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LAIMONAS MEŠKA

MODIFIED UNIVERSALITY THEOREMS FOR THE RIEMANN AND HURWITZ ZETA-FUNCTIONS

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VILNIAUS UNIVERSITETAS

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Introduction

In the thesis, the approximation of analytic functions by shifts of the Riemann zeta-function $\zeta(s+i\tau)$ and Hurwitz zeta-function $\zeta(s+i\tau,\alpha)$, $\tau\in\mathbb{R}$, is investigated, and the density of those shifts is considered.

We remind the definitions and basic properties of the Riemann zeta-function and Hurwitz zeta-function. The Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

or by the Euler product over prime numbers p

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}.$$

Moreover, the function $\zeta(s)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. The function $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ is the Euler gamma-function.

The function $\zeta(s)$ was already known to L. Euler, however, he considered it with a real variable s. In 1859, B. Riemann began to study $\zeta(s)$ as a function of a complex variable s, and applied it for the investigation of prime numbers in the set $\mathbb{N}$, more precisely, for the investigation of the function

$$\pi(x)=\sum_{p\leq x}1$$

as $x\to\infty$. Riemann proposed a new idea involving the function $\zeta(s)$ for the investigation of $\pi(s)$, however, he did not solve the problem completely. In 1896, Charles Jean de la Vallée-Poussin and Jacques Salomon Hadamard, using Riemann's idea obtained independently the asymptotic law of distribution of prime numbers. They proved that

$$\pi(x)=\int_2^x\frac{du}{\log u}(1+r(x)),\tag{1}$$

where $r(x)\to 0$ as $x\to\infty$, and even estimated $r(x)$. It turned out that the location of zeros of the function $\zeta(s)$ lying in the critical strip $\{s\in\mathbb{C}:0\leq\sigma\leq 1\}$ plays a crucial role in proving the asymptotic formula (1). To prove (1) with $r(x) = o(x)$, it suffices to know that $\zeta(s) \neq 0$ for $\sigma \geq 1$. The extension of a zero-free region to the left of the line $\sigma = 1$ gives estimates for $r(x)$. At the moment, it is known that there exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ in the region

$$\sigma > 1 - \frac{c}{\log^{\frac{2}{3}}(|t|+2)\log^{\frac{1}{3}}\log(|t|+2)}.$$

The zeros of $\zeta(s)$ lying in the critical strip are called non-trivial, while the zeros $s = -2m$, $m \in \mathbb{N}$, are called trivial. Riemann raised some conjectures on the zeros of $\zeta(s)$. The most important of them is the Riemann hypothesis (RH) which asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. Moreover, all these zeros are simple. Let $N(T)$ be the number of zeros in the rectangle $\{s \in \mathbb{C} : 0 \leq \sigma \leq 1, 0 < t < T\}$. Riemann conjectured that, as $T \to \infty$,

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

The latter formula was proved by Hans Carl Friedrich von Mangoldt in 1905. There are several equivalents of the RH. One of them, obtained by Helge von Koch, says that RH is equivalent to the estimate

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(\sqrt{x}\log x).$$

The function $\zeta(s)$ and its value-distribution is related to many problems of mathematics. The zero-distribution even appears in quantum mechanics. Thus, the investigation of the function $\zeta(s)$ is a very important problem of modern analytic number theory.

The Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter α, $0 < \alpha \leq 1$, was introduced by Hurwitz in [11]. The function $\zeta(s,\alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and, as the function $\zeta(s)$, has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s,\alpha)$ for $\alpha = 1$ becomes $\zeta(s)$, thus, the Hurwitz zeta-function is a generalization of the Riemann zeta-function $\zeta(s)$. The function $\zeta(s,\alpha)$ is not so important as $\zeta(s)$, however, it is an interesting analytic object depending on the parameter α and occurs in many formulas of analytic number theory. The Hurwitz zeta-function is not directly related to distribution of prime numbers, however, $\zeta(s,\alpha)$ with rational parameter α is closely related to Dirichlet L-functions which are the main tool in the investigation of prime numbers in arithmetical progressions. Let $\chi(m)$ be a Dirichlet character modulo q. The Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and can be analytically continued to an entire function if χ is not the principal character. If χ_0 is the principal character ($\chi_0(m) = 1$ for $(m,q) = 1$), the function $L(s,\chi_0)$ has the unique simple pole at $s = 1$ with residue

$$\prod_{p|q}\left(1 - \frac{1}{p}\right).$$

Let $\alpha = \frac{a}{q}$, $a, q \in \mathbb{N}$, $(a, q) = 1$. Then the functions $\zeta(s, \frac{a}{q})$ and $L(s, \chi)$ are connected by the equality

$$L(s, \chi) = \frac{1}{q^s} \sum_{m=1}^{q} \chi(m) \zeta(s, \frac{m}{q}).$$

Therefore, the properties of $L(s, \chi)$ depend on those of $\zeta(s, \frac{a}{q})$.

The function $\zeta(s, \alpha)$, except for the cases $\zeta(s, 1) = \zeta(s)$ and

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\, \zeta(s), \tag{2}$$

has no Euler product over primes. Therefore, the functions $\zeta(s)$ and $\zeta(s, \alpha)$ have different analytic properties. For example, the function $\zeta(s) \neq 0$ in the half-plane $\sigma \geq 1$, while the function $\zeta(s, \alpha)$ in that half-plane has infinitely many zeros [6], [7], [23]. The RH is not true for the function $\zeta(s, \alpha)$, it has infinitely many zeros in the critical strip provided that the parameter α is rational or transcendental [23]. We remind that the number α is transcendental if there are no polynomials $p(x) \not\equiv 0$ with rational coefficients such that $p(\alpha) = 0$.

On the other hand, the functions $\zeta(s)$ and $\zeta(s, \alpha)$ with transcendental or rational α have a common feature: they are universal in the Voronin sense, i.e., their shifts $\zeta(s + i\tau)$ and $\zeta(s + i\tau, \alpha)$ approximate wide classes of analytic functions.

Aims and problems

The aims of the thesis are modified universality theorems for the Riemann zeta-function and Hurwitz zeta-function as well as modified joint universality theorems for those functions. The problems are the following:

1. A modified universality theorem for the Riemann zeta-function and its consequences for universality of composite functions.
2. A modified discrete universality theorem for the Riemann zeta-function and its consequences for universality of composite functions.
3. A modified universality theorem for the Hurwitz zeta-function and its consequences for universality of composite functions.
4. A modified discrete universality theorem for the Hurwitz zeta-function and its consequences for universality of composite functions.
5. A modified version of the Mishou theorem.
6. A modified discrete version of the Mishou theorem.

Actuality

We have already mentioned that the investigation of value-distribution of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ is one of the most important problems of modern analytic number theory. In addition, universality of these functions is not only an interesting analytic property but also has numerous theoretical and practical applications. One of theoretical applications is related to the famous Hilbert problems. In the description of the 18th problem, Hilbert mentioned that the function $\zeta(s)$ cannot satisfy any algebraic-differential equation, and raised a similar problem for the function

$$\zeta(s,x)=\sum_{m=1}^{\infty}\frac{x^m}{m^s}.$$

The latter problem was solved succesfully by Ostrowski [41]. However, Voronin, using the universality property of the function $\zeta(s)$, proved much more: he obtained the functional independence of $\zeta(s)$. This means that if a polynomial, coefficients of which are continuous functions of $\zeta(s)$ and its derivatives, is identically equal to zero, all coefficients of this polynomial are identical zeros as well. It turned out that the universality method works for the proof of the functional independence of other zeta-functions.

One of the most important problems in the theory of zeta-functions is their zero distribution. Universality property also has applications in this very interesting and complicated problem, it allows to estimate the number of zeros lying in some regions. Finally, the RH is true if and only if the function $\zeta(s)$ can be approximated by its shifts $\zeta(s+i\tau)$.

Universality is applied to prove the denseness of various sets, see, for example, [1],[34],[48].

Obviously, practical applications of universality are connected to approximation of analytic functions. Some applications are given by physicists in [3], [9].

The above remarks show that it is not strange that universality of zeta-functions is widely studied by number theorists. Universality groups are working in Japan (K. Matsumoto, H. Mishou, T. Nakamura, H. Nagoshi), Germany (J. Steuding, T. Christ, H. Bauer, A. Reich, K. Grosse-Erdmann), Poland (J. Kaczorowski, L. Pankowski, M. Kulas, etc). Universality of zeta-functions is also studied in France, India, South Korea, Canada. However, universality has the deapest traditions in Lithuania (R. Garunkštis, A. Laurinčikas, A. Dubickas, R. Kačinskaitė, R. Macaitienė, D. Šiaučiūnas, E. Karikovas, etc). Therefore, there exists an obligation to continue one of the most popular and productive directions of analytic number theory in Lithuania and extend the investigations of universality in order for the inequality of the universality become stronger.

Methods

The probabilistic method is applied in order to prove modified universality theorems. Firstly, limit theorems on weak convergence of probability measures in the space of analytic functions are obtained, and then an equivalent of weak convergence of probability measures in terms of continuity sets is applied. Also, the Mergelyan theorem on approximation of analytic functions by polynomials plays an important role for proving the universality. The proofs of universality for composite functions involve elements of the operator theory.

Novelty

During the period of preparation of the thesis, Professor R. Garunkštis informed that J. L. Mauclaire has some results related to modification of the universality inequality. Indeed, in [35], a modified universality theorem for the Riemann zeta-function similar to Theorem 1.1 by a different method was obtained. All other theorems of the thesis are entirely new.

History of the problem and main results

Universality for zeta and L-functions was born on a day of 1975 when S. M. Voronin published the paper [50] on the universality of the Riemann zeta-function. He also mentioned that all Dirichlet L-functions have the universality property. The original form of the Voronin theorem is the following.

Theorem A. *Suppose that $0 < r < \frac{1}{4}$. Let $f(s)$ be a continuous non-vanishing function on the disc $|s| \le r$, and analytic in the disc $|s| < r$. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\le r}\left|\zeta(s+\frac{3}{4}+i\tau) - f(s)\right| < \varepsilon.$$

Theorem A shows that a wide class of analytic functions on the discs of the right-hand side of the critical strip can be approximated with a given accuracy by the Riemann zeta-function. This is the Voronin sense of universality. We notice that the function $\zeta(s)$ was the first explicitly given universal object in analysis. Until 1975 only the existence of various universal objects was known.

Theorem A and its applications and extensions form constitute the thesis of Voronin [51] which is a chapter of the monograph [13], and also is published in [52].

Voronin's paper [50] was observed by the mathematical community, and various authors improved the Voronin universality theorem and extended it for other zeta and L functions. For a modern version of the Voronin theorem, we need some notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements. Let $H_0(K)$, where $K \in \mathcal{K}$, be the class of continuous non-vanishing functons on K which are analytic in the interior of K. By *meas* A we denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then a stronger version of Theorem A is of the following form, for the proof, see [1], [15], [48].

Theorem B. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0.$$

We see that in Theorem B, differently from Theorem A, analytic functions are approximated by shifts $\zeta(s+i\tau)$ not only on the discs of the strip D but on the wide class $\mathcal{K}$ of compact subsets. Moreover, there exists not only one shift approximating the function $f(s)$ but infinitely many shifts with approximation property. The inequality of Theorem B shows that the set of such shifts has a positive lower density.

Theorem B poses a natural question: does the set of shifts $\zeta(s+i\tau)$ approximating a given function $f(s) \in H_0(K)$ have a positive density? In other words, is it possible to replace "lim inf" in Theorem B by "lim"? It turned out that the answer is essentially positive; however, some restriction for the set of $\varepsilon > 0$ is required. The first theorem of Chapter 1 gives the answer to the above question [30].

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 1.1 can be extended to a wide class of universal functions by considering the composite functions $F(\zeta(s))$. Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic functions on G endowed with the topology of uniform convergence on compacta. In this topology, $g_n(s) \in H(G)$ converges to $g(s) \in H(G)$ as $n \to \infty$ if and only if, for every compact subset $K \subset G$,

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

Universality of composite functions was began to study in [17] and was continued in [18]. Denote by $H(K)$, $K \in \mathcal{K}$, the set of continuous functions on K which are analytic in the interior of K, and let

$$S = \{g \in H(D) : g(s) \neq 0 \ \textit{or} \ g(s) \equiv 0\}.$$

Then, in [17], the following assertion has been obtained.

Theorem C. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{ \tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon \Big\} > 0.$$

Here, as usually, $F^{-1}G$ is the preimage of the set G. We note that the set G in Theorem C can be replaced by a polynomial. In Chapter 1, the analogues of Theorem C are presented.

Theorem 1.2. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas \Big\{ \tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

It is more convenient to deal with polynomials than with an open set. This is realised in Theorem 1.3.

Theorem 1.3. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 1.2 is true.*

Now let $a_1, ...a_r$ be distinct complex numbers, and let $F : H(D) \to H(D)$ be an operator. Define the set

$$H_{a_1,...,a_r;F}(D) = \{g \in H(D) : g(s) \neq a_j, j = 1, ..., r\} \cup \{F(0)\}.$$

The next theorem of Chapter 1 is devoted to the approximation of analytic functions related to the set $H_{a_1,...,a_r;F}(D)$.

Theorem 1.4. *Let $F : H(D) \to H(D)$ be a continuous operator for which $F(S) \supset H_{a_1,...,a_r;F}(D)$. In the case $r = 1$, let $K \in \mathcal{K}$, and $f(s)$ be a continuous function not taking the value a_1 on K and analytic in the interior of K. In the case $r \geq 2$, let $K \subset D$ be any compact set, and let $f(s) \in H_{a_1,...,a_r;F}(D)$. Then the same assertion as in Theorem 1.2 is true.*

For example, if $r = 1$ and $a_1 = 0$, Theorem 1.4 implies the universality of the function $\zeta^N(s)$, $n \in \mathbb{N}$. If $r = 2$, $a_1 = -1$, $a_2 = 1$, it implies the universality of the function $\sin\zeta(s)$, $\cos\zeta(s)$, $\sinh\zeta(s)$ and $\cosh\zeta(s)$.

Theorems B, C and 1.1-1.4 are of continuous type because τ can take any real value in shifts $\zeta(s + i\tau)$. Also, there exist the so-called discrete universality theorems when τ takes values from a certain discrete set, for example, from an arithmetical progression $\{kh : k = 0, 1, 2, ...\}$ with $h > 0$. Discrete universality theorems for zeta-functions were proposed by A. Reich. In [44] he proved such a theorem for the Dedekind zeta-function. The discrete version of Theorem B is a slightly different form as Theorem D was obtained in [1]. Denote by $\#A$ the cardinality of the set A.

Theorem D. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\} > 0.$$

We note that the proof of Theorem D is much more complicated than that of Theorem B. Two cases of $h > 0$ are considered. The case when $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z}\backslash\{0\}$ is similar to the case of Theorem B. However, the case when $\exp\left\{\frac{2\pi r}{h}\right\}$ is rational for some r requires additional algebraic investigations.

In [37], a modification of Theorem D was obtained. This is Theorem 1.5 of the thesis.

Theorem 1.5. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Discrete universality theorems for composite functions were obtained in [40], however, only in the case when $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z}\backslash\{0\}$. For example, the following assertion is true [43].

Theorem E. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Suppose that the number $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z}\backslash\{0\}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F\big(\zeta(s+ikh)\big) - f(s)| < \varepsilon \right\} > 0.$$

In Chapter 1 of the thesis, a series of modifications of Theorem E are presented. They were published in [37].

Theorem 1.6. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F\big(\zeta(s+ikh)\big) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The next theorem is a discrete analogue of Theorem 1.3.

Theorem 1.7. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 1.6 is true.*

It is well known that the non-vanishing of a polynomial in a bounded region can be controlled by the constant term. This suggests to consider, instead of the strip D, the rectangle

$$D_V = \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V \right\},$$

where $V > 0$ is an arbitrary fixed number. Let

$$S_V = \left\{ g \in H(D_V) : g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0 \right\}.$$

Then we have the following analogue of Theorem 1.7.

Theorem 1.8. *Suppose that $K \in \mathcal{K}$, $f(s) \in H(K)$, and $V > 0$ are such that $K \subset D_V$. Let $F : H(D_V) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the same assertion as in Theorem 1.6 is true.*

For example, we can take in Theorem 1.8

$$F(g) = c_1 g' + \ldots + c_r g^{(r)},\ g \in H(D_V),\ c_1, \ldots, c_r \in \mathbb{C}\backslash\{0\},$$

where $g^{(j)}$ denotes the jth derivative of g.

Now we state a discrete analogue of Theorem 1.4, and we preserve the notation of Theorem 1.4.

Theorem 1.9. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{a_1,\dots,a_r;F}(D)$. For $r = 1$, let $K \in \mathcal{K}$, and let $f(s)$ be an continuous function not taking the value a_1 on K and analytic in the interior of K. In the case $r \geq 2$, let $K \subset D$ be any compact set, and let $f(s) \in H_{a_1,\dots,a_r;F}(D)$. Then the same assertion as in Theorem 1.6 is true.*

The same examples of operators in Theorem 1.9 as in Theorem 1.4 can be taken.

We note that, differently from the paper [43], the above discrete universality theorems are valid for arbitrary fixed $h > 0$.

The results of Chapter 1 are published in [30] and [37].

The investigations of universality of zeta and L-functions did not stop after Voronin's paper. It turned out that the majority of the classical zeta and L-functions are also universal in the Voronin sense. We remind some classes of universal functions.

Let

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

denote the full modular group. Suppose that the function $F(z)$ is analytic in the upper half-plane $\Im z > 0$, and, for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

with some $\kappa \in 2\mathbb{N}$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z).$$

Then, clearly, the function $F(z)$ is periodic with period 1, thus, it has the Fourier series expansion

$$F(z) = \sum_{m=-\infty}^{\infty} c(m) e^{2\pi i m z}.$$

If $c(m) = 0$ for $m \leq 0$, then $F(z)$ is called a cusp form of weight κ for the full modular group. The zeta-function $\zeta(s,F)$ attached to the form F is defined, for $\sigma > \frac{\kappa+1}{2}$, by the Dirichlet series

$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and is analytically continued to an entire function. If, additionally, the form F is an eigen function of all Hecke operators

$$T_m f(z) = m^{\kappa-1} \sum_{a,d>0; ad=m} \frac{1}{d^{\kappa}} \sum_{b (\mod d)} f\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N},$$

then F can be normalized, i.e., it can be assumed that $c(1) = 1$. In this case, for $\sigma > \frac{\kappa+1}{2}$, the function $\zeta(s,F)$ has the Euler product expansion over primes

$$\zeta(s,F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p)+\beta(p)=c(p)$.

The universality theorem for the function $\zeta(s,F)$ was obtained in [25]. In this case, instead of the strip D, the strip

$$D_F=\left\{s\in\mathbb{C}:\frac{\kappa}{2}<\sigma<\frac{\kappa+1}{2}\right\}$$

is considered. We denote the class $\mathcal{K}$ by $\mathcal{K}_F$, and the class $H_0(K)$ by $H_{0F}(K)$. Then the universality of $\zeta(s,F)$ is of the form [25].

Theorem F. *Suppose that $K\in\mathcal{K}_F$ and $f(s)\in H_{0F}(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}meas\left\{\tau\in[0;T]:\sup_{s\in K}|\zeta(s+i\tau,F)-f(s)|<\varepsilon\right\}>0.$$

Let $N\in\mathbb{N}$ and

$$\Gamma_0(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z}):c\equiv 0(\bmod N)\right\}$$

be the so-called Hecke subgroup of $SL(2,\mathbb{Z})$. Then the cusp form F of weight κ with respect to $\Gamma_0(N)$ is called a cusp form of weight κ and level N. If F is not a cusp form of level less than N, then it is said to be a new form. Universality theorems for new forms were obtained in [26], [27],[28] and [29].

In [46], A. Selberg introduced a very important class S of L-functions defined as the set of all Dirichlet series

$$\mathcal{L}(s)=\sum_{m=1}^{\infty}\frac{a(m)}{m^s}$$

satisfying the following axioms:

1. Ramanujan hypothesis: $a(m)\ll m^{\varepsilon}$ for any $\varepsilon>0$;

2. Analytic continuation: there exists a number $k\in\mathbb{N}_0$, such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order, i.e., $(s-1)^k\mathcal{L}(s)=O\big(|s|^A\big)$ for some $A>0$ as $|s|\to\infty$;

3. Functional equation: $\mathcal{L}$ satisfies a functional equation of the type

   $$\Lambda_{\mathcal{L}}(s)=\omega\overline{\Lambda_{\mathcal{L}}(1-\bar{s})},$$

   where

   $$\Lambda_{\mathcal{L}}(s)=\mathcal{L}(s)Q^s\prod_{j=1}^{f}\Gamma(\lambda_j s+\mu_j)$$

   with the Euler gamma-function $\Gamma(s)$, positive real numbers Q, λ_j, and complex numbers μ_j and ω with $\Re\mu_j\geq 0$ and $|\omega|=1$;

4. Euler product: $\mathcal{L}$ has a product representation over primes

   $$\mathcal{L}(s)=\prod_p\mathcal{L}_p(s),$$

   where

   $$\log\mathcal{L}_p(s)=\sum_{\alpha=1}^{\infty}\frac{b\left(p^{\alpha}\right)}{p^{\alpha s}}$$

   with suitable coefficients $b\left(p^{\alpha}\right)$ satisfying $b\left(p^{\alpha}\right)\ll p^{\alpha\theta}$ for some $\theta<\frac{1}{2}$.

There exists a conjecture that the Selberg class S contains all classical zeta and L-functions. For example, the Riemann zeta-function $\zeta(s)$, Dirichlet L-functions, Dedekind zeta-functions of number fields, zeta functions attached to Hecke eigen forms are elements of the class S.

In [47], J. Steuding considered the universality of L-functions from a subclass of $\mathcal{S}$ having a polynomial Euler product satisfying the Ramanjan hypothesis. In [40], the latter requirement has been removed. In order to state the main result of [40], the degree $d_{\mathcal{L}}$ of $\mathcal{L} \in S$ is needed which is defined by

$$d_{\mathcal{L}} = 2\sum_{j=1}^{f} \lambda_j.$$

Moreover, let

$$\sigma_{\mathcal{L}} = \max\left\{\frac{1}{2}, 1 - \frac{1}{d_{\mathcal{L}}}\right\}.$$

In this case, the strip of universality is

$$D_{\mathcal{L}} = \{s \in \mathbb{C} : \sigma_{\mathcal{L}} < \sigma < 1\}.$$

Denote the class $\mathcal{K}$ by $\mathcal{K}_{\mathcal{L}}$, and the class $H_0(K)$, $K \in \mathcal{K}_{\mathcal{L}}$, by $H_{0\mathcal{L}}(K)$. Let, as usually,

$$\pi(x) = \sum_{p \le x} 1.$$

Then [40] the following universality theorem was proved.

Theorem G. *Suppose that $\mathcal{L}(s) \in S$ satisfies*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} |a(p)|^2 = \kappa,$$

where κ is some positive constant depending on $\mathcal{L}$. Let $K \in \mathcal{K}_{\mathcal{L}}$ and $f(s) \in H_{0\mathcal{L}}(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{\tau \in [0;T] : \sup_{s\in K} |\mathcal{L}(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The functions $\zeta(s,F)$ and $\mathcal{L}(s)$ in Theorems F and G have Euler product over primes. The simplest class of universal functions without Euler's product consists from Hurwitz zeta-functions $\zeta(s,\alpha)$, $0 < \alpha \le 1$.

Chapter 2 of the thesis is devoted to modified universal theorems for Hurwitz zeta-functions. It is well known that the function $\zeta(s,\alpha)$ for some classes of the parameter α is universal in the Voronin sense. More precisely, the following universality theorem is known.

Theorem H. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\Big\} > 0.$$

Theorem H was obtained in a slightly different form by Voronin [51], Gonek [8] and Bagchi [1], see also [23]. We note that the case of algebraic irrational parameter α remains an open problem. The cases $\alpha = 1$ and $\alpha = \frac{1}{2}$ are excluded because, as it was noted above, the equation $\zeta(s,1) = \zeta(s)$ and $(0,2)$ are valid. In these cases, the function $\zeta(s,\alpha)$ remains universal, however, the approximated function $f(s)$ must be non-vanishing on K. A joint universality theorem of [16] with $r = 1$ implies the following result.

Theorem I. *Suppose that the set $L(\alpha) = \{log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem H is true.*

A theorem of Cassel's asserts [6] that if $0 < \alpha \leq 1$, is an algebraic irrational number, at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linearly independent over $\mathbb{Q}$. Therefore, it is possible that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with algebraic irrational α, and the function $\zeta(s,\alpha)$ with this α is universal in the sense of Theorem I.

Theorems H and I are of continuous type. Also, the discrete universality of $\zeta(s,\alpha)$ has been considered. The following discrete universality theorems for $\zeta(s,\alpha)$ are known.

Theorem J. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$, $K \in \mathcal{K}$ and $f(s) \in H(K)$. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\} > 0.$$

For rational α, Theorem J was obtained in [1] and, by a different method, in [45]. For transcendental α, the theorem follows from more general discrete universality theorems of periodic Hurwitz zeta-function, proved in [24] $\zeta(s,\alpha,\mathfrak{a})$ which is defined, for $\sigma > 1$, by the series

$$\zeta(s,\alpha,\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

where $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers, and by analytic continuation elsewhere.

In [20], the following version of Theorem J was obtained.

Theorem K. *Suppose that the set*

$$L(\alpha,h,\pi) = \left\{ \big(\log(m+\alpha) : m \in \mathbb{N}_0\big), \frac{\pi}{h} \right\}$$

is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem J is true.

Chapter 2 of the thesis contains the modified versions of the above universality theorems for the Hurwitz zeta-function.

Theorem 2.1. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The second theorem of Chapter 2 involves the set $L(\alpha)$.

Theorem 2.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 2.1 is true.*

Other modified universality theorems of Chapter 2 are of discrete type.

Theorem 2.3. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$, $K \in \mathcal{K}$ and $f(s) \in H(K)$. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The next theorem involves the set $L(\alpha, h, \pi)$.

Theorem 2.4. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 2.3 is true.*

The results of Chapter 2 are published in [31].

Chapter 3 of the thesis connects in a certain sense the results of the previous chapters. Here modified joint universality theorems for the Riemann and Hurwitz zeta-functions are presented. Universality theorems for zeta-functions of such a kind are called mixed universality theorems because the Riemann zeta-function has Euler's product over primes, while the Hurwitz zeta-function has no Euler's product.

The first mixed joint universality theorem was obtained by H. Mishou in [38], and is called the Mishou theorem now. He proved the following theorem for $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental α.

Theorem L. *Suppose that α is a transcendental number, $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

In [12], Theorem L was generalized for periodic and periodic Hurwitz zeta-functions. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers. We remind that the periodic zeta-function $\zeta(s,\mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s,\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and can be meromorphically continued to the whole complex plane.

And extended modified version of Theorem L has the following form. Let

$$L(\alpha,\mathbb{P}) = \{(\log(m+\alpha) : m \in \mathbb{N}_0), (\log p : p \in \mathbb{P})\}.$$

Theorem 3.1. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For example, if α is transcendental, then the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$. Therefore, Theorem 3.1 extends the hypothesis of Theorem L.

In [19], universality theorems were proved for the function $F\big(\zeta(s),\zeta(s,\alpha)\big)$ with some operators $F : H^2(D) \to H(D)$. For example, in [19], the following assertion was obtained.

Theorem M. *Suppose that α is transcendental, and that $F : H^2(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau),\zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\Big\} > 0.$$

More general results were obtained in [21]. Here, universality theorems for composite functions of collections consisting from periodic and periodic Hurwitz zeta-functions were proved.

In the thesis, the following modification of Theorem M is proved.

Theorem 3.2. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, $F : H^2(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau),\zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Now let D_V and S_V be the same as in Theorem 1.8. Moreover, for brevity, we use the notation

$$H^2(D_V, D) = H(D_V) \times H(D).$$

Theorem 3.3. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and that K and $f(s)$ are the same as in Theorem 3.2, and $V > 0$ is such that $K \subset D_V$. Let $F : H^2(D_V, D) \to H(D_V)$ be a continuous operator such that , for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V \times H(D))$ is non-empty. Then the same assertion as in Theorem 3.2 is true.*

For example, Theorem 3.3 implies the modified universality of the functions

$$c_1\zeta(s)+c_2\zeta(s,\alpha) \text{ and } c_1\zeta'(s)+c_2\zeta'(s,\alpha) \quad \text{with } c_1,c_2\in\mathbb{C}\setminus\{0\}.$$

In Chapter 3, an analogue of Theorem 1.9 is also considered. Let $a_1,...,a_r$ be arbitrary distinct complex numbers, and

$$H_{a_1,\dots,a_r}(D)=\left\{g\in H(D):(g(s)-a_j)^{-1}\in H(D),\ j=1,...,r\right\}.$$

Theorem 3.4. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F:H^2(D)\to H(D)$ is a continuous operator such that $F(S\times H(D))\supset H_{a_1,\dots,a_r}(D)$. When $r=1$, let $K\in\mathcal{K}$, and $f(s)\in H(K)$ and $f(s)\neq a_1$ on K. If $r\geq 2$, let $K\subset D$ be an arbitrary compact subset, and $f(s)\in H_{a_1,\dots,a_r}(D)$. Then the same assertion as in Theorem 3.2 is true.*

The case $r=1$ with $a_1=0$ shows that $F(g_1(s),g_2(s))=e^{g_1(s)+g_2(s)}$ is universal in the sense of Theorem 3.4. If $r=2$ and $a_1=1$, $a_2=-1$, then, for example, for $F(g_1(s),g_2(s))=\cos(g_1(s)+g_2(s))$ and $f(s)\in H_{1,-1}(D)$, the limit of Theorem 3.2 exists for all but at most countably many $\varepsilon>0$.

The last continuous universality theorem of Chapter 3 is the following statement.

Theorem 3.5. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, $F:H^2(D)\to H(D)$ is a continuous operator, $K\subset D$ is a compact subset, and $f(s)\in F(S\times H(D))$. Then the same assertion as in Theorem 3.2 is true.*

Other theorems of Chapter 3 are devoted to modifications of discrete versions of the Mishou theorem. Let, for $h>0$,

$$L(\mathbb{P},\alpha,h,\pi)=\left\{(\log p:p\in\mathbb{P}),(\log(m+\alpha):m\in\mathbb{N}_0),\frac{\pi}{h}\right\}.$$

Then, in [4], the following discrete version of Theorem L was obtained.

Theorem N. *Suppose that the set $L(\mathbb{P},\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leq k\leq N:\sup_{s\in K_1}|\zeta(s+ikh)-f_1(s)|<\varepsilon,\ \sup_{s\in K_2}|\zeta(s+ikh,\alpha)-f_2(s)|<\varepsilon\right\}>0.$$

In [5], Theorem N was generalized by taking different arithmetical progressions for shifts $\zeta(s)$ and $\zeta(s,\alpha)$. For $h_1>0$ and $h_2>0$, define

$$L(\mathbb{P},\alpha,h_1,h_2,\pi)=\{(h_1\log p:p\in\mathbb{P}),(h_2\log(m+\alpha):m\in\mathbb{N}_0),\pi\}.$$

Theorem O. *Suppose that the set $L(\mathbb{P},\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leq k\leq N:\sup_{s\in K_1}|\zeta(s+ikh_1)-f_1(s)|<\varepsilon,\ \sup_{s\in K_2}|\zeta(s+ikh_2,\alpha)-f_2(s)|<\varepsilon\right\}>0.$$

In [22], arithmetical progressions were replaced by a more complicated set, and the uniform distribution of sequences modulo 1 was applied.

Theorem P. *Suppose that the number α is transcendental, and β, $0 < \beta < 1$, and $h > 0$ are fixed numbers. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ik^\beta h) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ik^\beta h, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

In Chapter 3 of the thesis, "liminf" in Theorems N-P is replaced by "lim". The following statements are true.

Theorem 3.6. *Suppose that the set $L(\mathbb{P}, \alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ikh) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ikh, \alpha) - f_2(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The next theorem is an analogue of Theorem O.

Theorem 3.7. *Suppose that the set $L(\mathbb{P}, \alpha, h_1, h_2, , \pi)$ is linearly independent over $\mathbb{Q}$ Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ikh_1) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ikh_2, \alpha) - f_2(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The last theorem of Chapter 3 is a modification of Theorem P.

Theorem 3.8. *Suppose that the number α is transcendental, and β, $0 < \beta < 1$, and $h > 0$ are fixed numbers. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ik^\beta h) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ik^\beta h, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

The results of Chapter 3 are published in [32] and [33].

Approbation

The results of the thesis were presented of the 11th International Vilnius Conference on Probability Theory and Mathematical Statistics (June 30 – July 4, 2014, Vilnius), the International Conference, Algebra, Number Theory and Discrete Geometry: Modern Problems and Applications (May 25-30, 2015, Tula, Russia), the MMA (Mathematical Modelling and Analysis) conferences (MMA2016, June 1-4, 2016, Tartu, Estonia), (MMA2017, May 30 – June 2, 2017, Druskininkai), the 6th Palanga Conference Analytic and Probabilistic Methods on Number Theory (September 11-17, 2016, Palanga), the Conferences of Lithuanian Mathematical Society (2014, 2015, 2016, 2017), as well as the Number Theory Seminar of Vilnius University.

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2. A. Laurinčikas and L. Meška, On the modification of the universality of the Hurwitz zeta-function, Nonlinear Analysis: Modeling and Control, 21 (2016), No4, 564-576.
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4. A. Laurinčikas and L. Meška, Modification of discrete versions of the Mishou theorem, in Anal. Probab. Mathem Number Theory, Proc. the 6th Palanga Conf (to appear)
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Chapter 1

Modified universality theorem for the Riemann zeta-function

In this chapter, we prove that the set of shifts of the Riemann zeta-function, approximating a given analytic function, with accuracy ε, has positive density for almost all $\varepsilon > 0$. We consider separately continuous shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, and discrete shifts $\zeta(s+ikh)$, $h > 0$ and $k \in \mathbb{N}_0$. Also, the chapter contains modified universality theorems for composite functions $F\big(\zeta(s)\big)$ for some classes of operators F in the space of analytic functions.

1.1 Continuous case

We remind that $\mathcal{K}$ is the class of compact subsets of the strip $D = \big\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\big\}$ with connected complements and $H_0(K)$ with $K \in \mathcal{K}$ denotes the class of continuous non-vanishing functions on K which are analytic in the interior of K. This section is devoted to the following theorem.

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas \left\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Proof of all universality theorems obtained in the thesis is based on limits theorems for weakly convergent probability measures in the space of analytic functions. This method was proposed in Bagchi's thesis [1] and developed in [15].

Let $\mathcal{B}(X)$ denote the Borel σ-field of the space X, and $H(G)$ be the space of analytic functions on the region $G \subset \mathbb{C}$ endowed with the topology of uniform convergence on compacta. Moreover, let

$\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane. We set

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. By the Tikhonov theorem [42], the infinite-dimensional torus Ω with the product topology and the operation of pointwise multiplication is a compact topological Abelian group. Therefore, on the measurable space $\big(\Omega, \mathcal{B}(\Omega)\big)$, the probability Haar measure m_H can be defined. The measure distinguishes from other probability measures by its invariance property. Namely, for all $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega$,

$$m_H(A) = m_H(\omega A) = m_H(A\omega).$$

Thus, we have the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordination space γ_p, $p \in \mathbb{P}$ ($\mathbb{P}$ is the set of all primes numbers), and on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$, where the $H(D)$-valued random element $\zeta(s, \omega)$ by the formula

$$\zeta(s, \omega) = \prod_p = \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

We note that the latter infinite product, for almost all $\omega \in \Omega$, on compact subsets of the strip D, defines the $H(D)$-valued random element. Let P_ζ be the distribution of the random element $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega) \in A), \ A \in \mathcal{B}\big(H(D)\big).$$

In other words, P_ζ is a probability measure on the space $\big(H(D), \mathcal{B}(H(D))\big)$.

We will deal with the weak convergence of probability measures in the thesis. Therefore, we recall the definition of that convergence. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $\big(X, \mathcal{B}(X)\big)$. We say that P_n converges weakly to P as $n \to \infty$ if, for every real bounded continuous function g on X,

$$\lim_{n \to \infty} \int_X g dP_n = \int_X g dP.$$

For P_T with continuous parameter T, the definition of the weak convergence remains the same.

In this section, we consider the weak convergence for

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\big\{\tau \in [0, T] : \zeta(s + i\tau) \in A\big\}, \ A \in \mathcal{B}\big(H(D)\big).$$

Lemma 1.1. *P_T converges weakly to the measure P_ζ as $T \to \infty$.*

Proof of the lemma is given in [15].

For the proof of Theorem 1.1, the support of the measure P_ζ is also needed. Let X be a separable metric space, i.e., X contains a countable and everywhere dense set, recall that the support of a probability measure P on $\big(X, \mathcal{B}(X)\big)$ is the minimal closed set $S_p \subset X$ such that $P(S_P) = 1$. The set S_P consists of all elements $x \in X$ such that for every open neighbourhood G of x we have $P(G) > 0$.

It is well known that the space of analytic functions is a separable one. Let

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Then the following assertion is known [15].

Lemma 1.2. *The support of the probability measure P_ζ is the set S.*

The definition of the weak convergence of probability measure has several equivalents in terms of various sets. For the proof of Theorem 1.1, we use such an equivalent in terms of continuity sets. We recall that the set $A \in \mathcal{B}(X)$ is called a continuity set of probability measure P on $\big(X, \mathcal{B}(X)\big)$ if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A.

Lemma 1.3. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $\big(X, \mathcal{B}(X)\big)$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 of [2].

For the proof of universality theorem, usually the Mergelyan theorem on the approximation of analytic functions by polynomials is applied. We state this important theorem as a separate lemma.

Lemma 1.4. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

The proof of the lemma is given in [36], see also [53].

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The function $f(s)$ has no zeros in the set K. Therefore, we take a continuity branch of $\log f(s)$. Therefore, according to Lemma 1.4, we can find a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}. \tag{1.1}$$

Consider the set

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s\in K} \big|f(s) - \mathrm{e}^{p(s)}\big| < \frac{\varepsilon}{2} \Big\}.$$

This set is an open neighbourhood of the function $e^{p(s)}$ which in virtue of Lemma 1.2, is an element of the support of the measure P_ζ. Therefore, we have

$$P_\zeta(G_\varepsilon) > 0. \tag{1.2}$$

Define one more set

$$A_\varepsilon = \Big\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then

$$\partial A_\varepsilon = \Big\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon \Big\}.$$

Hence, $\partial A_{\varepsilon_1} \cap \partial A_{\varepsilon_2} = \varnothing$ if $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Therefore, at most countably many sets ∂A_ε can have positive P_ζ-measure. Indeed, for any $n \in \mathbb{N}$, there are at most n sets A_ε for which

$$P_\zeta(\partial A_\varepsilon) > \frac{1}{n}.$$

Therefore, there are at most countably many sets ∂A_ε of positive P_ζ-measure. Hence, $P_\zeta(\partial A_\varepsilon) = 0$, i.e., A_ε is a continuity set of P_ζ for all but at most countably many $\varepsilon > 0$. Applying Lemmas 1.1 and 1.3, we obtain that

$$\lim_{T\to\infty} P_T(A_\varepsilon) = P_\zeta(A_\varepsilon) \tag{1.3}$$

for all but at most countably many $\varepsilon > 0$. Moreover, we observe that $G_\varepsilon \subset A_\varepsilon$ for all $\varepsilon > 0$. Indeed, suppose that $g \in G_\varepsilon$. Then, in view of (1.1),

$$\sup_{s\in K} \big|g(s) - f(s)\big| \leq \sup_{s\in K} \big|g(s) - \mathrm{e}^{p(s)}\big| + \sup_{s\in K} \big|f(s) - \mathrm{e}^{p(s)}\big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $g \in A_\varepsilon$. Now, in virtue of monotonicity of the measure and (1.2), we have $P_\zeta(A_\varepsilon) > 0$. By the definition of A_ε,

$$P_T(A_\varepsilon) = \frac{1}{T} meas \left\{ \tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\}.$$

Therefore, by (1.3), we obtain that the limit

$$\lim_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$. □

1.2 Universality of composite functions. Continuous case

In this section, we prove modified universality theorems for $F\big(\zeta(s)\big)$, where F is a certain continuous operator, $F : H(D) \to H(D)$. We start with the following assertion. Let $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K which are analytic in the interior of K.

Theorem 1.2. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \Big\{ \tau \in [0;T] : \sup_{s\in K} |F\big(\zeta(s+i\tau)\big) - f(s)| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Before proving of Theorem 1.2, we give some auxiliary probabilistic results.

Let X_1 and X_2 be two metric spaces. A function $u: X_1 \to X_2$ is a $\big(\mathcal{B}(X_1), (\mathcal{B}(X_2)\big)$-measurable if every $A \in \mathcal{B}(X)$, we have that $u^{-1}A \in \mathcal{B}(X_1)$. It is well known [2] that continuous function $u: X_1 \to X_2$ is $\big(\mathcal{B}(X_1), (\mathcal{B}(X_2)\big)$-measurable.

Now let $u: X_1 \to X_2$ be a $\big(\mathcal{B}(X_1), (\mathcal{B}(X_2)\big)$-measurable function. Then every probability measure P on $\big(X_1, \mathcal{B}(X_1)\big)$ induces the unique probability measure Pu^{-1} on $\big(X_2, \mathcal{B}(X_2)\big)$ defined by formula

$$Pu^{-1}(A) = P(u^{-1}A), \quad A \in \mathcal{B}(X_2).$$

The following property of weak convergence of probability measures sometimes is very useful.

Lemma 1.5. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $\big(X_1, \mathcal{B}(X_1)\big)$, P_n converges weakly to P as $n \to \infty$ and $u: X_1 \to X_2$ is a continuous function. Then $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$.*

Proof of the lemma can be found in [2].

For $A \in \mathcal{B}\big(H(D)\big)$ and $F: H(D) \to H(D)$, define

$$P_{T,F}(A) = \frac{1}{T}\Big\{\tau \in [0;T] : F\big(\zeta(s+i\tau)\big) \in A\Big\}$$

Then Lemmas 1.1 and 1.5 imply the following limit theorem.

Lemma 1.6. *Suppose that $F: H(D) \to H(D)$ is a continuous operator. Then $P_{T,F}$ converges weakly to $P_\zeta F^{-1}$ as $T \to \infty$.*

Proof. Clearly, we have that

$$P_{T,F}(A) = \frac{1}{T}\Big\{\tau \in [0;T] : \zeta(s+i\tau) \in F^{-1}A\Big\}$$

for all $A \in \mathcal{B}\big(H(D)\big)$. Therefore, $P_{T,F} = P_T F^{-1}$, where P_T is from Lemma 1.1. Hence, Lemmas 1.1 and 1.5 give the assertion of the lemma. □

Lemma 1.7. *Suppose that the operator $F: H(D) \to H(D)$ satisfies the hypothesis of Theorem 1.2. Then the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$.*

Proof. Let $g \in H(D)$ be an arbitrary element, and let G be an arbitrary open neighbourhood of g. Since F is continuous, the set $F^{-1}G$ is open as well. Since, by the hypothesis $(F^{-1}G) \cap S \neq \varnothing$, there exists an element $g_1 \in F^{-1}G$, such that $g_1 \in S$. This means by Lemma 1.2, that $F^{-1}G$ is an open neighbourhood of the element g_1, which is an element of the support S of P_ζ. Therefore, $P_\zeta(F^{-1}G) > 0$. Hence, $P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$. Since, g and G are arbitrary, this inequality proves the lemma □

Proof of Theorem 1.2. We use similar arguments to that of the proof of Theorem 1.1. By lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{1.4}$$

Define the set

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \frac{\varepsilon}{2} \Big\}.$$

Then we have that

$$P_\zeta F^{-1}(G_\varepsilon) > 0. \tag{1.5}$$

because the set G_ε, in view of Lemma 1.7, is an open neighbourhood of an element $p(s)$ the support of the measure P_ζ. Next we set

$$A_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then, as in the proof of Theorem 1.1, we have that the set A is a continuity set of the measure $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, according to Lemmas 1.6 and 1.7 the limit

$$\lim_{T \to \infty} P_{T,F}(A_\varepsilon) = P_\zeta F^{-1}(A_\varepsilon) \tag{1.6}$$

exists for all but at most countably many $\varepsilon > 0$, and the assertion of the theorem follows from the definition of $P_{T,F} \in A_\varepsilon$. □

Theorem 1.3. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 1.2 is true.*

For the proof of Theorem 1.3, we need a metric of the space $H(D)$. We define such a metric for a general region G on $\mathbb{C}$. It is known [15] that there exists a sequence of compact subsets $\{K_l : l \in \mathbb{N}\} \subset G$ such that

$$G = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, for all $l \in \mathbb{N}$, and if $K \subset G$ is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$. Now, for $g_1, g_2 \in H(G)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is the metric on $H(G)$ which induces the topology uniform convergence on compacta [15].

It is easily seen that, in the case of the space $H(D)$, $D = \big\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \big\}$, the sets K_l can be chosen with connected complements. For example, we can take a closed rectangle.

Proof of Theorem 1.3. In virtue of Theorem 1.2, it sufficient to show that the hypothesis $(F^{-1}\{p\}) \cap S \neq 0$ for every polynomial p implies that $(F^{-1}G) \cap S \neq 0$ for every open set of the space $H(D)$.

Let $\delta > 0$ be a fixed number. We take arbitrary element $g \in H(G)$ and its arbitrary open neighbourhood. We will prove that there exists a polynomial $p(s)$ which belongs to G. Let $\{K_l\} \subset D$ be square of compact subsets of the strip D with connected complements which occurs in the definition of the metric ρ. By Lemma 1.4, for some set $K_r \in \{K_l\}$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s \in K_r} |g(s) - p(s)| < \frac{\delta}{2}.$$

We can choose K_r to satisfy

$$\sum_{l>r}^{\infty} 2^{-l} \frac{\sup\limits_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s\in K_l} |g_1(s) - g_2(s)|} < \frac{\delta}{2}.$$

Then

$$\rho(g,p) < \sum_{l=1}^{r} 2^{-l} \frac{\sup\limits_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s\in K_l} |g_1(s) - g_2(s)|} + \frac{\delta}{2} < \frac{\delta}{2} \sum_{l=1}^{\infty} 2^{-l} + \frac{\delta}{2} = \delta$$

because $K_l \subset K$ for $l \leq r$. Hence, $p \in G$ if δ is small enough. Therefore, we have in view of inequality $(F^{-1}\{p\}) \cap S \neq \varnothing$, we obtain that $(F^{-1}G) \cap S \neq \varnothing$. Thus, we obtained the hypothesis of Theorem 1.2, and this proves Theorem 1.3. □

Let $a_1, ..., a_r \in \mathbb{C}$ and $F : H(D) \to H(D)$. We recall that

$$H_{a_1,\ldots,a_r;F}(D) = \big\{g \in H(D) : \ (g(s) - a_j)^{-1} \in H(D),\ j = 1, ..., r\big\} \cup \big\{F(0)\big\}.$$

Lemma 1.8. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{a_1,\ldots,a_r;F}(D)$. Then the support of the measure $P_\zeta F^{-1}$ contains the closure of the set $H_{a_1,\ldots,a_r;F}(D)$.*

Proof. Let g be an arbitrary element of $H_{a_1,\ldots,a_r;F}(D)$, and G be its open neighbourhood. Then, by the hypothesis of the lemma, there exists $g_1 \in S$ such that $F(g_1) = g$, and $F^{-1}G$ is an open neighbourhood of the g_1. In view of Lemma 1.2, we have that $P_\zeta(F^{-1}G) > 0.$. Hence,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0.$$

This inequality shows htat every element of the set $H_{a_1,\ldots,a_r;F}(D)$ is also an element of the support of the measure $P_\zeta F^{-1}$. Since the support is a closed set, we have that the support of $P_\zeta F^{-1}$ contains the closure of $H_{a_1,\ldots,a_r;F}(D)$. □

Theorem 1.4. *Let $F : H(D) \to H(D)$ be a continuous operator for which $F(S) \supset H_{a_1,\ldots,a_r;F}(D)$. In the case $r = 1$, let $K \in \mathcal{K}$, and $f(s)$ be a continuous function and not taking the value a_1 on K and analytic in the interior of K. In the case $r \geq 2$, let $K \subset D$ be any compact set, and let $f(s) \in H_{a_1,\ldots,a_r;F}(D)$. Then the same assertion as in Theorem 1.2 is true.*

Proof. The case $r = 1$. By Lemma 1.4, there exists a polynomial $p(s)$ for which

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{1.7}$$

Since $f(s) \neq a_1$ on K provided ε is small enough. Therefore, we can define a continuous branch of logarithm $\log\big(p(s) - a_1\big)$ which will be analytic in the interior of K. Again, by applying Lemma 1.4, we find a polynomial $p_1(s)$ such that

$$\sup_{s\in K} |p(s) - a_1 - \mathrm{e}^{p_1(s)}| < \frac{\varepsilon}{4}. \tag{1.8}$$

Let $h_{a_1}(s) = e^{p_1(s)} + a_1$. Then we have that $h_{a_1}(s) \in H(D)$ and $h_{a_1}(s) \neq a_1$. Therefore, according to Lemma 1.8, the function $h_{a_1}(s)$ is an element of the support of the measure $P_\zeta F^{-1}$. Moreover, it follows from the inequalities (1.7) and (1.8) that

$$\sup_{s\in K} \big|f(s) - h_{a_1}(s)\big| \leq \sup_{s\in K} \big|f(s) - p(s)\big| + \sup_{s\in K} \big|p(s) - h_{a_1}(s)\big| < \frac{\varepsilon}{2}. \tag{1.9}$$

Define the set

$$G_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |f(s) - h_{a_1}(s)| < \frac{\varepsilon}{2}\Big\}.$$

Then, in virtue of the above mentioned properties of the function $h_{a_1}(s)$, we have that

$$P_\zeta F^{-1}(G_\varepsilon) > 0. \tag{1.10}$$

Define one more set

$$A_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\Big\}.$$

The same argument as in the proof of Theorem 1.1 proves that A_ε is a continuity set of $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, according to Lemmas 1.6 and 1.3, we obtain that

$$\lim_{T\to\infty} P_{T,F}(A_\varepsilon) = P_\zeta F^{-1}(A_\varepsilon) \tag{1.11}$$

for all but at most countably many $\varepsilon > 0$. It follows from inequality (1.9) and the definition of the set G_ε and A_ε that $G_\varepsilon \subset A_\varepsilon$ for all $\varepsilon > 0$. Hence, (1.11) together with(1.10) implies

$$\lim_{T\to\infty} P_{T,F}(A_\varepsilon) = P_\zeta F^{-1}(G_\varepsilon) > 0,$$

which proves the required assertion in the case $r = 1$.

The case $r \geq 2$. Consider the set A_ε defined above. Since, $f(s) \in H_{a_1,\ldots,a_r;F}(D)$, it follows from Lemma 1.8 that $f(s)$ is an element of the support of the measure $P_\zeta F^{-1}$. Therefore, $P_\zeta F^{-1}(A_\varepsilon) > 0$. The same argument as in the proof of Theorem 1.1 proves that A_ε is a continuity set of $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. For $\varepsilon > 0$ such that $P_\zeta F^{-1}(\partial A_\varepsilon) = 0$, Lemmas 1.6 and 1.3 imply

$$P_{T,F}(A_\varepsilon) = P_\zeta F^{-1}(A_\varepsilon).$$

This and the definition of A_ε prove the theorem in the case $r \geq 2$. □

We give some explicit examples of Theorem 1.4.

Example 1. Let the operator $F : H(D) \to H(D)$ be given by the formula $F(g) = g^N$ with $N \in \mathbb{N}$. Then, obviously, $F(0) = 0$. If $g \in H(D)$ is a non-vanishing on D, then there exists a solution $f_1 \in S$ such that

$$F(f_1) = g,$$

i.e., $f_1 = \sqrt[N]{g}$. Therefore, we have that

$$F(s) \supset H_{0;F}(D),$$

and applying Theorem 1.4 with $r = 1$, $a_1 = 0$, we obtain the universality of $\zeta(s)$.

Example 2. Let the operator $F : H(D) \to H(D)$ be given by the formula

$$F(g) = \mathrm{e}^g, \quad g \in H(D).$$

Then $F(0) = 1$. If $g \in H(D)$ does not take the values 0 and 1 on D, then the equation

$$F(f) = g$$

has the solution $f = \log g$, where we may choose an arbitrary branch of logarithm which is continuous on D. Since $g(s) \neq 1$ on D, the function f is non-vanishing on D. Therefore, $g \in S$ and

$$F(s) \supset H_{0,1;F}(D).$$

Hence, by Theorem 1.4, the function $\mathrm{e}^{\zeta(s)}$ is universal with $f \in H_{0,1;F}(D)$.

Example 3. Let the operator $F : H(D) \to H(D)$ be given by the formula

$$F(g) = \sin g, \quad g \in H(D).$$

It is well known that

$$\sin g = \frac{\mathrm{e}^{ig} - \mathrm{e}^{-ig}}{2i}.$$

Therefore, $F(0) = 0$. We apply Theorem 1.4 with $r = 2$, $a_1 = 1$, $a_2 = -1$. Let $g \in H_{1,-1;0}(D)$. We solve the equation

$$\frac{\mathrm{e}^{if} - \mathrm{e}^{-if}}{2i} = g.$$

Denoting $y = \mathrm{e}^{if}$, we obtain the equation

$$y^2 - 2igy - 1 = 0,$$

and

$$y = ig \pm \sqrt{-g^2 + 1}.$$

Since $g \neq \pm 1$, we have that $-g^2 + 1 \neq 0$, and $ig + \sqrt{-g^2 + 1} \neq 0, 1$. Therefore,

$$f = \frac{1}{i} \log\left(ig + \sqrt{-g^2 + 1}\right) \in S$$

with a fixed branch of the logarithm. Thus, we have that

$$F(S) \supset H_{1,-1;0}(D),$$

and, by Theorem 1.4, the function $\sin \zeta(s)$ is universal.

Example 4. Similarly to Example 3, the universality of the functions $\cos \zeta(s)$, $\sin h\zeta(s)$ and $\cos h\zeta(s)$ are obtained. For this, the formulas

$$\cos g = \frac{\mathrm{e}^{ig} + \mathrm{e}^{-ig}}{2},$$

$$\sin hg = \frac{e^g - e^{-g}}{2}$$

and

$$\cos hg = \frac{e^g + e^{-g}}{2}$$

are applied.

The results of Sections 1.1 and 1.2 are published in [30], [31].

1.3 Discrete case

In this section, we prove a discrete version of Theorem 1.1. We always suppose that $h > 0$ is a fixed number.

Theorem 1.5. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proof of Theorem 1.5 is probabilistic, but, in our opinion, more complicated than that of Theorem 1.1. We will use the limit theorem from [1]. For the definition of the limit measure for

$$P_N(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \# \big\{ 0 \le k \le N : \zeta(s+ikh) \in A \big\}, \quad A \in \mathcal{B}\big(H(D)\big),$$

we need some notation and definitions. We say that the number $h > 0$ is of type 1 if the number $\mathtt{exp}\left\{\frac{2\pi m}{h}\right\}$ is irrational for all $m \in \mathbb{N}$, and h is of type 2 if it is not of type 1.

Suppose that $h > 0$ is of type 2. Then there exists the smallest $m_0 \in \mathbb{N}$ such that $\mathtt{exp}\left\{\frac{2\pi m_0}{h}\right\}$ is rational. Let

$$\mathtt{exp}\left\{\frac{2\pi m_0}{h}\right\} = \frac{a}{b}, \quad a, b \in \mathbb{N}, \quad (a, b) = 1.$$

Recall that $\mathbb{P}$ is the set of all prime numbers, and define

$$\mathbb{P}_0 = \Big\{ p \in \mathbb{P} : \alpha_p \ne 0 \ \ in \ \ \frac{a}{b} = \prod_{p\in\mathbb{P}} p^{\alpha_p} \Big\}.$$

Now, on a certain probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$, define a sequence of random variables $\big\{\theta_h(p) : p \in \mathbb{P}\big\}$. If $h > 0$ is of type 1, then $\big\{\theta_h(p) : p \in \mathcal{P}\big\}$ is a sequence of independent random variables which are uniformly distributed on the unit circle $\gamma = \big\{s \in \mathbb{C} : |s| = 1\big\}$. If $h > 0$ is of type 2, then the sequence $\big\{\theta_h(p) : p \in \mathbb{P}\big\}$ is defined in a more complicated manner. We fix $p_0 \in \mathbb{P}_0$ and suppose that $\{\theta_h(p) : p \in \mathbb{P} \setminus \{p_0\}\}$ is a sequence of independent random variables which are uniformly distributed on γ. Moreover, let the random variable $\theta_h(p_0)$ have the distribution

$$\mathbb{P}\left(\theta_h(p_0) = \mathtt{exp}\left\{ -\frac{1}{\alpha_{p_0}} \Big(2\pi i m + \sum_{p\in\mathcal{P}_0\setminus\{p_0\}} \alpha_p \log \theta_h(p) \Big) \right\}\right) = \frac{1}{|\alpha_{p_0}|},$$

where $0 \le m \le |\alpha_{p_0}|$. Having the sequence $\{\theta_h(p) : p \in \mathbb{P}\}$, define, on the probability space $(\Omega, \mathcal{A}, \mu)$, the $H(D)$-valued random element $\zeta_h(s)$ by the formula

$$\zeta_h(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\theta_h(p)}{p^s}\right)^{-1},$$

and denote by P_{ζ_h} its distribution, i.e.

$$P_{\zeta_h}(A) = \mu\big(\zeta_h(s) \in A\big), \quad A \in \mathcal{B}\big(H(D)\big).$$

Then the following limit theorem is true [1]. The set S is the same as in Theorem 1.2.

Lemma 1.9. *P_N converges weakly to the measure P_{ζ_h} as $N \to \infty$. Moreover, the support of the measure P_{ζ_h} is the set S.*

Proof of Theorem 1.5. We follow the proof of Theorem 1.1. Define the set

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then G_ε is an open set, and

$$\partial G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \Big\}.$$

This shows that, for positive $\varepsilon_1 \neq \varepsilon_2$, the sets $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ do not intersect. It follows from this that for all but at most a countably many $\varepsilon > 0$,

$$P_{\zeta_h}(G_\varepsilon) > 0,$$

i.e. the set G_ε is a continuity set of the measure P_{ζ_h} for all but at most a countably many $\varepsilon > 0$. This, Lemma 1.9 and Lemma 1.3 imply that

$$\begin{aligned} &\lim_{N \to \infty} \frac{1}{N+1} \#\big\{0 \le k \le N : \zeta(s + ikh) \in G_\varepsilon\big\} = \\ &= \lim_{N \to \infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s \in K} |\zeta(s+ikh) - f(s)| < \varepsilon\Big\} = P_{\zeta_h}(G_\varepsilon) > 0 \end{aligned} \tag{1.12}$$

for all but at most a countably many $\varepsilon > 0$. By Lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}. \tag{1.13}$$

Moreover, in view of second assertion of Lemma 1.9, $\mathrm{e}^{p(s)}$ is an element of the support of the measure P_{ζ_h}. Therefore,

$$P_{\zeta_h}(\widehat{G}_\varepsilon) > 0,$$

where

$$\widehat{G}_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2} \Big\}.$$

Since, for $g \in \widehat{G}_\varepsilon$,

$$\sup_{s \in K} \big|g(s) - f(s)\big| \le \sup_{s \in K} \big|g(s) - \mathrm{e}^{p(s)}\big| + \sup_{s \in K} \big|f(s) - \mathrm{e}^{p(s)}\big|$$

by (1.13), we have that $\widehat{G}_\varepsilon \subset G_\varepsilon$. Thus, $P_{\zeta_h}(G_\varepsilon) \ge P_{\zeta_h}(\widehat{G}_\varepsilon)$. This together with (1.12) proves the theorem. $\square$

1.4 Universality of composite functions. Discrete case

In this section, as in Section 1.2, we consider the approximation of analytic functions by shifts $F\big(\zeta(s+i\tau)\big)$, where $F : H(D) \to H(D)$ is a certain continuous operator, however τ takes values from the set $\{kh : k \in \mathbb{N}_0\}$ with fixed $h > 0$.

Theorem 1.6. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} \big|F\big(\zeta(s+ikh)\big) - f(s)\big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 1.6 is a discrete analogue of Theorem 1.2. We start with the limit theorem for

$$P_{N,F}(A) = \frac{1}{N+1} \#\Big\{0 \le k \le N : F\big(\zeta(s+ikh)\big) \in A\Big\}, \quad A \in \mathcal{B}\big(H(D)\big).$$

Lemma 1.10. *Suppose that $F : H(D) \to H(D)$ is a continuous operator. Then $P_{N,F}$ converges weakly to $P_{\zeta_h} F^{-1}$ as $N \to \infty$.*

Proof. From the definitions P_N and $P_{N,F}$, we have that for every $A \in \mathcal{B}\big(H(D)\big)$,

$$P_{N,F}(A) = \frac{1}{N+1} \#\Big\{0 \le k \le N : \zeta(s+ikh) \in F^{-1}A\Big\}.$$

Hence, $P_{N,F} = P_N F^{-1}$. Therefore, Lemmas 1.9 and 1.5 together with continuity of F give the assertion of the lemma. □

Lemma 1.11. *Suppose that the operator $F : H(D) \to H(D)$ satisfies the hypothesis of Theorem 1.6. Then the support of the measure $P_{\zeta_h} F^{-1}$ is the whole of $H(D)$.*

Proof. Let g be an arbitrary element of the space $H(D)$, and G be an open neighbourhood of the element g. From the continuity of the operator F, it follows that the set $F^{-1}G$ is open, too. Since the set $(F^{-1}G) \cap S$ is non-empty, there exists an element $\widehat{g} \in S$ which also belongs to $F^{-1}G$. Hence, $F^{-1}G$ is an open neighbourhood of the element $\widehat{g}$. In virtue of Lemma 1.9, the element $\widehat{g}$ belongs to the support of the measure P_{ζ_h}. Therefore, $P_{\zeta_h}(F^{-1}G) > 0$. Hence,

$$P_{\zeta_h} F^{-1}(G) = P_{\zeta_h}(F^{-1}G) > 0.$$

This proves the lemma, because the element g and its neighbourhood G are arbitrary. □

Proof of Theorem 1.6. By Lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{1.14}$$

Define the set

$$G_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\Big\}.$$

Then we have that

$$\partial G_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\Big\}.$$

Therefore, if $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_1 > 0$, then $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$. Hence, the set G_ε is a continuity set of measure $P_{\zeta_h} F^{-1}$ for all but at most a countably many $\varepsilon > 0$. This and Lemmas 1.9 and 1.3 show that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N+1} \#\big\{0 \le k \le N : F\big(\zeta(s+ikh)\big) \in G_\varepsilon\big\} = \\
&= \lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon\Big\} = P_{\zeta_h} F^{-1}(G_\varepsilon).
\end{aligned} \tag{1.15}$$

For all but at most countably many $\varepsilon > 0$.

Let

$$\widehat{G}_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\Big\}.$$

In view of (1.14), we have that, for $g \in \widehat{G}$,

$$\sup_{s\in K} |g(s) - f(s)| \le \sup_{s\in K} |g(s) - p(s)| + \sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

This shows that $\widehat{G}_\varepsilon \subset G_\varepsilon$. However, by Lemma 1.11, the polynomial $p(s)$ is an element of the support of the measure $P_{\zeta_h} F^{-1}$ and $\widehat{G}_\varepsilon$ is an open neighbourhood of $p(s)$. These remarks imply the inequality

$$P_{\zeta_h} F^{-1}(G_\varepsilon) \ge P_{\zeta_h} F^{-1}(\widehat{G}_\varepsilon) > 0.$$

Combining this with (1.15) proves the theorem. □

Now we state discrete analogue of Theorem 1.3.

Theorem 1.7. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 1.6 is true.*

The proof of Theorem 1.7 is based on Lemma 1.10 and the following statement.

Lemma 1.12. *Suppose that all hypotheses of Theorem 1.7 are satisfied. Then the support of the measure $P_{\zeta_h} F^{-1}$ is the whole of $H(D)$.*

Proof. We will show that the hypotheses of the lemma imply those of Lemma 1.11. In the proof of Theorem 1.3 , we have seen that the approximation in the space $H(D)$ is reduced to the approximation on compact subsets of the strip D with connected complements. Thus, let $K \subset D$ be a compact subset with connected complement. We take an arbitrary element $g \in H(D)$ and its open neighbourhood G. Then the set $F^{-1}G$ is open as well. By Lemma 1.4, for any $\varepsilon > 0$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Since $g \in G$, we may assume that $p \in G$, too, if ε is small enough. By the hypothesis of the lemma, we have that $\big(F^{-1}\{p\}\big) \cap S \neq \varnothing$. Therefore, $(F^{-1}G) \cap S \neq \varnothing$. Thus, we obtained the hypothesis of Lemma 1.11, and this proves the lemma. □

Proof of Theorem 1.7. We use Lemmas 1.10, 1.12 and 1.4, and repeat the proof of Theorem 1.6. □

Now we will prove a modification of Theorem 1.7. For an arbitrary $V > 0$, let

$$D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\},$$

and

$$S_V = \Big\{g \in H(D_V) : g^{-1}(s) \in H(D_V) \ \textit{or} \ g(s) \equiv 0\Big\}.$$

Theorem 1.8. *Suppose that $K \in \mathcal{K}$, $f(s) \in H(K)$, and $V > 0$ are such that $K \subset D_V$. Let $F : H(D_V) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the same assertion as in Theorem 1.6 is true.*

For $V > 0$, denote by $P_{N,V}$ and $P_{\zeta_h,V}$ the restrictions to the space $\big(H(D_V), \mathcal{B}(H(D_V))\big)$ for the measures P_N and P_{ζ_h}, respectively.

Lemma 1.13. *For every $V > 0$, $P_{N,V}$ converges weakly to $P_{\zeta_h,V}$ as $N \to \infty$.*

Proof. The function $u : H(D) \to H(D_V)$ given by the formula

$$u\big(g(s)\big) = g(s)\Big|_{s \in D_V}, \quad g \in H(D),$$

is continuous because $D_V \subset D$. Therefore, the lemma is a corollary to Lemmas 1.9 and 1.5. □

Lemma 1.14. *Suppose that $F : H(D_V) \to H(D_V)$ is a continuous operator, and*

$$P_{N,F,V}(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : F\big(\zeta(s+ikh)\big) \in A\Big\}, \quad A \in \mathcal{B}\big(H(D_V)\big).$$

Then $P_{N,F,V}$ converges weakly to $P_{\zeta_h,V}F^{-1}$ as $N \to \infty$.

Proof. The lemma follows from the continuity of F, and Lemmas 1.13 and 1.5. □

Now we will consider the support of the measure $P_{\zeta_h,V}F^{-1}$.

Lemma 1.15. *Suppose that all hypotheses of Theorem 1.8 are satisfied. Then the support of the measure $P_{\zeta_h,V}F^{-1}$ is the whole of $H(D_V)$.*

Proof. We follow the proofs of Lemmas 1.11 and 1.12. Let g be an arbitrary element of $H(D_V)$, and G be its open neighbourhood. Then the set $F^{-1}G$ is open as well because F is continuous one. Using Lemma 1.4, we obtain, as in the proof of Lemma 12, $(F^{-1}G) \cap S_V \neq \varnothing$. This shows that $F^{-1}G$ is an open neighbourhood of a certain element of the set S_V. The same arguments, as in the proof of Lemma 1.9, show that the set S_V is the support of the measure $P_{\zeta_h,V}$. Therefore, $P_{\zeta_h,V}(F^{-1}G) > 0$. Hence,

$$P_{\zeta_h,V}F^{-1}(G) = P_{\zeta_h,V}(F^{-1}G) > 0.$$

This proves the lemma because g and G are arbitrary. □

Proof of Theorem 1.8. Define the set

$$G_{\varepsilon,V} = \Big\{ g \in H(D_V) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then, as in the proofs of previous theorems, we have that the set $G_{\varepsilon,V}$ is a continuity set of the measure $P_{\zeta_h,V}F^{-1}$ for all but at most a countably many $\varepsilon > 0$. Therefore, by Lemmas 1.14 and 1.3,

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : F\big(\zeta(s+ikh)\big) \in G_{\varepsilon,V}\} = \\ &= \lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} \big|F\big(\zeta(s+ikh)\big) - f(s)\big| < \varepsilon\Big\} = P_{\zeta_h,V}F^{-1}(G_\varepsilon). \end{aligned} \tag{1.16}$$

for all but at most a countably many $\varepsilon > 0$.

Let

$$\widehat{G}_{\varepsilon,V} = \Big\{ g \in H(D_V) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \Big\},$$

where $p(s)$ is a polynomial satisfying inequality (1.14). Then we have that

$$\widehat{G}_{\varepsilon,V} \subset G_{\varepsilon,V}. \tag{1.17}$$

Since, in view of Lemma 1.15, the polynomial $p(s)$ is an element of the support of the measure $P_{\zeta_h,V}F^{-1}$, the inequality $P_{\zeta_h,V}F^{-1}(G_{\varepsilon,V}) > 0$ is true. Therefore, in virtue of (1.17),

$$P_{\zeta_h,V}F^{-1}(G_{\varepsilon,V}) > 0.$$

This and (1.16) prove the theorem. $\square$

Example. For $g \in H(D_V)$, let

$$F(g) = c_1 g' + ... + c_r g^{(r)}, \quad c_1, ..., c_r \in \mathbb{C} \setminus \{0\}.$$

The Cauchy integral formula shows that the operator F is continuous. We check the hypothesis that $\big(F\{p\}\big) \cap S_V \neq \varnothing$ for each polynomial $p = p(s)$. For this, we will prove that there exists a polynomial $q = q(s)$ such that $q \in F^{-1}\{p\}$ and $q(s) \neq 0$ for $s \in D_V$.

Let

$$p(s) = a_0 + a_1 s + ... + a_k s^k, \quad a_k \neq 0,$$

be arbitrary polynomial of degree k. We take

$$p(s) = b_0 + b_1 s + ... + b_{k+1} s^{k+1}, \quad b_{k+1} \neq 0.$$

First suppose that $r \le k+1$. Then we have

$$\begin{cases} q'(s) = b_1 + 2b_2 s + ... + (k+1) b_{k+1} s^k, \\ q''(s) = 2b_2 + ... + (k+1)k b_{k+1} s^{k-1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q^{(r)}(s) = r! b_r + ... + (k-1)k \cdot ... \cdot (k-r+2) b_{k+1} s^{k-r+1}, \end{cases}$$

and the equation $F(q) = p$ implies the equality

$$\begin{aligned}
&c_1b_1 + 2c_1b_2s + ... + (k+1)c_1b_{k+1}s^k \\
&+2c_2b_2 + ... + (k+1)kc_2b_{k+1}s^{k-1} + ... \\
&+r!c_rb_r + ... + (k+1)k \cdot ... \cdot (k-r+2)c_rb_{k+1}s^{k-r+1} \\
&=a_0 + a_1s + ... + a_ks^k.
\end{aligned}$$

Hence, we find that

$$\begin{cases}
(k+1)c_1b_{k+1} = a_k, \\
kc_1b_k + (k+1)kc_2b_{k+1} = a_{k-1}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
c_1b_1 + 2c_2b_2 + ... + r!c_rb_r = a_0.
\end{cases}$$

From this, we can determine the coefficients $b_1, ..., b_{k+1}$ of the polynomial $q(s)$.

The case $r > k+1$ is considered similarly. In this case, all derivetives $q^{(j)}$, for $j = k+2, ..., r$.

Now it remains to choose b_0 to be $|b_0|$ large enough so that $q(s) \neq 0$ for $s \in D_V$. This is possible because D_V is a bounded region.

Now we state the last theorem of this section and of Chapter 1. This theorem is a discrete analogue of Theorem 1.4. Let, as in Theorem 1.4, for $F : H(D) \to H(D)$ and $a_1, ..., a_r \in \mathbb{C}$,

$$H_{a_1,...,a_r;F}(D) = \{g \in H(D) : g(s) \neq a_j, j = 1, ..., r\} \cup \{F(0)\}.$$

Theorem 1.9. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{a_1,...,a_r;F}(D)$. For $r = 1$, let $K \in \mathcal{K}$, and let $f(s)$ be an continuous function not taking the value a_1 on K. In the case $r \geq 2$, let $K \subset D$ be any compact set, and let $f(s) \in H_{a_1,...,a_r;F}(D)$. Then the same assertion as in Theorem 1.6 is true.*

Lemma 1.16. *Suppose that all hypotheses of Theorem 1.9 are satisfied. Then the support of the measure $P_{\zeta_h}F^{-1}$ contains the closure of the set $H_{a_1,...,a_r;F}(D)$.*

Proof. By the property of the operator F that $F(S) \supset H_{a_1,...,a_r;F}$, for every element $g \in H_{a_1,...,a_r;F}(D)$, we can find an element $g_1 \in S$ such that $F(g_1) = g$. Let G be an open neighbourhood of the element g. Then $F^{-1}G$ is an open neighbourhood of the element g_1. Therefore, in virtue of Lemma 1.9, the inequality $P_{\zeta_h}(F^{-1}G) > 0$ is true. Therefore,

$$P_{\zeta_h}F^{-1}(G) = P_{\zeta_h}(F^{-1}G) > 0.$$

This shows that g is an element of the support of the measure $P_{\zeta_h}F^{-1}$. Therefore, the set $H_{a_1,...,a_r;F}(D)$ is a subset of the support of $P_{\zeta_h}F^{-1}$. Hence, the support of the measure $P_{\zeta_h}F^{-1}$ contains the closure of $H_{a_1,...,a_r;F}(D)$ because the support is a closed set.

□

Proof of Theorem 1.9. We consider two cases $r = 1$ and $r \geq 2$.
Let $r = 1$. Let

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then, as in the proof of Theorem 1.6, Lemmas 1.10 and 1.13 imply that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N+1} \#\big\{0 \leq k \leq N : F\big(\zeta(s+ikh)\big) \in G_\varepsilon\big\} = \\ &= \lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon\Big\} = P_{\zeta_h} F^{-1}(G_\varepsilon). \end{aligned} \tag{1.18}$$

By Lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{1.19}$$

Since $f(s) \neq a_1$ on K, we have that $p(s) \neq a_1$ on K as well if ε is small enough. Thus, we can define a continuous branch of $\log\big(p(s) - a_1\big)$ which will be analytic in the interior of K. Then, by Lemma 1.4 again, there exists a polynomial $q(s)$ such that

$$\sup_{s\in K} |p(s) - \varepsilon_1 - \mathrm{e}^{q(s)}| < \frac{\varepsilon}{4}. \tag{1.20}$$

Let $g_{a_1}(s) = \mathrm{e}^{q(s)} + a_1$. Then $g_{a_1}(s) \neq a_1$ because $\mathrm{e}^{q(s)} \neq a_1$. Hence, $g_{a_1}(s) \in H_{a_1}(D)$. Therefore, by Lemma 1.16, the function $g_{a_1}(s)$ is an element of the support of the measure $P_{\zeta_h} F^{-1}$. Hence,

$$P_{\zeta_h} F^{-1}(G_{1,\varepsilon}) > 0, \tag{1.21}$$

where

$$G_{1,\varepsilon} = \Big\{ g \in H(D) : \sup_{s\in K} |g(s) - g_{a_1}(s)| < \frac{\varepsilon}{2} \Big\}.$$

If $g \in G_{1,\varepsilon}$, then, in view of (1.19) and (1.20),

$$\sup_{s\in K} \big|g(s) - f(s)\big| \leq \sup_{s\in K} \big|g(s) - g_{a_1}(s)\big| + \sup_{s\in K} \big|g_{a_1}(s) - p(s)\big| + \sup_{s\in K} \big|f(s) - p(s)\big| < \varepsilon.$$

Consequently, $G_{1,\varepsilon} \subset G_\varepsilon$. Hence, $P_{\zeta_h} F^{-1}(G_{1,\varepsilon}) \leq P_{\zeta_h} F^{-1}(G_\varepsilon)$. This, (1.18) and (1.21) give the theorem in the case $r = 1$.

Now let $r \geq 2$. Then the set G_ε is an open neighbourhood of the element $f(s)$ of the support of the measure $P_{\zeta_h} F^{-1}$ by Lemma 1.16. Thus, $P_{\zeta_h} F^{-1}(G_\varepsilon) > 0$, and, the set G_ε is a continuity set of the measure $P_{\zeta_h} F^{-1}$ for all but at most a countably many $\varepsilon > 0$. Hence, by Lemmas 1.10 and 1.3,

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N+1} \#\big\{0 \leq k \leq N : F\big(\zeta(s+ikh)\big) \in G_\varepsilon\big\} = \\ &= \lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} \big|F\big(\zeta(s+ikh)\big) - f(s)\big| < \varepsilon\Big\} = P_{\zeta_h} F^{-1}(G_\varepsilon). \end{aligned} \tag{1.22}$$

for all but at most a countably many $\varepsilon > 0$.
The theorem is proved. $\square$

Chapter 2

Modified universality theorems for the Hurwitz zeta-function

Let α, $0 < \alpha \leq 1$, be a fixed parameter. We remind that the Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1.

This chapter is devoted to the approximation of analytic functions by shifts $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$. The classical theorems of such a kind assert that a given analytic function from a wide class can be approximated by shifts $\zeta(s + i\tau, \alpha)$ for some classes of the parameter α with accuracy $\varepsilon > 0$, and that the set of these shifts has a positive lower density. We prove that this set of shifts has a positive density for all but at most countably many $\varepsilon > 0$. As in Chapter 1, we consider the continuous and discrete cases separately.

2.1 Continuous case

The function $\zeta(s, \alpha)$ depends on the parameter α, and its value-distribution is influenced by the arithmetic of that parameter. This remark also concerns universality theorems for $\zeta(s, \alpha)$. Therefore, we consider separately some classes of the parameter α.

Let $\mathcal{K}$ be the same class of compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ as in Chapter 1, and $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K that are analytic in the interior of K.

Theorem 2.1. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proof of Theorem 2.1, as of universality theorems in Chapter 1, is also based on probabilistic limit theorems in the space of analytic functions $H(D)$. The case of transcendental α requires a new probability space.

Let, as in Chapter 1, γ be the unit circle on the complex plane. Define the set

$$\Omega_1 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With the product topology and the operation of pointwise multiplication, the torus Ω_1, similarly as Ω, is a compact topological Abelian group. Then the compactness of Ω_1 implies that, on $\big(\Omega_1, \mathcal{B}(\Omega_1)\big)$, the probability Haar measure m_{1H} can be defined, and we obtain the probability space $\big(\Omega_1, \mathcal{B}(\Omega_1), m_{1H}\big)$. Let $\omega_1(m)$ be the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. Now, on the probability space $\big(\Omega_1, \mathcal{B}(\Omega_1), m_{1H}\big)$, define the $H(D)$-valued random element $\zeta(s,\alpha,\omega_1)$ by the formula

$$\zeta(s,\alpha,\omega_1) = \sum_{m=0}^{\infty} \frac{\omega_1(m)}{(m+\alpha)^s}.$$

Let P_ζ be the distribution of the random element $\zeta(s,\alpha,\omega_1)$, i.e.

$$P_\zeta(A) = m_{1H}(\omega_1 \in \Omega_1 : \zeta(s,\alpha,\omega_1) \in A),\ A \in \mathcal{B}\big(H(D)\big).$$

For $A \in \mathcal{B}\big(H(D)\big)$, define

$$P_T(A) = \frac{1}{T} meas\{\tau \in [0,T] : \zeta(s+i\tau,\alpha) \in A\}.$$

Lemma 2.1. *Suppose that the number α is transcendental. Then P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of the measure P_ζ is the whole of $H(D)$.*

The proof of the lemma can be found in [23], Theorem 5.2.3 and Lemma 6.1.7.

To examine the case of rational parameter α, we need some auxiliary results for Dirichlet L-functions. Define

$$S_V = \Big\{g \in H(D_V) : g(s) \neq 0 \ \ or \ \ g(s) \equiv 0\Big\}.$$

For $A \in \mathcal{B}(H^v(D_V))$, let

$$P_{\underline{L}}(A) = m_H\left(\omega \in \Omega : \big(L(s,\omega,\chi_1), ..., L(s,\omega,\chi_v)\big) \in A\right),$$

where

$$L(s,\omega,\chi_l) = \prod_p \left(1 - \frac{\omega(p)\chi_l(p)}{p^s}\right)^{-1}, \quad l = 1,..,v.$$

Lemma 2.2. *The support of the measure $P_{\underline{L}}$ is the set S_V^v.*

Proof. The lemma is a particular case of more general Lemma 13 of [19ª] stated for arbitrary non-equivalent Dirichlet characters (Dirichlet characters which are not induced by the same primitive character). In our case, we consider different characters modulo b, thus, the problem of the non-equivalent does not exist. $\square$

Lemma 2.3. *Let $F : H^v(D_V) \to H(D_V)$ be a continuous operator such that, for each polynomial $q = q(s)$, the set $\left(F^{-1}\{q\}\right) \cap S_V$ is not empty. Then the support of the measure $P_{\underline{L}}F^{-1}$ is the whole of $H(D_V)$.*

Proof. The lemma is a particular case of Lemma 16 of [19ª] which is stated for arbitrary non-equivalent Dirichlet characters. $\square$

Now let the parameter α be rational, i.e., $\alpha = \frac{a}{b}$, $(a,b) = 1$, and let $\alpha \neq 1, \frac{1}{2}$. Then we have that $1 \leq a \leq b$ with $b \geq 3$, and, for $\sigma > 1$,

$$\zeta\left(s, \frac{a}{b}\right) = \sum_{m=0}^{\infty} \frac{1}{(m + \frac{a}{b})^s} = b^s \sum_{m=0}^{\infty} \frac{1}{(mb+a)^s} = f_1(s) f_2(s), \tag{2.1}$$

where $f_1(s) = b^s$ and

$$f_2(s) = \sum_{m=0}^{\infty} \frac{1}{(mb+a)^s}.$$

The function $f_2(s)$ can be written in a more convenient form

$$f_2(s) = \sum_{m \equiv a (\mathrm{mod} b)} \frac{1}{m^s}.$$

Now we use the probability space $\left(\Omega, \mathcal{B}(\Omega), m_H\right)$ defined in Section 1.1, i.e.

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p, and m_H is the probability Haar measure on $\left(\Omega, \mathcal{B}(\Omega)\right)$. On the probability space $\left(\Omega, \mathcal{B}(\Omega), m_H\right)$, define two $H(D)$-valued random elements

$$f_1(s, \omega) = \overline{\omega}(b) b^s,$$

where $\overline{\omega}(b)$ denotes the conjungate of $\omega(b)$, and

$$f_2(s, \omega) = \sum_{m \equiv a (\mathrm{mod} b)} \frac{\omega(m)}{m^s}.$$

Moreover, let

$$\zeta\left(s, \frac{a}{b}, \omega\right) = f_1(s, \omega) f_2(s, \omega), \tag{2.2}$$

and let P_ζ be the distribution of the random element $\zeta\left(s, \frac{a}{b}, \omega\right)$, i.e.

$$P_\zeta(A) = m_H\left(\omega \in \Omega : \zeta\left(s, \frac{a}{b}, \omega\right) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Lemma 2.4. *Suppose that the number α is rational $\neq 1, \frac{1}{2}$. Then P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of the measure P_ζ is the whole of $H(D)$.*

Proof. The function $f_1(s)$ is a polynomial, and the function $f_2(s)$ is given by an ordinary Dirichlet series. Therefore, by a standard method which was developed in [15] and applied in [48] can be proved that

$$\frac{1}{T} meas \left\{\tau \in [0;T] : \big(f_1(s+i\tau), f_2(s+i\tau)\big) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)) \tag{2.3}$$

converges weakly to the distribution of the $H^2(D)$-valued random element $\big(f_1(s,\omega_1), f_2(s,\omega_1)\big)$ as $T \to \infty$. Let $u : H^2(D) \to H(D)$ be given by the formula $u(g_1, g_2) = g_1 g_2$. Then the function u is continuous. Therefore, using Lemma 1.5, the weak convergence of (2.3) and equalities (2.1) and (2.2), show that P_T converges weakly to P_ζ as $T \to \infty$.

It remains to consider the support of the measure P_ζ. Since $(a,b) = 1$, we have that the random variable $\omega(b)$ and each random variable $\omega(m)$, $m \equiv a (\texttt{mod}\, b)$, are independent. From this, it follows that the random elements $f_1(s,\omega)$ and $f_2(s,\omega)$ are independent.

Define

$$a_m = \begin{cases} 1 & \text{if } m \equiv a(\texttt{mod}\, b), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a_m : m \in \mathbb{N}_0\}$ is a periodic sequence, and $(m,b) = 1$ when $m \equiv a(\texttt{mod}\, b)$, since $(a,b) = 1$. Therefore, we have that for $\sigma > 1$,

$$f_2(s) = \sum_{\substack{m=1 \\ (m,b)=1}}^{\infty} \frac{a_m}{m^s}.$$

Then, by a standard way, it follows that

$$\frac{1}{T} meas \left\{\tau \in [0;T] : f_2(s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(H(D)), \tag{2.4}$$

converges weakly to the distribution

$$P_{f_2}(A) = m_H \left\{\omega \in \Omega : f_2(s,\omega) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

of the random element $f_2(s,\omega)$ as $T \to \infty$. It remains to find the support of the measure P_{f_2}.

Having in mind that non-vanishing of a polynomial in a bounded region can be controlled by its constant term, we replace the strip D by a bounded rectangle. Let $V > 0$ be an arbitrary number, and

$$D_V = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\right\}.$$

Since the mapping $u : H(D) \to H(D_V)$ given by the formula $u\big(g(s)\big) = g(s)\Big|_{s \in D_V}$ is continuous, we find from the weak convergence of the measure (2.4) and Lemma 1.5 that

$$P_{T,V}(A) \stackrel{\text{def}}{=} \frac{1}{T} meas \left\{\tau \in [0;T] : f_2(s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(H(D_V)),$$

also converges weakly to $P_{f_2,V}$ as $T \to \infty$, where

$$P_{f_2,V}(A) = m_H \left\{\omega \in \Omega : f_2(s,\omega) \in A\right\}, \quad A \in \mathcal{B}(H(D_V)).$$

We will prove that the support of $P_{f_2,V}$ is the whole of $H(D_V)$. Denote by $\chi_1, ..., \chi_v$ all Dirichlet characters modulo b. Then, in view of properties of Dirichlet characters, there exist complex numbers $c_1, ..., c_v$ such that, for $1 \le m \le b$, $(m, b) = 1$,

$$a_m = \sum_{l=1}^{v} c_l \chi_l(m). \tag{2.5}$$

Since a_m and $\chi_v(m)$ are periodic, equality (2.5) remains true for all $m \in \mathbb{N}$, $(m, b) = 1$. Hence,

$$f_2(s) = \sum_{l=1}^{v} c_l L(s, \chi_l), \tag{2.6}$$

where $L(s, \chi_l)$ denote the Dirichlet L-functions.

Then, by Lemma 2.2, the support of the measure $P_{\underline{L}}$ is the set S_V^v. Without loss of generality, we can suppose that at least two numbers c_l in (2.5) are non-zeros. Actually, if only one of the numbers c_l is non-zero, then we have that a_m is a Dirichlet character, and the function $f_2(s)$ is a Dirichlet L-function. Therefore, this contradicts the condition that $b \ge 3$. The operator $F : H^v(D_V) \to H(D_V)$ given by the formula

$$F(g_1, ..., g_v) = \sum_{l=1}^{v} c_l g_l, \quad g_1, ..., g_v \in H(D_V),$$

is, of course, continuous. Moreover, for each polynomial $q = q(s)$, there exists $g_1, ..., g_v \in S_V$ such that

$$F(g_1, ..., g_v) = q. \tag{2.7}$$

For example, we may take that

$$g_1(s) = (q(s) + C)/c_1,$$

$$g_2(s) = -(C + c_2 + ... + c_v)/c_2$$

and $g_3(s) = ... = g_v(s) = 1$, where $|C|$ is rather large. Therefore, in view of (2.7) and Lemma 2.3, we have that the support of $P_{f_2,V}$ is the whole of $H(D_V)$. We remind that here $V > 0$ is an arbitrary number. Letting $V \to \infty$, we obtain that $H(D_V)$ coincides with $H(D)$, and $P_{f_2,V}$ becomes P_{f_2}. Thus, the support of the random element $f_2(s, \omega)$ is the whole of $H(D)$. Since $f_1(s, \omega)$ is not degenerated at zero, and $f_1(s, \omega)$ and $f_2(s, \omega)$ are independent random element, this shows that the support of the product $f_1(s, \omega) f_2(s, \omega)$ is the whole of $H(D)$. Therefore, in view of (2.2), the support of the measure $\hat{P}_\zeta$ is the whole of $H(D)$. $\square$

Proof of Theorem 2.1. Let

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then G_ε is an open set in $H(D)$, moreover,

$$\partial G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \Big\}.$$

Therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Hence, $P_\zeta(\partial G_\varepsilon) > 0$ for all but at most countably many $\varepsilon > 0$. Here by P_ζ we denote the limit measure in Lemmas 2.1 and 2.4. Thus, in view of Lemmas 2.1, 2.4 and 1.3,

$$\begin{gathered}\lim_{T\to\infty} \frac{1}{T} meas \{\tau \in [o;T] : \zeta(s+i\tau,\alpha) \in G_\varepsilon\} = \\ \lim_{T\to\infty} \frac{1}{T} meas \left\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| > \varepsilon\right\} = P_\zeta(G_\varepsilon)\end{gathered} \tag{2.8}$$

for all but at most countably many $\varepsilon > 0$. By Lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{2.9}$$

Since, by Lemmas 2.1 and 2.4, $p(s)$ is an element of the support of the measure P_ζ, we have that $P_\zeta(\hat{G}_\varepsilon) > 0$, where

$$\hat{G}_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\right\}.$$

Clearly, for $g \in \hat{G}_\varepsilon$, by (2.8),

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Therefore, $\hat{G}_\varepsilon \subset G_\varepsilon$. Hence, $P_\zeta(G_\varepsilon) \geq P_\zeta(\hat{G}_\varepsilon) > 0$, and the theorem follows from equality (2.7). □

We remind that

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}.$$

Theorem 2.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 2.1 is true.*

For the proof of Theorem 2.2, we apply the following statement.

Lemma 2.5. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then P_T converges weakly to the measure P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$.*

Proof. The lemma is a cases of Theorems 4 and 11 from [16] with $r = 1$. □

Proof of Theorem 2.2. We use Lemma 2.5 and follow the proof of Theorem 2.1 □

2.2 Discrete case

In this section, we prove discrete analogues of Theorem 2.1 and 2.2.

Theorem 2.3. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$, $K \in \mathcal{K}$ and $f(s) \in H(K)$. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

As in the proofs of previous theorems, we start with probabilistic limit theorems which state as separate lemmas.

Lemma 2.6. *Suppose that the parameter α is transcendental, and $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then*

$$P_N(A) \overset{\text{def}}{=} \frac{1}{N+1}\#\Big\{0 \le k \le N : \zeta(s+ikh,\alpha) \in A\Big\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the measure P_ζ as $N \to \infty$.

Proof. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers. The periodic Hurwitz zeta-function $\zeta(s,\alpha,\mathfrak{a})$, $0 \le \alpha \le 1$ is defined for $\sigma > 1$ by the Dirichlet series

$$\zeta(s,\alpha,\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and can be analytically continued to the whole complex plane, except for a possible pole at the point $s = 1$. Define the $H(D)$-valued random element $\zeta(s,\alpha,\omega,\mathfrak{a})$ by the formula

$$\zeta(s,\alpha,\omega,\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m\omega(m)}{(m+\alpha)^s},$$

and denote by $P_{\zeta,\mathfrak{a}}$ its distribution, i.e.

$$P_{\zeta,\mathfrak{a}}(A) = m_{1H}(\omega_1 \in \Omega_1 : \zeta(s,\alpha,\omega_1,\mathfrak{a}) \in A),\ A \in \mathcal{B}\big(H(D)\big).$$

Then in [24], Theorem 6.1, it was proved that, under hypotheses of the lemma,

$$\frac{1}{N+1}\#\Big\{0 \le k \le N : \zeta(s+ikh,\alpha,\mathfrak{a}) \in A\Big\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$. Obviously, if $a_m \equiv 1$, then the function $\zeta(s,\alpha,\mathfrak{a})$ becomes the Hurwitz zeta-function $\zeta(s,\alpha)$, and the random element $\zeta(s,\alpha,\omega_1,\mathfrak{a})$ becomes $\zeta(s,\alpha,\omega_1)$. Therefore, the lemma is a partial case of theorem 6.1 from [24]. $\square$

The case of rational α is more complicated, and we need more modifications of Lemma 2.6. We use the following notation. The number $h > 0$ is called of type 1, if $\exp\left\{\frac{2\pi m}{h}\right\}$ is an irrational number for all $m \in \mathbb{Z} \setminus \{0\}$, and type 2, if there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $\exp\left\{\frac{2\pi m}{h}\right\}$ is a rational number. Let Ω_h be the closed subgroup of Ω generated by $(p^{-ih} : p \in \mathbb{P})$. Then it is known [29], Lemma 1, that if h is of type 1, then $\Omega_h = \Omega$. Now suppose that $h > 0$ is of type 2. Then there exists the least $m_0 \in \mathbb{N}$ such that the number $\exp\left\{\frac{2\pi m}{h}\right\}$ is rational. Let

$$\exp\left\{\frac{2\pi m}{h}\right\} = \frac{u}{v}, \quad u, v \in \mathbb{N}, \quad (u,v) = 1. \tag{2.10}$$

Again by Lemma 1 of [29],

$$\Omega_h = \{\omega \in \Omega : \omega(u) = \omega(v)\}.$$

Denote by m_H^h the probability Haar measure on $(\Omega_h, \mathcal{B}(\Omega_h))$, and, on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_h^h))$, define the $H(D)$-valued random element

$$\zeta_h\left(s, \frac{a}{b}, \omega\right) = f_1(s,\omega) f_2(s,\omega), \quad \omega \in \Omega_h,$$

where $f_1(s,\omega)$ and $f_2(s,\omega)$ are the same as in (2.2). Let P_{ζ_h} stand for the distribution of $\zeta_h\left(s, \frac{a}{b}, \omega\right)$, i.e.

$$P_{\zeta_h}(A) = m_H^h(\omega \in \Omega_h : \zeta(s, \frac{a}{b}, \omega) \in A),\ A \in \mathcal{B}\big(H(D)\big).$$

Lemma 2.7. *Suppose that α is rational $\neq 1, \frac{1}{2}$, and $h > 0$ is an arbitrary number. Then P_N converges weakly to P_{ζ_h} as $N \to \infty$. Moreover, the support of P_{ζ_h} is the whole of $H(D)$.*

Proof. If h is of type 1, the proof, in view of the mentioned above equality $\Omega_h = \Omega$, coincides with that of Lemma 2.4. Therefore, it remains to consider the case of h of type 2. For this, we will apply the following assertion from [29], Lemma 2:

$$Q_{N,h}(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \# \left\{0 \le k \le N : \left(p^{-ikh} : p \in \mathbb{P}\right) \in A\right\}, \quad A \in \mathcal{B}(\Omega_h),$$

converges weakly to the Haar measure m_H^h as $N \to \infty$. To prove this, as usually, the method of Fourier transforms is applied. The main difficulty is to describe the characters of the group Ω_h. Define

$$\mathbb{P}_0 = \left\{ p \in \mathbb{P} : \alpha_p \neq 0 \text{ in } \frac{u}{v} = \prod_{p \in \mathbb{P}} p^{\alpha_p} \right\},$$

where u and v are defined in (2.9). Let Ω^* be the dual group (or character group) of the group Ω, the character $\chi_{m_0} \in \Omega^*$ be given by the formula

$$\chi_{m_0}(\omega) = \prod_{p \in \mathbb{P}} \omega^{\alpha_p}(p) = \frac{\omega(u)}{\omega(v)}, \tag{2.11}$$

and $\Omega_h^\perp = \left\{\chi \in \Omega^* : \chi(m) = 1, \omega \in \Omega_h\right\}$. If h is of type 2, then it is not difficult to see that

$$\Omega_h^\perp = \left\{\chi_{m_0}^l : l \in \mathbb{Z}\right\}. \tag{2.12}$$

In view of Theorem 27 from [39], we have that the factor group $\Omega^*/\Omega_h^\perp$ is the dual group of the group Ω_h. Hence, the characters of the group Ω_h are of the form

$$\chi(\omega) = \prod_{p \in \mathbb{P}\setminus\mathbb{P}_0} \omega^{k_p}(p) \prod_{p \in \mathbb{P}} \omega^{k_p + l\alpha_p}(p), \quad l \in \mathbb{Z},$$

where only a finite number of integers k_p are distinct from zero. Therefore, the Fourier transform $\varphi_{N,h}(\underline{k})$, $\underline{k} = (k_p, p \in \mathbb{P})$, of the measure $Q_{N,h}$ is of the form

$$\varphi_{N,h}(\underline{k}) = \int_{\Omega_h} \chi(\omega) dQ_{N,h} = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{p \in \mathbb{P}\setminus\mathbb{P}_0} p^{-ikk_p h} \prod_{p \in \mathbb{P}_0} p^{-ikh(k_p + l_{\alpha_p})}, \quad l \in \mathbb{Z}, \tag{2.13}$$

where only a finite number of integers k_p are distinct from zero.

Suppose that $k = 0$ for any $p \in \mathbb{P} \setminus \mathbb{P}_0$ and $k_p = r\alpha_p$ for any $p \in \mathbb{P}_0$ with some $r \in \mathbb{Z}$. Then, by (2.10) and (2.11),

$$\varphi_{N,h}(\underline{k}) = 1. \tag{2.14}$$

Now let $k_p \neq 0$ for some $p \in \mathbb{P} \setminus \mathbb{P}_0$ or there does not exist $r \in \mathbb{R}$ such that $k_p = r\alpha_p$ for all $p \in \mathbb{P}$. Then we observe that

$$\exp\left\{-ih\left(\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l\alpha_p)\log p\right)\right\} \neq 1 \tag{2.15}$$

In fact, if (2.14) is not true, the we have that

$$\exp\left\{\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l\alpha_p)\log p\right\} = \mathrm{e}^{\frac{2\pi \mathrm{l}_0}{\mathrm{h}}} \tag{2.16}$$

with some $l_0 \in \mathbb{Z}$. If l_0 is an multiple of m_0, then it follows that

$$\exp\left\{\frac{2\pi l_0}{h}\right\} = \prod_{p\in\mathbb{P}_0} p^{l_1\alpha_p}$$

with some $l_1 \in \mathbb{Z}$. Therefore, by (2.15)

$$\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l_2\alpha_p)\log p = 0$$

with some $l_2 \in \mathbb{Z}$, and this contradicts the linear independence over $\mathbb{Q}$ of the set $\{\log p : p \in \mathbb{P}\}$. If l_0 is not multiple of m_0, then the number $\exp\left\{\frac{2\pi l_0}{h}\right\}$ is irrational, and this is a contradiction because the left-hand side of (2.15) is a rational number. Therefore, inequality (2.14) is true in all cases. Now using the formula for the sum of the geometric progression, we derive from (2.12) that

$$\varphi_{N,h}(\underline{k}) = \frac{1}{N+1}\sum_{k=0}^{N} \exp\left\{-ikh\left(\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l_2\alpha_p)\log p\right)\right\}$$

$$= \frac{1 - \exp\left\{-ih(N+1)\left(\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l\alpha_p)\log p\right)\right\}}{(N+1)\left(1 - \exp\left\{-ih\left(\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} k_p \log p + \sum_{p\in\mathbb{P}_0} (k_p + l\alpha_p)\log p\right)\right\}\right)}$$

with $l \in \mathbb{Z}$. This together with (2.13) shows that

$$\lim_{N\to\infty} \varphi_{N,h}(\underline{k}) = \begin{cases} 1 & \text{if } k_p = 0 \text{ for } p \in \mathbb{P}\setminus\mathbb{P}_0 \text{ or } k_p = r\alpha_p \text{ for } p \in \mathbb{P}_0, \\ 0 & \text{otherwise.} \end{cases}$$

However,

$$\varphi(\underline{k}) = \begin{cases} 1 & \text{if } k_p = 0 \text{ for } p \in \mathbb{P}\setminus\mathbb{P}_0 \text{ or } k_p = r\alpha_p \text{ for } p \in \mathbb{P}_0, \\ 0 & \text{otherwise,} \end{cases}$$

is the Fourier transform of the Haar measure m_H^h. Therefore, $Q_{N,h}$, converges weakly to m_H^h by a continuity theorem for probability measures on compact topological groups, see, for example, Theorem

1.4.2 from [10]. Further, by a standard method, it follows that

$$\frac{1}{N+1}\#\left\{0\le k\le N:\big(f_1(s+ikh),f_2(s+ikh)\big)\in A\right\},\quad A\in\mathcal{B}(H^2(D)), \tag{2.17}$$

and, for $\omega\in\Omega_h$,

$$\frac{1}{N+1}\#\left\{0\le k\le N:\big(f_1(s+ikh,\omega),f_2(s+ikh,\omega)\big)\in A\right\},\quad A\in\mathcal{B}(H^2(D)), \tag{2.18}$$

converges weakly to the same probability measure P on $(H^2(D),\mathcal{B}H^2(D))$ as $N\to\infty$.

For the identification of the limit measure P, some elements of the ergodic theory are applied. Let, for brevity, $a_h=\left(p^{-ih}:p\in\mathbb{P}\right)$. Clearly, $a_h\in\Omega_h$, since

$$\frac{a_h(u)}{a_h(v)}=\prod_{p\in\mathbb{P}_0}\mathrm{e}^{-ih\alpha_p}=\left(\frac{u}{v}\right)^{-ih}=\left(\mathrm{e}^{\frac{2\pi i m_0}{h}}\right)^{-ih}=1.$$

Define the transformation $\varphi_h(\omega)$ of the group Ω_h by the formula

$$\varphi_h(\omega)=a_h\omega,\quad \omega\in\Omega_h.$$

Then φ_h is a measurable measure preserving transformation on the probability space $(\Omega_h,\mathcal{B}(\Omega_h),m_H^h)$. We remind that a set $A\in\mathcal{B}(\Omega_h)$ is called invariant with respect to the transformation φ_h if the sets A and $\varphi_h(A)$ can differ one from other at most by a set of m_H^h-measure zero. All invariant sets constitute a σ-field of invariant sets consists only from the sets having m_H^h-measure 1 or 0.

By Lemma 3 of [29], for h of type 2, the transformation φ_h is ergotic.

Let $A\in\mathcal{B}(H(D))$ be a fixed continuity set of the measure P. Then the weak convergence of the measure (2.17) and Lemma 1.3 imply the relation

$$\lim_{N\to\infty}\frac{1}{N+1}\#\left\{0\le k\le N:\big(f_1(s+ikh),f_2(s+ikh)\big)\in A\right\}=P(A). \tag{2.19}$$

On the probability space $(\Omega_h,\mathcal{B}(\Omega_h),m_H^h)$, define the random variable θ_h by the formula

$$\theta_h=\begin{cases}1 & \text{if } \big(f_1(s+ikh,\omega_1),f_2(s+ikh,\omega_1)\big)\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, we have that

$$\mathbb{E}\theta_h=\int_{\Omega_h}\theta_h dm_H^h=m_H^h\left\{\omega\in\Omega_h:\big(f_1(s+ikh,\omega),f_2(s+ikh,\omega)\big)\in A\right\}. \tag{2.20}$$

On the other hand, the ergodicity of the transformation φ_h and the classical Birkhoff-Khintchine theorem [14] show that

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\theta_h\big(\varphi_h^k(\omega)\big)=\mathbb{E}\theta_h \tag{2.21}$$

for almost all $\omega\in\Omega_h$. However, the definitions of θ_h and φ_h give

$$\frac{1}{N+1}\sum_{k=0}^{N}\theta_h\big(\varphi_h^k(\omega)\big)=\frac{1}{N+1}\#\left\{0\le k\le N:\big(f_1(s+ikh),f_2(s+ikh)\big)\in A\right\}.$$

From this, (2.19) and (2.20), we find that, for almost all $\omega \in \Omega_h$,

$$\lim_{N\to\infty} \frac{1}{N+1} \#\left\{0 \le k \le N : \big(f_1(s+ikh), f_2(s+ikh)\big) \in A\right\} = P_{\zeta_h}(A).$$

Therefore, in view of (2.18),

$$P(A) = P_{\zeta_h}(A) \tag{2.22}$$

for any continuity set A of the measure P. Since all continuity sets constitute the determining class, hence we obtain that (2.21) holds for all $A \in \mathcal{B}\big(H(D)\big)$. This proves the first part of the lemma.

For the proof that the support of the measure P_{ζ_h} is the whole of $H(D)$, it suffices to repeat the proof of Lemma 2.4 with obvious changes. □

Proof of Theorem 2.3. First let the parameter α be transcendental. Let, as in the proof of Theorem 2.1,

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\right\}.$$

Then, by Lemmas 2.6 and 1.3, we have that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N+1} \#\big\{0 \le k \le N : \zeta(s+ikh,\alpha) \in G_\varepsilon\big\} = \\ &= \lim_{N\to\infty} \frac{1}{N+1} \#\left\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} = P_\zeta(G_\varepsilon) \end{aligned} \tag{2.23}$$

for all but at most countably many $\varepsilon > 0$. Lemma 2.1 implies the inequality $P_\zeta(\hat{G}_\varepsilon) > 0$, where

$$\hat{G}_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\right\},$$

and $p(s)$ is a polynomial such that

$$\sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}.$$

This inequality shows that $\hat{G}_\varepsilon \subset G_\varepsilon$. Therefore, $P_\zeta(G_\varepsilon) \ge P_\zeta(\hat{G}_\varepsilon) > 0$. This and (2.22) prove the theorem.

If the parameter α is rational, then the proof is analogous to that of a transcendental α, and, in place of Lemmas 2.6 and 2.1, we apply Lemma 2.7. □

We remind that

$$L(\alpha, h, \pi) = \left\{\log(m+\alpha) : m \in \mathbb{N}_0, \frac{\pi}{h}\right\}.$$

Theorem 2.4. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 2.3 is true*

The proof of Theorem 2.4 is based on the following lemma.

Lemma 2.8. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then P_N converges weakly to the measure P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$.*

The proof of the lemma is given in [29], Theorems 2.1 and 3.1.

Proof of Theorem 2.4. The theorem follows from Lemma 2.8 in the same way as Theorem 2.3 from the analogues of Lemma 2.8. □

Chapter 3

Modified mixed joint universality theorems for the Riemann and Hurwitz zeta-functions

In this chapter, we approximate simultaneously a given pair $(f_1(s), f_2(s))$ of analytic functions by shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$ with $\tau \in \mathbb{R}$. We will show that the set of shifts with approximation property of the accuracy $\varepsilon > 0$ has a positive density for all but at most countably many $\varepsilon > 0$. This result improves in some sense the known theorems that the shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$ with approximation property have a positive lower density for all $\varepsilon > 0$. The theorems obtained are called joint because two functions are involved in the approximation process. The term "mixed" is derived from the fact that the function $\zeta(s)$ has the Euler product over primes, while the Hurwitz zeta-function $\zeta(s,\alpha)$, in general, has no a similar product.

3.1 Continuous case

We preserve the same notation for classes $H(K)$ and $H_0(K)$, $K \in \mathcal{K}$, as in previous chapters.

Theorem 3.1. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of Theorem 3.1, a limit theorem for weakly convergent probability measures in the space $(H^2(D, \mathcal{B}(H^2(D)))$ is applied. To state it, we need tori Ω and Ω_1 defined in Chapters 1 and 2, respectively. For the convenience only, we denote these tori by Ω_1 and Ω_2, i.e.

$$\Omega_1 = \prod_p \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=\mathbb{N}_0}^{\infty} \gamma_m.$$

Denote

$$\Omega = \Omega_1 \times \Omega_2.$$

Then Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote the elements of Ω by $\omega = (\omega_1, \omega_2)$, where $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. Let $\omega_1(p)$ be the projection of an element $\omega_1 \in \Omega_1$ to the circle γ_p, $p \in \mathbb{P}$, and $\omega(m)$ be the projection of an element $\omega_2 \in \Omega_2$ to the circle γ_m, $m \in \mathbb{N}_0$. Now, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element $\underline{\zeta}(s, \omega)$, $\omega = (\omega_1, \omega_2) \in \Omega$, by the formula

$$\underline{\zeta}(s, \alpha, \omega) = (\zeta(s, \omega_1), \zeta(s, \alpha, \omega_2)),$$

where

$$\zeta(s, \omega_1) = \prod_p \left(1 - \frac{\omega_1(p)}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s}.$$

Moreover, let

$$P_{\underline{\zeta}}(A) = m_H \left(\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega) \in A\right), \quad A \in \mathcal{B}(H^2(D)),$$

i.e., the measure $P_{\underline{\zeta}}$ is the distribution of the random element $\underline{\zeta}(s, \omega)$. For brevity, we set $\underline{\zeta}(s, \alpha) = (\zeta(s), \zeta(s, \alpha))$, and

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} meas \left\{\tau \in [0, T] : \underline{\zeta}(s + i\tau, \alpha) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)).$$

Lemma 3.1. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$. Then P converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof. The lemma for transcendental α is proved in [38], Theorem 1, however, the transcendence of α is used only to prove the linear independence of the set $L(\alpha, \mathbb{P})$. Therefore, it suffices to repeat the proof of the above mentioned theorem. $\square$

The next lemma is devoted to the support of the measure $P_{\underline{\zeta}}$. We remind that $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Lemma 3.2. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

Proof. Denote by m_{1H} and m_{2H} the probability Haar measures on $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively. Then we have that the measure m_H is the product of m_{1H} and m_{1H}, i.e., if $A = A_1 \times A_2$, where $A_1 \in \mathcal{B}(\Omega_1)$ and $A_2 \in \mathcal{B}(\Omega_2)$, then

$$m_H(A) = m_{1H}(A_1) m_{2H}(A_2). \tag{3.1}$$

The space $H^2(D)$ is separable, therefore, $\mathcal{B}(H^2(D)) = \mathcal{B}(H(D)) \times \mathcal{B}(H(D))$. Thus, it suffices to consider the measure $P_{\underline{\zeta}}$ on the sets $A = A_1 \times A_2$, $A_1, A_2 \in H(D)$.

It is well known that the space $H(D)$ is separable, therefore, the space $H^2(D)$ is separable as well. Thus [2],

$$\mathcal{B}(H^2(D)) = \mathcal{B}(H(D)) \times \mathcal{B}(H(D)).$$

In consequence, it suffices to consider the measure $P_{\underline{\zeta}}$ on the sets $A = A_1 \times A_2$, $A_1, A_2 \in H(D)$.

It is known [15] that the support of the measure

$$m_{1H} \{\omega_1 \in \Omega_1 : \zeta(s, \omega_1) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{3.2}$$

is the set S. Obviously, the linearly independence over $\mathbb{Q}$ of the set $L(\alpha, \mathbb{P})$ implies that of the set $L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}$. Therefore, the case $r = 1$ of Theorem 11 from [16] gives that the support of the measure

$$m_{2H} \{\omega_2 \in \Omega_2 : \zeta(s, \alpha, \omega_2) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{3.3}$$

is the set $H(D)$. Since

$$P_{\underline{\zeta}}(A) = m_H \left\{\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega) \in A\right\}, \ A \in \mathcal{B}(H^2(D)),$$

in view of (3.1), we have that, for $A = A_1 \times A_2$,

$$P_{\underline{\zeta}}(A) = m_{1H} \{\omega_1 \in \Omega_1 : \zeta(s, \omega_1) \in A_1\} \, m_{2H} \{\omega_2 \in \Omega_2 : \zeta(s, \alpha, \omega_2) \in A_2\}.$$

Since

$$m_{1H} \{\omega_1 \in \Omega_1 : \zeta(s, \omega_1) \in S\} = 1$$

and

$$m_{2H} \{\omega_2 \in \Omega_2 : \zeta(s, \alpha, \omega_2) \in H(D)\} = 1,$$

this shows that

$$P_{\underline{\zeta}}(S \times H(D)) = 1.$$

Moreover, if $A_1 \in \mathcal{B}(\Omega_1)$ with $A_1 \subset S$, or $A_2 \in \mathcal{B}(\Omega_2)$ with $A_2 \subset H(D)$, then, in view of the minimality of S and $H(D)$ for the measures (3.2) and (3.3), respectively, we have that

$$m_{1H} \{\omega_1 \in \Omega_1 : \zeta(s, \omega_1) \in A_1\} < 1$$

or

$$m_{2H} \{\omega_2 \in \Omega_2 : \zeta(s, \alpha, \omega_2) \in A_2\} < 1.$$

Thus, then $P_{\underline{\zeta}}(A_1 \times A_2) < 1$. Hence, the minimality of the set $S \times H(D)$ follows. □

Proof of Theorem 3.1. Put

$$G_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\}.$$

Then G_ε is an open set in the space $H^2(D)$. Moreover,

$$\begin{aligned} \partial G_\varepsilon = & \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| = \varepsilon \right\} \\ & \bigcup \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\} \\ & \bigcup \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| = \varepsilon \right\}. \end{aligned}$$

Therefore, if $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_1 \neq \varepsilon_2$, then $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$. Hence, at most countably many sets ∂G_ε can have a positive $P_{\underline{\zeta}}$-measure. Actually, there are at most $n-1$ sets G_ε such that

$$P_{\underline{\zeta}}(\partial G_\varepsilon) > \frac{1}{n}.$$

Therefore, there are at most countably many sets ∂G_ε with positive $P_{\underline{\zeta}}$-measure. This shows that set G_ε is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemmas 3.1 and 1.3,

$$\lim_{T \to \infty} \frac{1}{T} meas \left\{ \tau \in [0; T] : \underline{\zeta}(s + i\tau) \in G_\varepsilon \right\} = P_{\underline{\zeta}}(G_\varepsilon),$$

or, by the definition of G_ε,

$$\lim_{T \to \infty} \frac{1}{T} meas \left\{ \tau \in [0; T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} = P_{\underline{\zeta}}(G_\varepsilon) \quad (3.4)$$

for all but at most countably many $\varepsilon > 0$. By Lemma 1.4, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s \in K_1} \left| f_1(s) - \mathrm{e}^{p_1(s)} \right| < \frac{\varepsilon}{2} \quad (3.5)$$

and

$$\sup_{s \in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \quad (3.6)$$

In view of Lemma 3.2, $\left(\mathrm{e}^{p_1(s)}, p_2(s)\right)$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, putting

$$\hat{G}_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - \mathrm{e}^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s \in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2} \right\},$$

we have that $\hat{G}_\varepsilon$ is an open neighbourhood of $\left(\mathrm{e}^{p_1(s)}, p_2(s)\right)$ and $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$. Inequalities (3.5) and (3.6) show, that, for $(g_1, g_2) \in \hat{G}_\varepsilon$,

$$\sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon$$

and

$$\sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon.$$

Thus, we have that $\hat{G}_\varepsilon \subset G_\varepsilon$. Hence, in view of monotonicity of $P_{\underline{\zeta}}$,

$$P_{\underline{\zeta}}(G_\varepsilon) \geq P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0.$$

This together with (3.4) gives the inequality

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

The theorem is proved.

□

3.2 Universality of composite functions. Continuous case

In this section, we present few modified universality theorems for $F(\zeta(s),\zeta(s,\alpha))$, where F is a certain operator on the space of analytic functions.

Theorem 3.2. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, $F : H^2(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(D)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\Big\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau),\zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of the theorem, a limit theorem on a weakly convergent probability measures in $H(D)$ with an explicitly given limit measure is applied.

Lemma 3.3. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator. Then*

$$P_{T,F}(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\left\{\tau \in [0,T] : F\left(\underline{\zeta}(s+i\tau,\alpha)\right) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\underline{\zeta}}F^{-1}$ as $T \to \infty$.

Proof. Let P_T be same as in Lemma 3.1. Then the definitions of P_T and $P_{T,F}$ show that, for $A \in \mathcal{B}H(D)$,

$$P_{T,F}(A) = \frac{1}{T} meas\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau,\alpha) \in F^{-1}A\right\} = P_T(F^{-1}A) = P_T F^{-1}(A),$$

i.e., $P_{T,F} = P_T F^{-1}$. This equality, continuity of the operator F, and Lemmas 3.1 and 1.5 imply the assertion of the lemma. □

The next lemma is devoted to the support of the measure $P_{\underline{\zeta}}F^{-1}$.

Lemma 3.4. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty. Then the support of the measure $P_{\underline{\zeta}}F^{-1}$ is the whole of $H(D)$.*

Proof. Let $g \in H(D)$ be an arbitrary element, and G be its any open neighbourhood. Since the operator F is continuous, the set $F^{-1}G$ is open, too. Therefore, by the hypothesis of the lemma, $F^{-1}G$ is an open neighbourhood of a certain element of the set $S \times H(D)$. Since, by Lemma 3.2, the set $S \times H(D)$ is the support of the measure $P_{\underline{\zeta}}$, we have that $P_{\underline{\zeta}}(F^{-1}G) > 0$. Therefore,

$$P_{\underline{\zeta}}F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

Since the objects g and G are arbitrary, this proves the lemma. □

Proof of Theorem 3.2. Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, as in the proofs of previous universality theorems, we have that G_ε is a continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Hence, in view of Lemmas 1.3 and 3.3,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : F\left(\underline{\zeta}(s+i\tau, \alpha)\right) \in G_\varepsilon \right\} \\ &= \lim_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon \right\} = P_{\underline{\zeta}}F^{-1}(G_\varepsilon) \end{aligned} \tag{3.7}$$

for all but at most countably many $\varepsilon > 0$. By Lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.8}$$

Define one more set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \frac{\varepsilon}{2} \right\}.$$

The polynomial $p(s)$, by Lemma 3.4, is an element of the support of the measure $P_{\underline{\zeta}}F^{-1}$. Hence,

$$P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0. \tag{3.9}$$

It can be easily seen that, for $g \in \hat{G}_\varepsilon$, by (3.8),

$$\sup_{s\in K} |g(s) - f(s)| < \varepsilon.$$

Therefore, $\hat{G}_\varepsilon \subset G_\varepsilon$. Thus, by (3.9)

$$P_{\underline{\zeta}}F^{-1}(G_\varepsilon) \geq P_{\underline{\zeta}}F^{-1}(\hat{G}_\varepsilon) > 0,$$

and the theorem follows from (3.7). □

Let, as in previous chapters, $D_V = \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V \right\}$ and $S_V = \left\{ g \in H(D_V) : g^{-1}(s) \in H(D_V) \ \textit{or} \ g(s) \equiv 0 \right\}$ for every $V > 0$. Moreover, let

$$H^2(D_V, D) = H(D_V) \times H(D).$$

Theorem 3.3. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and that K and $f(s)$ are the same as in Theorem 3.2, and $V > 0$ is such that $K \subset D_V$. Let $F : H^2(D_V, D) \to H(D_V)$ be a continuous operator such that , for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S \times H(D_V))$ is non-empty. Then the same assertion as in Theorem 3.2 is true.*

As in the case of previous universality theorems, we will deal with the weak convergence of probability measures. Let, for $A \in \mathcal{B}\left(H^2(D_V, D)\right)$,

$$P_{T,V}(A) = \frac{1}{T} meas\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau, \alpha) \in A\right\}$$

and

$$P_{\underline{\zeta},V}(A) = m_H\left\{\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega) \in A\right\}.$$

Lemma 3.5. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D_V, D) \to H(D_V)$ is a continuous operator. Then*

$$P_{T,F,V}(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\left\{\tau \in [0,T] : F\left(\underline{\zeta}(s+i\tau, \alpha)\right) \in A\right\}, \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to $P_{\underline{\zeta},V}F^{-1}$ as $T \to \infty$.

Proof. Clearly, $D_V \subset D$. Therefore, the function $u_V : H^2(D) \to H^2(D_V, D)$ given by the formula

$$u_V(g_1(s), g_2(s)) = \left(g_1(s)\Big|_{s \in D_V}, g_2(s)\right), \quad g_1, g_2 \in H(D),$$

is continuous, and

$$P_{T,V}(A) = \frac{1}{T} meas\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau, \alpha) \in u_V^{-1}A\right\},$$

i.e. $P_{T,V} = P_T u_V^{-1}$. Therefore, Lemmas 1.3 and 1.5 imply that $P_{T,V}$ converges weakly to $P_{\underline{\zeta},V}$ as $T \to \infty$. Moreover,

$$P_{T,F,V}(A) = \frac{1}{T} meas\left\{\tau \in [0,T] : F\underline{\zeta}(s+i\tau, \alpha) \in F^{-1}A\right\}, \quad A \in \mathcal{B}(H(D_V)),$$

i.e. $P_{T,F,V} = P_{T,V}F^{-1}$. This, the weak convergence of $P_{\underline{\zeta},V}$, shows that $P_{T,F,V}$ converges weakly to $P_{\underline{\zeta},V}F^{-1}$ as $T \to \infty$. □

Now we consider the support of the measure $P_{\underline{\zeta},V}F^{-1}$. We begin with the support of the measure $P_{\underline{\zeta},V}$.

Lemma 3.6. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $V > 0$. Then the support of $P_{\underline{\zeta},V}$ is the set $S_V \times H(D)$.*

Proof. Let g be an arbitrary element of $S_V \times H(D)$, and G be its open neighbourhood. The function u_V defined in the proof of Lemma 3.5 is continuous. Therefore, by the definition of u_V, the set $u_V^{-1}G$ is open and non-empty. Actually, we already have seen in section 1.2 that the approximation in the space $H(D)$ coincides with the uniform approximation on compact sets with connected complements.

Therefore, by Lemma 1.4, there exists a polynomial $p(s)$ such that $p(s) \in G$. Since the polynomial $p(s)$ is an entire function, $p(s)$ also belongs to $u_V^{-1}G$. Thus, the set $u_V^{-1}G$ is non-empty, and is an open neighbourhood of an element from the set $S \times H(D)$. Therefore, by Lemma 3.2, $P_{\underline{\zeta}}(F^{-1}G) > 0$. Hence,

$$P_{\underline{\zeta},V}(G) = P_{\underline{\zeta}} u_V^{-1}(G) = P_{\underline{\zeta}}(u_V^{-1}G) > 0.$$

Clearly, if $(g_1, g_2) \in S \times H(D)$, then also $(g_1, g_2) \in S_V \times H(D)$, i.e., $S \times H(D) \subset S_V \times H(D)$. Therefore, by Lemma 3.2 again,

$$m_H \left\{\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega) \in S_V \times H(D)\right\} \geq m_H \left(\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega) \in S \times H(D)\right) = 1.$$

Hence,

$$P_{\underline{\zeta},V}\left(S_V \times H(D)\right) = 1,$$

and the lemma is proved. □

Lemma 3.7. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$. Let $F : H^2(D_V, D) \to H(D_V)$ be a continuous operator such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V \times H(D))$ is non-empty. Then the support of the measure $P_{\underline{\zeta},V}F^{-1}$ is the whole of $H(D_V)$.*

Proof. Let g be an arbitrary element of $H(D_V)$, and G be its arbitrary open neighbourhood. Then, by Lemma 1.4, there exists a polynomial $p(s) \in G$. Therefore, the hypotheses of the lemma imply that the set $F^{-1}G$ is open and contains an element of the set $S_V \times H(D)$. Thus, in virtue of Lemma 3.6, $P_{\underline{\zeta},V}(F^{-1}G) > 0$. From this, it follows that

$$P_{\underline{\zeta},V}F^{-1}(G) = P_{\underline{\zeta},V}(F^{-1}G) > 0,$$

and the lemma is proved because g and G are arbitrary. □

Proof of Theorem 3.3. We follow the proof of Theorem 3.2, and use Lemma 3.5 in place of Lemma 3.3, and Lemma 3.7 in place of Lemma 3.4. □

Example 1. Let $F(g_1, g_2) = c_1 g_1 + c_2 g_2$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. The F is a continuous operator. We take arbitrary polynomial $p = p(s)$ and choose

$$g_1 = \frac{1}{c_1}, \qquad g_2 = \frac{p-1}{c_2}.$$

Then

$$c_1 g_1 + c_2 g_2 = c_1 \cdot \frac{1}{c_1} + c_2 \cdot \frac{p-1}{c_2} = p.$$

Therefore,

$$(g_1, g_2) \in F^{-1}\{p\} \quad \text{and} \quad (g_1, g_2) \in S_V \times H(D_V).$$

Hence, by Theorem 3.3, the function

$$c_1 \zeta(s) + c_2 \zeta(s, \alpha)$$

is universal.

Example 2. Let $F(g_1, g_2) = c_1 g_1' + c_2 g_2'$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then, in view of the Cauchy integral formula, F is a continuous operator. Let $p = p(s)$ be arbitrary polynomial, and $P = P(s)$ be a primitive function of $p(s)$. We take

$$g = 1 \quad \text{and} \quad g_2 = \frac{P}{c_2}.$$

Then

$$c_1 g_1' + c_2 g_2' = 0 + c_2 \cdot \frac{p}{c_2} = p.$$

Thus,

$$(g_1, g_2) \in F^{-1}\{p\} \quad \text{and} \quad (g_1, g_2) \in S_V \times H(D_V).$$

Therefore, by Theorem 3.3, the function

$$c_1 \zeta'(s) + c_2 \zeta'(s, \alpha)$$

is universal.

Now let $a_1, ..., a_r$ be arbitrary complex numbers, and

$$H_{a_1,...,a_r}(D) = \left\{ g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\ j = 1, ..., r \right\}.$$

Theorem 3.4. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator such that $F(S \times H(D)) \supset H_{a_1,...,a_r}(D)$. When $r = 1$, let $K \in \mathcal{K}$, and $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. If $r \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{a_1,...,a_r}(D)$. Then the same assertion as in Theorem 3.2 is true.*

We will present some examples of operators satisfying the hypothesis of Theorem 3.4.

Let $F(g_1(s), g_2(s)) = \mathrm{e}^{\mathrm{g}_1(\mathrm{s}) + \mathrm{g}_2(\mathrm{s})}$. Then we have that $F(S \times H(D)) \supset H_0(D)$, i.e., $r = 1$ and $r = 0$. Actually, let $g(s) \in H_0(D)$. We consider the operator

$$\mathrm{e}^{g_1(s) + g_2(s)} = g(s).$$

Hence,

$$g_1(s) + g_2(s) = \log g(s) + 2k\pi i.$$

We fix $k = 0$ and take $g_2(s) = \log g(s) - 1$. Then $g_1(s) \equiv 1$. Thus, $g_1(s) \in S$ and $g_2(s) \in H(D)$. This shows that, for every $g(s) \in H_0(D)$, there exists $g_1(s) \in S$ and $g_2(s) \in H(D)$ such that $F(g_1(s), g_2(s)) = g(s)$. Therefore, $F(S \times H(D)) \supset H_0(D)$. Moreover, we can take transcendental α, then the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$. Thus, the assertion of the theorem is true in the case $r = 1$.

Not let $F(g_1(s), g_2(s)) = \cos(g_1(s) + g_2(s))$. Using the formula

$$\cos s = \frac{\mathrm{e}^{is} + \mathrm{e}^{-is}}{2},$$

we will prove that $F(S\times H(D))\supset H_{-1,1}(D)$, i.e., $r=2$ and $a_1=-1$, $a_2=1$. So, let $g(s)\in H_{-1,1}(D)$. We consider the equation

$$\frac{\mathrm{e}^{ih(s)}+\mathrm{e}^{-ih(s)}}{2}=g(s).$$

Hence,

$$\mathrm{e}^{ih(s)}+\mathrm{e}^{-ih(s)}-2g(s)=0.$$

We set $y(s)=\mathrm{e}^{ih(s)}$. Then we obtain the equation

$$y^2(s)-2y(s)g(s)+1=0.$$

Thus,

$$y(s)=g(s)\pm\sqrt{g^2(s)-1}.$$

Since $g(s)\neq\pm1$, we have that $g^2(s)-1\neq0$, and $g(s)+\sqrt{g^2(s)-1}\neq0$. Therefore,

$$h(s)=\frac{1}{i}\log(g(s)+\sqrt{g^2(s)-1})\in H(D).$$

Now if $h(s)=g_1(s)+g_2(s)$ and

$$g_2(s)=\frac{1}{i}\log(g(s)+\sqrt{g^2(s)-1})-1\in H(D),$$

then $g(s)=1\in S$. Therefore, for every $g(s)\in H_{-1,1}(D)$ there exists $g_1(s)\in S$ and $g_2(s)\in H(D)$ such that

$$\cos(g_1(s)+g_2(s))=g(s).$$

This shows that $F(S\times H(D))\supset H_{-1,1}(D)$ and the case $r=2$ of assertion of Lemma 3.4 is true with transcendental α, for example, we can take $\alpha=\frac{1}{\pi}$.

For the proof of Theorem 3.4, we need the following lemma.

Lemma 3.8. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and the operator $F: H^2(D)\to H(D)$ satisfies the hypotheses of Theorem 3.4. Then the support of the measure $P_{\underline{\zeta}}F^{-1}$ contains the closure of the set $H_{a_1,\dots,a_r}(D)$.*

Proof. Since $F(S\times H(D))\supset H_{a_1,\dots,a_r}(D)$, for each element $g\in H_{a_1,\dots,a_r}(D)$, there exists a pair $(g_1,g_2)\in S\times H(D))$ such that $F(g_1,g_2)=g$. If G is an arbitrary open neighbourhood of g, then we have that the open set $F^{-1}G$ is an open neighbourhood of a certain element of $S\times H(D)$. Therefore, in view of Lemma 3.2, $P_{\underline{\zeta}}(F^{-1}G)>0$. Hence,

$$P_{\underline{\zeta}}F^{-1}(G)=P_{\underline{\zeta}}(F^{-1}G)>0.$$

This shows that the element g lies in the support of the measure $P_{\underline{\zeta}}F^{-1}$. Since g is an arbitrary element of $H_{a_1,\dots,a_r}(D)$, we have that the support of $P_{\underline{\zeta}}F^{-1}$ contains the set $H_{a_1,\dots,a_r}(D)$. However, the support is a closed set, therefore, it contains the closure of $H_{a_1,\dots,a_r}(D)$. $\square$

Proof of Theorem 3.4. The case $r = 1$. By Lemma 1.4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4} \tag{3.10}$$

By the hypotheses of the theorem, $f(s) \neq a_1$ on K. Therefore, in view of (3.10), the polynomial $p(s) \neq 0$ on K as well if ε is small enough. Thus, we can define a continuous branch of $\log(p(s) - a_1)$ which will be an analytic function in the interior of K. Using Lemma 1.4 once more, we find a polynomial $p_1(s)$ such that

$$\sup_{s\in K} |p(s) - a_1 - \mathrm{e}^{p_1(s)})| < \frac{\varepsilon}{4}. \tag{3.11}$$

Now we put $f_1(s) = \mathrm{e}^{p_1(s)} + a_1$. Then $f_1(s) \in H(D)$ and $f_1(s) \neq a_1$. Therefore, by Lemma 3.8, $f_1(s)$ is an element of the support of the measure $P_{\underline{\zeta}}F^{-1}$. Define

$$G_{1,\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f_1(s)| < \frac{\varepsilon}{2} \right\}.$$

Then $G_{1,\varepsilon}$ is an open neighbourhood of the function $f_1(s)$, thus, $P_{\underline{\zeta}}F^{-1}G_{1,\varepsilon}) > 0$. Now consider the set

$$\hat{G}_{1,\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Similarly as in the proof of the above theorems, we observe that $\hat{G}_{1,\varepsilon}$ is a continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, taking into account Lemmas 3.3 and 1.3, we have that

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : F\left(\underline{\zeta}(s + i\tau, \alpha)\right) \in \hat{G}_{1,\varepsilon} \right\} \\ &= \lim_{T\to\infty} \frac{1}{T} meas \Big\{ \tau \in [0;T] : \sup_{s\in K} |F(\underline{\zeta}(s + i\tau, \alpha)) - f(s)| < \varepsilon \Big\} = P_{\underline{\zeta}}F^{-1}(\hat{G}_{1,\varepsilon}). \end{aligned} \tag{3.12}$$

Clearly, by (3.10) and (3.11),

$$\sup_{s\in K} |f(s) - f_1(s)| < \frac{\varepsilon}{2}.$$

Therefore, if $g \in G_{1,\varepsilon}$, then $g \in \hat{G}_{1,\varepsilon}$, i.e., $G_{1,\varepsilon} \subset \hat{G}_{1,\varepsilon}$. Since $P_{\underline{\zeta}}F^{-1}(G_{1,\varepsilon}) > 0$, hence we have that $P_{\underline{\zeta}}F^{-1}(\hat{\mathcal{G}}_{1,\varepsilon}) > 0$ as well. This inequality together with (3.12) proves the theorem in the case $r = 1$.

Now let $r \geq 2$. Define

$$G_{2,\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Since $f(s) \in H_{a_1,\dots,a_r}(D)$, we have, by Lemma 3.8, that $f(s)$ is an element of the support of $P_{\underline{\zeta}}F^{-1}$. Moreover, $G_{2,\varepsilon}$ is an open neighbourhood of $f(s)$. Therefore,

$$P_{\underline{\zeta}}F^{-1}(G_{2,\varepsilon}) > 0. \tag{3.13}$$

On the other hand, $G_{2,\varepsilon}$ is a continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, in view of Lemmas 3.3 and 1.3, and (3.13).

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} meas \Big\{ \tau \in [0;T] : \sup_{s\in K} |F(\underline{\zeta}(s + i\tau, \alpha)) - f(s)| < \varepsilon \Big\} \\ &= \lim_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0;T] : F\left(\underline{\zeta}(s + i\tau, \alpha)\right) \in G_{2,\varepsilon} \right\} = P_{\underline{\zeta}}F^{-1}(G_{2,\varepsilon}) > 0. \end{aligned}$$

The theorem is proved. □

Theorem 3.5. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, $F : H^2(D) \to H(D)$ is a continuous operator, $K \subset D$ is a compact subset, and $f(s) \in F(S \times H(D))$. Then the same assertion as in Theorem 3.2 is true.*

The proof of the theorem is based on the following lemma.

Lemma 3.9. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator. Then the support of $P_{\underline{\zeta}} F^{-1}$ is the closure of $F(S \times H(D))$.*

Proof. Let g be an arbitrary element of $F(S \times H(D))$, and G be its any neighbourhood. Then, by Lemma 3.2, $P_{\underline{\zeta}}(F^{-1}G) > 0$. Hence,

$$P_{\underline{\zeta}} F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

Moreover, by Lemma 3.2,

$$P_{\underline{\zeta}} F^{-1} \left(F(S \times H(D))\right) = P_{\underline{\zeta}} \left(S \times H(D)\right) = 1.$$

Therefore, the support of $P_{\underline{\zeta}} F^{-1}$, as a closed set, is the closure of $F(S \times H(D))$. □

Proof of Theorem 3.5. We repeat the proof of the case $r \geq 2$ of Theorem 3.4, and, in place of Lemma 3.8, we apply Lemma 3.9. □

3.3 Discrete case

In this section, we prove discrete analogues of theorems of section 3.1. We remind that, for $h > 0$,

$$L(\mathbb{P}, \alpha, h, \pi) = \left\{ (\log p : p \in \mathbb{P}), (\log(m + \alpha) : m \in \mathbb{N}_0), \frac{\pi}{h} \right\}.$$

Theorem 3.6. *Suppose that the set $L(\mathbb{P}, \alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\zeta(s + ikh) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + ikh, \alpha) - f_2(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proof of Theorem 3.6 follows that of Theorem 3.1. We also preserve the notation of section 3.1.

Lemma 3.10. *Suppose that the set $L(\mathbb{P}, \alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$\frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \underline{\zeta}(s + ikh, \alpha) \in A \right\}, \quad A \in \mathcal{B}\big(H^2(D)\big),$$

converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.

Proof of the lemma is given in [4], Theorem 7.

Lemma 3.11. *Suppose that the set $L(\mathbb{P}, \alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

Proof of the lemma is given in [4].

Proof of Theorem 3.6. We apply the same arguments as in the proof of Theorem 3.1. Lemma 1.4 implies the existence of polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} \left|f_1(s) - \mathrm{e}^{p_1(s)}\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{3.14}$$

Define the set

$$G_{\frac{\varepsilon}{2}} = \left\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - \mathrm{e}^{\mathrm{p}_1(\mathrm{s})}| < \frac{\varepsilon}{2}, \sup_{\mathrm{s}\in \mathrm{K}_2} |\mathrm{g}_2(\mathrm{s}) - \mathrm{p}_2(\mathrm{s})| < \frac{\varepsilon}{2}\right\}.$$

Then $G_{\frac{\varepsilon}{2}}$ is an open neighbourhood of an element $\left(\mathrm{e}^{p_1(s)}, p_2(s)\right)$ which, by Lemma 3.11, belongs to the support of the measure $P_{\underline{\zeta}}$. Therefore,

$$P_{\underline{\zeta}}(G_{\frac{\varepsilon}{2}}) > 0. \tag{3.15}$$

Define one more set

$$\hat{G}_\varepsilon = \left\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \varepsilon\right\}.$$

Then, as in the proof of Theorem 3.1, we have that $\hat{G}_\varepsilon$ is a continuity of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemmas 3.10 and 1.3,

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{0 \le k \le N : \underline{\zeta}(s + ikh, \alpha) \in \hat{G}_\varepsilon\right\} = P_{\underline{\zeta}}(\hat{G}_\varepsilon) \tag{3.16}$$

for all but at most countably many $\varepsilon > 0$. Taking into account (3.14), we find that $G_{\frac{\varepsilon}{2}} \subset \hat{G}_\varepsilon$. Therefore, $P_{\underline{\zeta}}(\hat{G}_\varepsilon) \ge P_{\underline{\zeta}}(G_{\frac{\varepsilon}{2}})$. This, (3.15), (3.16) and the definition of the set $\hat{G}_\varepsilon$ show that the limit

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{0 \le k \le N : \sup_{s\in K_1} |\zeta(s + ikh) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s + ikh) - f_2(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$. □

Now let, for $h_1 > 0$, $h_2 > 0$,

$$L(\mathbb{P}, \alpha, h_1, h_2, \pi) = \left\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m + \alpha) : m \in \mathbb{N}_0), \pi\right\}.$$

Theorem 3.7. *Suppose that the set $L(\mathbb{P}, \alpha, h_1, h_2, , \pi)$ is linearly independent over $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{0 \le k \le N : \sup_{s\in K_1} |\zeta(s + ikh_1) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s + ikh_2, \alpha) - f_2(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The assertion of Theorem 3.7 follows from the next two lemmas.

Lemma 3.12. *Suppose that the set $L(\mathbb{P}, \alpha, h_1 2, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$\frac{1}{N+1}\#\left\{0 \le k \le N : \big(\underline{\zeta}(s+ikh_1), \underline{\zeta}(s+ikh_2, \alpha)\big) \in A\right\}, \quad A \in \mathcal{B}\big(H^2(D)\big),$$

converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.

Proof of the lemma is given in [5].

Lemma 3.13. *Suppose that the set $L(\mathbb{P}, \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

Proof of the lemma is given in [5].

Proof of Theorem 3.7. We repeat the proof of Theorem 3.6, and, instead of Lemmas 3.10 and 3.11, we use Lemmas 3.12 and 3.13. □

Now we state the last theorem of the thesis.

Theorem 3.8. *Suppose that the number α is transcendental, and β, $0 < \beta < 1$, and $h > 0$ are fixed numbers. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1}\#\left\{0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ik^\beta h) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ik^\beta h, \alpha) - f_2(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Lemma 3.14. *Suppose that the number α is transcendental, and β, $0 < \beta < 1$, and $h > 0$ are fixed numbers. Then*

$$\frac{1}{N+1}\#\left\{0 \le k \le N : \underline{\zeta}(s+ik^\beta h, \alpha) \in A\right\}, \quad A \in \mathcal{B}\big(H^2(D)\big),$$

converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.

Proof of the lemma is given in [22], Theorem 4. In the proof, the uniform distribution modulo 1 of the sequence $\{ak^\beta : k \in \mathbb{N}\}$ is essentially applied. We remind that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for each interval $I = [a, b) \subset [a, 1)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi\big(\{x_k\}\big) = b - a,$$

where $\{x_k\}$ denotes the fractional part of x_k, and χ_I is the indicator function of the interval I, i.e.

$$\chi_I(x) = \begin{cases} 1 & \text{if} \quad x \in I, \\ 0 & \text{if} \quad x \notin I. \end{cases}$$

Lemma 3.15. *Suppose that α is transcendental. Then the support of the measure $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

Proof. The lemma follows from Lemma 3.2 because the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$ if α is transcendental. $\square$

Proof of Theorem 3.8. We repeat the proof of the Theorem 3.6, and, instead of Lemmas 3.10 and 3.11, we use Lemmas 3.14 and 3.15. $\square$

Conclusions

1. The set of shifts of the Riemann zeta-function that approximate on compact sets a given analytic function with accuracy ε has a positive density for all but at most countably many $\varepsilon > 0$. This is true for continuous and discrete shifts.

2. The set of shifts of the Hurwitz zeta-function, with the parameter satisfying some natural independence hypotheses, that approximate on compact sets a given analytic function with accuracy ε has a positive density for all but at most countably many $\varepsilon > 0$. This is true for continuous and discrete shifts.

3. The set of shifts of the Riemann zeta-function and Hurwitz zeta-function, with the parameter satisfying some natural independence hypotheses, that approximate on compact sets a pair of given analytic functions with accuracy ε has a positive density for all but at most countably many $\varepsilon > 0$. This is true for continuous and discrete shifts.

4. The statement on the positivity of a density is valid for the sets of shifts of composite functions of functions discussed in Conclusions 1-3.

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Notation

p	prime number
k, l, m, n, q	non-negative integers
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$i = \sqrt{-1}$	imaginary unity
$s = \sigma + it$, $\sigma, t \in \mathbb{R}$	complex variable
$H(G)$	space of analytic functions on G
$\mathcal{B}(X)$	Borel σ-field of the space X
$\chi(m)$	Dirichlet character
$\zeta(s)$	Riemann zeta-function defined, for $\sigma > 1$, by $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$, and by analytic continuation elsewhere
$L(s, \chi)$	Dirichlet L-function defined, for $\sigma > 1$ by $L(s, \chi) = \sum\limits_{m=1}^{\infty} \frac{\chi(m)}{m^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha)$	Hurwitz zeta-function defined, for $\sigma > 1$, by $\zeta(s, \alpha) = \sum\limits_{m=1}^{\infty} \frac{1}{(m+\alpha)^s}$, and by analytic continuation elsewhere
$\Gamma(s)$	Euler gamma-function defined, for $\sigma > 1$, by $\Gamma(s) = \int\limits_0^{\infty} \mathrm{e}^{-\mathrm{u}} \mathrm{u}^{\mathrm{s}-1} \mathrm{du}$, and by analytic continuation elsewhere
meas A	Lebesgue measure of $A \subset \mathbb{R}$

$\#A$	cardinality of A
$F^{-1}G$	preimage of a set G
$F^{-1}\{p\}$	preimage of a polynomial p