_2\to\infty}\limsup_{k_1,k_2\to\infty}\Big|\mathbb{E}\exp\left\{\mathrm{i}\lambda\bar{S}_{n_1,n_2}\right\} - \left(\mathbb{E}\exp\left\{\mathrm{i}\lambda\,(k_1k_2)^{-1/\alpha}\,\bar{S}_{m_1,m_2}\right\}\right)^{k_1k_2}\Big| = 0,\ \ \lambda\in\mathbb{R}. \tag{5.19}$$

Here and in the sequel, we assume that $n_i = m_i k_i,\ i = 1, 2$, with integer m_i and k_i. Since the second term in the difference in (5.19) is the power of a ch.f., it corresponds to the sum of i.i.d. summands. Thus, using the assumption $\{\mathbf{X}_\mathbf{i}\} \in \mathscr{D}_{\text{sn}}(\{\mathbf{Y}_\mathbf{i}\})$, we have

$$\left(\mathbb{E}\exp\left\{\mathrm{i}\lambda\,(k_1k_2)^{-1/\alpha}\,\bar{S}_{m_1,m_2}\right\}\right)^{k_1k_2} \to \mathbb{E}\exp\left\{\mathrm{i}\lambda\bar{Z}_{m_1,m_2}\right\} \tag{5.20}$$

as $(k_1, k_2) \to \infty$. Since $\bar{Z}_{m_1,m_2}$ is a stable random field, as a last step, we apply Theorem 5.1 to get (5.7). Finally, (5.7), together with (5.19) and (5.20), implies (5.6). Now we return to the main step in the proof. For $b > 0$, let us define the function $f_b : \mathbb{R} \to \mathbb{R}$ by $f_b(x) = b\mathbb{1}_{(b,\infty)}(x) + x\mathbb{1}_{[-b,b]}(x) - b\mathbb{1}_{(-\infty,-b)}(x)$. Since $f_b(x)$ is a non-decreasing function in x, the random field $\{f_b(X_\mathbf{i}),\ \mathbf{i} \in \mathbb{Z}^2\}$ is also associated. We use the decomposition

$$\bar{S}_{n_1,n_2} = Z^{(1)}_{n_1,n_2} + Z^{(2)}_{n_1,n_2},$$

where

$$Z^{(1)}_{n_1,n_2} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} f_b\left((n_1n_2)^{-1/\alpha}X_{i,j}\right),$$

$$Z^{(2)}_{n_1,n_2} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\left((n_1n_2)^{-1/\alpha}X_{i,j} - f_b\left((n_1n_2)^{-1/\alpha}X_{i,j}\right)\right).$$

Since the random variable $X_{i,j}$ belongs to the strict normal domain of attraction of a stable random variable $Y_{i,j}$, there exists $C > 0$ such that $\mathbb{P}(|X_{i,j}| > x) \leqslant Cx^{-\alpha}$. Let us take $\varepsilon > 0$ and $b > (C/\varepsilon)^{1/\alpha}$. Then we get

$$\begin{aligned} &\mathbb{P}(Z^{(2)}_{n_1,n_2} \neq 0)\\ &\leqslant \mathbb{P}(\exists 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2 : |X_{i,j}| > b(n_1n_2)^{1/\alpha})\\ &\leqslant n_1n_2\mathbb{P}(|X_{1,1}| > b(n_1n_2)^{1/\alpha})\\ &\leqslant Cn_1n_2b^{-\alpha}(n_1n_2)^{-1} = Cb^{-\alpha} < \varepsilon. \end{aligned} \tag{5.21}$$

We will show that

$$\left|\mathbb{E}\exp\left\{\mathrm{i}\lambda\bar{S}_{n_1,n_2}\right\} - \mathbb{E}\exp\left\{\mathrm{i}\lambda Z^{(1)}_{n_1,n_2}\right\}\right| < 2\varepsilon, \tag{5.22}$$

$$\left|\left(\mathbb{E}\exp\left\{\mathrm{i}\lambda\,(k_1k_2)^{-1/\alpha}\,\bar S_{m_1,m_2}\right\}\right)^{k_1k_2} - \right.$$
$$\left. - \left(\mathbb{E}\exp\left\{\mathrm{i}\lambda\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} f_b\left((n_1n_2)^{-1/\alpha}X_{i,j}\right)\right\}\right)^{k_1k_2}\right| < 2\varepsilon. \quad (5.23)$$

Inequality (5.22) follows from (5.21):

$$\begin{aligned}
&\left|\mathbb{E}\exp\left\{\mathrm{i}\lambda\left(Z^{(1)}_{n_1,n_2}+Z^{(2)}_{n_1,n_2}\right)\right\}-\mathbb{E}\exp\left\{\mathrm{i}\lambda Z^{(1)}_{n_1,n_2}\right\}\right|\\
&\leqslant \mathbb{E}\left|\exp\left\{\mathrm{i}\lambda Z^{(2)}_{n_1,n_2}\right\}-1\right|\\
&= \mathbb{E}\left|\left(\exp\left\{\mathrm{i}\lambda Z^{(2)}_{n_1,n_2}\right\}-1\right)\mathbb{1}_{\{Z^{(2)}_{n_1,n_2}\neq 0\}}\right|\\
&\leqslant 2\mathbb{P}\left(Z^{(2)}_{n_1,n_2}\neq 0\right) < 2\varepsilon.
\end{aligned} \quad (5.24)$$

For simplicity, let us introduce the notation

$$U_{m_1,m_2} := U_{m_1,m_2}(n_1,n_2) = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} f_b\left((n_1n_2)^{-1/\alpha}X_{i,j}\right),$$

$$U^{(1)}_{m_1,m_2} = (k_1k_2)^{-1/\alpha}\,\bar S_{m_1,m_2} - U_{m_1,m_2}.$$

Then (5.23) becomes

$$\Delta_1 := \left|\left(\mathbb{E}\exp\left\{\mathrm{i}\lambda\left(U_{m_1,m_2}+U^{(1)}_{m_1,m_2}\right)\right\}\right)^{k_1k_2} - \right.$$
$$\left. - \left(\mathbb{E}\exp\left\{\mathrm{i}\lambda U_{m_1,m_2}\right\}\right)^{k_1k_2}\right| < 2\varepsilon. \quad (5.25)$$

Using the inequality $|a^n-b^n|\leqslant n|a-b|$, $|a|\,,|b|\leqslant 1$, we can estimate

$$\Delta_1 \leqslant k_1k_2\left|\mathbb{E}\exp\left\{\mathrm{i}\lambda\left(U_{m_1,m_2}+U^{(1)}_{m_1,m_2}\right)\right\}-\mathbb{E}\exp\left(\mathrm{i}\lambda U_{m_1,m_2}\right)\right|. \quad (5.26)$$

Similarly to inequalities (5.21) and (5.24), we have

$$\mathbb{P}(U^{(1)}_{m_1,m_2}\neq 0)\leqslant m_1m_2\mathbb{P}(|X_{1,1}|>b(n_1n_2)^{1/\alpha})$$
$$\leqslant Cb^{-\alpha}(k_1k_2)^{-1} < \varepsilon(k_1k_2)^{-1} \quad (5.27)$$

and

$$\left|\mathbb{E}\exp\left\{\mathrm{i}\lambda\left(U_{m_1,m_2}+U^{(1)}_{m_1,m_2}\right)\right\}-\mathbb{E}\exp\left\{\mathrm{i}\lambda U_{m_1,m_2}\right\}\right|$$
$$\leqslant 2\mathbb{P}(U^{(1)}_{m_1,m_2}\neq 0) < 2(k_1k_2)^{-1}\varepsilon. \quad (5.28)$$

Collecting (5.26)–(5.28), we get (5.25) and, at the same time, (5.23).

Since the random variables in the sum U_{m_1,m_2} are bounded, we use (as in [17]) Newman's inequality to get the following estimate:

$$\Delta_2 := \left|\mathbb{E}\exp\{\mathrm{i}\lambda U_{n_1,n_2}\} - (\mathbb{E}\exp\{\mathrm{i}\lambda U_{m_1,m_2}\})^{k_1k_2}\right|$$

$$= \left|\mathbb{E}\exp\left\{\mathrm{i}\lambda\sqrt{n_1n_2}\frac{1}{\sqrt{n_1n_2}}U_{n_1,n_2}\right\}\right.$$
$$\left. - \left(\mathbb{E}\exp\left\{\mathrm{i}\lambda\sqrt{n_1n_2}\frac{1}{\sqrt{k_1k_2}}\frac{1}{\sqrt{m_1m_2}}U_{m_1,m_2}\right\}\right)^{k_1k_2}\right|$$
$$\leqslant \frac{\lambda^2 n_1n_2}{2}\left(\mathrm{Var}\left(\frac{1}{\sqrt{n_1n_2}}U_{n_1,n_2}\right) - \mathrm{Var}\left(\frac{1}{\sqrt{m_1m_2}}U_{m_1,m_2}\right)\right)$$
$$= \frac{\lambda^2 n_1n_2}{2}\left(\frac{1}{n_1n_2}\mathrm{Var}\left(U_{n_1,n_2}\right) - \frac{1}{m_1m_2}\mathrm{Var}\left(U_{m_1,m_2}\right)\right). \tag{5.29}$$

We must show that the last quantity can be made arbitrarily small if we take m_1, m_2 sufficiently large, and to this aim, we consider separately $\mathrm{Var}\left(U_{n_1,n_2}\right)$. Let us denote

$$C_{d_1,d_2}^{n_1,n_2} = \mathrm{Cov}\left(f_b\left((n_1n_2)^{-1/\alpha}X_{d_1+1,d_2+1}\right), f_b\left((n_1n_2)^{-1/\alpha}X_{1,1}\right)\right).$$

Then it is not difficult to verify that

$$\mathrm{Var}\left(U_{n_1,n_2}\right)$$
$$= \mathrm{Cov}\left(\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} f_b\left((n_1n_2)^{-1/\alpha}X_{i_1,i_2}\right), \sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2} f_b\left((n_1n_2)^{-1/\alpha}X_{j_1,j_2}\right)\right)$$
$$= \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\mathrm{Cov}\left(f_b\left((n_1n_2)^{-1/\alpha}X_{i_1,i_2}\right), f_b\left((n_1n_2)^{-1/\alpha}X_{j_1,j_2}\right)\right)$$
$$= \sum_{d_1=1-n_1}^{n_1-1}\sum_{d_2=1-n_2}^{n_2-1}(n_1-|d_1|)(n_2-|d_2|)C_{d_1,d_2}^{n_1,n_2}.$$

Similarly, we calculate $\mathrm{Var}\left(U_{m_1,m_2}\right)$:

$$\mathrm{Var}\left(U_{m_1,m_2}\right) = \sum_{d_1=1-m_1}^{m_1-1}\sum_{d_2=1-m_2}^{m_2-1}(m_1-|d_1|)(m_2-|d_2|)C_{d_1,d_2}^{n_1,n_2}. \tag{5.30}$$

From (5.29)–(5.30) we get $\Delta_2 \leqslant \lambda^2 n_1n_2\left(\Sigma_1+\Sigma_2\right)/2$, where

$$\Sigma_1 = \sum_{(d_1,d_2)\in B_1}\left(\frac{n_1-|d_1|}{n_1}\frac{n_2-|d_2|}{n_2} - \frac{m_1-|d_1|}{m_1}\frac{m_2-|d_2|}{m_2}\right)C_{d_1,d_2}^{n_1,n_2},$$

$$\Sigma_2 = \sum_{(d_1,d_2)\in B_2} \frac{n_1 - |d_1|}{n_1}\frac{n_2 - |d_2|}{n_2} C^{n_1,n_2}_{d_1,d_2}.$$

Here $B_1 = \{(d_1, d_2) \in \mathbb{Z}^2 : |d_1| < m_1,\ |d_2| < m_2\}$ and $B_2 = \{(d_1, d_2) \in \mathbb{Z}^2 : m_1 \leqslant |d_1| < n_1, |d_2| < n_2$ or $|d_1| < n_1, m_2 \leqslant |d_2| < n_2\}$. Since $n_i \geqslant m_i$ implies $(n_i - |d_i|)n_i^{-1} \geqslant (m_i - |d_i|)m_i^{-1}$, we have that the coefficients at $C^{n_1,n_2}_{d_1,d_2}$ in both sums are non-negative. Therefore, it suffices to estimate the quantity $C^{n_1,n_2}_{d_1,d_2}$ from above as follows:

$$\begin{aligned}
C^{n_1,n_2}_{d_1,d_2} &= \operatorname{Cov}\left(f_b\left((n_1n_2)^{-1/\alpha}X_{d_1+1,d_2+1}\right), f_b\left((n_1n_2)^{-1/\alpha}X_{1,1}\right)\right)\\
&= (n_1n_2)^{-2/\alpha}\operatorname{Cov}\left(f_{b(n_1n_2)^{1/\alpha}}\left(X_{d_1+1,d_2+1}\right), f_{b(n_1n_2)^{1/\alpha}}\left(X_{1,1}\right)\right)\\
&= (n_1n_2)^{-2/\alpha}\int_{-b(n_1n_2)^{1/\alpha}}^{b(n_1n_2)^{1/\alpha}}\int_{-b(n_1n_2)^{1/\alpha}}^{b(n_1n_2)^{1/\alpha}} H_{(X_{1,1},X_{d_1+1,d_2+1})}(x,y)\mathrm{d}x\mathrm{d}y\\
&\leqslant (n_1n_2)^{-2/\alpha}\left(b(n_1n_2)^{1/\alpha}\right)^{2-\alpha} I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1})\\
&= b^{2-\alpha}(n_1n_2)^{-1}I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1}).
\end{aligned}$$

From the last two obtained inequalities (for Δ_2 and $C^{n_1,n_2}_{d_1,d_2}$) it is not difficult to get $\Delta_2 \leqslant \lambda^2 b^{2-\alpha}(\Delta_{21} + \Delta_{22})/2$, where

$$\begin{aligned}
\Delta_{21} &= \sum_{(d_1,d_2)\in B_1}\left(1 - \frac{m_1 - |d_1|}{m_1}\frac{m_2 - |d_2|}{m_2}\right) I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1}),\\
\Delta_{22} &= \sum_{|d_1|\geqslant m_1 \text{ or } |d_2|\geqslant m_2} I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1}).
\end{aligned}$$

Here it is appropriate to note that both quantities $\Delta_{2i},\ i = 1, 2$, do not depend on n_1, n_2, and therefore, it is legitimate to consider the first lim sup in the main relation (5.19). Now we will show that these two quantities converge to zero as $(m_1, m_2) \to \infty$, and the main tool for doing this is our assumption (5.5). For any fixed d_1, d_2, we have $1 - (m_1 - |d_1|)(m_2 - |d_2|)(m_1m_2)^{-1} \to 0$ as $m_1, m_2 \to \infty$, the terms in the double sum of Δ_{21} can be dominated

$$\left|\left(1 - \frac{m_1 - |d_1|}{m_1}\frac{m_2 - |d_2|}{m_2}\right) I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1})\right| \leqslant I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1}),$$

so that the double sum $\sum_{(d_1,d_2)\in\mathbb{Z}^2} I^A_\alpha(X_{1,1}, X_{d_1+1,d_2+1})$ is convergent due to (5.5). Therefore, a standard application of dominated convergence theorem yields $\Delta_{21} \to 0$ as $m_1, m_2 \to \infty$. The relation $\Delta_{22} \to 0$ as $m_1, m_2 \to \infty$ follows simply from (5.5). From these two relations we get

$$\lim_{m_1,m_2\to\infty} \limsup_{k_1,k_2\to\infty} \Delta_2 = 0. \tag{5.31}$$

Collecting (5.22), (5.23), and (5.31), we get (5.19).

It remains to prove (5.7). For this, we use Theorem 5.1. In the case $0 < \alpha \le 1$, there is nothing to prove, and in the case $1 < \alpha < 2$, we show that (5.5) implies (3.17). We have that $(X_{\mathbf{i}}, X_{\mathbf{j}}) \in \mathcal{D}_{\rm sn}(Y_{\mathbf{i}}, Y_{\mathbf{j}})$. Let $\Gamma_{\mathbf{i},\mathbf{j}}$ be the spectral measure of $(Y_{\mathbf{i}}, Y_{\mathbf{j}})$, $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^2$. We will prove that

$$\int_{\mathbb{S}^1} s_1 s_2 \Gamma_{\mathbf{i},\mathbf{j}}(\mathrm{d}\mathbf{s}) \leqslant \frac{2-\alpha}{\alpha} I_\alpha^A(X_{\mathbf{i}}, X_{\mathbf{j}}). \tag{5.32}$$

Let $\nu_{\mathbf{i},\mathbf{j}}$ denote the Lévy measure of the stable vector $(Y_{\mathbf{i}}, Y_{\mathbf{j}})$. Using polar coordinates on the plane, we have

$$\int_{\|x\|\leqslant b} x_1 x_2 \nu_{\mathbf{i},\mathbf{j}}(\mathrm{d}x_1, \mathrm{d}x_2) = \int_0^b \int_{\mathbb{S}^1} r s_1 r s_2 \frac{\alpha \mathrm{d}r}{r^{\alpha+1}} \Gamma_{\mathbf{i},\mathbf{j}}(\mathrm{d}\mathbf{s}) = \frac{\alpha b^{2-\alpha}}{2-\alpha} \int_{\mathbb{S}^1} s_1 s_2 \Gamma_{\mathbf{i},\mathbf{j}}(\mathrm{d}\mathbf{s}).$$

Therefore,

$$\int_{\mathbb{S}^1} s_1 s_2 \Gamma_{\mathbf{i},\mathbf{j}}(\mathrm{d}\mathbf{s}) = \frac{2-\alpha}{\alpha} b^{\alpha-2} \int_{\|x\|\leqslant b} x_1 x_2 \nu_{\mathbf{i},\mathbf{j}}(\mathrm{d}x_1, \mathrm{d}x_2).$$

Since the stable random variables $Y_{\mathbf{i}}, Y_{\mathbf{j}}$ are associated, we can write

$$\int_{\|x\|\leqslant b} x_1 x_2 \nu_{\mathbf{i},\mathbf{j}}(\mathrm{d}x_1, \mathrm{d}x_2) \leqslant \int_{\mathbb{R}^2} f_b(x_1) f_b(x_2) \nu_{\mathbf{i},\mathbf{j}}(\mathrm{d}x_1, \mathrm{d}x_2).$$

Using Lemma 3.1 from [63] (see also (38) in [17]), denoting $b_n = b n^{1/\alpha}$, it is not difficult to show that, for all sufficiently large n and for A from condition (5.5), we get

$$\begin{aligned} & n\mathrm{Cov}\left(f_b(n^{-1/\alpha} X_{\mathbf{i}}), f_b(n^{-1/\alpha} X_{\mathbf{j}})\right) \\ = \; & n \int_{-b}^{b} \int_{-b}^{b} H_{\left(n^{-1/\alpha} X_{\mathbf{i}}, n^{-1/\alpha} X_{\mathbf{j}}\right)}(x, y) \,\mathrm{d}x \mathrm{d}y \\ = \; & n^{(\alpha-2)/\alpha} \int_{-b_n}^{b_n} \int_{-b_n}^{b_n} H_{(X_{\mathbf{i}}, X_{\mathbf{j}})}(u, v) \,\mathrm{d}u \mathrm{d}v \\ \leqslant \; & n^{(\alpha-2)/\alpha} b_n^{2-\alpha} I_\alpha^A(X_{\mathbf{i}}, X_{\mathbf{j}}) = b^{2-\alpha} I_\alpha^A(X_{\mathbf{i}}, X_{\mathbf{j}}). \end{aligned} \tag{5.33}$$

As in [17] (see (43) therein), we use the relation

$$n\mathrm{Cov}\left(f_b(n^{-1/\alpha}X_{\mathbf{i}}), f_b(n^{-1/\alpha}X_{\mathbf{j}})\right) \to \int_{\mathbb{R}^2} f_b(x_1)f_b(x_2)\nu_{\mathbf{i},\mathbf{j}}(\mathrm{d}x_1, \mathrm{d}x_2),$$

which, together with (5.33) (taking $b = 1$), gives (5.32). Thus, we proved (5.7), which, together with (5.19) and (5.20), proves (5.6). □

Proof of Theorem 5.5. Suppose $m \in \mathbb{N}$ and $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)} \in [0,\infty)^d$. In what follows we assume $\mathbf{t}^{(0)} = \mathbf{0}$. We need to show that

$$A_{\mathbf{n}}^{-1}\left(S_{\mathbf{n}}(\mathbf{t}^{(1)}), \dots, S_{\mathbf{n}}(\mathbf{t}^{(m)})\right) \xrightarrow{\mathscr{D}} \left(I(\mathbf{t}^{(1)}), \dots, I(\mathbf{t}^{(m)})\right),$$

as $\min(n_1, \dots, n_d) \to \infty$, and we do this by investigating the convergence of characteristic functions.

Before finding the characteristic function of

$$A_{\mathbf{n}}^{-1}\left(S_{\mathbf{n}}(\mathbf{t}^{(1)}), \dots, S_{\mathbf{n}}(\mathbf{t}^{(m)})\right) \tag{5.34}$$

let us express $S_{\mathbf{n}}(\mathbf{t})$ in a more convenient way:

$$\begin{aligned}
S_{\mathbf{n}}(\mathbf{t}) &= \sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}}\sum_{\mathbf{j}\geqslant\mathbf{0}} c_{\mathbf{j}}\xi_{\mathbf{k}-\mathbf{j}} \\
&= \sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}}\sum_{\mathbf{j}\geqslant\mathbf{0}} c_{\mathbf{j}-\mathbf{k}+\mathbf{k}}\xi_{\mathbf{k}-\mathbf{j}} \\
&= \sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}}\sum_{\mathbf{j}-\mathbf{k}\geqslant-\mathbf{k}} c_{\mathbf{j}-\mathbf{k}+\mathbf{k}}\xi_{\mathbf{k}-\mathbf{j}} \\
&= \sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}}\sum_{\mathbf{j}\leqslant\mathbf{k}} c_{\mathbf{k}-\mathbf{j}}\xi_{\mathbf{j}} \\
&= \sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}}\sum_{\mathbf{j}} c_{\mathbf{k}-\mathbf{j}}\xi_{\mathbf{j}}\prod_{l=1}^{d}\mathbb{1}_{\{j_l\leqslant k_l\}} \\
&= \sum_{\mathbf{j}}\xi_{\mathbf{j}}\sum_{\mathbf{0}\leqslant\mathbf{k}\leqslant\mathbf{nt}} c_{\mathbf{k}-\mathbf{j}}\prod_{l=1}^{d}\mathbb{1}_{\{j_l\leqslant k_l\}} \\
&= \sum_{\mathbf{j}}\xi_{\mathbf{j}}\sum_{\mathbf{k}} c_{\mathbf{k}-\mathbf{j}}\prod_{l=1}^{d}\mathbb{1}_{\{(j_l\vee 0)\leqslant k_l\leqslant n_l t_l\}}.
\end{aligned}$$

Characteristic function of (5.34) is

$$\phi_{\mathbf{n}}(x_1, \dots, x_m) = \mathbb{E}\exp\left(\mathrm{i}\sum_{i=1}^{m} x_i A_{\mathbf{n}}^{-1}S_{\mathbf{n}}(\mathbf{t}^{(i)})\right),$$

using the recently obtained expression of $S_{\mathbf{n}}(\mathbf{t})$ we can write

$$\sum_{i=1}^{m} x_i A_{\mathbf{n}}^{-1} S_{\mathbf{n}}(\mathbf{t}^{(i)}) = \sum_{\mathbf{j}} \xi_{\mathbf{j}} A_{\mathbf{n}}^{-1} \sum_{i=1}^{m} x_i \sum_{\mathbf{k}} c_{\mathbf{k}-\mathbf{j}} \prod_{l=1}^{d} \mathbb{1}_{\{(j_l \vee 0) \leqslant k_l \leqslant n_l t_l^{(i)}\}}$$

$$= \sum_{\mathbf{j}} \xi_{\mathbf{j}} A_{\mathbf{n}}^{-1} \sum_{i=1}^{m} x_i \sum_{\mathbf{k}} \prod_{l=1}^{d} a_{k_l - j_l}(\gamma_l, l) \mathbb{1}_{\{(j_l \vee 0) \leqslant k_l \leqslant n_l t_l^{(i)}\}}$$

$$= \sum_{\mathbf{j}} \xi_{\mathbf{j}} A_{\mathbf{n}}^{-1} \sum_{i=1}^{m} x_i \prod_{l=1}^{d} \sum_{k_l} a_{k_l - j_l}(\gamma_l, l) \mathbb{1}_{\{(j_l \vee 0) \leqslant k_l \leqslant n_l t_l^{(i)}\}}$$

$$= \sum_{\mathbf{j}} \xi_{\mathbf{j}} A_{\mathbf{n}}^{-1} \sum_{i=1}^{m} x_i \prod_{l=1}^{d} \sum_{k_l} a_{k_l}(\gamma_l, l) \mathbb{1}_{\{(0 \vee (-j_l)) \leqslant k_l \leqslant n_l t_l^{(i)} - j_l\}}.$$

It is convenient to introduce notation

$$S_{\gamma_l, l, t}(j, n) = \sum_{k} a_k(\gamma_l, l) \mathbb{1}_{\{(0 \vee (-j)) \leqslant k \leqslant nt - j\}},$$

$$C_{\mathbf{j},\mathbf{n}} = \left(\prod_{j=1}^{d} s_{n_j, \gamma_j, j} \right)^{-1} \sum_{i=1}^{m} x_i \prod_{l=1}^{d} S_{\gamma_l, l, t_l^{(i)}}(j_l, n_l)$$

and

$$D_{\mathbf{j},\mathbf{n}} = A_{\mathbf{n}}^{-1} \sum_{i=1}^{m} x_i \prod_{l=1}^{d} S_{\gamma_l, l, t_l^{(i)}}(j_l, n_l).$$

Since $\xi_{\mathbf{j}}$ are independent and have common characteristic function (2.13), we obtain

$$\phi_{\mathbf{n}}(x_1, \ldots, x_m)$$
$$= \exp\left(-\sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) \left(1 + r\left(D_{\mathbf{j},\mathbf{n}}\right)\right) |D_{\mathbf{j},\mathbf{n}}|^{\alpha} \left(1 - \mathrm{i}\beta\tau_\alpha \mathrm{sign}\left(D_{\mathbf{j},\mathbf{n}}\right)\right) \right).$$

The following lemma is proved on page 125:

Lemma 5.9. $D_{\mathbf{j},\mathbf{n}} \to 0$ *uniformly for all* $\mathbf{j}$, *as* $\min(n_1, \ldots, n_d) \to \infty$.

Due to this Lemma, it suffices to investigate the asymptotic behaviour of

$$\sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) |D_{\mathbf{j},\mathbf{n}}|^{\alpha} \left(1 - \mathrm{i}\beta\tau_\alpha \mathrm{sign}\left(D_{\mathbf{j},\mathbf{n}}\right)\right)$$
$$= \sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) |D_{\mathbf{j},\mathbf{n}}|^{\alpha} - \mathrm{i}\beta\tau_\alpha \sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) D_{\mathbf{j},\mathbf{n}}^{<\alpha>}. \quad (5.35)$$

We continue by investigating

$$J_{\mathbf{n}} := \sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) f\left(D_{\mathbf{j},\mathbf{n}}\right)$$

with $f(x) = |x|^{\alpha}$ and $f(x) = x^{<\alpha>}$.

In what follows $\delta = \min(\delta_1, \ldots, \delta_d)$, where

$$\delta_l = \begin{cases} \frac{\alpha - 1/\gamma_l}{2} \text{ if } \gamma_l > 1 \text{ and } \sum_k a_k(\gamma_l, l) \neq 0, \text{ or } 1/\alpha < \gamma_l \leqslant 1 \\ \frac{\min(\alpha - 1/\gamma_l,\ 1/(\gamma_l - 1) - \alpha)}{2} \text{ if } 1 < \gamma_l < 1 + 1/\alpha \text{ and } \sum_k a_k(\gamma_l, l) \neq 0. \end{cases}$$

For sets $G_i \subset \mathbb{R}, i = 1, \ldots, d$, we introduce notation

$$J_{\mathbf{n}}(G_1, \ldots, G_d) = \sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) f\left(D_{\mathbf{j},\mathbf{n}}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\mathbf{j})$$

and split $J_{\mathbf{n}}$ as

$$J_{\mathbf{n}} = \sum_{G_i \in \{\mathcal{A}_i, \mathcal{A}_i^c\}, i=1,\ldots,d} J_{\mathbf{n}}(G_1, \ldots, G_d), \tag{5.36}$$

where

$$\mathcal{A}_i = \mathcal{A}_i(\epsilon, n_i) = \bigcap_{j=0}^{m} \left(n_i t_i^{(j)} - n_i^{\epsilon}, n_i t_i^{(j)} + n_i^{\epsilon}\right)^c, i = 1, \ldots, d,$$

with $\epsilon = \min(\epsilon_1, \ldots, \epsilon_d)$, where

$$\epsilon_l = \begin{cases} \frac{1 - (\gamma_l - 1)(\alpha + \delta)}{2} \text{ if } 1 < \gamma_l < 1 + 1/\alpha \text{ and } \sum_{k=0}^{\infty} a_k(\gamma_l, l) = 0, \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

We shall show that

$$J_{\mathbf{n}}(\mathcal{A}_1, \ldots, \mathcal{A}_d) \to \int_{\mathbb{R}^d} f\left(\sum_{i=1}^{m} x_i \mathcal{H}(\mathbf{u}, \mathbf{t}^{(i)})\right) \mathrm{d}\mathbf{u} \tag{5.37}$$

and

$$J_{\mathbf{n}}(G_1, \ldots, G_d) \to 0 \tag{5.38}$$

if $G_i = \mathcal{A}_i^c$, for some $i = 1, \ldots, d$.

We have

$$J_{\mathbf{n}}(G_1, \ldots, G_d) = \sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) f\left(D_{\mathbf{j},\mathbf{n}}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\mathbf{j})$$

$$= \int_{\mathbb{R}^d} h\left(\left|D_{\lfloor \mathbf{u} \rfloor, \mathbf{n}}\right|^{-1}\right) f\left(D_{\lfloor \mathbf{u} \rfloor, \mathbf{n}}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{u} \rfloor) \mathrm{d}\mathbf{u}$$

$$= \int_{\mathbb{R}^d} h\left(\left|D_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{-1}\right) f\left(D_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{nu} \rfloor) \mathrm{d}\mathbf{nu}$$

$$= \left(\prod_{l=1}^d n_l\right) \int_{\mathbb{R}^d} h\left(\left|D_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{-1}\right) f\left(D_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{nu} \rfloor) \mathrm{d}\mathbf{u},$$

and, since

$$\left(\prod_{l=1}^d n_l\right) f\left(D_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right)$$

$$= f\left(\left(\prod_{l=1}^d n_l\right)^{1/\alpha} D_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right)$$

$$= f\left(h_{1/\alpha}^{-1/\alpha}\left(\prod_{l=1}^d n_l\right) C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right)$$

$$= \frac{1}{h_{1/\alpha}\left(\prod_{l=1}^d n_l\right)} f\left(C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right)$$

$$= \frac{h\left(\left(\prod_{l=1}^d n_l\right)^{1/\alpha} h_{1/\alpha}^{1/\alpha}\left(\prod_{l=1}^d n_l\right)\right)}{h_{1/\alpha}\left(\prod_{l=1}^d n_l\right)} \frac{f\left(C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right)}{h\left(\left(\prod_{l=1}^d n_l\right)^{1/\alpha} h_{1/\alpha}^{1/\alpha}\left(\prod_{l=1}^d n_l\right)\right)},$$

we obtain

$$J_{\mathbf{n}}(G_1, \ldots, G_d) = \frac{h\left(\left(\prod_{l=1}^d n_l\right)^{1/\alpha} h_{1/\alpha}^{1/\alpha}\left(\prod_{l=1}^d n_l\right)\right)}{h_{1/\alpha}\left(\prod_{l=1}^d n_l\right)} \int_{\mathbb{R}^d} F_{\mathbf{n}}(\mathbf{u}, G_1, \ldots, G_d) \mathrm{d}\mathbf{u}, \quad (5.39)$$

where

$$F_{\mathbf{n}}(\mathbf{u}, G_1, \ldots, G_d) = \frac{h\left(q_{\mathbf{n}}\left|C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{-1}\right)}{h\left(q_{\mathbf{n}}\right)} f\left(C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{nu} \rfloor),$$

with $q_{\mathbf{n}} = \left(\prod_{l=1}^d n_l\right)^{1/\alpha} h_{1/\alpha}^{1/\alpha}\left(\prod_{l=1}^d n_l\right)$.

The function $h_{1/\alpha}$ satisfies (2.2), therefore

$$\frac{h\left(\left(\prod_{l=1}^d n_l\right)^{1/\alpha} h_{1/\alpha}^{1/\alpha}\left(\prod_{l=1}^d n_l\right)\right)}{h_{1/\alpha}\left(\prod_{l=1}^d n_l\right)} \to 1, \text{ as } \min(n_1, \ldots, n_d) \to \infty,$$

and, due to (5.39), it is sufficient to investigate

$$I_{\mathbf{n}}(G_1, \ldots, G_d) := \int_{\mathbb{R}^d} F_{\mathbf{n}}(\mathbf{u}, G_1, \ldots, G_d) \mathrm{d}\mathbf{u}.$$

If $G_i = \mathcal{A}_i,\ i = 1, \ldots, d$, we will show that $F_{\mathbf{n}}$ converges point-wise and is dominated by an integrable function. We provide the proof of the following lemma on page 127.

Lemma 5.10. *Suppose $l \in \{1, \ldots, d\}$ and $t \in \{t_l^{(1)}, \ldots, t_l^{(m)}\}$, then*

$$\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor) \frac{S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)}{s_{n,\gamma_l,l}} \to H_{\gamma_l}(u,t,l), \ \text{as } n \to \infty, \tag{5.40}$$

and

$$\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor) \frac{|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)|}{s_{n,\gamma_l,l}} \leqslant G_{\gamma_l}(u,t,l),$$

where function G is such that

$$\int_{-\infty}^{\infty} G_{\gamma_l}^{\alpha+\delta}(u,t,l)\mathrm{d}u < \infty,\ \int_{-\infty}^{\infty} G_{\gamma_l}^{\alpha-\delta}(u,t,l)\mathrm{d}u < \infty. \tag{5.41}$$

Since

$$C_{\mathbf{nu},\mathbf{n}} = \sum_{i=1}^{m} x_i \prod_{l=1}^{d} \left(s_{n_l,\gamma_l,l}^{-1} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \right),$$

we have

$$\mathbb{1}_{\mathcal{A}_1 \times \cdots \times \mathcal{A}_d}(\lfloor \mathbf{nu} \rfloor) C_{\mathbf{nu},\mathbf{n}} \to \sum_{i=1}^{m} x_i \prod_{l=1}^{d} H_{\gamma_l}(u_l, t_l^{(i)}, l) = \sum_{i=1}^{m} x_i \mathcal{H}(\mathbf{u}, \mathbf{t}^{(i)}),$$

and, applying Lemma 2.4, we obtain

$$F_{\mathbf{n}}(\mathbf{u}, \mathcal{A}_1, \ldots, \mathcal{A}_d) \to f\left(\sum_{i=1}^{m} x_i \mathcal{H}(\mathbf{u}, \mathbf{t}^{(i)})\right). \tag{5.42}$$

Lemmas 5.9 and 2.3 imply that

$$\begin{aligned}
&|F_{\mathbf{n}}(\mathbf{u}, G_1, \ldots, G_d)| \\
&\qquad \leqslant 2 \max\left(\left|C_{\lfloor \mathbf{nu} \rfloor,\mathbf{n}}\right|^{\alpha-\delta}, \left|C_{\lfloor \mathbf{nu} \rfloor,\mathbf{n}}\right|^{\alpha+\delta} \right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{nu} \rfloor) \\
&\qquad\qquad \leqslant 2 \left(\left|C_{\lfloor \mathbf{nu} \rfloor,\mathbf{n}}\right|^{\alpha-\delta} + \left|C_{\lfloor \mathbf{nu} \rfloor,\mathbf{n}}\right|^{\alpha+\delta} \right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{nu} \rfloor). \quad (5.43)
\end{aligned}$$

We wish to show that the function on the right-hand side is dominated by an integrable function if $G_i = \mathcal{A}_i, i = 1, \ldots, d$.

We have

$$\mathbb{1}_{\mathcal{A}_1 \times \cdots \times \mathcal{A}_d}(\lfloor \mathbf{nu} \rfloor) \left| C_{\mathbf{nu},\mathbf{n}} \right|$$

$$= \left|\sum_{i=1}^m x_i \prod_{l=1}^d \left(s_{n_l,\gamma_l,l}^{-1} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{\mathcal{A}_l}(\lfloor n_l u_l \rfloor)\right)\right|$$

$$\leqslant \sum_{i=1}^m |x_i| \prod_{l=1}^d \left|s_{n_l,\gamma_l,l}^{-1} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{\mathcal{A}_l}(\lfloor n_l u_l \rfloor)\right|$$

$$\leqslant \sum_{i=1}^m |x_i| \prod_{l=1}^d G_{\gamma_l}\left(u_l, t_l^{(i)}, l\right),$$

according to Lemma 5.10. Thus

$$\left|C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{\alpha-\delta} \leqslant \left(|x_i| \prod_{l=1}^d G_{\gamma_l}\left(u_l, t_l^{(i)}, l\right)\right)^{\alpha-\delta}$$

$$\leqslant \left(m \max_i \left(|x_i| \prod_{l=1}^d G_{\gamma_l}\left(u_l, t_l^{(i)}, l\right)\right)\right)^{\alpha-\delta}$$

$$\leqslant m^{\alpha-\delta} \sum_{i=1}^m |x_i|^{\alpha-\delta} \prod_{l=1}^d G_{\gamma_l}^{\alpha-\delta}\left(u_l, t_l^{(i)}, l\right),$$

and, similarly,

$$\left|C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{\alpha+\delta} \leqslant m^{\alpha+\delta} \sum_{i=1}^m |x_i|^{\alpha+\delta} \prod_{l=1}^d G_{\gamma_l}^{\alpha+\delta}\left(u_l, t_l^{(i)}, l\right).$$

Hence,

$$|F_{\mathbf{n}}(\mathbf{u}, \mathcal{A}_1, \dots, \mathcal{A}_d)|$$
$$\leqslant 2m^{\alpha+\delta} \sum_{i=1}^m \left(|x_i|^{\alpha+\delta} \prod_{l=1}^d G_{\gamma_l}^{\alpha+\delta}\left(u_l, t_l^{(i)}, l\right) + |x_i|^{\alpha-\delta} \prod_{l=1}^d G_{\gamma_l}^{\alpha-\delta}\left(u_l, t_l^{(i)}, l\right)\right),$$

and the dominating function is integrable due to (5.41). This, together with (5.42), enables us to use the dominated convergence theorem. We obtain

$$I_{\mathbf{n}}(\mathcal{A}_1, \dots, \mathcal{A}_d) \to \int_{\mathbb{R}^d} f\left(\sum_{i=1}^m x_i \mathcal{H}(\mathbf{u}, \mathbf{t}^{(i)})\right) \mathrm{d}\mathbf{u}. \tag{5.44}$$

It remains to show that

$$I_{\mathbf{n}}(G_1, \dots, G_d) \to 0 \tag{5.45}$$

if $G_l \neq \mathcal{A}_l$ for some l. Inequality (5.43) implies

$$|I_{\mathbf{n}}(G_1, \dots, G_d)|$$
$$\leqslant \int_{\mathbb{R}^d} 2\left(\left|C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{\alpha-\delta} + \left|C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}}\right|^{\alpha+\delta}\right) \mathbb{1}_{G_1 \times \cdots \times G_d}(\lfloor \mathbf{nu} \rfloor) \mathrm{d}\mathbf{u}. \tag{5.46}$$

Let us show that

$$\int_{\mathbb{R}^d} \left| C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}} \right|^{\alpha-\delta} \mathbb{1}_{G_1\times\cdots\times G_d}(\lfloor \mathbf{nu} \rfloor) \mathrm{d}\mathbf{u} \to 0. \tag{5.47}$$

We have

$$\begin{aligned}
&\left| C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}} \right|^{\alpha-\delta} \mathbb{1}_{G_1\times\cdots\times G_d}(\lfloor \mathbf{nu} \rfloor) \\
&= \left| \sum_{i=1}^m x_i \prod_{l=1}^d \left(s^{-1}_{n_l,\gamma_l,l} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{G_l}(\lfloor n_l u_l \rfloor) \right) \right|^{\alpha-\delta} \\
&\leqslant \sum_{i=1}^m m^{\alpha-\delta} \left| x_i \prod_{l=1}^d \left(s^{-1}_{n_l,\gamma_l,l} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{G_l}(\lfloor n_l u_l \rfloor) \right) \right|^{\alpha-\delta} \\
&= \sum_{i=1}^m m^{\alpha-\delta} |x_i|^{\alpha-\delta} \prod_{l=1}^d \left| s^{-1}_{n_l,\gamma_l,l} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{G_l}(\lfloor n_l u_l \rfloor) \right|^{\alpha-\delta},
\end{aligned}$$

therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}} \right|^{\alpha-\delta} \mathbb{1}_{G_1\times\cdots\times G_d}(\lfloor \mathbf{nu} \rfloor) \mathrm{d}\mathbf{u} \\
&\quad \leqslant \sum_{i=1}^m m^{\alpha-\delta} |x_i|^{\alpha-\delta} \prod_{l=1}^d \int_{\mathbb{R}} \left| s^{-1}_{n_l,\gamma_l,l} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{G_l}(\lfloor n_l u_l \rfloor) \right|^{\alpha-\delta} \mathrm{d}u_l,
\end{aligned}$$

If $G_l = \mathcal{A}_l$, then, due to Lemma 5.10,

$$\begin{aligned}
&\int_{\mathbb{R}} \left| s^{-1}_{n_l,\gamma_l,l} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{G_l}(\lfloor n_l u_l \rfloor) \right|^{\alpha-\delta} \mathrm{d}u_l \\
&\qquad\qquad \leqslant \int_{\mathbb{R}} G_{\gamma_l}^{\alpha-\delta}(u_l, t_l^{(i)}, l) \mathrm{d}u_l < \infty.
\end{aligned}$$

and (5.47) follows from the following lemma, which is proved on page 141:

Lemma 5.11. *If $G_l = \mathcal{A}_l^c$, then*

$$\int_{\mathbb{R}} \left| s^{-1}_{n_l,\gamma_l,l} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \mathbb{1}_{G_l}(\lfloor n_l u_l \rfloor) \right|^{\alpha-\delta} \mathrm{d}u_l \to 0.$$

Similarly we can show that

$$\int_{\mathbb{R}^d} \left| C_{\lfloor \mathbf{nu} \rfloor, \mathbf{n}} \right|^{\alpha+\delta} \mathbb{1}_{G_1\times\cdots\times G_d}(\lfloor \mathbf{nu} \rfloor) \mathrm{d}\mathbf{u} \to 0,$$

therefore, the right-hand side of (5.46) converges to 0, implying (5.45).

Relations (5.44) and (5.45) imply (5.37) and (5.38). Recalling (5.36) we see that

$$J_{\mathbf{n}} \to \int_{\mathbb{R}^d} f\left(\sum_{i=1}^m x_i \mathcal{H}(\mathbf{u}, \mathbf{t}^{(i)}) \right) \mathrm{d}\mathbf{u}.$$

We are now in a position to find the limit of (5.35):

$$\sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right)|D_{\mathbf{j},\mathbf{n}}|^{\alpha} - \mathrm{i}\beta\tau_\alpha \sum_{\mathbf{j}} h\left(|D_{\mathbf{j},\mathbf{n}}|^{-1}\right) D_{\mathbf{j},\mathbf{n}}^{<\alpha>}$$
$$\to \int_{\mathbb{R}^d}\left|\sum_{i=1}^m x_i\mathcal{H}(\mathbf{u},\mathbf{t}^{(i)})\right|^{\alpha}\mathrm{d}\mathbf{u} - \mathrm{i}\beta\tau_\alpha\int_{\mathbb{R}^d}\left(\sum_{i=1}^m x_i\mathcal{H}(\mathbf{u},\mathbf{t}^{(i)})\right)^{<\alpha>}\mathrm{d}\mathbf{u},$$

which implies

$$\phi_{\mathbf{n}}(x_1,\ldots,x_m)$$
$$\to \exp\left(-\int_{\mathbb{R}^d}\left|\sum_{i=1}^m x_i\mathcal{H}(\mathbf{u},\mathbf{t}^{(i)})\right|^{\alpha}\left(1-\mathrm{i}\beta\tau_\alpha\mathrm{sign}\left(\sum_{i=1}^m x_i\mathcal{H}(\mathbf{u},\mathbf{t}^{(i)})\right)\right)\right)\mathrm{d}\mathbf{u}.$$

The limit is ch.f. of the vector $(I(\mathbf{t}^{(1)}),\ldots,I(\mathbf{t}^{(m)}))$. The proof is complete.

□

We provide proofs of lemmas used in the previous proof.

Proof of Lemma 5.9. Let us demonstrate that for every l there exists $K_l,\kappa_l > 0$ such that

$$n_l^{-1/\alpha}s_{n_l,\gamma_l,l}^{-1}\left|S_{\gamma_l,l,t^{(i)}}(\lfloor n_l u\rfloor, n_l)\right| \leqslant K_l n_l^{-\kappa_l}$$

for all $i=1,\ldots,m$, $u\in\mathbb{R}$ and $n_l\in\mathbb{N}$.

If $\gamma_l>1$ and $\sum_{j=0}^{\infty}a_j(\gamma_l,l)\neq 0$, then

$$n_l^{-1/\alpha}s_{n_l,\gamma_l,l}^{-1}\left|S_{\gamma_l,l,t}(\lfloor n_l u\rfloor, n_l)\right|$$
$$= n_l^{-1/\alpha}\left|S_{\gamma_l,l,t}(\lfloor n_l u\rfloor, n_l)\right|$$
$$\leqslant n_l^{-1/\alpha}\sum_{j=0}^{\infty}|a_j(\gamma_l,l)|,$$

and we can set $K_l=\sum_{j=0}^{\infty}|a_j(\gamma_l,l)|$, $\kappa_l = 1/\alpha$.

If $1<\gamma_l<1+1/\alpha$ and $\sum_{j=0}^{\infty}a_j(\gamma_l,l)=0$, then

$$n_l^{-1/\alpha}s_{n_l,\gamma_l,l}^{-1}\left|S_{\gamma_l,l,t}(\lfloor n_l u\rfloor, n_l)\right|$$
$$= n_l^{-1/\alpha}n_l^{\gamma_l-1}L_l^{-1}(n_l)\left|S_{\gamma_l,l,t}(\lfloor n_l u\rfloor, n_l)\right|$$
$$\leqslant n_l^{\gamma_l-1-1/\alpha}L_l^{-1}(n_l)\sum_{j=0}^{\infty}|a_j(\gamma_l,l)|.$$

As L_l is a s.v.f. and $\gamma_l - 1 - 1/\alpha < 0$, there exists a constant $p_l > 0$ such that $L_l(n_l) \geqslant p_l n_l^{(\gamma_l - 1 - 1/\alpha)/2}$ for $n_l \in \mathbb{N}$. Therefore,

$$n_l^{-1/\alpha} s_{n_l,\gamma_l,l}^{-1} \left| S_{\gamma_l,l,t}(\lfloor n_l u \rfloor, n_l) \right| \leqslant n_l^{(\gamma_l - 1 - 1/\alpha)/2} p_l^{-1} \sum_{j=0}^{\infty} |a_j(\gamma_l, l)|$$

and we can set $K_l = p_l^{-1} \sum_{j=0}^{\infty} |a_j(\gamma_l, l)|$ and $\kappa_l = (1 + 1/\alpha - \gamma_l)/2$.

It remains to deal with the case $1/\alpha < \gamma \leqslant 1$. The sequence $s_{n_l,\gamma_l,l}$ is of the form $n_l^{1-\gamma} \tilde{L}_l(n_l)$, with a s.v.f. $\tilde{L}_l$, thus,

$$\begin{aligned}
& n_l^{-1/\alpha} s_{n_l,\gamma_l,l}^{-1} \left| S_{\gamma_l,l,t}(\lfloor n_l u \rfloor, n_l) \right| \\
&= n_l^{\gamma_l - 1 - 1/\alpha} \tilde{L}_l^{-1}(n_l) \left| \sum_k a_k(\gamma_l, l) \mathbb{1}_{\{(0 \vee (-\lfloor n_l u \rfloor)) \leqslant k \leqslant n_l t - \lfloor n_l u \rfloor\}} \right| \\
&\leqslant n_l^{\gamma_l - 1 - 1/\alpha} \tilde{L}_l^{-1}(n_l) \sum_k |a_k(\gamma_l, l)| \mathbb{1}_{\{(0 \vee (-\lfloor n_l u \rfloor)) \leqslant k \leqslant n_l t - \lfloor n_l u \rfloor\}}.
\end{aligned}$$

Using the fact that there exists a constant $p_l > 0$ such that $\tilde{L}_l(n_l) \geqslant p_l n_l^{-1/(4\alpha)}$, and applying Lemma 2.2 with $\eta = 1/(4\alpha)$, we obtain

$$\begin{aligned}
& n_l^{-1/\alpha} s_{n_l,\gamma_l,l}^{-1} \left| S_{\gamma_l,l,t}(\lfloor n_l u \rfloor, n_l) \right| \\
&\leqslant p_l^{-1} E n_l^{\gamma_l - 1 - 1/\alpha} n_l^{1/(4\alpha)} \sum_k (1 + k)^{1/(4\alpha) - \gamma_l} \mathbb{1}_{\{(0 \vee (-\lfloor n_l u \rfloor)) \leqslant k \leqslant n_l t - \lfloor n_l u \rfloor\}} \\
&\leqslant p_l^{-1} E n_l^{\gamma_l - 1 - 3/(4\alpha)} \int_{-\infty}^{\infty} (1 + \lfloor v \rfloor)^{1/(4\alpha) - \gamma_l} \mathbb{1}_{\{(0 \vee (-\lfloor n_l u \rfloor)) \leqslant v \leqslant n_l t - \lfloor n_l u \rfloor + 1\}} \mathrm{d}v \\
&\leqslant p_l^{-1} E n_l^{\gamma_l - 1 - 3/(4\alpha)} \int_{-\infty}^{\infty} v^{1/(4\alpha) - \gamma_l} \mathbb{1}_{\{(0 \vee (-n_l u)) \leqslant v \leqslant n_l t - n_l u + 2n_l\}} \mathrm{d}v \\
&= p_l^{-1} E n_l^{\gamma_l - 1 - 3/(4\alpha)} \int_{-\infty}^{\infty} (n_l v)^{1/(4\alpha) - \gamma_l} \mathbb{1}_{\{(0 \vee (-n_l u)) \leqslant n_l v \leqslant n_l t - n_l u + 2n_l\}} \mathrm{d}n_l v \\
&= p_l^{-1} E n_l^{-1/(2\alpha)} \int_{-\infty}^{\infty} v^{1/(4\alpha) - \gamma_l} \mathbb{1}_{\{(0 \vee (-u)) \leqslant v \leqslant t - u + 2\}} \mathrm{d}v \\
&= p_l^{-1} E n_l^{-1/(2\alpha)} \frac{(t - u + 2)_+^{1 + 1/(4\alpha) - \gamma_l} - (-u)_+^{1 + 1/(4\alpha) - \gamma_l}}{1 + 1/(4\alpha) - \gamma_l},
\end{aligned}$$

here E is from (2.3). Therefore, we can set $\kappa_l = 1/(2\alpha)$ and $K_l = \max(K_l(i), i = 1, \ldots, m)$, where

$$K_l(i) = \max_u \left(p_l^{-1} E \frac{(t_l^{(i)} - u + 2)_+^{1 + 1/(4\alpha) - \gamma_l} - (-u)_+^{1 + 1/(4\alpha) - \gamma_l}}{1 + 1/(4\alpha) - \gamma_l} \right).$$

The maximum exists and is finite, since the function is continuous and its limits as $u \to \pm\infty$ are 0.

Let us denote $K = \max(K_1, \dots, K_d)$ and $\kappa = \min(\kappa_1, \dots, \kappa_d)$, then

$$n_l^{-1/\alpha} s_{n_l,\gamma_l,l}^{-1} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u \rfloor, n_l) \leqslant K n_l^{-\kappa},$$

thus

$$\begin{aligned}\left|D_{\lfloor \mathbf{nu} \rfloor,\mathbf{n}}\right| &= q_{\mathbf{n}}^{-1} \left|C_{\lfloor \mathbf{nu} \rfloor,\mathbf{n}}\right| \\ \leqslant h_{1/\alpha}^{-1/\alpha} \left(\prod_{l=1}^{d} n_l\right) &\sum_{i=1}^{m} |x_i| \prod_{l=1}^{d} \left(n_l^{1/\alpha} s_{n_l,\gamma_l,l}^{-1} \left|S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l)\right|\right) \\ &\leqslant \sum_{i=1}^{m} |x_i| K^d h_{1/\alpha}^{-1/\alpha} \left(\prod_{l=1}^{d} n_l\right) \left(\prod_{l=1}^{d} n_l\right)^{-\kappa} \to 0,\end{aligned}$$

as $\min(n_1, \dots, n_d) \to \infty$, since $h_{1/\alpha}$ is a s.v.f. and $\kappa > 0$. The limit is uniform for all $\mathbf{u} \in \mathbb{R}^d$. □

Proof of Lemma 5.10. It is easy to see that $S_{\gamma_l,l,t}(\lfloor nu \rfloor, n) = 0$ if $u \geqslant t+1$, therefore, in what follows we assume $u < t + 1$.

If $\gamma_l > 1$ we have $\sum_{j=0}^{\infty} |a_j(\gamma_l, l)| < \infty$. Assuming $\sum_{j=0}^{\infty} a_j(\gamma_l, l) \neq 0$, almost surely we have

$$S_{\gamma_l,l,t}(\lfloor nu \rfloor, n) \to \mathbb{1}_{[0,t)}(u) \sum_{k=0}^{\infty} a_k(\gamma_l, l) = H_{\gamma_l}(u, t, l).$$

If $u \geqslant -1$, we can bound

$$|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)| = \left| \sum_{k=0 \vee (-\lfloor nu \rfloor)}^{nt - \lfloor nu \rfloor} a_k(\gamma_l, l) \right| \leqslant \mathbb{1}_{[-1,t+1)}(u) \sum_{k=0}^{\infty} |a_k(\gamma_l, l)|.$$

Suppose $\eta = \min(\gamma_l - 1,\ \gamma_l - 1/(\alpha - \delta))/2$. Notice that the choice of δ implies $\eta > 0$. Applying Lemma 2.2, for $u < -1$ we get

$$|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)| \leqslant \sum_{k=-\lfloor nu \rfloor}^{nt - \lfloor nu \rfloor} |a_k(\gamma_l, l)|$$

$$\begin{aligned}&= \int_{-\lfloor nu \rfloor}^{nt - \lfloor nu \rfloor + 1} \left|a_{\lfloor v \rfloor}(\gamma_l, l)\right| \mathrm{d}v \leqslant E \int_{-\lfloor nu \rfloor}^{nt - \lfloor nu \rfloor + 1} (1 + \lfloor v \rfloor)^{\eta - \gamma_l} \mathrm{d}v \\ &\leqslant E \int_{-nu}^{nt - nu + 2} v^{\eta - \gamma_l} \mathrm{d}v \leqslant E \int_{-nu}^{n(t - u + 2)} v^{\eta - \gamma_l} \mathrm{d}v\end{aligned}$$

$$= E\frac{n^{1+\eta-\gamma_l}}{\gamma_l - 1 - \eta}\left((-u)^{1+\eta-\gamma_l} - (t-u+2)^{1+\eta-\gamma_l}\right)$$

$$\leqslant \frac{E}{\gamma_l - 1 - \eta}\left((-u)^{1+\eta-\gamma_l} - (t-u+2)^{1+\eta-\gamma_l}\right).$$

Therefore, the following inequality holds

$$|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)| \leqslant G_{\gamma_l}(u,t,l) := \begin{cases} \mathbb{1}_{[-1,t+1)}(u)\sum_{k=0}^{\infty}|a_k(\gamma_l,l)|, & u \geqslant -1, \\ \frac{E\left((-u)^{1+\eta-\gamma_l}-(t-u+2)^{1+\eta-\gamma_l}\right)}{\gamma_l-1-\eta}, & u < -1. \end{cases}$$

Let us show that (5.41) holds. Function G_{γ_l} is bounded on $[-1, t+1)$, and equals 0 if $u \geqslant t+1$, therefore

$$\int_{-1}^{\infty} G_{\gamma_l}^{\alpha\pm\delta}(u,t,l)\mathrm{d}u < \infty.$$

Function G_{γ_l} is continuous on $(-\infty, -1)$,

$$\lim_{u\uparrow -1} G_{\gamma_l}(u,t,l) = \frac{E\left(1-(t+3)^{1+\eta-\gamma_l}\right)}{\gamma_l - 1 - \eta},$$

and

$$G_{\gamma_l}(u,t,l) \sim E_1(-u)^{\eta-\gamma_l}, \text{ as } u \to -\infty,$$

with some constant E_1, therefore, in order for (5.41) to hold, we must have $(\eta-\gamma_l)(\alpha-\delta) < -1$ and $(\eta-\gamma_l)(\alpha+\delta) < -1$. But those inequalities do hold, since $\eta < \gamma_l - 1/(\alpha-\delta)$.

Next we look at the case $1/\alpha < \gamma_l < 1$. We begin by investigating the point-wise convergence of $s_{n,\gamma_l,l}^{-1}S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)$, with $s_{n,\gamma_l,l}$ defined by (5.10). For convenience of writing we introduce the notation

$$b(k,u,n) = a_k(\gamma_l,l)\mathbb{1}_{\{(0\vee(-\lfloor nu \rfloor))\leqslant k\leqslant nt-\lfloor nu \rfloor\}}.$$

We split $S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)$ into two terms

$$S_{\gamma_l,l,t}(\lfloor nu \rfloor, n) = \sum_{k=0}^{\lfloor n^\varepsilon \rfloor - 1} b(k,u,n) + \sum_{k=\lfloor n^\varepsilon \rfloor}^{\infty} b(k,u,n) =: Z_1 + Z_2, \quad (5.48)$$

here $0 < \varepsilon < 1 - \gamma_l$ is a fixed number. Applying Lemma 2.2 with $\eta = \gamma_l$ we obtain the following bound

$$|Z_1| \leqslant \sum_{k=0}^{\lfloor n^\varepsilon \rfloor - 1} |b(k,u,n)| \leqslant \sum_{k=0}^{\lfloor n^\varepsilon \rfloor - 1} |a_k(\gamma_l, l)|$$
$$\leqslant E \sum_{k=0}^{\lfloor n^\varepsilon \rfloor - 1} (1+k)^{\gamma_l - \gamma_l} = E \lfloor n^\varepsilon \rfloor \leqslant E n^\varepsilon,$$

which implies

$$\frac{Z_1}{s_{n,\gamma_l,l}} \to 0, \ n \to \infty, \tag{5.49}$$

uniformly for all u.

We now turn to Z_2:

$$Z_2 = \sum_{k=\lfloor n^\varepsilon \rfloor}^{\infty} b(k,u,n) = \int_{\lfloor n^\varepsilon \rfloor}^{\infty} b(\lfloor v \rfloor, u, n) \mathrm{d}v = n \int_{\frac{\lfloor n^\varepsilon \rfloor}{n}}^{\infty} b(\lfloor nv \rfloor, u, n) \mathrm{d}v$$
$$= n \int_0^\infty \mathbb{1}_{\left(\frac{\lfloor n^\varepsilon \rfloor}{n}, \infty\right)}(v) b(\lfloor nv \rfloor, u, n) \mathrm{d}v.$$

It is easy to see that almost surely

$$\mathbb{1}_{\left(\frac{\lfloor n^\varepsilon \rfloor}{n}, \infty\right)}(v) \to \mathbb{1}_{(0,\infty)}(v),$$

$$\mathbb{1}_{\{(0 \vee (-\lfloor nu \rfloor)) \leqslant \lfloor nv \rfloor \leqslant nt - \lfloor nu \rfloor\}} \to \mathbb{1}_{\{(0 \vee (-u)) \leqslant v \leqslant t-u\}}, \ n \to \infty.$$

From (5.9) we know that

$$\frac{a_{\lfloor nv \rfloor}(\gamma_l, l)}{(1 + \lfloor nv \rfloor)^{-\gamma_l} L_l(\lfloor nv \rfloor)} \to 1, \ n \to \infty, \tag{5.50}$$

and by Lemma 2.4, for $v > 0$, we have

$$\frac{(1 + \lfloor nv \rfloor)^{-\gamma_l} L_l(\lfloor nv \rfloor)}{n^{-\gamma_l} L_l(n)} = \left(\frac{1 + \lfloor nv \rfloor}{n} \right)^{-\gamma_l} \frac{L_l(n \frac{\lfloor nv \rfloor}{n})}{L_l(n)} \to v^{-\gamma_l}, \tag{5.51}$$

as $n \to \infty$. From the above we conclude that almost surely

$$\frac{1}{n^{-\gamma_l} L_l(n)} \mathbb{1}_{\left(\frac{\lfloor n^\varepsilon \rfloor}{n}, \infty\right)}(v) b(\lfloor nv \rfloor, u, n) \tag{5.52}$$

$$= \frac{(1 + \lfloor nv \rfloor)^{-\gamma_l} L_l(\lfloor nv \rfloor)}{n^{-\gamma_l} L_l(n)} \mathbb{1}_{\left(\frac{\lfloor n^\varepsilon \rfloor}{n}, \infty\right)}(v) \frac{a_{\lfloor nv \rfloor}(\gamma_l, l) \mathbb{1}_{\{(0 \vee (-\lfloor nu \rfloor)) \leqslant \lfloor nv \rfloor \leqslant nt - \lfloor nu \rfloor\}}}{(1 + \lfloor nv \rfloor)^{-\gamma_l} L_l(\lfloor nv \rfloor)}$$

$$\to v^{-\gamma_l}\mathbb{1}_{\{(0\vee(-u))\leqslant v\leqslant t-u\}}.$$

Our next objective is to show that the absolute value of (5.52) is bounded above by an integrable function.

If $\lfloor nv\rfloor \geqslant 0$, then $v \geqslant 0$. Also, if $\lfloor nv\rfloor \leqslant nt - \lfloor nu\rfloor$, then $nv \leqslant \lfloor nv\rfloor + 1 \leqslant nt - \lfloor nu\rfloor + 1 \leqslant nt - nu + 2 \leqslant n(t-u+1)$ for $n \geqslant 2$. Therefore, $v \leqslant t - u + 1$. By the above

$$\mathbb{1}_{\{(0\vee(-\lfloor nu\rfloor))\leqslant\lfloor nv\rfloor\leqslant nt-\lfloor nu\rfloor\}} \leqslant \mathbb{1}_{\{0\leqslant v\leqslant t-u+1\}}.$$

Relation (5.9) implies the existence of N such that

$$\frac{|a_k(\gamma_l,l)|}{(1+k)^{-\gamma_l}L_l(k)} \leqslant 2$$

for $k > N$. If $n > (N+2)^{1/\varepsilon}$, we have $\lfloor nv\rfloor > N$ for all $v > \lfloor n^\varepsilon\rfloor/n$ and, thus,

$$\mathbb{1}_{\left(\frac{\lfloor n^\varepsilon\rfloor}{n},\infty\right)}(v)\frac{\left|a_{\lfloor nv\rfloor}(\gamma_l,l)\right|}{(1+\lfloor nv\rfloor)^{-\gamma_l}L_l(\lfloor nv\rfloor)} \leqslant 2$$

for all $v > 0$. It is clear that

$$\frac{(1+\lfloor nv\rfloor)^{-\gamma_l}}{n^{-\gamma_l}} \leqslant v^{-\gamma_l},$$

and it remains to deal with $L_l(\lfloor nv\rfloor)/L_l(n)$. Theorem 2.1, applied with $f = L_l$, $A = 2$ and $\eta = \min(1-\gamma_l, \gamma_l - 1/(\alpha-\delta))/2$, implies the existence of B such that

$$\frac{L_l(y)}{L_l(x)} \leqslant 2\max\left(\left(\frac{y}{x}\right)^\eta, \left(\frac{y}{x}\right)^{-\eta}\right), \; x,y > B. \tag{5.53}$$

For $v \in (\lfloor n^\varepsilon\rfloor/n, \infty)$ we have $\lfloor nv\rfloor > n^\varepsilon - 2$, therefore, if $n > (B+2)^{1/\varepsilon}$ we have $\lfloor nv\rfloor > B$. If, additionally, $n > B$, (5.53) implies

$$\mathbb{1}_{\left(\frac{\lfloor n^\varepsilon\rfloor}{n},\infty\right)}(v)\frac{L_l(\lfloor nv\rfloor)}{L_l(n)} \leqslant 2\mathbb{1}_{\left(\frac{\lfloor n^\varepsilon\rfloor}{n},\infty\right)}(v)\max\left(\left(\frac{\lfloor nv\rfloor}{n}\right)^\eta, \left(\frac{\lfloor nv\rfloor}{n}\right)^{-\eta}\right)$$

$$\leqslant 2\max\left(v^\eta, \left(\frac{v}{2}\right)^{-\eta}\right) \leqslant 2^{1+\eta}\max\left(v^\eta, v^{-\eta}\right)$$

for all $v > 0$. We have thus obtained the following inequality

$$\frac{|b(\lfloor nv \rfloor, u, n)|}{n^{-\gamma_l} L_l(n)} \mathbb{1}_{\left(\frac{\lfloor n^\varepsilon \rfloor}{n}, \infty\right)}(v) \leqslant 2^{2+\eta} v^{-\gamma_l} \max\left(v^\eta, v^{-\eta}\right) \mathbb{1}_{\{0 \leqslant v \leqslant t-u+1\}}. \quad (5.54)$$

The function on the right hand side is integrable in $(0, \infty)$, therefore the dominated convergence theorem implies

$$\frac{Z_2}{s_{n,\gamma_l,l}} \to \int_0^\infty v^{-\gamma_l} \mathbb{1}_{\{(0 \vee (-u)) \leqslant v \leqslant t-u\}} \mathrm{d}v$$

$$= \frac{(t-u)_+^{1-\gamma_l} - (-u)_+^{1-\gamma_l}}{1-\gamma_l} = H_{\gamma_l}(u, t, l).$$

Recalling (5.48) and (5.49), we get

$$\frac{S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)}{s_{n,\gamma_l,l}} \to H_{\gamma_l}(u, t, l).$$

We can now proceed to showing that

$$s_{n,\gamma_l,l}^{-1} \left|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)\right| \quad (5.55)$$

is bounded by a function $G_{\gamma_l}(u, t, l)$ satisfying (5.41).

For $-1 \leqslant u < t+1$ we split $S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)$ as in (5.48). We conclude from (5.49) that $s_{n,\gamma_l,l}^{-1} |Z_1| < 1$ for large n. From (5.54) we obtain

$$s_{n,\gamma_l,l}^{-1} |Z_2| \leqslant 2^{2+\eta} \int_0^\infty v^{-\gamma_l} \max\left(v^\eta, v^{-\eta}\right) \mathbb{1}_{\{0 \leqslant v \leqslant t-u+1\}} \mathrm{d}v$$

$$\leqslant 2^{2+\eta} \int_0^{t+2} v^{-\gamma_l} \max\left(v^\eta, v^{-\eta}\right) \mathrm{d}v.$$

As the integral is finite and does not depend on u, we conclude that, for $-1 \leqslant u < t+1$, (5.55) is bounded by a finite constant C.

Suppose $u < 1$, then

$$\left|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)\right| \leqslant \sum_{k=0}^\infty |b(k, u, n)| = \sum_{k=0}^\infty \mathbb{1}_{\{-\lfloor nu \rfloor \leqslant k \leqslant nt - \lfloor nu \rfloor\}} |a_k(\gamma_l, l)|$$

$$= \int_0^\infty \mathbb{1}_{\{-\lfloor nu \rfloor \leqslant \lfloor v \rfloor \leqslant nt - \lfloor nu \rfloor\}} \left|a_{\lfloor v \rfloor}(\gamma_l, l)\right| \mathrm{d}v$$

$$= n \int_0^\infty \mathbb{1}_{\{-\lfloor nu \rfloor \leqslant \lfloor nv \rfloor \leqslant nt - \lfloor nu \rfloor\}} \left|a_{\lfloor nv \rfloor}(\gamma_l, l)\right| \mathrm{d}v.$$

Since $\lfloor nv \rfloor \geqslant - \lfloor nu \rfloor \geqslant n \to \infty$, similarly to (5.54) we obtain

$$\frac{\left|a_{\lfloor nv \rfloor}\right|}{n^{-\gamma_l} L_l(n)} \mathbb{1}_{\{-\lfloor nu \rfloor \leqslant \lfloor nv \rfloor \leqslant nt - \lfloor nu \rfloor\}} \leqslant 2^{2+\eta} v^{\eta - \gamma_l} \mathbb{1}_{\{-u \leqslant v \leqslant t-u+2\}},$$

therefore

$$\frac{S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)}{s_{n,\gamma_l,l}} \leqslant 2^{2+\eta} \int_0^\infty v^{\eta-\gamma_l} \mathbb{1}_{\{-u \leqslant v \leqslant t-u+2\}} \mathrm{d}v$$
$$= 2^{2+\delta} \frac{(t-u+2)^{1+\eta-\gamma_l} - (-u)^{1+\eta-\gamma_l}}{1+\eta-\gamma_l}.$$

Denoting

$$G_{\gamma_l}(u,t,l) = \begin{cases} C \text{ if } -1 \leqslant u < t+1, \\ 2^{2+\eta} \frac{(t-u+2)^{1+\eta-\gamma_l} - (-u)^{1+\eta-\gamma_l}}{1+\eta-\gamma_l} \text{ if } u < -1, \\ 0 \text{ elsewhere,} \end{cases}$$

we get a function dominating (5.55). Let us show that (5.41) holds. Function G_{γ_l} is constant on $[-1, t+1)$, and equals 0 if $u \geqslant t+1$, therefore

$$\int_{-1}^\infty G_{\gamma_l}^{\alpha \pm \delta}(u,t,l) \mathrm{d}u < \infty.$$

Function G_{γ_l} is continuous on $(-\infty, -1)$,

$$\lim_{u \uparrow -1} G_{\gamma_l}(u,t,l) = \frac{2^{2+\eta}\left((t+3)^{1+\eta-\gamma_l} - 1\right)}{1+\eta-\gamma_l},$$

and

$$G_{\gamma_l}(u,t,l) \sim E_2(-u)^{\eta-\gamma_l}, \text{ as } u \to -\infty,$$

with some constant E_2, therefore, in order for (5.41) to hold, we must have $(\eta - \gamma_l)(\alpha - \delta) < -1$ and $(\eta - \gamma_l)(\alpha + \delta) < -1$. But those inequalities do hold, since $\eta < \gamma_l - 1/(\alpha - \delta)$.

We now turn to the case $\gamma_l = 1$. As was previously mentioned, in this case we make the assumption $L_l \equiv 1$. It follows that

$$|a_j(\gamma_l, l)|\,(1+j) \leqslant E_3$$

for some constant E_3, since by (5.9) we have $a_j(\gamma_l, l)(1+j) \to 1$.

We separate the first term in $S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)$,

$$S_{\gamma_l,l,t}(\lfloor nu \rfloor, n) = \sum_{k=0\vee(-\lfloor nu \rfloor)}^{nt-\lfloor nu \rfloor} a_k(\gamma_l, l)$$

$$= a_{0\vee(-\lfloor nu \rfloor)}(\gamma_l, l)\mathbb{1}_{\{0\vee(-\lfloor nu \rfloor)\leqslant nt-\lfloor nu \rfloor\}} + \sum_{k=1+0\vee(-\lfloor nu \rfloor)}^{nt-\lfloor nu \rfloor} a_k(\gamma_l, l)$$

and denote

$$\tilde{S}_{\gamma_l,l,t}(\lfloor nu \rfloor, n) = \sum_{k=1+0\vee(-\lfloor nu \rfloor)}^{nt-\lfloor nu \rfloor} a_k(\gamma_l, l). \tag{5.56}$$

The quantity $\left|a_{0\vee(-\lfloor nu \rfloor)}(\gamma_l, l)\mathbb{1}_{\{0\vee(-\lfloor nu \rfloor)\leqslant nt-\lfloor nu \rfloor\}}\right|$ can be bounded by

$$R(u) = \begin{cases} 0 \text{ if } u \geqslant t+1, \\ E_3 \text{ if } -2 \leqslant u < t+1, \\ E_3(-u)^{-1}, \text{ if } u < -2. \end{cases}$$

This bound implies

$$\frac{\left|a_{0\vee(-\lfloor nu \rfloor)}(\gamma_l, l)\mathbb{1}_{\{0\vee(-\lfloor nu \rfloor)\leqslant nt-\lfloor nu \rfloor\}}\right|}{\ln n} \to 0. \tag{5.57}$$

We continue by examining (5.56). Introducing notation

$$\tilde{b}(k, u, n) = a_k(\gamma_l, l)\mathbb{1}_{\{1+(0\vee(-\lfloor nu \rfloor))\leqslant k\leqslant nt-\lfloor nu \rfloor\}},$$

we have

$$\begin{aligned}\tilde{S}_{\gamma_l,l,t}(\lfloor nu \rfloor, n) &= \sum_{k=0}^{\infty} \tilde{b}(k, u, n) = \int_0^{\infty} \tilde{b}(\lfloor v \rfloor, u, n)\mathrm{d}v \\ &= \int_{-\infty}^{\infty} \tilde{b}(\lfloor n^v \rfloor, u, n)\mathrm{d}n^v \\ &= \ln n \int_{-\infty}^{\infty} \tilde{b}(\lfloor n^v \rfloor, u, n)n^v\mathrm{d}v \\ &= \ln n \int_0^{\infty} \tilde{b}(\lfloor n^v \rfloor, u, n)n^v\mathrm{d}v,\end{aligned}$$

the last equality holds since for $v < 0$ we have $\tilde{b}(\lfloor n^v \rfloor, u, n) = 0$. Let us investigate a.s. convergence of the function under the integral.

Suppose $u \notin \{0,t\}$. We have

$$\mathbb{1}_{\{1+(0\vee(-\lfloor nu\rfloor))\leqslant\lfloor n^v\rfloor\leqslant nt-\lfloor nu\rfloor\}} \to \begin{cases} \mathbb{1}_{(0,t)}(u),\ 0<v<1, \\ 0,\ v>1, \end{cases}$$

therefore, by (5.9),

$$\tilde{b}(\lfloor n^v\rfloor, u, n)n^v \to \mathbb{1}_{(0,t)}(u)\mathbb{1}_{(0,1)}(v).$$

If $\lfloor n^v\rfloor \leqslant nt - \lfloor nu\rfloor$, for $n \geqslant 2$ we have

$$n^v \leqslant \lfloor n^v\rfloor + 1 \leqslant nt - \lfloor nu\rfloor + 1 \leqslant nt - nu + 2 \leqslant n(t-u+1),$$

and if $1 + (0 \vee (-\lfloor nu\rfloor)) \leqslant \lfloor n^v\rfloor$,

$$n^v \geqslant \lfloor n^v\rfloor \geqslant 1 + (0 \vee (-\lfloor nu\rfloor)) \geqslant 1 + (0 \vee (-nu)) \geqslant (1 \vee (-nu)),$$

therefore, for $n \geqslant 3$,

$$\begin{aligned} \mathbb{1}_{\{1+(0\vee(-\lfloor nu\rfloor))\leqslant\lfloor n^v\rfloor\leqslant nt-\lfloor nu\rfloor\}} &\leqslant \mathbb{1}_{\{1\vee(-nu)\leqslant n^v\leqslant n(t-u+1)\}} \\ &= \mathbb{1}_{\{\ln(1\vee(-nu))\leqslant v\ln n\leqslant \ln n+\ln(t-u+1)\}} \\ &\leqslant \mathbb{1}_{\{0\leqslant v\leqslant 1+\ln(t-u+1)\}}, \end{aligned} \tag{5.58}$$

This implies

$$\left|\tilde{b}(\lfloor n^v\rfloor, u, n)n^v\right| \leqslant E_3\mathbb{1}_{\{0\leqslant v\leqslant 1+\ln(t-u+1)\}}.$$

The dominating function is integrable, therefore, the dominated convergence theorem implies

$$\frac{\tilde{S}_{\gamma_l,l,t}(\lfloor nu\rfloor, n)}{\ln n} \to \int_0^\infty \mathbb{1}_{(0,t)}(u)\mathbb{1}_{(0,1)}(v)\mathrm{d}v = \mathbb{1}_{(0,t)}(u).$$

Recalling (5.57) we obtain

$$\frac{S_{\gamma_l,l,t}(\lfloor nu\rfloor, n)}{\ln n} \to \mathbb{1}_{(0,t)}(u).$$

With the help of (5.58) we can also bound $(\ln n)^{-1}\left|\tilde{S}_{\gamma_l,l,t}(\lfloor nu\rfloor, n)\right|$. For $-2 \leqslant u < t+1$ and $n \geqslant 3$ we have

$$\frac{\left|\tilde{S}_{\gamma_l,l,t}(\lfloor nu\rfloor, n)\right|}{\ln n} \leqslant E_3\int_0^\infty \mathbb{1}_{\{0\leqslant v\ln n\leqslant \ln n+\ln(t-u+2)\}}\mathrm{d}v$$

$$= E_3 \frac{\ln n + \ln(t-u+2)}{\ln n} \leqslant E_3(1+\ln(t+2-u)),$$

and for $u < -2$,

$$\frac{\left|\tilde{S}_{\gamma_l,l,t}(\lfloor nu \rfloor, n)\right|}{\ln n} \leqslant \int_0^\infty \left|\tilde{b}(\lfloor n^v \rfloor, u, n)\right| n^v \mathrm{d}v$$
$$\leqslant E_3 \int_0^\infty \mathbb{1}_{\{\ln(-nu) \leqslant v \ln n \leqslant \ln n + \ln(t-u+1)\}} \mathrm{d}v$$
$$= E_3 \frac{\ln n + \ln(t-u+1) - \ln(-nu)}{\ln n}$$
$$= E_3 \frac{\ln(t-u+1) - \ln(-u)}{\ln n}$$
$$\leqslant E_3 \left(\ln(t-u+1) - \ln(-u)\right).$$

Let us denote

$$G_{\gamma_l}(u,t,l) = \begin{cases} 0 \text{ if } u \geqslant t+1, \\ E_3(1+\ln(t+2-u)) + R(u) \text{ if } -2 \leqslant u < t+1, \\ E_3\left(\ln(t-u+1) - \ln(-u)\right) + R(u) \text{ if } u < -2. \end{cases}$$

This function satisfies

$$s_{n,\gamma_l,l}^{-1} \left|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)\right| \leqslant G_{\gamma_l}(u,t,l),$$

let us show that it also satisfies (5.41). Function $G_{\gamma_l}(\cdot,t,l)$ is bounded on $[-2, t+1)$, and equals 0 if $u \geqslant t+1$, therefore

$$\int_{-2}^\infty G_{\gamma_l}^{\alpha \pm \delta}(u,t,l) \mathrm{d}u < \infty.$$

Function $G_{\gamma_l}(\cdot,t,l)$ is continuous on $(-\infty,-2)$,

$$\lim_{u \uparrow -2} G_{\gamma_l}(u,t,l) = E_3\left(\ln(t+3) - \ln(2)\right) + R(-2),$$

and

$$G_{\gamma_l}(u,t,l) \sim E_4(-u)^{-1}, \text{ as } u \to -\infty,$$

with some constant E_4. Hence, in order for (5.41) to hold, we must have $-(\alpha-\delta) < -1$ and $-(\alpha+\delta) < -1$. These inequalities do hold, since $\delta < \alpha - 1$.

We now move to the case $1 < \gamma_l < 1 + 1/\alpha$, $\sum_{k=0}^{\infty} a_k(\gamma_l, l) = 0$, and examine

$$\frac{S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)}{n^{1-\gamma_l} L_l(n)} \mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor),$$

assuming that $t \in \left\{t_l^{(j)}, j = 1, \ldots, m\right\}$.

If $u \geqslant t$, and $\lfloor nu \rfloor \in \mathcal{A}_l$ we have $\lfloor nu \rfloor \geqslant nt + n^{\epsilon}$, therefore $nt - \lfloor nu \rfloor \leqslant -n^{\epsilon} < 0$, and $S_{\gamma_l,l,t}(\lfloor nu \rfloor, n) = 0$.

Suppose $0 < u < t$, $u \notin \left\{t_l^{(j)}, j = 1, \ldots, m\right\}$, then

$$\begin{aligned} S_{\gamma_l,l,t}(\lfloor nu \rfloor, n) = \sum_{k=0}^{nt-\lfloor nu \rfloor} a_k(\gamma_l, l) &= -\sum_{k=\lfloor nt \rfloor - \lfloor nu \rfloor + 1}^{\infty} a_k(\gamma_l, l) \\ &= -\int_0^{\infty} a_{\lfloor v \rfloor}(\gamma_l, l) \mathbb{1}_{\{\lfloor nt \rfloor - \lfloor nu \rfloor + 1 \leqslant \lfloor v \rfloor\}} \mathrm{d}v \\ &= -n \int_0^{\infty} a_{\lfloor nv \rfloor}(\gamma_l, l) \mathbb{1}_{\{\lfloor nt \rfloor - \lfloor nu \rfloor + 1 \leqslant \lfloor nv \rfloor\}} \mathrm{d}v \\ &= -n^{1-\gamma_l} L_l(n) \int_0^{\infty} \frac{a_{\lfloor nv \rfloor}(\gamma_l, l)}{n^{-\gamma_l} L_l(n)} \mathbb{1}_{\{\lfloor nt \rfloor - \lfloor nu \rfloor + 1 \leqslant \lfloor nv \rfloor\}} \mathrm{d}v \\ &= -n^{1-\gamma_l} L_l(n) \int_0^{\infty} \kappa_n^{(1)}(v) \mathrm{d}v, \end{aligned}$$

where

$$\kappa_n^{(1)}(v) = \frac{a_{\lfloor nv \rfloor}(\gamma_l, l)}{n^{-\gamma_l} L_l(n)} \mathbb{1}_{\{\lfloor nt \rfloor - \lfloor nu \rfloor + 1 \leqslant \lfloor nv \rfloor\}}.$$

Almost surely we have

$$\mathbb{1}_{\{\lfloor nt \rfloor - \lfloor nu \rfloor + 1 \leqslant \lfloor nv \rfloor\}} = \mathbb{1}_{\{(\lfloor nt \rfloor - \lfloor nu \rfloor + 1)/n \leqslant \lfloor nv \rfloor / n\}} \to \mathbb{1}_{(t-u,\infty)}(v),$$

and for $v > 0$

$$\begin{aligned} &\frac{a_{\lfloor nv \rfloor}(\gamma_l, l)}{n^{-\gamma_l} L_l(n)} \\ &\qquad = \frac{a_{\lfloor nv \rfloor}(\gamma_l, l)}{(1 + \lfloor nv \rfloor)^{-\gamma_l} L_l(\lfloor nv \rfloor)} \frac{(1 + \lfloor nv \rfloor)^{-\gamma_l} L_l(\lfloor nv \rfloor)}{n^{-\gamma_l} L_l(n)} \to v^{-\gamma_l} \end{aligned} \tag{5.59}$$

by (5.50) and (5.51). Thus,

$$\kappa_n^{(1)}(v) \to \mathbb{1}_{(t-u,\infty)}(v) v^{-\gamma_l}.$$

Also, we have

$$\mathbb{1}_{\mathcal{A}_l}(\lfloor nu\rfloor) = \prod_{j=0}^{m} \mathbb{1}_{\left(nt_l^{(j)}-n^\epsilon, nt_l^{(j)}+n^\epsilon\right)^c}(\lfloor nu\rfloor) \to 1,$$

since $|u - t_l^{(j)}| > 0$ for all $j = 0, \dots, m$.

Suppose

$$\eta = \frac{\min(\gamma_l - 1, \gamma_l - 1/(\alpha-\delta), 1-\gamma_l + 1/(\alpha+\delta))}{2},$$

Theorem 2.1, applied with $f = L_l$ and $A = 2$, implies the existence of B such that

$$\frac{L_l(y)}{L_l(x)} \leqslant 2\max\left(\left(\frac{y}{x}\right)^\eta, \left(\frac{y}{x}\right)^{-\eta}\right), \quad x, y > B. \tag{5.60}$$

If $\lfloor nu\rfloor \in \mathcal{A}_l(\epsilon, n)$, we have $nt - \lfloor nu\rfloor \geqslant n^\epsilon$. For such u, and v satisfying $\lfloor nv\rfloor \geqslant \lfloor nt\rfloor - \lfloor nu\rfloor + 1$ we have $\lfloor nv\rfloor \geqslant nt - \lfloor nu\rfloor \geqslant n^\epsilon$. Therefore, if $n^\epsilon > B$, we have

$$\begin{aligned}
&\mathbb{1}_{\mathcal{A}_l}(\lfloor nu\rfloor)\mathbb{1}_{\{\lfloor nt\rfloor - \lfloor nu\rfloor + 1 \leqslant \lfloor nv\rfloor\}}\frac{L_l(\lfloor nv\rfloor)}{L_l(n)} \\
&\leqslant 2\mathbb{1}_{\{\lfloor nt\rfloor - \lfloor nu\rfloor + 1 \leqslant \lfloor nv\rfloor\}} \max\left(\left(\frac{\lfloor nv\rfloor}{n}\right)^\eta, \left(\frac{\lfloor nv\rfloor}{n}\right)^{-\eta}\right) \\
&\leqslant 2^{1+\eta}\mathbb{1}_{(t-u,\infty)}(v)\max\left(v^\eta, v^{-\eta}\right) \\
&\leqslant 2^{1+\eta}\mathbb{1}_{(t-u,\infty)}(v)\left(v^\eta + v^{-\eta}\right).
\end{aligned}$$

There exists a constant E_5 such that

$$\left|\frac{a_{\lfloor nv\rfloor}(\gamma_l, l)}{(1+\lfloor nv\rfloor)^{-\gamma_l}L_l(\lfloor nv\rfloor)}\right| \leqslant E_5. \tag{5.61}$$

The inequalities above imply that for large n and all $0 < u < t,\ v > 0$ we have

$$\mathbb{1}_{\mathcal{A}_l}(\lfloor nu\rfloor)\left|\kappa_n^{(1)}(v)\right| \leqslant E_5 2^{1+\eta}\mathbb{1}_{(t-u,\infty)}(v)v^{-\gamma_l}\left(v^\eta + v^{-\eta}\right). \tag{5.62}$$

The dominating function is integrable, thus

$$\begin{aligned}
\frac{S_{\gamma_l,l,t}(\lfloor nu\rfloor, n)}{n^{1-\gamma_l}L_l(n)}\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu\rfloor) &= -\mathbb{1}_{\mathcal{A}_l}(\lfloor nu\rfloor)\int_0^\infty \kappa_n^{(1)}(v)\mathrm{d}v \\
&\to -\int_0^\infty \mathbb{1}_{(t-u,\infty)}(v)v^{-\gamma_l}\mathrm{d}v = \frac{(t-u)^{1-\gamma_l}}{1-\gamma_l} = H_{\gamma_l}(u,t,l),
\end{aligned}$$

by the dominated convergence theorem.

From (5.62) we also obtain the following inequality:

$$\frac{|S_{\gamma_l,l,t}(\lfloor nu\rfloor,n)|}{n^{1-\gamma_l}L_l(n)}\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu\rfloor)\leqslant \mathbb{1}_{\mathcal{A}_l}(\lfloor nu\rfloor)\int_0^\infty\left|\kappa_n^{(1)}(v)\right|\mathrm{d}v$$

$$\leqslant E_5 2^{1+\eta}\int_0^\infty \mathbb{1}_{(t-u,\infty)}(v)v^{-\gamma_l}\left(v^\eta+v^{-\eta}\right)\mathrm{d}v$$

$$= E_5 2^{1+\eta}\left(\frac{(t-u)^{1-\gamma_l+\eta}}{1-\gamma_l+\eta}+\frac{(t-u)^{1-\gamma_l-\eta}}{1-\gamma_l-\eta}\right)$$

$$= E_5 2^{1+\eta}\left(H_{\gamma_l-\eta}(u,t,l)+H_{\gamma_l+\eta}(u,t,l)\right).$$

Suppose $u<0$, then

$$S_{\gamma_l,l,t}(\lfloor nu\rfloor,n)=\sum_{k=-\lfloor nu\rfloor}^{nt-\lfloor nu\rfloor}a_k(\gamma_l,l)$$

$$=\int_0^\infty a_{\lfloor v\rfloor}(\gamma_l,l)\mathbb{1}_{\{-\lfloor nu\rfloor\leqslant\lfloor v\rfloor\leqslant\lfloor nt\rfloor-\lfloor nu\rfloor\}}\mathrm{d}v$$

$$=n\int_0^\infty a_{\lfloor nv\rfloor}(\gamma_l,l)\mathbb{1}_{\{-\lfloor nu\rfloor\leqslant\lfloor nv\rfloor\leqslant\lfloor nt\rfloor-\lfloor nu\rfloor\}}\mathrm{d}v$$

$$=n^{1-\gamma_l}L_l(n)\int_0^\infty \frac{a_{\lfloor nv\rfloor}(\gamma_l,l)}{n^{-\gamma_l}L_l(n)}\mathbb{1}_{\{-\lfloor nu\rfloor\leqslant\lfloor nv\rfloor\leqslant\lfloor nt\rfloor-\lfloor nu\rfloor\}}\mathrm{d}v$$

$$=n^{1-\gamma_l}L_l(n)\int_0^\infty \kappa_n^{(2)}(v)\mathrm{d}v,$$

where

$$\kappa_n^{(2)}(v)=\frac{a_{\lfloor nv\rfloor}(\gamma_l,l)}{n^{-\gamma_l}L_l(n)}\mathbb{1}_{\{-\lfloor nu\rfloor\leqslant\lfloor nv\rfloor\leqslant\lfloor nt\rfloor-\lfloor nu\rfloor\}}.$$

Almost surely we have

$$\mathbb{1}_{\{-\lfloor nu\rfloor\leqslant\lfloor nv\rfloor\leqslant\lfloor nt\rfloor-\lfloor nu\rfloor\}}=\mathbb{1}_{\{-\lfloor nu\rfloor/n\leqslant\lfloor nv\rfloor/n\leqslant(\lfloor nt\rfloor-\lfloor nu\rfloor)/n\}}\to\mathbb{1}_{(-u,t-u)}(v),$$

and for $v>0$ (5.59) holds, therefore, almost surely,

$$\kappa_n^{(2)}(v)\to\mathbb{1}_{(-u,t-u)}(v)v^{-\gamma_l}.$$

If $\lfloor nu \rfloor \in \mathcal{A}_l(\epsilon, n)$, we have $\lfloor nu \rfloor \leqslant -n^\epsilon$, therefore $-\lfloor nu \rfloor \geqslant n^\epsilon$. From (5.60), if $n^\epsilon > B$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\mathcal{A}_l}(\lfloor nu \rfloor)\mathbb{1}_{\{-\lfloor nu \rfloor \leqslant \lfloor nv \rfloor \leqslant \lfloor nt \rfloor - \lfloor nu \rfloor\}} \frac{L_l(\lfloor nv \rfloor)}{L_l(n)} \\
&\qquad \leqslant 2\mathbb{1}_{\{-\lfloor nu \rfloor \leqslant \lfloor nv \rfloor \leqslant \lfloor nt \rfloor - \lfloor nu \rfloor\}} \max\left(\left(\frac{\lfloor nv \rfloor}{n}\right)^\eta, \left(\frac{\lfloor nv \rfloor}{n}\right)^{-\eta}\right) \\
&\qquad\qquad \leqslant 2^{1+\eta}\mathbb{1}_{(-u,t-u+1)}(v) \max\left(v^\eta, v^{-\eta}\right). \\
&\qquad\qquad\qquad \leqslant 2^{1+\eta}\mathbb{1}_{(-u,t-u+1)}(v)\left(v^\eta + v^{-\eta}\right).
\end{aligned}$$

The inequalities above, with (5.61), imply that for large n and all $u < 0$, $v > 0$ we have

$$\mathbb{1}_{\mathcal{A}_l}(\lfloor nu \rfloor)\left|\kappa_n^{(2)}(v)\right| \leqslant E_5 2^{1+\eta}\mathbb{1}_{(-u,t-u+1)}(v) v^{-\gamma_l}\left(v^\eta + v^{-\eta}\right). \tag{5.63}$$

The dominating function is integrable, thus

$$\begin{aligned}
&\frac{S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)}{n^{1-\gamma_l} L_l(n)}\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor) = \mathbb{1}_{\mathcal{A}_l}(\lfloor nu \rfloor)\int_0^\infty \kappa_n^{(2)}(v)\mathrm{d}v \\
&\qquad \to \int_0^\infty \mathbb{1}_{(-u,t-u)}(v) v^{-\gamma_l}\mathrm{d}v = \frac{(t-u)^{1-\gamma_l} - (-u)^{1-\gamma_l}}{1-\gamma_l} = H_{\gamma_l}(u,t,l).
\end{aligned}$$

From (5.63) we obtain

$$\begin{aligned}
\frac{|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)|}{n^{1-\gamma_l} L_l(n)}\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor) &\leqslant \int_0^\infty \mathbb{1}_{\mathcal{A}_l}(\lfloor nu \rfloor)\left|\kappa_n^{(2)}(v)\right|\mathrm{d}v \\
&\leqslant E_5 2^{1+\eta}\int_0^\infty \mathbb{1}_{(-u,t-u+1)}(v)\left(v^{-\gamma_l+\eta} + v^{-\gamma_l-\eta}\right)\mathrm{d}v \\
&= E_5 2^{1+\eta}\left(H_{\gamma_l-\eta}(u,t+1,l) + H_{\gamma_l+\eta}(u,t+1,l)\right).
\end{aligned}$$

Summarizing, we have shown that for almost every u

$$\frac{S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)}{n^{1-\gamma_l} L_l(n)}\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor) \to H_{\gamma_l}(u,t,l),$$

and

$$\frac{|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)|}{n^{1-\gamma_l} L_l(n)}\mathbb{1}_{\mathcal{A}_l(\epsilon,n)}(\lfloor nu \rfloor) \leqslant G_{\gamma_l}(u,t,l),$$

where

$$G_{\gamma_l}(u,t,l) = \begin{cases} E_5 2^{1+\eta}\left(H_{\gamma_l-\eta}(u,t+1,l) + H_{\gamma_l+\eta}(u,t+1,l)\right) \text{ if } u<0, \\ E_5 2^{1+\eta}\left(H_{\gamma_l-\eta}(u,t,l) + H_{\gamma_l+\eta}(u,t,l)\right) \text{ if } 0<u<t, \\ 0 \text{ if } u>t. \end{cases}$$

It remains to show that (5.41) holds. Function $G_{\gamma_l}(\cdot,t,l)$ is continuous on the intervals $(-\infty,0)$ and $(0,t)$, and $G_{\gamma_l}(u,t,l)=0$ for $u>t$. Therefore, we only need to investigate the behaviour of $G_{\gamma_l}(\cdot,t,l)$ at the endpoints of intervals $(-\infty,0)$ and $(0,t)$.

We begin with the interval $(-\infty,0)$. As $u\to-\infty$ we have

$$G_{\gamma_l}(u,t,l) \sim E_6(-u)^{-\gamma_l+\eta},$$

and, as $u\uparrow 0$,

$$G_{\gamma_l}(u,t,l) \sim E_7(-u)^{1-\gamma_l-\eta},$$

with some constants E_6, E_7. In order to have

$$\int_{-\infty}^{0} G_{\gamma_l}^{\alpha\pm\delta}(u,t,l)\mathrm{d}u < \infty,$$

we need inequalities $(-\gamma_l+\eta)(\alpha\pm\delta)<-1$ and $(1-\gamma_l-\eta)(\alpha\pm\delta)>-1$ to hold. They do hold, since

$$\eta<\gamma_l-\frac{1}{\alpha-\delta},\ \eta<1-\gamma_l+\frac{1}{\alpha+\delta}$$

by the choice of η.

Let us investigate $(0,t)$ now. As $u\uparrow t$ we have

$$G_{\gamma_l}(u,t,l) \sim E_8(t-u)^{1-\gamma_l-\eta},$$

with some $E_8\in\mathbb{R}$, and, as $u\downarrow 0$, $G_{\gamma_l}(u,t,l)$ converges to a constant. Hence,

$$\int_0^t G_{\gamma_l}^{\alpha\pm\delta}(u,t,l)\mathrm{d}u < \infty$$

holds if $(1-\gamma_l-\eta)(\alpha\pm\delta)>-1$. This is the same inequality as before, and it was already shown that it holds.

The proof is complete.

$\square$

Proof of Lemma 5.11. Due to the fact that

$$G_l=\mathcal{A}_l^c=\bigcup_{j=0}^{m}(n_lt_l^{(j)}-n_l^{\epsilon},n_lt_l^{(j)}+n_l^{\epsilon}),$$

we have

$$\begin{aligned}\int_{\mathbb{R}}\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\mathbb{1}_{G_l}(\lfloor n_lu_l\rfloor)\right|^{\alpha-\delta}\mathrm{d}u_l\\ \leqslant\sum_{j=0}^{m}\int_{\mathbb{R}}\mathbb{1}_{(n_it_i^{(j)}-n_i^{\epsilon},n_it_i^{(j)}+n_i^{\epsilon})}(\lfloor n_lu_l\rfloor)\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\right|^{\alpha-\delta}\mathrm{d}u_l.\end{aligned}$$

Inequality

$$\begin{aligned}\mathbb{1}_{(n_lt_l^{(j)}-n_l^{\epsilon},n_lt_l^{(j)}+n_l^{\epsilon})}(\lfloor n_lu_l\rfloor)\leqslant\mathbb{1}_{(n_lt_l^{(j)}-n_l^{\epsilon},n_lt_l^{(j)}+n_l^{\epsilon}+1)}(n_lu_l)\\ \leqslant\mathbb{1}_{(n_lt_l^{(j)}-n_l^{\epsilon},n_lt_l^{(j)}+2n_l^{\epsilon})}(n_lu_l)=\mathbb{1}_{(t_l^{(j)}-n_l^{\epsilon-1},t_l^{(j)}+2n_l^{\epsilon-1})}(u_l)\end{aligned}$$

implies

$$\begin{aligned}\int_{\mathbb{R}}\mathbb{1}_{(n_lt_l^{(j)}-n_l^{\epsilon},n_lt_l^{(j)}+n_l^{\epsilon})}(\lfloor n_lu_l\rfloor)\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\right|^{\alpha-\delta}\mathrm{d}u_l\\ \leqslant\int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}}\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\right|^{\alpha-\delta}\mathrm{d}u_l,\end{aligned}$$

therefore

$$\begin{aligned}\int_{\mathbb{R}}\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\mathbb{1}_{G_l}(\lfloor n_lu_l\rfloor)\right|^{\alpha-\delta}\mathrm{d}u_l\\ \leqslant\sum_{j=0}^{m}\int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}}\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\right|^{\alpha-\delta}\mathrm{d}u_l,\end{aligned}$$

and the proof will be complete if we show that for $j=0,\dots,m$

$$\int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}}\left|s^{-1}_{n_l,\gamma_l,l}S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_lu_l\rfloor,n_l)\right|^{\alpha-\delta}\mathrm{d}u_l\to0,\ n_l\to\infty.\tag{5.64}$$

We proceed by separately investigating the cases $\gamma_l>1$ with $\sum_j a_j(\gamma_l,l)\neq0$, $1/\alpha<\gamma_l\leqslant1$, and $1<\gamma_l<1+1/\alpha$ with $\sum_j a_j(\gamma_l,l)=0$.

If $\gamma_l > 1$ with $\sum_j a_j(\gamma_l, l) \neq 0$, or $1/\alpha < \gamma_l \leqslant 1$ we will use the results obtained while proving Lemma 5.10. In these cases the set $\mathcal{A}_l$ played no role, therefore, we have shown that

$$\frac{|S_{\gamma_l,l,t}(\lfloor nu \rfloor, n)|}{s_{n,\gamma_l,l}} \leqslant G_{\gamma_l}(u,t,l)$$

with a function $G_{\gamma_l}(u,t,l)$ satisfying

$$\int_{-\infty}^{\infty} G_{\gamma_l}^{\alpha-\delta}(u,t,l)\mathrm{d}u < \infty.$$

Now

$$\begin{aligned}
\int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}} & \left| s_{n_l,\gamma_l,l}^{-1} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \right|^{\alpha-\delta} \mathrm{d}u_l \\
& \leqslant \int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}} G_{\gamma_l}^{\alpha-\delta}(u_l, t_l^{(i)}, l)\mathrm{d}u_l \\
& = \int_{-\infty}^{+\infty} \mathbb{1}_{\left(t_l^{(j)}-n_l^{\epsilon-1}, t_l^{(j)}+2n_l^{\epsilon-1}\right)}(u_l) G_{\gamma_l}^{\alpha-\delta}(u_l, t_l^{(i)}, l)\mathrm{d}u_l.
\end{aligned}$$

If $u \neq t_l^{(j)}$, the function under the integral converges to 0, it is bounded from above by an integrable function $G_{\gamma_l}^{\alpha-\delta}(u_l, t_l^{(i)}, l)$, therefore, the dominated convergence theorem implies (5.64).

Next, we turn to the case $1 < \gamma_l < 1 + 1/\alpha$ with $\sum_j a_j(\gamma_l, l) = 0$. As $\gamma_l > 1$, we have

$$\left| S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \right| \leqslant \sum_{k=0}^{\infty} |a_k(\gamma_l, l)| < \infty,$$

therefore

$$\begin{aligned}
\int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}} & \left| s_{n_l,\gamma_l,l}^{-1} S_{\gamma_l,l,t_l^{(i)}}(\lfloor n_l u_l \rfloor, n_l) \right|^{\alpha-\delta} \mathrm{d}u_l \\
& \leqslant \int_{t_l^{(j)}-n_l^{\epsilon-1}}^{t_l^{(j)}+2n_l^{\epsilon-1}} \left(s_{n_l,\gamma_l,l}^{-1} \sum_{k=0}^{\infty} |a_k(\gamma_l, l)| \right)^{\alpha-\delta} \mathrm{d}u_l \\
& = 3n_l^{\epsilon-1} \left(s_{n_l,\gamma_l,l}^{-1} \sum_{k=0}^{\infty} |a_k(\gamma_l, l)| \right)^{\alpha-\delta} \\
& = 3 \left(\sum_{k=0}^{\infty} |a_k(\gamma_l, l)| \right)^{\alpha-\delta} n_l^{\epsilon-1} s_{n_l,\gamma_l,l}^{\delta-\alpha} \\
& = 3 \left(\sum_{k=0}^{\infty} |a_k(\gamma_l, l)| \right)^{\alpha-\delta} n_l^{\epsilon-1+(\gamma-1)(\delta-\alpha)} L_l^{\delta-\alpha}(n_l) \to 0,
\end{aligned}$$

since $\epsilon - 1 + (\gamma - 1)(\delta - \alpha) < 0$ and L_l is a s.v.f.

□

Proof of Theorem 5.7. Suppose $d \in \mathbb{N}$ and $0 = t_0 < t_1 < t_2 < \cdots < t_d$ are real numbers. In order to prove the finite-dimensional convergence of $\bar{S}_n(t)$ we will investigate the convergence of characteristic functions. Ch.f. of $\left(\bar{S}_n(t_1), \ldots, \bar{S}_n(t_d)\right)$ is

$$\begin{aligned}\varphi_{t_1,\ldots,t_d}(x_1, \ldots, x_d) &= \mathbb{E}\exp\left(\mathrm{i}\sum_{l=1}^{d} \bar{S}_n(t_l)x_l\right) \\ &= \mathbb{E}\exp\left(\mathrm{i}A_n^{-1}\sum_{l=1}^{d} x_l \sum_{k=0}^{\lfloor nt_l\rfloor}\sum_{i=0}^{\infty} b_i\xi_{k-i}\right). \quad (5.65)\end{aligned}$$

We have

$$\begin{aligned}\sum_{l=1}^{d} x_l \sum_{k=0}^{\lfloor nt_l\rfloor}\sum_{i=0}^{\infty} b_i\xi_{k-i} &= \sum_{k=0}^{\lfloor nt_d\rfloor}\sum_{i=0}^{\infty}\sum_{l=1}^{d} x_l \mathbb{1}_{[0,\lfloor nt_l\rfloor]}(k) b_i\xi_{k-i} \\ &= \sum_{k=0}^{\lfloor nt_d\rfloor}\sum_{i=-\infty}^{k} B_{k,i}\xi_i = \sum_{i=-\infty}^{\lfloor nt_d\rfloor}\sum_{k=0}^{\lfloor nt_d\rfloor} B_{k,i}\mathbb{1}_{\{k\geqslant i\}}\xi_i,\end{aligned}$$

where $B_{k,i} = \sum_{l=1}^{d} x_l \mathbb{1}_{[0,\lfloor nt_l\rfloor]}(k) b_{k-i}$. Let us denote

$$C_i = C_i(n) = \sum_{k=0}^{\lfloor nt_d\rfloor} B_{k,i}\mathbb{1}_{\{k\geqslant i\}}.$$

It follows that

$$\begin{aligned}&\varphi_{t_1,\ldots,t_d}(x_1, \ldots, x_d) \\ &= \exp\Bigg(-A_n^{-\alpha}\sum_{i=-\infty}^{\lfloor nt_d\rfloor} |C_i|^\alpha h\left(\left|A_n C_i^{-1}\right|\right) \times \\ &\qquad \times (1 - \mathrm{i}\beta\mathrm{sign}(C_i)\tau_\alpha)\left(1 + r\left(A_n^{-1}C_i\right)\right) + \mathrm{i}A_n^{-1}\sum_{i=-\infty}^{\lfloor nt_d\rfloor} C_i\mu\Bigg) \\ &= \exp\left(-A_n^{-\alpha}\sum_{i=-\infty}^{\lfloor nt_d\rfloor} |C_i|^\alpha h\left(\left|A_n C_i^{-1}\right|\right)(1 - \mathrm{i}\beta\mathrm{sign}(C_i)\tau_\alpha)\left(1 + r\left(A_n^{-1}C_i\right)\right)\right),\end{aligned}$$

since

$$\sum_{i=-\infty}^{\lfloor nt_d\rfloor} C_i = \sum_{i=-\infty}^{\lfloor nt_d\rfloor}\sum_{k=0}^{\lfloor nt_d\rfloor}\sum_{l=1}^{d} x_l \mathbb{1}_{[0,\lfloor nt_l\rfloor]}(k) b_{k-i}\mathbb{1}_{\{k\geqslant i\}}$$

$$= \sum_{l=1}^{d} x_l \sum_{k=0}^{\lfloor nt_l \rfloor} \sum_{i=-\infty}^{k} b_{k-i} = \sum_{l=1}^{d} x_l \sum_{k=0}^{\lfloor nt_l \rfloor} \sum_{i=0}^{\infty} b_i = 0.$$

$C_{\lfloor nu \rfloor}$ is uniformly bounded for all u:

$$\left| C_{\lfloor nu \rfloor} \right| \leqslant \sum_{k=0}^{\lfloor nt_d \rfloor} \sum_{l=1}^{d} |x_l| \, \mathbb{1}_{[\lfloor nu \rfloor, \lfloor nt_l \rfloor]}(k) \left| b_{k-\lfloor nu \rfloor} \right| \leqslant \sum_{l=1}^{d} |x_l| \sum_{k=0}^{\infty} |b_k| \,,$$

and $A_n = n^{1/\alpha + 1/\theta - \gamma} h_{1/\alpha}^{1/\alpha}(n) \to \infty$, since $1/\alpha + 1/\theta - \gamma > 0$ and $h_{1/\alpha}$ is a s.v.f. Therefore

$$A_n^{-1} C_{\lfloor nu \rfloor} \to 0, \text{ uniformly for all } u. \tag{5.66}$$

This implies that $\left(1 + r\left(A_n^{-1} C_i\right)\right)$ uniformly converges to 1.

It remains to find the limit of

$$A_n^{-\alpha} \sum_{i=-\infty}^{\lfloor nt_d \rfloor} |C_i|^{\alpha} \, h\left(\left|A_n C_i^{-1}\right|\right) (1 - \mathrm{i}\beta \mathrm{sign}(C_i) \tau_\alpha) \,.$$

We do this by investigating $I_n := A_n^{-\alpha} \sum_{i=-\infty}^{\lfloor nt_d \rfloor} f(C_i) h\left(\left|A_n C_i^{-1}\right|\right)$ with $f(x) = |x|^{\alpha}$ and $f(x) = x^{<\alpha>}$, which we split as $I_n = \sum_{j=0}^{d} Z_{j,n}$, with

$$Z_{j,n} = \sum_{i=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} f(A_n^{-1} C_i) h\left(\left|A_n C_i^{-1}\right|\right), j = 0, 1, \ldots, d,$$

where $t_{-1} = -\infty$. We assume n is large enough so that $\lfloor nt_j \rfloor > \lfloor nt_{j-1} \rfloor$ for all $j = 1, \ldots, d$.

Let us further split $Z_{j,n} = W_{j,n} + f\left(A_n^{-1} C_{\lfloor nt_j \rfloor}\right) h\left(\left|A_n C_{\lfloor nt_j \rfloor}^{-1}\right|\right)$, $j = 0, 1, \ldots, d$, where

$$W_{j,n} = \sum_{i=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor - 1} f(A_n^{-1} C_i) h\left(\left|A_n C_i^{-1}\right|\right).$$

Formula (5.66) implies that

$$\sum_{j=0}^{d} f\left(A_n^{-1} C_{\lfloor nt_j \rfloor}\right) h\left(\left|A_n C_{\lfloor nt_j \rfloor}^{-1}\right|\right) \to 0, n \to \infty,$$

so we can concentrate on investigating $W_{j,n}$.

We have

$$W_{j,n} = \sum_{i=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor - 1} f\left(A_n^{-1} C_i\right) h\left(\left|A_n C_i^{-1}\right|\right)$$

$$= \int_{\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor} f\left(A_n^{-1}C_{\lfloor u\rfloor}\right) h\left(\left|A_n C_{\lfloor u\rfloor}^{-1}\right|\right) \mathrm{d}u$$

$$= \int_{(\lfloor nt_{j-1}\rfloor+1)/n}^{\lfloor nt_j\rfloor/n} n f\left(A_n^{-1}C_{\lfloor nu\rfloor}\right) h\left(\left|A_n C_{\lfloor nu\rfloor}^{-1}\right|\right) \mathrm{d}u$$

$$= \int_{(\lfloor nt_{j-1}\rfloor+1)/n}^{\lfloor nt_j\rfloor/n} \left(h_{1/\alpha}(n)\right)^{-1} f\left(n^{\gamma-\frac{1}{\theta}}C_{\lfloor nu\rfloor}\right) h\left(\left|n^{\frac{1}{\alpha}+\frac{1}{\theta}-\gamma} h_{1/\alpha}^{\frac{1}{\alpha}}(n) C_{\lfloor nu\rfloor}^{-1}\right|\right) \mathrm{d}u$$

$$= \int_{(\lfloor nt_{j-1}\rfloor+1)/n}^{\lfloor nt_j\rfloor/n} f\left(n^{\gamma-\frac{1}{\theta}}C_{\lfloor nu\rfloor}\right) \frac{h\left(\left|n^{\frac{1}{\alpha}+\frac{1}{\theta}-\gamma} h_{1/\alpha}^{\frac{1}{\alpha}}(n) C_{\lfloor nu\rfloor}^{-1}\right|\right)}{h\left(n^{\frac{1}{\alpha}} h_{1/\alpha}^{\frac{1}{\alpha}}(n)\right)} \frac{h\left(n^{\frac{1}{\alpha}} h_{1/\alpha}^{\frac{1}{\alpha}}(n)\right)}{h_{1/\alpha}(n)} \mathrm{d}u$$

$$= \frac{h\left(n^{\frac{1}{\alpha}} h_{1/\alpha}^{\frac{1}{\alpha}}(n)\right)}{h_{1/\alpha}(n)} \int_{(\lfloor nt_{j-1}\rfloor+1)/n}^{\lfloor nt_j\rfloor/n} \kappa_n(u)\mathrm{d}u,$$

where

$$\kappa_n(u) = f\left(n^{\gamma-\frac{1}{\theta}}C_{\lfloor nu\rfloor}\right) \frac{h\left(\left|n^{\frac{1}{\alpha}} h_{1/\alpha}^{\frac{1}{\alpha}}(n) n^{\frac{1}{\theta}-\gamma} C_{\lfloor nu\rfloor}^{-1}\right|\right)}{h\left(n^{\frac{1}{\alpha}} h_{1/\alpha}^{\frac{1}{\alpha}}(n)\right)}.$$

Since the fraction $h\left(n^{1/\alpha}h_{1/\alpha}^{1/\alpha}(n)\right)/h_{1/\alpha}(n)$ converges to 1 by the choice of the function $h_{1/\alpha}$, it remains to find the limits of

$$J_{j,n} = \int_{(\lfloor nt_{j-1}\rfloor+1)/n}^{\lfloor nt_j\rfloor/n} \kappa_n(u)\mathrm{d}u, \ \ j = 0, 1, \ldots, d.$$

We begin by studying $J_{0,n}$. Suppose $u \in (-\infty, 0)$. We have

$$C_{\lfloor nu\rfloor} = \sum_{k=0}^{\lfloor nt_d\rfloor} \sum_{l=1}^{d} x_l \mathbb{1}_{[0,\lfloor nt_l\rfloor]}(k) b_{k-\lfloor nu\rfloor} \mathbb{1}_{\{k\geqslant\lfloor nu\rfloor\}}$$

$$= \sum_{l=1}^{d} x_l \sum_{k=0}^{\lfloor nt_l\rfloor} b_{k-\lfloor nu\rfloor} = \sum_{l=1}^{d} x_l \sum_{k=-\lfloor nu\rfloor}^{\lfloor nt_l\rfloor-\lfloor nu\rfloor} b_k.$$

In what follows, a_k stands for $(-1)^k k^{-\gamma}$, $k \in \mathbb{N}$. Writing $b_k = a_k + 2k^{-\gamma}\mathbb{1}_T(k)$ we obtain

$$\sum_{k=-\lfloor nu\rfloor}^{\lfloor nt_l\rfloor-\lfloor nu\rfloor} b_k = \sum_{k=-\lfloor nu\rfloor}^{\lfloor nt_l\rfloor-\lfloor nu\rfloor} a_k + 2\sum_{k=-\lfloor nu\rfloor}^{\lfloor nt_l\rfloor-\lfloor nu\rfloor} k^{-\gamma}\mathbb{1}_T(k).$$

We observe that

$$\left|\sum_{k=-\lfloor nu\rfloor}^{\lfloor nt_l\rfloor-\lfloor nu\rfloor} a_k\right| \leqslant 2\left|a_{-\lfloor nu\rfloor}\right| = 2(-\lfloor nu\rfloor)^{-\gamma},$$

therefore

$$R_{u,n}^{(0)} := n^{\gamma-\frac{1}{\theta}} \sum_{l=1}^{d} x_l \sum_{k=-\lfloor nu \rfloor}^{\lfloor nt_l \rfloor - \lfloor nu \rfloor} a_k \to 0,\ n \to \infty,$$

and

$$\left|R_{u,n}^{(0)}\right| \leqslant R^{(0)}(u) := \begin{cases} 2\sum_{l=1}^{d} |x_l| \, (-u)^{\frac{1}{\theta}-\gamma},\ -1 \leqslant u < 0, \\ 2\sum_{l=1}^{d} |x_l| \, (-u)^{-\gamma},\ u < -1. \end{cases} \tag{5.67}$$

The inequality above holds since

$$(-\lfloor nu \rfloor)^{-\gamma} \leqslant n^{1/\theta-\gamma}(-u)^{-\gamma}$$

and

$$(-\lfloor nu \rfloor)^{-\gamma} \leqslant (-\lfloor nu \rfloor)^{1/\theta-\gamma} \leqslant n^{1/\theta-\gamma}(-u)^{1/\theta-\gamma}.$$

It is convenient to introduce the notation $i_l := 2\left\lfloor l^\theta \right\rfloor - 1,\ l \in \mathbb{N}$. Suppose y_n and z_n are some sequences. We continue by examining the quantity

$$M(y_n, z_n) := \sum_{y_n \leqslant k \leqslant z_n} k^{-\gamma} \mathbb{1}_T(k) = \sum_{l \geqslant 3: y_n \leqslant i_l \leqslant z_n} i_l^{-\gamma} = \sum_{l \geqslant 3: \frac{y_n+1}{2} \leqslant \lfloor l^\theta \rfloor \leqslant \frac{z_n+1}{2}} i_l^{-\gamma}.$$

Since

$$\left\{l \in \mathbb{N} : a+1 \leqslant l^\theta \leqslant b\right\} \subset \left\{l \in \mathbb{N} : a \leqslant \left\lfloor l^\theta \right\rfloor \leqslant b\right\} \subset \left\{l \in \mathbb{N} : a \leqslant l^\theta \leqslant b+1\right\},$$

the following inequalities hold:

$$M(y_n, z_n) \leqslant \sum_{l \geqslant 3: \frac{y_n+1}{2} \leqslant l^\theta \leqslant \frac{z_n+1}{2}+1} i_l^{-\gamma} = \sum_{l \geqslant 3: \left(\frac{y_n+1}{2}\right)^{\frac{1}{\theta}} \leqslant l \leqslant \left(\frac{z_n+3}{2}\right)^{\frac{1}{\theta}}} i_l^{-\gamma}, \tag{5.68}$$

$$M(y_n, z_n) \geqslant \sum_{l \geqslant 3: \frac{y_n+1}{2}+1 \leqslant l^\theta \leqslant \frac{z_n+1}{2}} i_l^{-\gamma} = \sum_{l \geqslant 3: \left(\frac{y_n+3}{2}\right)^{\frac{1}{\theta}} \leqslant l \leqslant \left(\frac{z_n+1}{2}\right)^{\frac{1}{\theta}}} i_l^{-\gamma}.$$

Let us investigate $\sum_{l=q_n}^{w_n} i_l^{-\gamma}$, where q_n and w_n are sequences of integers (we allow infinite values as well), $w_n \geqslant q_n \geqslant 3, n \in \mathbb{N}$. We have

$$\begin{aligned}\sum_{l=q_n}^{w_n} i_l^{-\gamma} &= \int_{q_n}^{w_n+1} i_{\lfloor v \rfloor}^{-\gamma} \mathrm{d}v = \int_{n^{-\frac{1}{\theta}} q_n}^{n^{-\frac{1}{\theta}}(w_n+1)} i_{\left\lfloor n^{\frac{1}{\theta}} v \right\rfloor}^{-\gamma} \mathrm{d}n^{\frac{1}{\theta}} v \\ &= n^{\frac{1}{\theta}-\gamma} \int_{n^{-\frac{1}{\theta}} q_n}^{n^{-\frac{1}{\theta}}(w_n+1)} n^{\gamma} i_{\left\lfloor n^{\frac{1}{\theta}} v \right\rfloor}^{-\gamma} \mathrm{d}v = n^{\frac{1}{\theta}-\gamma} \int_0^{\infty} g_n(v) \mathrm{d}v,\end{aligned} \tag{5.69}$$

with $g_n(v) = n^{\gamma} i_{\lfloor n^{1/\theta} v \rfloor}^{-\gamma} \mathbb{1}_{\left(n^{-1/\theta} q_n, n^{-1/\theta}(w_n+1)\right)}(v)$. Notice that $g_n(v) > 0$ implies $n^{1/\theta} v \geqslant 3$, hence we have

$$n^{\gamma} i_{\left\lfloor n^{\frac{1}{\theta}} v \right\rfloor}^{-\gamma} = n^{\gamma} \left(2 \left\lfloor \left\lfloor n^{\frac{1}{\theta}} v \right\rfloor^{\theta} \right\rfloor - 1 \right)^{-\gamma}$$

$$\leqslant n^{\gamma} \left(2 \left\lfloor n^{\frac{1}{\theta}} v \right\rfloor^{\theta} - 3 \right)^{-\gamma} \leqslant n^{\gamma} \left(2 \left(n^{\frac{1}{\theta}} v - 1 \right)^{\theta} - 3 \right)^{-\gamma}$$

$$\leqslant n^{\gamma} \left(\frac{1}{2} \left(n^{\frac{1}{\theta}} v - 1 \right)^{\theta} \right)^{-\gamma} \leqslant n^{\gamma} \left(\frac{2^{\theta-1}}{3^{\theta}} \left(n^{\frac{1}{\theta}} v \right)^{\theta} \right)^{-\gamma} = K_0 v^{-\gamma\theta}.$$

Therefore,

$$g_n(v) \leqslant K_0 v^{-\gamma\theta} \mathbb{1}_{\left(n^{-\frac{1}{\theta}} q_n, n^{-\frac{1}{\theta}}(w_n+1)\right)}(v). \tag{5.70}$$

Assuming the sequences $n^{-1/\theta} q_n$ and $n^{-1/\theta} w_n$ converge to $q, w > 0$, respectively, we have $g_n(v) \to 2^{-\gamma} v^{-\gamma\theta} \mathbb{1}_{(q,w)}(v)$ almost surely. Also, there exists a number $n_0 \in \mathbb{N}$ such that

$$\mathbb{1}_{\left(n^{-\frac{1}{\theta}} q_n, n^{-\frac{1}{\theta}}(w_n+1)\right)}(v) \leqslant \mathbb{1}_{\left(\frac{q}{2}, \frac{3w}{2}\right)}(v), \ n \geqslant n_0.$$

As the function $K_0 v^{-\gamma\theta} \mathbb{1}_{(q/2,\ 3w/2)}(v)$ is integrable, the dominated convergence theorem implies

$$n^{\gamma-\frac{1}{\theta}} \sum_{l=q_n}^{w_n} i_l^{-\gamma} \to \int_q^w 2^{-\gamma} v^{-\gamma\theta} \mathrm{d}v = \frac{2^{-\gamma}}{\gamma\theta - 1} \left(q^{1-\gamma\theta} - w^{1-\gamma\theta} \right).$$

By selecting $y_n = -\lfloor nu \rfloor$ and $z_n = \lfloor nt_l \rfloor - \lfloor nu \rfloor$, for any fixed number c we have

$$n^{-\frac{1}{\theta}} \left(\frac{y_n + c}{2} \right)^{\frac{1}{\theta}} \to \left(-\frac{u}{2} \right)^{\frac{1}{\theta}},$$

$$n^{-\frac{1}{\theta}} \left(\frac{z_n + c}{2} \right)^{\frac{1}{\theta}} \to \left(\frac{t_l - u}{2} \right)^{\frac{1}{\theta}},$$

as $n \to \infty$, therefore the previous calculations imply

$$n^{\gamma-\frac{1}{\theta}} M(y_n, z_n)$$
$$\to \frac{2^{-\gamma}}{\gamma\theta - 1}\left(\left(-\frac{u}{2}\right)^{\frac{1}{\theta}-\gamma} - \left(\frac{t_l - u}{2}\right)^{\frac{1}{\theta}-\gamma}\right) = \phi_{\alpha,\frac{1}{\alpha}+\frac{1}{\theta}-\gamma}\left(\frac{-2^{-\frac{1}{\theta}}}{\gamma\theta-1}, 0; t_l, u\right).$$

Returning to $n^{\gamma-1/\theta}C_{\lfloor nu \rfloor}$, we obtain, as $n \to \infty$,

$$n^{\gamma-\frac{1}{\theta}} C_{\lfloor nu \rfloor} = R^{(0)}_{u,n} + 2\sum_{l=1}^{d} x_l \frac{M(-\lfloor nu \rfloor, \lfloor nt_l \rfloor - \lfloor nu \rfloor)}{n^{\frac{1}{\theta}-\gamma}}$$
$$\to \sum_{l=1}^{d} x_l \phi_{\alpha,\frac{1}{\alpha}+\frac{1}{\theta}-\gamma}\left(\frac{-2^{1-\frac{1}{\theta}}}{\gamma\theta-1}, 0; t_l, u\right) =: F(u). \quad (5.71)$$

Applying Lemma 2.4 with $q_n = n^{1/\alpha} h_{1/\alpha}^{1/\alpha}(n)$ and $y_n = n^{\gamma-1/\theta}C_{\lfloor nu \rfloor}$ we obtain

$$\kappa_n(u) \to f(F(u)). \quad (5.72)$$

In order to apply the dominated convergence theorem we need to show that $|\kappa_n(u)|$ is bounded above by an integrable function. It follows from (5.66) that $\left|n^{1/\alpha} h_{1/\alpha}^{1/\alpha}(n) n^{1/\theta-\gamma} C^{-1}_{\lfloor nu \rfloor}\right| \to \infty$, uniformly for all u. Let us suppose that $0 < \delta < \alpha$ and denote $\Lambda_{\alpha,\delta}(x) = \max\{x^{\alpha+\delta}, x^{\alpha-\delta}\}$. Applying Lemma 2.3 and recalling from (5.71) the expression of $n^{\gamma-\frac{1}{\theta}}C_{\lfloor nu \rfloor}$ we obtain

$$|\kappa_n(u)| \leqslant D\Lambda_{\alpha,\delta}\left(\left|n^{\gamma-\frac{1}{\theta}} C_{\lfloor nu \rfloor}\right|\right)$$
$$\leqslant D\Lambda_{\alpha,\delta}\left(\left|R^{(0)}_{u,n}\right| + 2\sum_{l=1}^{d} |x_l| \frac{M(-\lfloor nu \rfloor, \lfloor nt_l \rfloor - \lfloor nu \rfloor)}{n^{\frac{1}{\theta}-\gamma}}\right).$$

Inequality (5.68) together with (5.69) and (5.70) gives us

$$\frac{M(-\lfloor nu \rfloor, \lfloor nt_l \rfloor - \lfloor nu \rfloor)}{n^{\frac{1}{\theta}-\gamma}} \leqslant \frac{K_0}{\gamma\theta - 1}\left(q_n^{1-\gamma\theta} - w_n^{1-\gamma\theta}\right)$$

with

$$q_n = n^{-\frac{1}{\theta}}\left\lceil\left(\frac{-\lfloor nu \rfloor + 1}{2}\right)^{\frac{1}{\theta}}\right\rceil,$$
$$w_n = n^{-\frac{1}{\theta}}\left(\left\lfloor\left(\frac{\lfloor nt_l \rfloor - \lfloor nu \rfloor + 3}{2}\right)^{\frac{1}{\theta}}\right\rfloor + 1\right).$$

We have

$$q_n \geqslant \left(\frac{-u}{2}\right)^{\frac{1}{\theta}}, \quad w_n \leqslant \left(\left(\frac{t_l - u + 4}{2}\right)^{\frac{1}{\theta}} + 1\right),$$

therefore,

$$\frac{M(-\lfloor nu \rfloor, \lfloor nt_l \rfloor - \lfloor nu \rfloor)}{n^{\frac{1}{\theta}-\gamma}} \leqslant \frac{K_0}{\gamma\theta - 1}\left(\left(\frac{-u}{2}\right)^{\frac{1}{\theta}-\gamma} - \left(\left(\frac{t_l - u + 4}{2}\right)^{\frac{1}{\theta}} + 1\right)^{1-\gamma\theta}\right) =: h(t_l, u).$$

The obtained estimates for $M(-\lfloor nu \rfloor, \lfloor nt_l \rfloor - \lfloor nu \rfloor)$ and $R^{(0)}_{u,n}$ (see (5.67)) enable us to estimate

$$|\kappa_n(u)| \leqslant D\Lambda_{\alpha,\delta}\left(R^{(0)}(u) + 2\sum_{l=1}^{d} |x_l|\, h(t_l, u)\right) =: G_\delta(u). \qquad (5.73)$$

As $u \to -\infty$ we have $R^{(0)}(u) \sim c_1(-u)^{-\gamma}$, $h(t_l, u) \sim c_2(-u)^{1/\theta-\gamma-1}$, therefore $G_\delta(u) \sim c_3(-u)^{-\gamma(\alpha-\delta)}$. As $u \uparrow 0$, $R^{(0)}(u) \sim c_4(-u)^{1/\theta-\gamma}$, $h(t_l, u) \sim c_5(-u)^{1/\theta-\gamma}$, therefore $G_\delta(u) \sim c_6(-u)^{(1/\theta-\gamma)(\alpha+\delta)}$. Since, as $\delta \to 0$, $-\gamma(\alpha-\delta) \to -\alpha\gamma < -1$ and $(1/\theta - \gamma)\,(\alpha+\delta) \to (1/\theta-\gamma)\alpha > -1$, there exists $\delta > 0$ such that the function G_δ is integrable on $(-\infty, 0)$.

(5.72) and (5.73) enable us to apply the dominated convergence theorem, which implies

$$J_{0,n} = \int_{-\infty}^{0} \kappa_n(u)\mathrm{d}u \to \int_{-\infty}^{0} f\left(\sum_{l=1}^{d} x_l \phi_{\alpha,\frac{1}{\alpha}+\frac{1}{\theta}-\gamma}\left(\frac{-2^{1-\frac{1}{\theta}}}{\gamma\theta - 1}, 0; t_l, u\right)\right)\mathrm{d}u, \ n \to \infty.$$

Let us now investigate the convergence of $J_{j,n}, j = 1, \dots, d$. We start with point-wise convergence of $n^{\gamma-1/\theta}C_{\lfloor nu \rfloor}$, $u \in (t_{j-1}, t_j)$. We have

$$
\begin{aligned}
C_{\lfloor nu \rfloor} &= \sum_{k=0}^{\lfloor nt_d \rfloor} \sum_{l=1}^{d} x_l \mathbb{1}_{[\lfloor nu \rfloor, \lfloor nt_l \rfloor]}(k) b_{k-\lfloor nu \rfloor} \\
&= \sum_{l=1}^{d} x_l \sum_{k=0}^{\lfloor nt_d \rfloor - \lfloor nu \rfloor} \mathbb{1}_{[0, \lfloor nt_l \rfloor - \lfloor nu \rfloor]}(k) b_k \\
&= \sum_{l=j}^{d} x_l \sum_{k=0}^{\lfloor nt_l \rfloor - \lfloor nu \rfloor} b_k \\
&= -\sum_{l=j}^{d} x_l \sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} b_k
\end{aligned}
$$

the last equality was obtained using the property $\sum_{k=0}^{\infty} b_k = 0$. By splitting $b_k = a_k + 2k^{-\gamma}\mathbb{1}_T(k)$ we obtain

$$
\sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} b_k = \sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} a_k + 2 \sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} k^{-\gamma}\mathbb{1}_T(k).
$$

Since

$$
\left| \sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} a_k \right| \leqslant \left| a_{\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1} \right| = (\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1)^{-\gamma},
$$

we get

$$
R_{u,n}^{(j)} := n^{\gamma - \frac{1}{\theta}} \sum_{l=j}^{d} x_l \sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} a_k \to 0, \;\; n \to \infty,
$$

and

$$
\left| R_{u,n}^{(j)} \right| \leqslant R^{(j)}(u) := \sum_{l=j}^{d} |x_l| \, (t_l - u)^{\frac{1}{\theta} - \gamma} .
$$

Notice that $\sum_{k=\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1}^{\infty} k^{-\gamma}\mathbb{1}_T(k) = M(\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1, \infty)$. Previous analysis implies

$$
\frac{M(\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1, \infty)}{n^{\frac{1}{\theta} - \gamma}} \to \frac{2^{-\gamma}}{\gamma\theta - 1} \left(\frac{t_l - u}{2} \right)^{\frac{1}{\theta} - \gamma} = \phi_{\alpha, \frac{1}{\alpha} + \frac{1}{\theta} - \gamma} \left(\frac{2^{-\frac{1}{\theta}}}{\gamma\theta - 1}, 0; t_l, u \right)
$$

and

$$
\frac{M(\lfloor nt_l \rfloor - \lfloor nu \rfloor + 1, \infty)}{n^{\frac{1}{\theta} - \gamma}} \leqslant \frac{K_0}{\gamma\theta - 1} \left(\frac{t_l - u}{2} \right)^{\frac{1}{\theta} - \gamma} .
$$

We have thus obtained

$$n^{\gamma-\frac{1}{\theta}}C_{\lfloor nu\rfloor} = -R^{(j)}_{u,n} - 2\sum_{l=j}^{d}\frac{M(\lfloor nt_l\rfloor - \lfloor nu\rfloor + 1,\infty)}{n^{\frac{1}{\theta}-\gamma}}$$
$$\to \phi_{\alpha,\frac{1}{\alpha}+\frac{1}{\theta}-\gamma}\left(\frac{-2^{1-\frac{1}{\theta}}}{\gamma\theta-1},0;t_l,u\right)$$

and

$$\left|n^{\gamma-\frac{1}{\theta}}C_{\lfloor nu\rfloor}\right|\mathbb{1}_{\left(\frac{\lfloor nt_{j-1}\rfloor+1}{n},\frac{\lfloor nt_j\rfloor}{n}\right)}(u)$$
$$\leqslant \left|R^{(j)}(u) + \sum_{l=j}^{d}\frac{K_0}{\gamma\theta-1}\left(\frac{t_l-u}{2}\right)^{\frac{1}{\theta}-\gamma}\right|\mathbb{1}_{(t_{j-1},t_j)}(u),$$

which, as before, imply

$$\kappa_n(u)\to f(F(u)),$$

and

$$\left|\kappa_n(u)\mathbb{1}_{\left(\frac{\lfloor nt_{j-1}\rfloor+1}{n},\frac{\lfloor nt_j\rfloor}{n}\right)}(u)\right|$$
$$\leqslant D\Lambda_{\alpha,\delta}\left(\left|R^{(j)}(u) + \sum_{l=j}^{d}\frac{K_0}{\gamma\theta-1}\left(\frac{t_l-u}{2}\right)^{\frac{1}{\theta}-\gamma}\right|\right)\mathbb{1}_{(t_{j-1},t_j)}(u).$$

The dominating function is integrable in (t_{j-1},t_j) for small enough δ, therefore the dominated convergence theorem applies giving us

$$W_{j,n} = \int_{(\lfloor nt_{j-1}\rfloor+1)/n}^{\lfloor nt_j\rfloor/n}\kappa_n(u)\mathrm{d}u$$
$$\to \int_{t_{j-1}}^{t_j} f\left(\sum_{l=1}^{d}x_l\phi_{\alpha,\frac{1}{\alpha}+\frac{1}{\theta}-\gamma}\left(\frac{-2^{1-\frac{1}{\theta}}}{\gamma\theta-1},0;t_l,u\right)\right)\mathrm{d}u,\ n\to\infty.$$

In conclusion, we see that

$$\varphi_{t_1,\dots,t_d}(x_1,\dots,x_d)$$
$$\to \exp\left(-\sigma^{\alpha}\int_{-\infty}^{\infty}\left|\sum_{l=1}^{d}x_l v(t_l,u)\right|^{\alpha}\left(1-\mathrm{i}\beta\tau_{\alpha}\mathrm{sign}\left(\sum_{l=1}^{d}x_l v(t_l,u)\right)\right)\mathrm{d}u\right),$$

where $v(t,u)=\phi_{\alpha,1/\alpha+1/\theta-\gamma}\left(\frac{-2^{1-1/\theta}}{\gamma\theta-1},0;t,u\right)$, which is a ch. f. of

$$(Z_{\alpha,H}(a,b;t_1),\dots,Z_{\alpha,H}(a,b;t_d))$$

with parameters $H = 1/\alpha + 1/\theta - \gamma$, $a = -2^{1-1/\theta}/(\gamma\theta - 1)$, $b = 0$ and skewness intensity $\beta(u) \equiv \beta$. The proof is complete.

□

Proof of Corollary 5.8. The corollary follows directly from Theorem 5.7 by choosing $\theta = \alpha/(\alpha\gamma - 1 + \delta)$. □

6 Conclusions

Based on the results obtained while writing this thesis, we can draw the following conclusions:

- For linear processes $X(n) = \sum_{j=0}^{\infty} c_j \epsilon_{n-j}, \;\; n \in \mathbb{N}$, the condition $\sum_i c_i = 0$ has no effect on asymptotic rate of decay of the spectral covariance in the case $\alpha < 2$, this is explained after the formulation of Theorem 4.7. The relation between spectral covariance and memory, as defined by Definition 5, is not as strong as in the case of finite variance.

- The rate of decay of the spectral covariance for linear processes with asymptotically regularly varying coefficients, linear fractional stable noise, log-fractional stable noise is similar to that of codifference and covariation.

- Newly introduced measure of dependence – α-spectral covariance – displays simpler asymptotic dependence structure of investigated linear fields.

- There exists an analogue of Theorem 3.2 for spectral covariance and α-spectral covariance, see Corollary 4.9.

- Theorems 3.8 and 3.9 generalize to stationary associated random fields, see Theorem 5.1 and Theorem 5.3.

- In Theorems 3.8 and 5.1 one can substitute codifference for spectral covariance to obtain an equivalent statement. If one uses α-spectral covariance or covariation, a weaker statement is obtained.

- Definition of memory for stationary fields in [52] needs revision as it does not apply to linear fields with innovations belonging to the domain of attraction of α-stable random variable.

- Consider linear process $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$ and suppose ξ_i belong to the normal domain of attraction of α-stable law. Given $\gamma \in (\max(1/\alpha, 1), 1 + 1/\alpha)$ and any $\lambda \in (0, 1/\alpha - \gamma + 1)$, it is possible to choose the signs of the coefficients $c_k, k \in \mathbb{N}$, satisfying

$$|c_k| = k^{-\gamma}, k \in \mathbb{N}, \text{ and } \sum_{k=0}^{\infty} c_k = 0$$

  so that $n^{-\lambda} \sum_{k=1}^{n} X_k$ would converge to a non-degenerate distribution.

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