VILNIUS UNIVERSITY

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LIMIT THEOREMS FOR SPATIO-TEMPORAL MODELS WITH LONG-RANGE DEPENDENCE

Doctoral dissertation
Physical sciences, Mathematics (01 P)

Vilnius, 2017

The dissertation was written in 2013–2017 at Vilnius University and University of Nantes.

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VILNIAUS UNIVERSITETAS

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RIBINĖS TEOREMOS ERDVĖS IR LAIKO MODELIAMS SU ILGĄJA ATMINTIMI

Daktaro disertacija
Fiziniai mokslai, matematika (01 P)

Vilnius, 2017

Disertacija rengta 2013–2017 metais Vilniaus universitete ir Nanto universitete (Prancūzija).

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Acknowledgements

First and foremost, I would like to express my sincere gratitude to my PhD supervisor Donatas Surgailis who introduced me to exciting research problems and shared his immense knowledge with me. I am deeply indebted to him for all his help and patience throughout the years.

I would also like to thank my PhD supervisor Anne Philippe who made me welcome in Nantes. I appreciate her hospitality, guidance and advice, invaluable suggestions for my drafts and R scripts.

I am grateful to Hermine Biermé and Jan Beran who agreed to review my thesis, and to all the members of my thesis committee. I feel greatly honoured by their interest in my work and appreciate their insightful comments.

My joint PhD studies were made possible by the scholarship of the French government that the Embassy of France in Lithuania awarded me with. Their support is gratefully acknowledged.

My special thanks go to Remigijus Leipus for his help and illuminating discussions about problems on which I was fortunate to work jointly.

I also thank Ingemar Kaj for sharing his knowledge and insights during my short-term visit to Uppsala University.

I am very grateful to Antanas Apynis and Rita Vinskūnaitė who introduced me to mathematics while I was studying at Pakruojis ‘Atžalynas‘ Gymnasium.

I thank the Institute of Mathematics and Informatics of Vilnius University for the flexible work environment.

Many thanks to the members of the Department of Econometric Analysis and the Department of Mathematical Analysis where I always felt welcome and found a helpful advice.

For good times and conversations covering the most divergent topics many thanks go to fellow PhD students Gražvydas Šemetulskis, Ieva Grublytė, Julius Damarackas, Šarūnas Dirmeikis. My thanks extend to Radek Novák and all (former) PhD students that I had pleasure to meet in Nantes. Especially, I thank Christophe A.N. Biscio for his care.

I am grateful to my godparents for their encouragement to travel during my studies.

Finally, I wish to thank my beloved mother who kindly accepted to spend less time together for the sake of my extra work on the thesis. Her full and unconditional support is indispensable for me.

Vytautė Pilipauskaitė
Vilnius
October 26, 2017

Contents

Notation and abbreviations

$:=$	by definition
$\mathbb{Z}$	$:= \{0, \pm 1, \pm 2, \dots\}$ the set of integers
$\mathbb{N}$	$:= \{1, 2, \dots\}$ the set of positive integers
$\mathbb{R}$	$:= (-\infty, \infty)$ the set of real numbers
$\mathbb{R}_+$	$:= (0, \infty)$
$C(A)$	the space of continuous functions defined on a set A
C	positive constant, which may change from line to line
i	$:= \sqrt{-1}$
$\mathbf{1}(A)$	indicator function of a set A
$x \wedge y$	$:= \min(x, y)$
$x \vee y$	$:= \max(x, y)$
x_+	$:= \max(x, 0)$
$\lfloor x \rfloor$	the greatest integer less than or equal to x (floor)
$\lceil x \rceil$	the least integer greater than or equal to x (ceiling)
$\Gamma(x)$	the gamma function
$\mathrm{B}(x, y)$	the beta function
$\mathrm{E}X$	mean of a random variable X
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$\stackrel{\mathrm{d}}{=}$	equality in distribution
$\stackrel{\mathrm{fdd}}{=}$	equality of finite-dimensional distributions
$\stackrel{\mathrm{p}}{\to}$	convergence in probability
$\stackrel{\mathrm{d}}{\to}$	convergence in distribution
$\stackrel{\mathrm{fdd}}{\to}$	weak convergence of finite-dimensional distibutions
$(\mathrm{fdd})\lim$	limit of finite-dimensional distributions
a.s.	almost surely
d.f.	(cumulative) distribution function
i.i.d.	independent identically distributed
r.v.	random variable

w.r.t.	with respect to
w.l.g.	without loss of generality
AR(1)	autoregressive process of order 1
LRD	long-range dependence
SRD	short-range dependence
RF	random field

Chapter 1

Introduction

Long-range dependence (also called long memory) is a well-established empirical fact observed in diverse scientific disciplines and applied fields, including hydrology, astronomy, environmental sciences, economics and finance, communication networks, see [7, 8, 29] for data examples and numerous references on the subject. It refers to the persistence of dependence between observations that are far apart in time or space. In mathematical framework, long-range dependence usually means the property of a stationary process, when its covariance series is not absolutely convergent. To develop statistical methodology for long-range dependent data is of great importance. Given the difficulty to specify the law of sample statistics, a significant part of statistical procedures relies on limit theorems for sums of observations or their functions. However, asymptotic results and thus statistical inference under long-range dependence may differ very much from the classical case of i.i.d. random variables. This thesis is devoted to limit theorems for spatio-temporal models with long-range dependence.

Aims and problems. We summarize briefly the problems studied in this doctoral thesis.

In Chapter 3 we discuss a joint temporal and contemporaneous aggregation of N independent copies $X_1, \ldots, X_N$ of AR(1) process X with random autoregressive coefficient a. Given the point-wise sum of $X_1, \ldots, X_N$, we look for the limit distribution of its normalized partial sums process as both N and time scale n tend to infinity. Under assumption that a has a density, regularly varying near the unit root $a = 1$ with index $\beta \in (-1, 1)$, we show that different limit processes exist if $N^{1/(1+\beta)}/n$ tends to (i) ∞, (ii) 0, (iii) $\mu \in (0, \infty)$. We compare our results to those obtained by [35,70], where three distinct limit regimes appear for cumulative network traffic generated by N independent sources at time scale n.

Chapter 4 complements Chapter 3 as we solve the identical problem for AR(1) processes with i.i.d. random autoregressive coefficients, but all driven by common innovations. Under the same assumption on the density of autoregressive coefficient, for $\beta \in (-1/2, 0)$ we obtain different limit distribution of normalized aggregated partial sums process if $N^{1/(1+\beta)}/n$ tends to (i) ∞, (ii) 0, (iii) $\mu \in (0, \infty)$.

In Chapter 5 we discuss estimation of the distribution function G of the autoregressive coefficient from N random-coefficient AR(1) series each of length n. And contrary to [9, 92], we take a nonparametric approach to the problem. We estimate G by the empirical distribution function of lag 1 sample autocorrelations of individual AR(1) processes, which are themselves estimates of unobservable autoregressive coefficients. We study the limit of the corresponding empirical process under some conditions on regularity of G and on the relative rate how fast N and n tend to infinity. We apply the obtained result to testing with Kolmogorov–Smirnov statistic both simple and composite hypotheses that G equals the beta distribution function. We perform a simulation study to compare the finite-sample performance of our test and its parametric analogue due to [9].

In Chapter 6 we consider a random field X defined as a nonlinear function (Appell polynomial) of a $\mathbb{Z}^2$-indexed random field Y. Let Y itself be linear, more precisely, a moving average of $\mathbb{Z}^2$-indexed standardized i.i.d. r.v.s with deterministic coefficients decaying slowly (so as to induce long-range dependence in Y) and possibly at different rate along horizontal and vertical directions. For a nonlinear random field X, we study the limiting distribution of its normalized partial sums over rectangles with sides growing at rates $O(\lambda)$ and $O(\lambda^\gamma)$ as $\lambda \to \infty$ for arbitrary $\gamma > 0$. We aim to find the limiting random fields for all $\gamma > 0$. The main question is if there exists a change-point $\gamma_0 > 0$ such that the limiting random fields do not depend on γ but differ for $\gamma > \gamma_0$ and $\gamma < \gamma_0$ and if so under what conditions. We extend the results of [89, 90], where this phenomenon, referred to as scaling transition, appears for some linear long-range dependent random fields.

In Chapter 7 we consider a $\mathbb{R}^2$-indexed random field X, the so-called random grain model, that counts sets, which are uniformly scattered on the plane and of infinite variance in area so that to induce long-range dependence in X. Our aim is to obtain the limiting random field of normalized partial integrals of the centered X over rectangles with sides growing at rates $O(\lambda)$ and $O(\lambda^\gamma)$ as $\lambda \to \infty$ for arbitrary $\gamma > 0$. We thus extend results on isotropic scaling ($\gamma = 1$ case) due to [53]. Moreover, we investigate the total accumulated workload from a generalized M/G/∞ model and relate its asymptotics to the results obtained for

X.

The novelty of the results in the thesis:

- three different limit regimes identified in the scheme of joint temporal-contemporaneous aggregation for random-coefficient AR(1) processes; new limit process in the 'intermediate' regime and its properties;
- proof that the empirical process, based on lag 1 sample autocorrelations of individual random-coefficient AR(1) processes, converges weakly to a generalized Brownian bridge;
- proof that nonlinear random fields may exhibit scaling transition;
- proof that two change-points may exist in the family of scaling limits (for a random grain model).

Publications. This doctoral thesis contains the following research articles, which have been co-authored:

1. V. Pilipauskaitė, D. Surgailis. Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. *Stochastic Process. Appl.* 124(2):1011–1035, 2014.
2. V. Pilipauskaitė, D. Surgailis. Joint aggregation of random-coefficient AR(1) processes with common innovations. *Statist. Probab. Lett.* 101:73–82, 2015.
3. V. Pilipauskaitė, D. Surgailis. Anisotropic scaling of the random grain model with application to network traffic. *J. Appl. Probab.* 53(3):857–879, 2016.
4. R. Leipus, A. Philippe, V. Pilipauskaitė, D. Surgailis. Nonparametric estimation of the distribution of the autoregressive coefficient from panel random-coefficient AR(1) data. *J. Multivar. Anal.* 153:121–135, 2017.
5. V. Pilipauskaitė, D. Surgailis. Scaling transition for nonlinear random fields with long-range dependence. *Stochastic Process. Appl.* 127(8):2751–2779, 2017.

Talks and posters. The main results of the thesis were presented at the following conferences and seminars:

- 55th Conference of Lithuanian Mathematical Society, Vilnius, Lithuania, June 26–27, 2014.

- 11th International Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius, Lithuania, June 30–July 4, 2014.
- Zürich Spring School on Lévy Processes, Zürich, Switzerland, March 29–April 2, 2015.
- Journée des Doctorants de l'ED STIM, Nantes, France, April 21, 2016.
- 57th Conference of Lithuanian Mathematical Society, Vilnius, Lithuania, June 20–21, 2016.
- Conference on Ambit Fields and Related Topics, Aarhus, Denmark, August 15–18, 2016.
- Séminaire de Mathématiques Appliquées, Université de Nantes, France, November 10, 2016.
- Séminaire Probabilités, Statistique et Applications, Université de Poitiers, France, November 17, 2016.
- 9th International Conference of the ERCIM WG on Computational and Methodological Statistics, Seville, Spain, December 9–11, 2016.
- Thiele Seminar, Aarhus University, Denmark, January 19, 2017.
- 2nd Conference on Ambit Fields and Related Topics, Aarhus, Denmark, August 14–16, 2017.
- 34th International Seminar on Stability Problems for Stochastic Models, Debrecen, Hungary, August 25–29, 2017.

The results of the thesis were also presented at the seminar on Probability Theory and Statistics held at the Institute of Mathematics and Informatics of Vilnius University, and at the seminar on Econometrics held at the Faculty of Mathematics and Informatics of Vilnius University.

Chapter 2

Literature review

In this chapter we review the most important concepts and some results related to the later parts of the thesis and give references for their in-depth coverage.

2.1 Long-range dependence

A T-indexed stochastic process X is a collection $\{X(t),\, t \in T\}$ of random variables (r.v.s), where $T \subseteq \mathbb{R}^d$ is the set of indices with $d = 1$ or 2 in the thesis. If $d \geq 2$, X is usually called a random field (RF) on T. The law (distribution) of X is completely determined by its finite-dimensional distributions $\mathrm{P}(X(t_1) \in A_1, \dots, X(t_m) \in A_m)$ for all Borel sets $A_i \subset \mathbb{R}$, $t_i \in T$, $i = 1, \dots, m$ and $m \in \mathbb{N}$. A stochastic process X indexed by $T = \mathbb{Z}^d$ or $\mathbb{R}^d$ is called stationary if X and $\{X(t + t_0),\, t \in T\}$ have the same law for any $t_0 \in T$. In case of stationary X with $\mathrm{E}X^2(0) < \infty$, its mean function $t \mapsto \mathrm{E}X(t)$ is constant and its covariance function $(t, s) \mapsto \mathrm{Cov}(X(t), X(s))$ depends only on the difference $t - s$, since $\mathrm{Cov}(X(t), X(s)) = \mathrm{Cov}(X(0), X(t - s))$ for any $t, s \in T$.

In the thesis, we study several examples of stationary X that may have long-range dependence. We also refer to this property as long memory, which is a more common term for processes indexed by $\mathbb{Z}$ in literature for time series analysis.

Definition 2.1. A stationary stochastic process $\{X(t),\, t \in \mathbb{Z}^d\}$ with $\mathrm{E}X^2(0) < \infty$ and covariance function $r(t) := \mathrm{Cov}(X(0), X(t))$, $t \in \mathbb{Z}^d$, is said to be long-range dependent (LRD) if

$$\sum_{t \in \mathbb{Z}^d} |r(t)| = \infty,$$

and short-range dependent (SRD) if

$$\sum_{t \in \mathbb{Z}^d} |r(t)| < \infty \text{ and } \sum_{t \in \mathbb{Z}^d} r(t) \neq 0.$$

The case when

$$\sum_{t\in\mathbb{Z}^d} |r(t)| < \infty \text{ and } \sum_{t\in\mathbb{Z}^d} r(t) = 0$$

is referred to as negative dependence.

This definition easily extends to the $\mathbb{R}^d$-indexed X.

By Bochner's theorem, the covariance function of a stationary RF X on $\mathbb{Z}^d$ with $\mathrm{E}X^2(0) < \infty$ has the following spectral representation:

$$r(t) = \int_{[-\pi,\pi)^d} \mathrm{e}^{\mathrm{i}<t,x>} F(\mathrm{d}x), \quad t \in \mathbb{Z}^d,$$

where $F(\mathrm{d}x)$ is a nonnegative finite measure on $[-\pi,\pi)^d$ called the spectral measure of X and $< t, x >$ is the scalar product of t and x. In most cases of interest, the spectral measure is absolutely continuous w.r.t. the Lebesgue measure and is determined by its density function $f(x)$, $x \in [-\pi,\pi)^d$ called the spectral density of X, which can also describe the dependence of X. In particular, the fact that the spectral density f is unbounded implies that X has LRD, since the absolute convergence of the covariance series results in bounded f.

Definition 2.1 being limited to stationary processes with finite second moment, there are other notions of LRD, see e.g. [22, 44, 94, 95] and [77, 86] with references therein. In the thesis we refer to LRD in the sense of Definition 2.1, unless stated otherwise. In case $d = 2$, the dependence of X may vary when quantified along different directions. This leads to a more detailed classification of LRD/SRD properties by Definition 7.1 on page 128.

2.2 Random-coefficient AR(1) process

In Chapters 3–5 of the thesis, the following stochastic model plays an important role. The process $X = \{X(t),\, t \in \mathbb{Z}\}$ is said to be an autoregressive process of order 1 (or AR(1)) with random coefficient if it is stationary and for every t satisfies

$$X(t) = aX(t-1) + \zeta(t), \tag{2.1}$$

where innovations $\{\zeta(t),\, t \in \mathbb{Z}\}$ are i.i.d. r.v.s with $\mathrm{E}\zeta(0) = 0$, $\mathrm{E}\zeta^2(0) = 1$ and AR coefficient $a \in (-1,1)$ is a r.v., independent of $\{\zeta(t),\, t \in \mathbb{Z}\}$. There exists a unique stationary solution to this equation, given by the series

$$X(t) = \sum_{s\le t} a^{t-s}\zeta(s), \tag{2.2}$$

see [87, Proposition 1]. The series converges conditionally a.s. and in L^2 for almost every $a \in (-1, 1)$. Moreover, if

$$\mathrm{E}\Big[\frac{1}{1-a^2}\Big] < \infty,$$

then the series in (2.2) converges unconditionally in L^2 and X has zero-mean and covariance function

$$r(t) := \mathrm{E}X(0)X(t) = \mathrm{E}\Big[\frac{a^{|t|}}{1-a^2}\Big], \quad t \in \mathbb{Z}.$$

See [92] for spectral properties of $r(t)$ and some other properties of X.

Next, we discuss two types of aggregation considered in the thesis jointly for copies of the process X in (2.2) and a related problem of estimating the underlying distribution of the AR coefficient. Recent developments in aggregation and statistical inference for AR models with focus on the long memory property can be found in the review [64] and the thesis [86].

2.2.1 Aggregation

Contemporaneous (or cross-sectional) aggregation refers to the point-wise summation of processes $X_i = \{X_i(t),\, t \in \mathbb{Z}\}$, $i = 1, 2, \dots$. The limit aggregated process $\mathcal{X}$, if exists, is defined as

$$\{\mathcal{X}(t),\, t \in \mathbb{Z}\} := (\mathrm{fdd}) \lim_{N\to\infty} \Big\{A_N^{-1} \sum_{t=1}^{N} X_i(t),\, t \in \mathbb{Z}\Big\}, \tag{2.3}$$

where A_N is some normalization. Granger [40] originated the idea that contemporaneous aggregation may be a reason for the long memory phenomenon observed in macro-level economic time series $\mathcal{X}$.

To be specific, consider a huge population of heterogeneous ‘micro-agents’ (such as households or firms), each of which evolves according to a short memory AR(1) process X_i with its own deterministic coefficient a_i. Drawing a random sample from this population leads to the assumption that AR coefficients a_i are i.i.d. r.v.s. Following Granger [40], assume that the common distribution of a_i is continuous with the following density function of beta type

$$g(x) = \frac{2}{\mathrm{B}(\alpha, \beta)} x^{2\alpha-1}(1-x^2)^{\beta-1}, \quad x \in (0, 1), \tag{2.4}$$

where $\alpha > 0$, $\beta > 0$. To rephrase this, the squared AR coefficient a_i is beta distributed with parameters (α, β).

Let X_i, $i = 1, 2, \dots$ be independent copies of a random-coefficient AR(1) process X in (2.2) under assumption (2.4) with $\beta > 1$, which guarantees $\mathrm{E}X^2(0) <$

∞. Then by the classical CLT, the limit in (2.3) exists for $A_N = \sqrt{N}$ with $\mathcal{X}$ being a stationary Gaussian process with the same second-order characteristics as those of the individual 'micro-agent', i.e. $\mathcal{X}$ has zero-mean and covariance function $\mathrm{E}\mathcal{X}(0)\mathcal{X}(t) = \mathrm{E}X(0)X(t) = r(t)$. If $\beta \in (1,2)$ in (2.4), then $r(t) \sim \mathrm{const}\, t^{1-\beta}$ as $t \to \infty$, implying that $\mathcal{X}$ has long memory. (Note that long memory of single X is not observable since X is indistinguishable from AR(1) series with the same deterministic AR coefficient.)

Another limit arises in aggregation of dependent time series. Let X_i, $i = 1, 2, \dots$, be random-coefficient AR(1) processes as X in (2.1), that have i.i.d. AR coefficients a_i, but are all driven by the same innovations $\{\zeta_i(t),\, t \in \mathbb{Z}\} \equiv \{\zeta(t),\, t \in \mathbb{Z}\}$. Assume (2.4) with $\beta > 1/2$. In this case, under normalization $A_N = N$ the limit (in probability) aggregated process $\mathcal{X}$ exists and can be written as $\mathcal{X}(t) := \sum_{s \le t} \mathrm{E}[a^{t-s}]\zeta(s)$, $t \in \mathbb{Z}$, see [87, 111]. If $\beta \in (1/2, 1)$, the limit $\mathcal{X}$ has long memory, since $\mathrm{E}\mathcal{X}(0)\mathcal{X}(t) \sim \mathrm{const}\, t^{-2\beta+1}$ as $t \to \infty$.

Following Granger [40], many authors took up the topic of contemporaneous aggregation, extending it to more general processes. We refer to [37, 39, 74, 87, 88, 111, 112], for instance.

Let us now introduce another type of aggregation. Temporal aggregation occurs when the frequency at which we observe a variable is lower than the frequency of its generating model. For a process $Y = \{Y(t),\, t \in \mathbb{Z}\}$ accumulating over time, we define its 'stock' as a partial sums process

$$S_n(\tau) := \sum_{t=1}^{[n\tau]} Y(t), \quad \tau \ge 0,$$

with $S_n(0) := 0$, whereas

$$S_n(\tau) - S_n(\tau - 1) = \sum_{t=n(\tau-1)+1}^{n\tau} Y(t), \quad \tau = 1, 2, \dots,$$

represents a 'flow', measured per unit of time. Note that evolution of the partial sum process S_n during a time interval $[0, \tau]$ corresponds to an interval $[0, n\tau]$ on the original finer time scale for Y.

We may wonder which processes may occur as limits in temporal aggregation of Y as $n \to \infty$. Under reasonably weak assumptions, Lamperti [58] showed that all possible limiting processes of suitably normalized S_n are self-similar. Recall that a process $V = \{V(\tau),\, \tau \ge 0\}$ is called self-similar, if for some $H > 0$,

$$\{V(\lambda\tau),\, \tau \ge 0\} \stackrel{\mathrm{fdd}}{=} \{\lambda^H V(\tau),\, \tau \ge 0\} \quad \text{for all } \lambda > 0.$$

In other words, V is invariant in distribution under certain simultaneous scaling of time and space.

Theorem 2.1 (Lamperti [58]). *Let $\{Y(t), t \in \mathbb{Z}\}$ be a stationary process and assume there exist a sequence of positive numbers $A_n \to \infty$ such that*

$$A_n^{-1} \sum_{t=1}^{[n\tau]} Y(t) \overset{\text{fdd}}{\to} V(\tau), \quad \tau \geq 0,$$

as $n \to \infty$, where the limit process $V := \{V(\tau), \tau \geq 0\}$ is not identically zero and is stochastically continuous. Then V is a H-self-similar process having stationary increments, where $H > 0$ and the normalization $A_n = n^H \ell(n)$ for some slowly varying function ℓ at infinity.

The only H-self-similar Gaussian process with stationary increments is a fractional Brownian motion. Let $H \in (0, 1]$. A Gaussian process $\{B_H(\tau), \tau \geq 0\}$ with $\mathrm{E}B_H(\tau) \equiv 0$ and covariance function given by

$$\mathrm{E}B_H(\tau_1)B_H(\tau_2) = \frac{1}{2}(\tau_1^{2H} + \tau_2^{2H} - |\tau_1 - \tau_2|^{2H}), \quad \tau_1 \geq 0, \tau_2 \geq 0,$$

is called a standard fractional Brownian motion with (Hurst) index H.

Recall the limit $\mathcal{X}$ in (2.3) for independent copies of random-coefficient AR(1) process under assumption (2.4) with $\beta \in (1, 2)$. By [88, Theorem 3.1],

$$n^{-H} \sum_{t=1}^{[n\tau]} \mathcal{X}(t) \overset{\text{fdd}}{\to} \sigma B_H(\tau), \quad \tau \geq 0,$$

where $\sigma > 0$ is a certain constant and B_H is a standard fractional Brownian motion with index $H = (3 - \beta)/2 \in (1/2, 1)$. A similar fact holds for random-coefficient AR(1) processes driven by common innovations under assumption (2.4) with $\beta \in (1/2, 1)$, see e.g. [87].

There are other classes of long memory processes Y. Limit theory for their partial sums $\{S_n(\tau), \tau \geq 0\}$ can be found in books [8, Chapter 4], [36, Chapter 4]. The methods and results differ significantly from the case when Y has short memory.

In the thesis temporal and contemporaneous aggregation are treated jointly. We look for the limit distribution of the normalized joint aggregate (contemporaneously aggregated partial sums) of random-coefficient AR(1) copies $X_1, \ldots, X_N$ as N and the time scale n tend to infinity simultaneously.

2.2.2 Estimation of the distribution of the AR coefficient

A statistical problem naturally arises, such as recovering the distribution function $G(x)$, $x \in [-1, 1]$, of the random AR coefficient. Estimation of G from the limit aggregated series $\{\mathcal{X}(0), \dots, \mathcal{X}(n)\}$ was treated in [20,61] and some related results were obtained in [19, 48, 50]. However, we may expect a much more accurate estimate if individual series (panel data) are available.

Consider N random-coefficient AR(1) series, each of length $n+1$: $\{X_i(0), \dots, X_i(n)\}$, $i = 1, \dots, N$, which are independent copies of X in (2.1). Robinson [92] suggested to estimate the parameters characterizing G by the method of moments. [92] identified moments of G in terms of autocovariances of individual random-coefficient AR(1) processes:

$$\mu^{(u)} := \int_{-1}^{1} x^u \mathrm{d}G(x) = \frac{r(u) - r(u+2)}{r(0) - r(2)}$$

where $r(u) := \mathrm{E}X(0)X(u)$, $u = 0, \dots, n$, can be estimated by

$$\frac{1}{(n-u+1)N} \sum_{t=0}^{n-u} \sum_{i=1}^{N} X_i(t)X_i(t+u),$$

and proved the asymptotic normality of the corresponding estimators of $\mu^{(u)}$, $u = 1, \dots, n-2$, as $N \to \infty$, whereas n remains fixed, under assumption

$$\int_{-1}^{1} \frac{\mathrm{d}G(x)}{(1-x^2)^2} < \infty,$$

which does not allow for long memory in X and the limit aggregated process $\mathcal{X}$.

Beran et al. [9] considered independent copies X_i, $i = 1, 2, \dots$, of the process $X = \{X(t),\, t = 0, 1, \dots\}$, satisfying the AR(1) equation (2.1) for all $t \in \mathbb{N}$ with initial value $|X(0)| \le C$, $\mathrm{E}X(0) = 0$, independent of the AR coefficient a and i.i.d. standard normal innovations $\{\zeta(t),\, t \in \mathbb{N}\}$. Assume that a has a density function g given by (2.4) with $(\alpha, \beta) \in (1, \infty)^2$. (Recall X and the limit aggregated process $\mathcal{X}$ in (2.3) have long memory if $\beta \in (1, 2)$.) Given the panel random-coefficient AR(1) data $\{X_i(t),\, t = 0, \dots, n,\, i = 1, \dots, N\}$, [9] estimated (α, β) by the method of maximum likelihood. The idea of [9] about the likelihood is to replace each unobservable a_i, $i = 1, \dots, N$, by its estimate, which in turn is a truncated version of lag 1 sample autocorrelation of the individual AR(1) process:

$$\begin{aligned} \widehat{a}_{i,n,\kappa} &:= \min(\max(\widehat{a}_{i,n}, \kappa), 1-\kappa), \quad \text{where} \\ \widehat{a}_{i,n} &:= \frac{\sum_{i=0}^{n-1} X_i(t)X_i(t+1)}{\sum_{i=0}^{n} X_i^2(t)}, \quad \kappa > 0. \end{aligned}$$

[9] proved the consistency of the corresponding maximum likelihood estimator of (α, β) and its asymptotic normality with the convergence rate $\sqrt{N}$ under the following conditions on the length of series n and the truncation parameter κ: $(\log \kappa)^2 N^{-1/2} \to 0$, $\sqrt{N}\kappa^{\min(\alpha,\beta)} \to 0$ and $\sqrt{N}\kappa^{-2}n^{-1} \to 0$ as $N, n \to \infty$, $\kappa \to 0$.

[9] is the closest in spirit to Chapter 5, where we discuss nonparametric estimation of the distribution function G of the AR coefficient from panel random-coefficient AR(1) data. Furthermore, employing the idea of [9], we consider a different estimator for moments of G and prove its asymptotic normality as $N, n \to \infty$ under less restrictive condition on G in contrast to [92].

2.3 Aggregation of network traffic models

To explain the observed self-similarity and LRD in network traffic measurements, the following model has been proposed. Consider cumulative network traffic as an aggregate of data streams from a large number of independent sources, where each source alternates between ON and OFF states depending if it transmits data (at a constant rate 1) or not. Then it is natural to analyze the total workload of high-speed network accumulated over time and study the distribution of its fluctuations around cumulative average.

But firstly, let us introduce the ON/OFF process in a mathematical framework. Assume the lengths of ON-periods $\{X_{\mathrm{on}}, X_1, X_2, \dots\}$ are i.i.d. non-negative r.v.s and the lengths of OFF-periods $\{Y_{\mathrm{off}}, Y_1, Y_2, \dots\}$ are i.i.d. non-negative r.v.s with the Pareto distribution functions:

$$\begin{aligned} F_{\mathrm{on}}(x) &:= \mathrm{P}(X_{\mathrm{on}} \le x) = 1 - C_{\mathrm{on}} x^{-\alpha_{\mathrm{on}}}, \quad \alpha_{\mathrm{on}} \in (1,2), \\ F_{\mathrm{off}}(y) &:= \mathrm{P}(Y_{\mathrm{off}} \le y) = 1 - C_{\mathrm{off}} y^{-\alpha_{\mathrm{off}}}, \quad \alpha_{\mathrm{off}} \in (1,2), \end{aligned} \tag{2.5}$$

where C_{on}, C_{off} are finite positive constants (which can be replaced by arbitrary slowly varying functions at infinity). Note that X_{on}, Y_{off} have finite means μ_{on}, μ_{off} respectively, but their variances are infinite so that to induce LRD in the ON/OFF process. We define the renewal sequence $\{T_k,\ k = 0, 1, \dots\}$ by

$$\begin{aligned} T_k &:= \sum_{j=0}^{k} (X_j + Y_j), \\ X_0 &:= B\tilde{X}_{\mathrm{on}}, \quad Y_0 := BY_{\mathrm{off}} + (1-B)\tilde{Y}_{\mathrm{off}}, \end{aligned}$$

where B is a Bernoulli r.v. with $\mathrm{P}(B=1) = 1 - \mathrm{P}(B=0) = \mu_{\mathrm{on}}/(\mu_{\mathrm{on}} + \mu_{\mathrm{off}})$ and $\tilde{X}_{\mathrm{on}}$, $\tilde{Y}_{\mathrm{off}}$ have distribution functions

$$\mathrm{P}(\tilde{X}_{\mathrm{on}} \le x) = \frac{1}{\mu_{\mathrm{on}}} \int_0^x (1 - F_{\mathrm{on}}(u))\mathrm{d}u, \quad \mathrm{P}(\tilde{Y}_{\mathrm{off}} \le y) = \frac{1}{\mu_{\mathrm{off}}} \int_0^y (1 - F_{\mathrm{off}}(u))\mathrm{d}u,$$

respectively. Assume all B, $\tilde{X}_{\mathrm{on}}$, $\tilde{Y}_{\mathrm{off}}$, $\{X_{\mathrm{on}}, X_1, X_2, \dots\}$, $\{Y_{\mathrm{off}}, Y_1, Y_2, \dots\}$ are mutually independent. Finally, we define a stationary ON/OFF process $W =$

$\{W(t),\, t \geq 0\}$ as

$$W(t) := \mathbf{1}(0 \leq t < X_0) + \sum_{k=0}^{\infty} \mathbf{1}(T_k \leq t < T_k + X_{k+1}). \tag{2.6}$$

In other words, $W(t) = 1$ if time t is in the ON-period, $W(t) = 0$ if time t is in the OFF-period. For $\alpha_{\text{on}} < \alpha_{\text{off}}$, we have $r(t) := \operatorname{Cov}(W(0), W(t)) \sim \text{const}\, t^{-(\alpha_{\text{on}}-1)}$ as $t \to \infty$, see [43]. With the covariance function being absolutely nonintegrable, W exhibits LRD.

Now consider N independent copies W_i, $i = 1, \ldots, N$, of the ON/OFF proces $W = \{W(t),\, t \geq 0\}$ in (2.6). Let

$$S_{N,n}(\tau) := \int_0^{n\tau} \sum_{i=1}^{N} W_i(t) \mathrm{d}t, \quad \tau \geq 0, \tag{2.7}$$

be the total accumulated workload from N i.i.d. ON/OFF sources by time τ at scale n. Taqqu et al. [105] studied the asymptotic behavior of $S_{N,n} = \{S_{N,n}(\tau),\, \tau \geq 0\}$ in the sequential scheme. More precisely, [105] proved that finite-dimensional distributions of properly normalized and centered $S_{N,n}$ converge weakly to those of a fractional Brownian motion as first the number N of sources goes to infinity and then the time scale n converges to infinity. If limits are taken in reversed order, the limit distribution of properly normalized and centered $S_{N,n}$ corresponds to an infinite variance α_{on}-stable Lévy motion. The increment process of fractional Brownian motion, fractional Gaussian noise, exhibits LRD. This is in contrast to stable Lévy motion, which while self-similar too, has independent increments. In Mikosch et al. [70], the double limits are replaced by a single scheme as N and n go to infinity simultaneously. Two limit regimes of fast connection rate and slow connection rate (see Theorem 2.2(i) and (ii) below, respectively) are identified. In these two regimes fractional Brownian motion and α_{on}-stable Lévy motion reappear as limit processes of the scaled centered total ON/OFF workload accumulated over time. [27] complemented the results of [70] by showing that a third limit process arises at intermediate connection rate (see Theorem 2.2(iii)).

Theorem 2.2 (Mikosch et al. [70], Dombry, Kaj [27])**.** *Let $\alpha := \alpha_{\text{on}} < \alpha_{\text{off}}$ in (2.5). If $N \to \infty$ and $n \to \infty$ so that*

(i) $N/n^{\alpha-1} \to \infty$, then

$$\frac{S_{N,n}(\tau) - \mathrm{E}S_{N,n}(\tau)}{N^{1/2} n^{(3-\alpha)/2}} \xrightarrow{\text{fdd}} \sigma_\infty B_H(\tau), \quad \tau \geq 0,$$

where $B_H = \{B_H(\tau),\, \tau \geq 0\}$ is a standard fractional Brownian motion with index $H = (3-\alpha)/2$ and

$$\sigma_\infty^2 := C_{\text{on}} \frac{2\mu_{\text{off}}^2 \Gamma(2-\alpha)/(\alpha-1)}{(\mu_{\text{on}} + \mu_{\text{off}})^3 \Gamma(4-\alpha)};$$

(ii) $N/n^{\alpha-1} \to 0$, *then*

$$\frac{S_{N,n}(\tau) - \mathrm{E}S_{N,n}(\tau)}{N^{1/\alpha}n^{1/\alpha}} \overset{\mathrm{fdd}}{\to} \sigma_0 L(\tau), \quad \tau \geq 0,$$

where $L = \{L(\tau),\, \tau \geq 0\}$ *is an* α*-stable Lévy motion with* $\mathrm{E}e^{\mathrm{i}\theta L(1)} = \exp\{-|\theta|^\alpha(1 - \mathrm{i}\,\mathrm{sgn}\,\theta\tan(\pi\alpha/2))\}$, $\theta \in \mathbb{R}$, *and*

$$\sigma_0 := \frac{\mu_{\mathrm{off}}(C_{\mathrm{on}}/C_\alpha)^{1/\alpha}}{(\mu_{\mathrm{on}} + \mu_{\mathrm{off}})^{1+1/\alpha}}, \quad C_\alpha := \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)};$$

(iii) $N/n^{\alpha-1} \to c^{\alpha-1}(\mu_{\mathrm{on}} + \mu_{\mathrm{off}})/C_{\mathrm{on}}$, *then*

$$\frac{S_{N,n}(\tau) - \mathrm{E}S_{N,n}(\tau)}{n} \overset{\mathrm{fdd}}{\to} \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}} + \mu_{\mathrm{off}}} cZ(\tau/c), \quad \tau \geq 0,$$

where $Z = \{Z(\tau),\, \tau \geq 0\}$ *is characterized by the cumulant generating function of its finite-dimensional distributions.*

The 'intermediate' process $Z = \{Z(\tau),\, \tau \geq 0\}$ is zero-mean, non-Gaussian and non-stable with stationary increments. It is not self-similar and has a representation as

$$Z(\tau) = \int_{\mathbb{R}\times\mathbb{R}_+} \Big\{ \int_0^\tau \mathbf{1}(u \leq t < u + x)\mathrm{d}t \Big\} \widetilde{M}(\mathrm{d}x, \mathrm{d}u),$$

where $\widetilde{M}(\mathrm{d}x, \mathrm{d}u) := M(\mathrm{d}x, \mathrm{d}u) - \alpha x^{-\alpha-1}\mathrm{d}x\mathrm{d}u$ and $M(\mathrm{d}x, \mathrm{d}u)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\alpha x^{-\alpha-1}\mathrm{d}x\mathrm{d}u$. Other properties of Z are discussed in [34, 35]. In particular, employing the stochastic-integral representation of Z, Gaigalas [34] showed that the process is locally and globally asymptotically self-similar with B_H and L as its tangent limits. So Z can be viewed as a bridge between the limiting processes in cases (i) and (ii).

Similar limit theorems as for the ON/OFF hold for other network traffic models, e.g. renewal-reward process, M/G/∞ queue (or infinite source Poisson process), see [54, 55, 68, 70, 83], with Gaigalas, Kaj [35] being the first to obtain the limit Z at intermediate connection rate for the sum of independent scaled renewal processes.

In Chapters 3, 4 of the thesis we discuss joint aggregation of type (2.7) for copies of random-coefficient AR(1) process, which has a very different dependence structure from the above-mentioned models.

Finally, let us introduce another popular network traffic model related to the thesis. M/G/∞ queue $W_\lambda = \{W_\lambda(t),\, t \geq 0\}$ describes a system where the arrivals are Markovian, the service times follow some general distribution and there are infinitely many servers, so jobs do not need to wait. More precisely, let $W_\lambda(t)$ count the number of active sessions (or sources in the network system) at time t. The sessions start at times $\{T_k,\ k \in \mathbb{Z}\}$, which are the points of a rate λ homogeneous Poisson process on $\mathbb{R}$, and throughout each session data are transmitted at rate 1. Assume the transmission durations (session lengths) $\{X_k,\ k \in \mathbb{Z}\}$ are i.i.d. r.v.s with the distribution function F_{on} given by (2.5), independent of the starting points of sessions. Then we define the workload process W_λ by

$$W_\lambda(t) := \sum_{k=-\infty}^{\infty} \mathbf{1}(T_k \le t < T_k + X_k), \quad t \ge 0. \tag{2.8}$$

Similarly to the ON/OFF case, high variability in transmission durations causes LRD in the rate at which work is offered: $\mathrm{Cov}(W_\lambda(0), W_\lambda(t)) \sim \mathrm{const}\, t^{-(\alpha_{\mathrm{on}}-1)}$ as $t \to \infty$. The so-called infinite source Poisson process

$$S_{\lambda,n}(\tau) = \int_0^{n\tau} W_\lambda(t)\mathrm{d}t, \quad \tau \ge 0,$$

represents the total accumulated workload by time τ at scale n. Note, as the session intensity $\lambda \to \infty$ and $n \to \infty$ simultaneously, normalized and centered $S_{\lambda,n}$ admits the same limits as the total accumulated ON/OFF workload.

In Chapter 7 of the thesis we generalize the network traffic model W_λ in (2.8) for the situation when the transmission rate is random and bound with its duration and then study asymptotic behavior of the corresponding aggregated workload. Moreover, Chapter 7 of the thesis treats a random grain model, whose analogue in dimension 1 the M/G/∞ queue is.

2.4 Anisotropic scaling of random fields

Let $X = \{X(t,s),\ (t,s) \in \mathbb{Z}^2\}$ be a stationary random field (RF). For arbitrary $\gamma > 0$ we study the limit

$$A_{\lambda,\gamma}^{-1} \sum_{t=1}^{[\lambda x]} \sum_{s=1}^{[\lambda^\gamma y]} X(t,s) \overset{\mathrm{fdd}}{\to} V_\gamma(x,y), \quad (x,y) \in \mathbb{R}_+^2, \tag{2.9}$$

of the partial sums of X over increasing rectangles $(0, \lambda x] \times (0, \lambda^\gamma y] \cap \mathbb{Z}^2$ as $\lambda \to \infty$, where $A_{\lambda,\gamma} \to \infty$ is a normalization. Provided $\gamma \ne 1$, the sides of the rectangles grow at different rates $O(\lambda)$ and $O(\lambda^\gamma)$ as $\lambda \to \infty$, thus $\gamma > 0$ characterizes the anisotropy of scaling procedure. Next, let us introduce some general properties of scaling limits in (2.9).

Proposition 2.3 (Puplinskaitė, Surgailis [90])**.** *Let $X = \{X(t,s),\ (t,s) \in \mathbb{Z}^2\}$ be a stationary RF satisfying* (2.9) *for some $\gamma > 0$ and $A_{\lambda,\gamma} = \lambda^H \ell(\lambda)$, where $H > 0$*

and $\ell : [1,\infty) \to \mathbb{R}_+$ *is a slowly varying function at infinity. Then the limit RF* $V_\gamma = \{V_\gamma(x,y),\, (x,y) \in \mathbb{R}^2_+\}$ *in* (2.9) *satisfies the operator-scaling property:*

$$\{V_\gamma(\lambda x, \lambda^\gamma y),\, (x,y) \in \mathbb{R}^2_+\} \stackrel{\mathrm{fdd}}{=} \{\lambda^H V_\gamma(x,y),\, (x,y) \in \mathbb{R}^2_+\} \quad \textit{for all } \lambda > 0. \tag{2.10}$$

Moreover, V_γ *has stationary rectangular increments: for any fixed* $(x_0, y_0) \in \mathbb{R}^2_+$,

$$\begin{aligned} &\{V_\gamma((x_0,x] \times (y_0,y]),\, x > x_0, y > y_0\} \\ \stackrel{\mathrm{fdd}}{=}\ &\{V_\gamma((0, x-x_0] \times (0, y-y_0]),\, x > x_0, y > y_0\} \\ \equiv\ &\{V_\gamma(x-x_0, y-y_0),\, x > x_0, y > y_0\}, \end{aligned}$$

where the increment of V_γ *on a rectangle* $(x_0,x] \times (y_0,y] \subset \mathbb{R}^2_+$ *is defined as* $V_\gamma((x_0,x] \times (y_0,y]) := V_\gamma(x,y) - V_\gamma(x_0,y) - V_\gamma(x,y_0) + V_\gamma(x_0,y_0)$.

Note (2.10) is a particular case of the operator-scaling RF property introduced in Biermé et al. [12].

For many RFs, nontrivial V_γ in (2.9) exists for any $\gamma > 0$. In that case, with a given RF X we can associate a one-parameter family $\{V_\gamma,\, \gamma > 0\}$ of scaling limits, which characterizes the dependence structure and large-scale properties of the underlying X. If $\{X(t,s),\, (t,s) \in \mathbb{Z}^2\}$ are i.i.d. standardized r.v.s, the scaling limit V_γ coincides with a standard Brownian sheet for all $\gamma > 0$, i.e. $\{V_\gamma,\, \gamma > 0\}$ consists of a single element. A similar fact holds for SRD RFs, see e.g. [15, 31] and the references therein. Actually, limit theorems for SRD RFs often assume a general shape of summation domain, the limit distribution being independent of the way in which this region tends to $\mathbb{Z}^2$. However, a surprising phenomenon appears for many LRD RFs X, which exhibit a dramatic change of their scaling behavior at some point $\gamma_0 > 0$ in the following sense.

Definition 2.2 (Puplinskaitė, Surgailis [90])**.** We say that the RF $X = \{X(t,s),$ $(t,s) \in \mathbb{Z}^2\}$ exhibits a scaling transition at $\gamma_0 > 0$ such that

$$\begin{gathered} V_\gamma \stackrel{\mathrm{fdd}}{=} V_+ \text{ for all } \gamma > \gamma_0, \quad V_\gamma \stackrel{\mathrm{fdd}}{=} V_- \text{ for all } 0 < \gamma < \gamma_0, \\ V_+ \stackrel{\mathrm{fdd}}{\neq} aV_- \text{ for any } a > 0. \end{gathered} \tag{2.11}$$

If V_γ is the same for all $\gamma > 0$, then X does not exhibit scaling transition.

In other words, (2.11) says that scaling limits V_γ do not depend on γ for $\gamma > \gamma_+$ and $\gamma < \gamma_-$ and are different up to a multiplicative constant (the last condition is needed to exclude a trivial change of the scaling limit by a linear change of normalization).

Scaling transition arises under joint temporal and contemporaneous aggregation of independent LRD processes in communication networks and economics, see [27,35,55,70,79], also [90, Remark 2.3]. E.g., let $\{X_i(t) := W_i(t) - \mathrm{E}W_i(t),\, t \ge 0\}$, $i = 1, 2, \dots$, be independent copies of a centered ON/OFF process given by (2.6) with $\alpha_{\mathrm{on}} < \alpha_{\mathrm{off}}$. Align them vertically to define a RF $X = \{X_i(t),\, t \ge 0,\, i = 1, 2, \dots\}$ with 'one-dimensional dependence', exhibiting scaling transition, since the limit distribution in

$$A_{\lambda,\gamma}^{-1} \int_0^{\lambda\tau} \sum_{i=1}^{[\lambda^\gamma y]} X_i(t)\mathrm{d}t \overset{\mathrm{fdd}}{\to} V_\gamma(\tau, y), \quad (\tau, y) \in \mathbb{R}_+^2, \quad \text{as } \lambda \to \infty,$$

changes from a stable Lévy sheet for $0 < \gamma < \gamma_0$ to a fractional Brownian sheet for $\gamma > \gamma_0 := \alpha_{\mathrm{on}} - 1$. Indeed, since V_γ has stationary rectangular increments, which are independent in the vertical direction, the limit process $\{V_\gamma(\tau, 1),\, \tau \ge 0\}$ arising in Theorem 2.2(i)–(iii) determines the distribution of V_γ for $\gamma > \gamma_0$, $0 < \gamma < \gamma_0$ and $\gamma = \gamma_0$, respectively. We observe that for the individual ON/OFF process W, its properly normalized and centered partial sums process tends to a stable Lévy motion, though W itself has a finite second moment and exhibits LRD. We can expect to obtain scaling transition for independent copies of other LRD processes if asymptotic behaviour of their partial sums differs from fractional Brownian motion.

Recently, scaling transition has been observed for some classes of LRD Gaussian and aggregated nearest-neighbor autoregressive RFs on $\mathbb{Z}^2$ in [89, 90]. In a more general way, the phenomenon has appeared for some RF on $\mathbb{Z}^d$ with $d \ge 2$ in [10]. We summarize briefly the results of [89].

Let $X = \{X(t, s),\, (t, s) \in \mathbb{Z}^2\}$ be a real-valued stationary Gaussian RF, having the spectral representation

$$X(t, s) := \int_{[-\pi,\pi]^2} \mathrm{e}^{\mathrm{i}(tu+sv)} \sqrt{f(u, v)} Z(\mathrm{d}u, \mathrm{d}v), \tag{2.12}$$

where $Z(\mathrm{d}u, \mathrm{d}v)$ is a complex-valued Gaussian random measure on $[-\pi, \pi]^2$ with zero mean and variance $\mathrm{E}|Z(\mathrm{d}u, \mathrm{d}v)|^2 = \mathrm{d}u\mathrm{d}v$ and the spectral density is given by

$$f(u, v) = \frac{g(u, v)}{(|u|^2 + |v|^{2h_2/h_1})^{h_1/2}}, \quad (u, v) \in [-\pi, \pi]^2,$$

where $0 < h_1 \le h_2 < \infty$, $h_i \ne 1$, $i = 1, 2$, $\sum_{i=1}^2 1/h_i > 1$ and $g \ge 0$ is bounded and continuous at the origin with $g(0, 0) > 0$.

Recall that a zero-mean Gaussian RF $B_{H_1,H_2} = \{B_{H_1,H_2}(x, y),\, (x, y) \in \mathbb{R}_+^2\}$ is a standard fractional Brownian sheet with Hurst index $(H_1, H_2) \in (0,1]^2$ if

$$\mathrm{E}B_{H_1,H_2}(x_1,x_2)B_{H_1,H_2}(y_1,y_2) = \frac{1}{2^2}\prod_{i=1}^{2}(x_i^{2H_i} + y_i^{2H_i} - |x_i - y_i|^{2H_i}).$$

Theorem 2.4 (Puplinskaitė, Surgailis [89]). *Assume $X = \{X(t,s), (t,s) \in \mathbb{Z}^2\}$ in (2.12) and set $\gamma_0 := h_1/h_2$. Then for any $\gamma > 0$ the limit in (2.9) exists with*

$$V_\gamma := \begin{cases} V_+, & \gamma > \gamma_0, \\ V_-, & \gamma < \gamma_0, \\ V_{\gamma_0}, & \gamma = \gamma_0, \end{cases} \qquad H(\gamma) := \begin{cases} H_1^+ + \gamma/2, & h_1 < 1,\ \gamma \ge \gamma_0, \\ 1 + \gamma H_2^+, & h_1 > 1,\ \gamma \ge \gamma_0, \\ 1/2 + \gamma H_2^-, & h_2 < 1,\ \gamma \le \gamma_0, \\ H_1^- + \gamma, & h_2 > 1,\ \gamma \le \gamma_0; \end{cases}$$

where $H_1^+ = (1+h_1)/2$, $H_2^+ = (1+h_2-h_2/h_1)/2$, $H_2^- = (1+h_2)/2$, $H_1^- = (1+h_1-h_1/h_2)/2$ and

$$V_+ := \sigma^+_{h_1,h_2}\begin{cases} B_{H_1^+,1/2}, & h_1 < 1, \\ B_{1,H_2^+}, & h_1 > 1; \end{cases} \qquad V_- := \sigma^-_{h_1,h_2}\begin{cases} B_{1/2,H_2^-}, & h_2 < 1, \\ B_{H_1^-,1}, & h_2 > 1; \end{cases}$$

and $\sigma^\pm_{h_1,h_2}$ are some positive constants and V_{γ_0} is a Gaussian RF given by its spectral representation. As a consequence, the RF X exhibits scaling transition at $\gamma_0 = h_1/h_2$.

In Theorem 2.4 the RFs V_+ and V_- have either independent, or invariant (completely dependent) rectangular increments along one of the coordinate axes, whereas V_{γ_0} inherits the dependence structure of the underlying X. This property of increments is characteristic of limits $V_\pm$ in the presence of scaling transition.

However, for another class of LRD Gaussian RFs, [89] proved the absence of scaling transition. To be precise, if X is defined by (2.12) with a spectral density given by

$$f(u,v) = \frac{g(u,v)}{|u|^{2d_1}|v|^{2d_2}}, \quad (u,v) \in [-\pi,\pi]^2,$$

where $0 < d_i < 1/2$, $i = 1,2$, and $g \ge 0$ is bounded and continuous at the origin with $g(0,0) > 0$, then, for all $\gamma > 0$, the scaling limit of X in (2.9) coincides with a fractional Brownian sheet with Hurst index $(d_1 + 1/2, d_2 + 1/2)$.

Results of this type contribute to the large-sample theory of strongly dependent spatial data by showing that the limit distribution of simple statistics such as the sample mean may depend on the relation between γ and γ_0. And if so, these quantities need to be estimated or decided in advance before applying the limit theorem.

Although general properties of $\{V_\gamma, \gamma > 0\}$ are of interest, in the thesis we focus on describing the anisotropic scaling limits for specific classes of RFs, see Chapters 6, 7.

Chapter 3

Aggregation of independent AR(1) processes

This chapter contains the article [79]. We discuss joint temporal and contemporaneous aggregation of N independent copies of AR(1) process with random-coefficient $a \in [0, 1)$ when N and time scale n increase at different rate. Assuming that a has a density, regularly varying at $a = 1$ with exponent $-1 < \beta < 1$, different joint limits of normalized aggregated partial sums are shown to exist when $N^{1/(1+\beta)}/n$ tends to (i) ∞, (ii) 0, (iii) $0 < \mu < \infty$. The limit process arising under (iii) admits a Poisson integral representation on $(0, \infty) \times C(\mathbb{R})$ and enjoys 'intermediate' properties between fractional Brownian motion or random line limit in (i) and sub-Gaussian limit in (ii).

3.1 Introduction

Since macroeconomic time series are obtained by aggregation of microeconomic variables, an important issue in econometrics is establishing the relationship between individual (micro) and aggregate (macro) models. One of the simplest aggregation schemes deals with contemporaneous aggregation of N independent copies $X_i := \{X_i(t),\, t \in \mathbb{Z}\}$, $i = 1, \dots, N$, of stationary random-coefficient AR(1) process

$$X(t) = aX(t-1) + \varepsilon(t), \quad t \in \mathbb{Z}, \tag{3.1}$$

with standardized i.i.d. innovations $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ and a random coefficient $a \in [0, 1)$, independent of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ and such that $\mathrm{E}(1-a)^{-1} < \infty$. The limit aggregated process

$$N^{-1/2} \sum_{i=1}^{N} X_i(t) \xrightarrow{\mathrm{fdd}} \mathcal{X}(t), \quad t \in \mathbb{Z}, \tag{3.2}$$

exists in the sense of weak convergence of finite-dimensional distributions, and is a Gaussian process with zero mean and covariance function

$$\mathrm{E}[\mathcal{X}(0)\mathcal{X}(t)] = \mathrm{E}[X(0)X(t)] = \mathrm{E}\Big[\frac{a^{|t|}}{1-a^2}\Big], \quad t \in \mathbb{Z}. \tag{3.3}$$

Granger [40] observed that for a particular type of beta-distributed random coefficient a, the processes X and $\mathcal{X}$ may have slowly decaying autocovariance functions similarly as in the case of ARFIMA models while normalized partial sums of $\mathcal{X}$ tend to a fractional Brownian motion. Further results on aggregation of autoregressive models with finite variance were obtained in Gonçalves and Gouriéroux [39], Zaffaroni [111], Oppenheim and Viano [74], Celov et al. [19] and other papers. In economic interpretation, individual processes X_i, $i = 1, \dots, N$, in (3.2) are obtained by random sampling from a huge and heterogeneous 'population' of independent 'microagents', each evolving according to a short memory AR(1) process with its own deterministic parameter $a \in [0,1)$, the population being characterized by the distribution (frequency) of a across the population. Thus, aggregation of (randomly sampled) short memory processes may provide an explanation of long memory in observed macroeconomic time series. See also [8, page 85], [111], [112, page 238].

In this chapter we consider the limit behavior of sums

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0, \tag{3.4}$$

where X_i, $i = 1, \dots, N$, are the same as in (3.2). The sum in (3.4) represents joint temporal and contemporaneous aggregate of N individual AR(1) evolutions (3.1) at time scale n. Our main object is the joint aggregation limit of $\{A_{N,n}^{-1} S_{N,n}(\tau),\, \tau \geq 0\}$ in distribution, where $A_{N,n}$ are some normalizing constants and both N and n increase to infinity, possibly at different rate. We also discuss the iterated limits of $\{A_{N,n}^{-1} S_{N,n}(\tau),\, \tau \geq 0\}$ when first $n \to \infty$ and then $N \to \infty$, and vice-versa. Related problems for some network traffic models were studied in Willinger et al. [110], Taqqu et al. [105], Mikosch et al. [70], Gaigalas and Kaj [35], Pipiras et al. [83], Dombry and Kaj [27] and other papers. In these papers, the role of AR(1) processes X_i in (3.4) is played by independent and centered ON/OFF processes, renewal or renewal-reward processes, or M/G/∞ queues with heavy-tailed activity periods.

Let us describe the main results of this chapter. Similarly to [88, 111], we assume that the r.v. $a \in [0,1)$ in (3.1), or the mixing distribution, has a probability density function ϕ such that

$$\phi(x) = \psi(x)(1-x)^{\beta}, \quad x \in [0,1), \tag{3.5}$$

where $\beta > -1$ and ψ is an integrable function on $[0,1)$ having a limit $\lim_{x\to 1}\psi(x) = \psi_1 > 0$. Under the above condition with $0 < \beta < 1$, it immediately follows from the Tauberian theorem [33, Chapter 13, §5, Theorem 3] that the covariance in (3.3) decays as $ct^{-\beta}$, $t \to \infty$, with $c = (\psi_1/2)\Gamma(\beta)$, implying that partial sums of the Gaussian process $\mathcal{X}$ in (3.2) normalized by n^H with $H := 1-(\beta/2) \in (1/2,1)$, tend to a fractional Brownian motion B_H with Hurst parameter H (see [103]). Hence it follows that B_H coincides with the iterated limit of $\{n^{-H}N^{-1/2}S_{N,n}(\tau),\, \tau \ge 0\}$ when $N \to \infty$ first, followed by $n \to \infty$. However, when the order of the above limits is reversed, the limit is a sub-Gaussian $(1+\beta)$-stable process defined in (3.11). See Theorem 3.1 for rigorous formulations.

Let now N, n increase simultaneously so as

$$\frac{N^{1/(1+\beta)}}{n} \to \mu \in [0,\infty], \tag{3.6}$$

leading to the three cases (i)–(iii):

$$\text{Case (i): } \mu = \infty, \quad \text{Case (ii): } \mu = 0, \quad \text{Case (iii): } 0 < \mu < \infty. \tag{3.7}$$

Our main result is Theorem 3.2 which says that under (3.5) and (3.6), the 'simultaneous limit' of $\{A_{N,n}^{-1}S_{N,n}(\tau),\, \tau \ge 0\}$ exists and is different in all three Cases (i)–(iii), namely, it agrees with the above iterated limits in the extreme Cases (i) and (ii), while in Case (iii) it is written as $\{\mu^{1/2}\mathcal{Z}(\tau/\mu),\, \tau \ge 0\}$, where the process $\mathcal{Z}$ corresponding to 'intermediate scaling' in (iii) is defined in (3.31) as a stochastic integral w.r.t. a Poisson random measure on the product space $\mathbb{R}_+ \times C(\mathbb{R})$ with mean $\psi_1 x^\beta \mathrm{d}x \times \mathrm{P}_B$, where P_B is the Wiener measure on $C(\mathbb{R})$. This process enjoys several 'intermediate' properties between the limits in (i) and (ii) and is discussed in Section 3.3 in detail.

Theorems 3.1 and 3.2 can be compared to the results in [27, 35, 55, 70, 83] and other papers, with [55] probably being the closest in spirit to the present work. In particular, Mikosch et al. [70] discuss the 'total accumulated input' $\mathcal{A}_{N,n} := \{\sum_{i=1}^N \int_0^{n\tau} W_i(t)\mathrm{d}t,\, \tau \ge 0\}$ from N independent 'sources' at time scale n. The aggregated inputs W_i, $i = 1, \dots, N$, are i.i.d. copies of ON/OFF process $W := \{W(t),\, t \ge 0\}$, alternating between 1 and 0 and taking value 1 if t is in an ON-period and 0 if t is in an OFF-period, the ON- and OFF-periods forming a stationary renewal process having heavy-tailed lengths with respective tail parameters $\alpha_{\mathrm{on}}, \alpha_{\mathrm{off}} \in (1,2)$, $\alpha_{\mathrm{on}} < \alpha_{\mathrm{off}}$, see [70] for details. The role of condition (3.6) is played in the above papers by

$$\frac{N}{n^{\alpha_{\mathrm{on}}-1}} \to \mu \in [0,\infty],$$

leading to the three cases analogous to (3.7):

$$\text{Case (i'): } \mu = \infty, \quad \text{Case (ii'): } \mu = 0, \quad \text{Case (iii'): } 0 < \mu < \infty,$$

known as the 'fast growth condition', the 'slow growth condition' and the 'intermediate growth condition', respectively. The limit of (normalized) 'accumulated input' $\mathcal{A}_{N,n}$ in Cases (i') and (ii') was obtained in [70], as fractional Brownian motion and α_{on}-stable Lévy process, respectively. The 'intermediate' limit in Case (iii') was identified in [34, 35] and [27] who showed that this process can be regarded as a 'bridge' between the limiting processes in Cases (i') and (ii'), and can be represented as a stochastic integral w.r.t. a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$, see (3.49), although distinctly different from the corresponding process $\mathcal{Z}$ arising in Case (iii). Related results for some other heavy-tailed duration-based models were obtained in [52, 55, 68, 83] and elsewhere.

The differences between the respective limiting processes in this chapter and the above mentioned works can be partially explained by the fact that the 'memory mechanism', or dependence structure, of the AR(1) model is very different from that of telecommunication models. Contrary to the latter models, the random-coefficient AR(1) process is non-ergodic (each individual X_i picks a random value a and sticks to it forever), the long memory being a consequence of a sufficiently high concentration of a's near the unit root 1. This 'memory mechanism' is very different from the M/G/∞ model where each session gets its own duration and the long memory is essentially due to the occurrence of a few very long durations. The above differences are reflected in different limit behaviors of the partial sums of the individual processes (a discontinuous stable Lévy process with independent increments in the latter case and a continuous sub-Gaussian process with conditionally independent and unconditionally dependent increments in the former case), extending also to the 'slow growth' limits in (ii) and (ii'). On the other hand, the 'fast growth' limits in (i) and (i') coincide up to a choice of parameters, since in both cases Gaussian fluctuations play a dominating role.

In the above context, an interesting open problem concerns possible existence and description of an 'intermediate limit regime' for double sums (3.4), where X_i, $i = 1, 2, \dots$, are general independent and identically distributed processes such that the iterated limits of (3.4) exist and are different. A particular case of such X_i is the regime-switching AR(1) process with covariance long memory and Lévy stable partial sums behavior studied in [62, 65]. This process is of particular interest since it combines the dependence structures of random-coefficient AR(1) and duration models. Other possible generalizations of our results concern random-coefficient AR(1) process with infinite variance [88] and/or common innovations [87, 111], autoregressive random fields [90]. See also Remark 3.5.

3.2 Main results

Let $\{\varepsilon,\, \varepsilon(t),\, t \in \mathbb{Z}\}$ be i.i.d. r.v.s with $\mathrm{E}\varepsilon = 0$, $\mathrm{E}\varepsilon^2 = 1$, and $a \in [0,1)$ be a r.v. independent of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$. It is easy to show [88, Proposition 2.1] that there exists a unique stationary solution to the AR(1) equation (3.1) given by

$$X(t) = \sum_{k=0}^{\infty} a^k \varepsilon(t-k), \quad t \in \mathbb{Z}. \tag{3.8}$$

The series in (3.8) converges conditionally almost surely and in L^2 given $a \in [0,1)$. Moreover, if

$$\mathrm{E}\Big[\frac{1}{1-a}\Big] < \infty,$$

then the series in (3.8) converges in L^2 and defines a stationary process with zero mean and covariance in (3.3).

Consider the following stochastic integral representation of a fractional Brownian motion $B_{1-(\beta/2)} := \{B_{1-(\beta/2)}(\tau),\, \tau \geq 0\}$:

$$\begin{aligned} B_{1-(\beta/2)}(\tau) &:= \int_{\mathbb{R}_+ \times \mathbb{R}} (\mathfrak{f}(x, \tau - s) - \mathfrak{f}(x, -s)) Z(\mathrm{d}x, \mathrm{d}s), \text{ where} \\ \mathfrak{f}(x,t) &:= \begin{cases} (1 - \mathrm{e}^{-xt})/x, & \text{if } x > 0 \text{ and } t > 0, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{3.9}$$

w.r.t. a Gaussian random measure $Z(\mathrm{d}x, \mathrm{d}s)$ on $\mathbb{R}_+ \times \mathbb{R}$ with zero mean, variance $\nu(\mathrm{d}x, \mathrm{d}s) := \psi_1 x^\beta \mathrm{d}x\mathrm{d}s$ and the characteristic function $\mathrm{E}\exp\{\mathrm{i}\theta Z(A)\} = \exp\{-\theta^2 \nu(A)/2\}$ for each Borel set $A \subset \mathbb{R}_+ \times \mathbb{R}$ with $\nu(A) < \infty$. Here $0 < \beta < 1$ and ψ_1 is the asymptotic constant from (3.5). The representation (3.9) appeared in Puplinskaitė and Surgailis [88, equation (1.5)], as a particular case of a new class of stable self-similar processes. It is related to the superposition of Ornstein-Uhlenbeck processes discussed in Barndorff-Nielsen [6, Section 6], see also Section 3.3. It easily follows that

$$\begin{aligned} &\mathrm{E}B_{1-(\beta/2)}(u) B_{1-(\beta/2)}(v) \\ &\quad = \int_{\mathbb{R}_+ \times \mathbb{R}} (\mathfrak{f}(x, u-s) - \mathfrak{f}(x,-s))(\mathfrak{f}(x, v-s) - \mathfrak{f}(x,-s)) \nu(\mathrm{d}x, \mathrm{d}s) \\ &\quad = \frac{\Gamma(\beta)\psi_1}{2(2-\beta)(1-\beta)} (u^{2-\beta} + v^{2-\beta} - |u-v|^{2-\beta}), \quad u, v \geq 0. \end{aligned} \tag{3.10}$$

When $-1 < \beta < 0$, let V_β be a symmetric $2(1+\beta)$-stable r.v. with characteristic function given in Proposition 3.5(ii). Let $V_0 \stackrel{\mathrm{d}}{=} \mathcal{N}(0, \psi_1/2)$ be a normal r.v. Thus, the process $\{V_\beta \tau,\, \tau \geq 0\}$ is a random $2(1+\beta)$-stable line for any $-1 < \beta \leq 0$.

Next, let $W_\beta > 0$, $-1 < \beta < 1$, be a $(1+\beta)/2$-stable r.v. with Laplace transform $\mathrm{E}\mathrm{e}^{-\theta W_\beta} = \exp\{-k_\beta \theta^{(1+\beta)/2}\}$, $\theta \geq 0$, and $k_\beta > 0$ defined at (3.60). Let $\{B(\tau),\, \tau \geq 0\}$ be a standard Brownian motion, $\mathrm{E}B^2(\tau) = \tau$, independent of r.v. W_β. The process

$$\mathcal{W}_\beta(\tau) := W_\beta^{1/2} B(\tau), \quad \tau \geq 0, \tag{3.11}$$

has $(1+\beta)$-stable finite-dimensional distributions and stationary increments. According to the terminology in [96, Section 3.8], $\mathcal{W}_\beta$ is called a sub-Gaussian process.

Finally, we define a random process $\mathcal{Z}_\beta := \{\mathcal{Z}_\beta(\tau),\, \tau \geq 0\}$ depending on parameter $-1 < \beta < 1$, through its finite-dimensional characteristic function:

$$\begin{aligned} &\mathrm{E}\exp\Big\{\mathrm{i}\sum_{j=1}^m \theta_j \mathcal{Z}_\beta(\tau_j)\Big\} \\ &\qquad = \exp\Big\{\psi_1 \int_{\mathbb{R}_+} \Big(\mathrm{e}^{-\frac{1}{2}\int_{\mathbb{R}}(\sum_{j=1}^m \theta_j(\mathfrak{f}(x,\tau_j - s) - \mathfrak{f}(x,-s)))^2 \mathrm{d}s} - 1\Big) x^\beta \mathrm{d}x\Big\}, \end{aligned} \tag{3.12}$$

where $\theta_j \in \mathbb{R}$, $\tau_j \in \mathbb{R}_+$, $j = 1, \ldots, m$, $m \in \mathbb{N}$, and $\mathfrak{f}$ is given in (3.9). A Poisson stochastic integral representation and various properties of $\mathcal{Z}_\beta$ are discussed in Section 3.3.

In Theorems 3.1 and 3.2, $S_{N,n}(\tau)$ is the double sum in (3.4) of independent copies of the random-coefficient AR(1) process X (3.8) and the mixing density satisfies (3.5).

Theorem 3.1. *The iterated limits of the normalized aggregated partial sums process $S_{N,n}$ are given by*

$$\begin{aligned} \text{(fdd)}\lim_{n\to\infty}\lim_{N\to\infty} n^{(\beta/2)-1}N^{-1/2}S_{N,n}(\tau) &= B_{1-(\beta/2)}(\tau) \text{ if } \beta \in (0,1), &(3.13)\\ \text{(fdd)}\lim_{n\to\infty}\lim_{N\to\infty} n^{-1}N^{-1/2(1+\beta)}S_{N,n}(\tau) &= V_\beta \tau \text{ if } \beta \in (-1,0), &(3.14)\\ \text{(fdd)}\lim_{n\to\infty}\lim_{N\to\infty} n^{-1}(N\log N)^{-1/2}S_{N,n}(\tau) &= V_0 \tau \text{ if } \beta = 0, &(3.15)\\ \text{(fdd)}\lim_{N\to\infty}\lim_{n\to\infty} N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(\tau) &= \mathcal{W}_\beta(\tau) \text{ if } \beta \in (-1,1). &(3.16) \end{aligned}$$

Theorem 3.2. *The simultaneous limits of the normalized aggregated partial sums process $S_{N,n}$ when $N, n \to \infty$ as in* (3.6) *are given in respective Cases (i)–(iii) of*

(3.7) *by*

$$N^{-1/2}n^{-1+(\beta/2)}S_{N,n}(\tau) \overset{\text{fdd}}{\to} B_{1-(\beta/2)}(\tau) \quad \textit{in Case (i) if } \beta \in (0,1), \tag{3.17}$$

$$N^{-1/2(1+\beta)}n^{-1}S_{N,n}(\tau) \overset{\text{fdd}}{\to} V_\beta \tau \quad \textit{in Case (i) if } \beta \in (-1,0), \tag{3.18}$$

$$(N\log(N/n))^{-1/2}n^{-1}S_{N,n}(\tau) \overset{\text{fdd}}{\to} V_0 \tau \quad \textit{in Case (i) if } \beta = 0, \tag{3.19}$$

$$N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(\tau) \overset{\text{fdd}}{\to} \mathcal{W}_\beta(\tau) \quad \textit{in Case (ii) if } \beta \in (-1,1), \tag{3.20}$$

$$N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(\tau) \overset{\text{fdd}}{\to} \mu^{1/2}\mathcal{Z}_\beta(\tau/\mu) \quad \textit{in Case (iii) if } \beta \in (-1,1). \tag{3.21}$$

Since higher β means smaller chances for the individual AR(1) process being close to the unit root $a = 1$, hence having less memory, it is natural to expect that this tendency should be reflected in the limit behavior of the partial sums $S_{N,n}$. It is most clearly seen in (3.17), as the 'memory' of the fractional Brownian motion $B_{1-(\beta/2)}$ decreases with β increasing. On the other hand, in (3.18) and (3.20), a change of β does not alter the dependence structure of the limit processes $V_\beta\tau$ and $\mathcal{W}_\beta$ but rather affects their variability since β is directly related to the stability index of these processes. These tendencies can be also observed although less clearly in the 'intermediate' limit of (3.21). However, when $\beta > 1$ these differences disappear and the joint limit of the partial sums process is a usual Brownian motion independent of β and the mutual increase rate of N and n; see below.

Theorem 3.3. *Let $\beta > 1$. Then, as $N, n \to \infty$ in arbitrary way,*

$$N^{-1/2}n^{-1/2}S_{N,n}(\tau) \overset{\text{fdd}}{\to} \sigma B(\tau) \quad \textit{with } \sigma^2 := \mathrm{E}(1-a)^{-2} < \infty.$$

Remark 3.1. The question about weak convergence in the Skorohod space $D[0,1]$ in Theorems 3.1–3.3 remains generally open, although in some cases ((3.13), (3.17)) the weak convergence follows rather easily by the Kolmogorov criterion.

3.3 The 'intermediate' process

This section discusses the definition and various properties of the process $\mathcal{Z}_\beta$ arising in the 'intermediate' limit (iii) of Theorem 3.2. Let $\mathfrak{X}$ be a measurable space with a σ-finite measure μ defined on a σ-algebra $\mathcal{F}(\mathfrak{X})$ of measurable subsets of $\mathfrak{X}$. Let $\mathcal{M}$ be a Poisson random measure on $\mathfrak{X}$ with mean μ and $\widetilde{\mathcal{M}} = \mathcal{M} - \mu$ be the centered Poisson random measure. The stochastic integrals $\int_{\mathfrak{X}} f(x)\mathcal{M}(\mathrm{d}x) \equiv \int f(x)\mathcal{M}(\mathrm{d}x)$ and $\int_{\mathfrak{X}} f(x)\widetilde{\mathcal{M}}(\mathrm{d}x) \equiv \int f(x)\widetilde{\mathcal{M}}(\mathrm{d}x)$ are defined for any measurable function $f : \mathfrak{X} \to \mathbb{R}$ with $\int 1 \wedge |f(x)|\mu(\mathrm{d}x) < \infty$

and $\int |f(x)| \wedge |f(x)|^2 \mu(\mathrm{d}x) < \infty$, respectively, as limits in probability of suitable integral sums, and their characteristic functions are given by

$$\begin{aligned}
\mathrm{E}\mathrm{e}^{\mathrm{i}\theta \int f(x)\mathcal{M}(\mathrm{d}x)} &= \exp\Big\{ \int (\mathrm{e}^{\mathrm{i}\theta f(x)} - 1)\mu(\mathrm{d}x)\Big\}, \\
\mathrm{E}\mathrm{e}^{\mathrm{i}\theta \int f(x)\widetilde{\mathcal{M}}(\mathrm{d}x)} &= \exp\Big\{ \int (\mathrm{e}^{\mathrm{i}\theta f(x)} - 1 - \mathrm{i}\theta f(x))\mu(\mathrm{d}x)\Big\}, \quad \theta \in \mathbb{R}. \qquad (3.22)
\end{aligned}$$

We have $\mathrm{E}\int f(x)\widetilde{\mathcal{M}}(\mathrm{d}x) = 0$, $f \in L^1(\mathfrak{X})$, and $\mathrm{E}[\int f(x)\widetilde{\mathcal{M}}(\mathrm{d}x)]^2 = \int f^2(x)\mu(\mathrm{d}x)$, $f \in L^2(\mathfrak{X})$. Moreover,

$$\mathrm{E}\Big|\int f(x)\widetilde{\mathcal{M}}(\mathrm{d}x)\Big|^p < \infty \Longleftrightarrow \begin{cases} \int (|f(x)|^2 \wedge |f(x)|^p)\mu(\mathrm{d}x) < \infty, & 1 \le p \le 2, \\ \int (|f(x)|^2 \vee |f(x)|^p)\mu(\mathrm{d}x) < \infty, & p \ge 2; \end{cases} \qquad (3.23)$$

see Rajput and Rosinski [91], while

$$\begin{aligned}
\mathrm{E}\Big|\int f(x)\mathcal{M}(\mathrm{d}x)\Big|^p &\le C(p)\|f\|_p^p, \quad 0 < p \le 1, \\
\mathrm{E}\Big|\int f(x)\widetilde{\mathcal{M}}(\mathrm{d}x)\Big|^p &\le C(p)\|f\|_p^p, \quad 1 \le p \le 2, \qquad (3.24)
\end{aligned}$$

where $\|f\|_p^p := \int_{\mathfrak{X}} |f(x)|^p \mu(\mathrm{d}x)$ and the constant $C(p) < \infty$ depends only on p. For step functions $f = \sum_{j=1}^\infty f_j \mathbf{1}(\cdot \in A_j) \in L^p(\mathfrak{X})$ taking values $f_j \in \mathbb{R}$ on $A_j \in \mathcal{F}(\mathfrak{X})$ with $\mu(A_j) < \infty$, the first inequality of (3.24) with $C(p) = 1$ follows from $|\sum_{j=1}^\infty f_j \mathcal{M}(A_j)|^p \le \sum_{j=1}^\infty |f_j|^p |\mathcal{M}(A_j)|^p$ and $\mathrm{E}|\mathcal{M}(A_j)|^p \le \mathrm{E}\mathcal{M}(A_j) = \mu(A_j)$. The second inequality of (3.24) with $C(p) = 2^{2-p} \le 2$ is obtained in [99, Theorem 3.1], by interpolation between $L^1(\mathfrak{X})$ and $L^2(\mathfrak{X})$.

Consider a family of 'elementary' integrated Ornstein-Uhlenbeck processes

$$\begin{aligned}
z(\tau; x) &:= \int_0^\tau \mathrm{d}u \int_{\mathbb{R}} \mathrm{e}^{-x(u-s)} \mathbf{1}(u > s)\mathrm{d}B(s) \\
&= \int_{\mathbb{R}} \big(\mathfrak{f}(x, \tau - s) - \mathfrak{f}(x, -s)\big)\mathrm{d}B(s), \quad \tau \in \mathbb{R},\ x > 0, \qquad (3.25)
\end{aligned}$$

where $B := \{B(s),\ s \in \mathbb{R}\}$ is a standard Brownian motion and $\mathfrak{f}$ is defined in (3.9). For each $x > 0$ the process $\{z(\tau; x),\ \tau \in \mathbb{R}\}$ is a.s. continuously differentiable on $\mathbb{R}$ and its derivative $z'(\tau; x) = \mathrm{d}z(\tau; x)/\mathrm{d}\tau$ satisfies the Langevin equation

$$\mathrm{d}z'(\tau; x) = -xz'(\tau; x)\mathrm{d}\tau + \mathrm{d}B(\tau). \qquad (3.26)$$

Accordingly, the joint characteristic function of $z(\tau_j; x_j)$, $\tau_j \in \mathbb{R}$, $x_j \in \mathbb{R}$, $j = 1, \dots, m$, $m \in \mathbb{N}$, is given by

$$\begin{aligned}
&\mathrm{E}\exp\Big\{\mathrm{i}\sum_{j=1}^m \theta_j z(\tau_j; x_j)\Big\} \\
&= \exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} \Big(\sum_{j=1}^m \theta_j\big(\mathfrak{f}(x_j, \tau_j - s) - \mathfrak{f}(x_j, -s)\big)\Big)^2 \mathrm{d}s\Big\}. \qquad (3.27)
\end{aligned}$$

W.l.g., we may assume that the process $\{z(\tau;x),\ \tau\in\mathbb{R},\ x>0\}$ is defined on the space $C(\mathbb{R})$ equipped with the Wiener measure P_B induced by the Brownian motion B. In other words, for any cylinder set $A=\{\zeta(\cdot)\in C(\mathbb{R}):\zeta(\tau_j)\in I_j,\ j=1,\dots,m\}$, $\tau_j\in\mathbb{R}$, and intervals $I_j\subset\mathbb{R}$, $j=1,\dots,m$, we have

$$\mathrm{P}_B(A)=\mathrm{P}(B(\tau_j)\in I_j,\ j=1,\dots,m). \tag{3.28}$$

Let $\mathcal{M}(\mathrm{d}x,\mathrm{d}B)$ be a Poisson random measure on the product space $\mathbb{R}_+\times C(\mathbb{R})$ with the mean

$$\mu(\mathrm{d}x,\mathrm{d}B)=\mathrm{E}\mathcal{M}(\mathrm{d}x,\mathrm{d}B)=\psi_1 x^\beta\mathrm{d}x\times\mathrm{P}_B(\mathrm{d}B(\cdot)), \tag{3.29}$$

where P_B is defined at (3.28), and let

$$\widetilde{\mathcal{M}}(\mathrm{d}x,\mathrm{d}B):=\mathcal{M}(\mathrm{d}x,\mathrm{d}B)-\mu(\mathrm{d}x,\mathrm{d}B)$$

be the centered Poisson measure. Then, $\mathcal{Z}_\beta$ is defined as a stochastic integral with respect to the above Poisson measure:

$$\mathcal{Z}_\beta(\tau):=\int_{(0,1)\times C(\mathbb{R})}z(\tau;x)\mathcal{M}(\mathrm{d}x,\mathrm{d}B)+\int_{[1,\infty)\times C(\mathbb{R})}z(\tau;x)\widetilde{\mathcal{M}}(\mathrm{d}x,\mathrm{d}B),\quad \tau\ge 0, \tag{3.30}$$

As shown below, for $-1/2<\beta<1$ the two integrals can be combined into a single one:

$$\begin{aligned}\mathcal{Z}_\beta(\tau)\ &=\ \int_{\mathbb{R}_+\times C(\mathbb{R})}z(\tau;x)\widetilde{\mathcal{M}}(\mathrm{d}x,\mathrm{d}B) \qquad (3.31)\\ &=\ \int_{\mathbb{R}_+\times C(\mathbb{R})}\Big\{\int_{\mathbb{R}}\big(\mathfrak{f}(x,\tau-s)-\mathfrak{f}(x,-s)\big)\mathrm{d}B(s)\Big\}\widetilde{\mathcal{M}}(\mathrm{d}x,\mathrm{d}B).\end{aligned}$$

Proposition 3.4. *(i) The process $\mathcal{Z}_\beta$ in* (3.30) *is well-defined for any $-1<\beta<1$. It has stationary increments, infinitely divisible finite-dimensional distributions and the joint characteristic function is given in* (3.12).
(ii) $\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p<\infty$ for any $p<2(1+\beta)$, $0<p<4$, and $\mathrm{E}\mathcal{Z}_\beta(\tau)=0$ for $-1/2<\beta<1$.
(iii) If $0<\beta<1$ then $\mathrm{E}\mathcal{Z}^2_\beta(\tau)<\infty$ and

$$\mathrm{E}[\mathcal{Z}_\beta(\tau_1)\mathcal{Z}_\beta(\tau_2)]=\frac{\Gamma(\beta)\psi_1}{2(2-\beta)(1-\beta)}(\tau_1^{2-\beta}+\tau_2^{2-\beta}-|\tau_1-\tau_2|^{2-\beta}),\quad \tau_1,\tau_2\ge 0. \tag{3.32}$$

(iv) For $-1/2<\beta<1$, the process $\mathcal{Z}_\beta$ in (3.31) *has a.s. continuous trajectories.*

Proof. (i) It suffices to check that $I_1:=\int_{(0,1)\times C(\mathbb{R})}\mu(\mathrm{d}x,\mathrm{d}B)=C\int_0^1 x^\beta\mathrm{d}x<\infty$ and $I_2:=\int_{[1,\infty)\times C(\mathbb{R})}|z(\tau;x)|^2\ \mu(\mathrm{d}x,\mathrm{d}B)=C\int_1^\infty \mathrm{E}_B|z(\tau;x)|^2x^\beta\mathrm{d}x<\infty$. We have

$\mathrm{E}_B|z(\tau;x)|^2 = \sigma^2(\tau;x)$, where

$$\begin{aligned}\sigma^2(\tau;x) \ &:= \ \int_{-\infty}^{\tau}\big(\mathfrak{f}(x,\tau-s)-\mathfrak{f}(x,-s)\big)^2\mathrm{d}s \le C\Big(\frac{(x\tau)\wedge(x\tau)^3}{x^3}+\frac{(1-\mathrm{e}^{-x\tau})^2}{x^3}\Big)\\ &\le \ C\Big(\frac{(x\tau)\wedge(x\tau)^3}{x^3}+\frac{1\wedge(x\tau)^2}{x^3}\Big)\le\frac{C\tau}{x^2}(1\wedge(\tau x)). \qquad (3.33)\end{aligned}$$

Thus, $I_2 < \infty$ when $\beta < 1$. (3.12) follows from (3.22), (3.27) and $\mathrm{E}z(\tau;x) = 0$. The stationarity of increments is immediate from (3.12).

(ii) Obviously, it suffices to show $\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p < \infty$ for $p < 2(1+\beta)$ sufficiently close to $2(1+\beta)$ such that $1+\beta < p < 2(1+\beta)$. Let first $0 < p \le 2$. Then using (3.24) we have $\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p \le C\int_0^\infty \mathrm{E}_B|z(\tau;x)|^p x^\beta \mathrm{d}x$. Since $z(\tau;x) \stackrel{\mathrm{d}}{=} \mathcal{N}(0,\sigma^2(\tau;x))$, we have $\mathrm{E}_B|z(\tau;x)|^p \le C|\sigma(\tau;x)|^p$ and hence from (3.33) we obtain

$$\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p \le C\tau^{p/2}\int_0^\infty\Big[\frac{1\wedge(x\tau)}{x^2}\Big]^{p/2}x^\beta\mathrm{d}x = C\tau^{(3p/2)-1-\beta} < \infty. \qquad (3.34)$$

Next, let $p > 2$. Then $\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p < \infty$ follows from (3.23) and

$$\int_0^\infty x^\beta\mathrm{d}x\mathrm{E}_B\big[|z(\tau;x)|^p\vee|z(\tau;x)|^2\big]\le C\int_0^\infty\big(|\sigma(\tau;x)|^p+|\sigma(\tau;x)|^2\big)x^\beta\mathrm{d}x<\infty$$

according to (3.34). The fact that (3.34) holds for $p = 1 < 2(1+\beta)$ implies $\mathrm{E}\mathcal{Z}_\beta(\tau) = 0$ for $-1/2 < \beta < 1$.

(iii) From (3.31), (3.25) and (3.10) we have that for any $\tau \ge 0$

$$\begin{aligned}\mathrm{E}\mathcal{Z}_\beta^2(\tau) \ &= \ \int_{\mathbb{R}_+\times\mathbb{R}}(\mathfrak{f}(x,\tau-s)-\mathfrak{f}(x,-s))^2\nu(\mathrm{d}x,\mathrm{d}s)\\ &= \ \mathrm{E}B^2_{1-\beta/2}(\tau) = \frac{\Gamma(\beta)\psi_1}{(2-\beta)(1-\beta)}\tau^{2-\beta},\end{aligned}$$

hence (3.32) follows from (3.10).

(iv) From (3.34) and stationarity of increments, for $-1/2 < \beta < 1$ and $1 < p < 2(1+\beta)$ sufficiently close to $2(1+\beta)$, we have that $\mathrm{E}|\mathcal{Z}_\beta(\tau+h)-\mathcal{Z}_\beta(\tau)|^p \le Ch^{(3p/2)-1-\beta}$ for any $\tau, h \ge 0$. Since $(3p/2)-1-\beta > 1$, the Kolmogorov criterion applies, yielding the a.s. continuity of (3.31). Proposition 3.4 is proved. □

Remark 3.2. Let $\mathcal{M}_2(\mathrm{d}x,\mathrm{d}B)$ be a Gaussian random measure on $\mathbb{R}_+\times C(\mathbb{R})$ with zero mean and variance $\mu(\mathrm{d}x,\mathrm{d}B)$ in (3.29). From Proposition 3.4(iii) it follows that for $0 < \beta < 1$ the corresponding Gaussian integral of (3.31) is a representation of fractional Brownian motion:

$$B_{1-\beta/2}(\tau) \stackrel{\mathrm{fdd}}{=} \int_{\mathbb{R}_+\times C(\mathbb{R})} z(\tau;x)\mathcal{M}_2(\mathrm{d}x,\mathrm{d}B).$$

Remark 3.3. As noted in [96], the sub-Gaussian process $\mathcal{W}_\beta$ (3.11) admits a stochastic integral representation

$$\mathcal{W}_\beta(\tau) \overset{\mathrm{fdd}}{=} \int_{C(\mathbb{R}_+)} B(\tau)\mathcal{N}(\mathrm{d}B)$$

w.r.t. symmetric $(1+\beta)$-stable random measure $\mathcal{N}$ on $C(\mathbb{R}_+)$ with control measure $\nu := c_\beta \mathrm{P}_B$, where P_B is the Wiener measure, see (3.28), and $c_\beta := \psi_1\pi/2\sin(\pi(1+\beta)/2)\Gamma(2+\beta)$. The process $\mathcal{W}_\beta$ can be further represented as a stochastic integral w.r.t. the Poisson random measure $\mathcal{M}(\mathrm{d}x,\mathrm{d}B)$ on $\mathbb{R}_+\times C(\mathbb{R}_+)$ in (3.29), viz.,

$$\mathcal{W}_\beta(\tau) \overset{\mathrm{fdd}}{=} \int_{(0,1)\times C(\mathbb{R}_+)} \frac{B(\tau)}{x}\mathcal{M}(\mathrm{d}x,\mathrm{d}B) + \int_{[1,\infty)\times C(\mathbb{R}_+)} \frac{B(\tau)}{x}\widetilde{\mathcal{M}}(\mathrm{d}x,\mathrm{d}B).$$

Remark 3.4. Curiously enough, the $2(1+\beta)$-stable r.v. V_β in Theorem 3.1(ii) (see Proposition 3.5(ii) below) can be also represented as a stochastic integral w.r.t. the Poisson measure $\mathcal{M}(\mathrm{d}x,\mathrm{d}B)$:

$$V_\beta \overset{\mathrm{d}}{=} \int_{(0,1)\times C(\mathbb{R})} z'(1;x)\mathcal{M}(\mathrm{d}x,\mathrm{d}B) + \int_{[1,\infty)\times C(\mathbb{R})} z'(1;x)\widetilde{\mathcal{M}}(\mathrm{d}x,\mathrm{d}B),$$

where $\{z'(\tau;x)\}$ is the stationary Ornstein-Uhlenbeck process in (3.26).

The following proposition describes local and global scaling behavior of the process $\mathcal{Z}_\beta$.

Proposition 3.5. *Let $\mathcal{Z}_\beta$ be defined as in* (3.31)*.*
(i) Let $0<\beta<1$. Then

$$b^{-1+\beta/2}(\mathcal{Z}_\beta(\tau+bu)-\mathcal{Z}_\beta(\tau)) \overset{\mathrm{fdd}}{\to} B_{1-\beta/2}(u) \quad \text{as } b\to 0.$$

(ii) Let $-1<\beta<0$. Then

$$b^{-1}(\mathcal{Z}_\beta(\tau+bu)-\mathcal{Z}_\beta(\tau)) \overset{\mathrm{fdd}}{\to} uV_\beta \quad \text{as } b\to 0,$$

where V_β is a $2(1+\beta)$-stable r.v. with characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta V_\beta} = \mathrm{e}^{-K_\beta|\theta|^{2(1+\beta)}}, \quad K_\beta := \frac{\psi_1\Gamma(-\beta)}{4^{1+\beta}(1+\beta)}.$$

(iii) Let $\beta=0$. Then

$$(b\log^{1/2}(1/b))^{-1}(\mathcal{Z}_\beta(\tau+bu)-\mathcal{Z}_\beta(\tau)) \overset{\mathrm{fdd}}{\to} uV_0 \quad \text{as } b\to 0,$$

where $V_0 \overset{\mathrm{d}}{=} \mathcal{N}(0,\psi_1/2)$ is a Gaussian r.v. with variance $\psi_1/2$.
(iv) Let $-1<\beta<1$. Then

$$b^{-1/2}\mathcal{Z}_\beta(b\tau) \overset{\mathrm{fdd}}{\to} \mathcal{W}_\beta(\tau) \quad \text{as } b\to\infty.$$

Proof. (i) By stationarity of increments, it suffices to prove the convergence for $\tau = 0$, or $\mathrm{E}\exp\{\mathrm{i}\sum_{j=1}^m \theta_j b^{-1+\beta/2}\mathcal{Z}_\beta(bu_j)\} \to \mathrm{E}\exp\{\mathrm{i}\sum_{j=1}^m \theta_j B_{1-\beta/2}(u_j)\}$ for any $u_j \in \mathbb{R}_+$, $\theta_j \in \mathbb{R}$, $m \in \mathbb{N}$. Using (3.12) and (3.9), the last convergence follows from

$$\begin{aligned}\int_0^\infty \Big(1 - \exp\Big\{ -\frac{b^{\beta-2}}{2}\int_{\mathbb{R}}\Big|\sum_{j=1}^m \theta_j(\mathfrak{f}(x, bu_j - s) - \mathfrak{f}(x,-s))\Big|^2 \mathrm{d}s\Big\}\Big) x^\beta \mathrm{d}x \\ \to \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}}\Big|\sum_{j=1}^m \theta_j(\mathfrak{f}(x, u_j - s) - \mathfrak{f}(x,-s))\Big|^2 x^\beta \mathrm{d}x\mathrm{d}s. \qquad (3.35)\end{aligned}$$

Using the scaling property $\mathfrak{f}(x/b, bs) = b\mathfrak{f}(x,s)$ of $\mathfrak{f}$ in (3.9), the l.h.s. of (3.35) can be rewritten as

$$b^{-1-\beta}\int_0^\infty \Big(1 - \exp\Big\{ -\frac{b^{1+\beta}}{2}\int_{\mathbb{R}}\Big|\sum_{j=1}^m \theta_j(\mathfrak{f}(x, u_j - s) - \mathfrak{f}(x,-s))\Big|^2 \mathrm{d}s\Big\}\Big) x^\beta \mathrm{d}x$$

and the convergence in (3.35) follows from $b^{-1-\beta}(1-\mathrm{e}^{-b^{1+\beta}I}) \to I$ $(b \to 0)$ and the dominated convergence theorem, since $0 \le 1 - \mathrm{e}^{-x} \le x$ for any $x \ge 0$ and the integral on the r.h.s. of (3.35) converges. This proves part (i).

(ii) Using the notation in (3.35), it suffices to show that

$$\begin{aligned}\int_0^\infty \Big(1 - \exp\Big\{ -\frac{b^{-2}}{2}\int_{\mathbb{R}}\Big|\sum_{j=1}^m \theta_j(\mathfrak{f}(x, bu_j - s) - \mathfrak{f}(x,-s))\Big|^2 \mathrm{d}s\Big\}\Big) \psi_1 x^\beta \mathrm{d}x \\ \to \int_0^\infty \Big(1 - \exp\Big\{ -\frac{1}{4x}\Big|\sum_{j=1}^m \theta_j u_j\Big|^2\Big\}\Big)\psi_1 x^\beta \mathrm{d}x = K_\beta\Big|\sum_{j=1}^m \theta_j u_j\Big|^{2(1+\beta)} \qquad (3.36)\end{aligned}$$

Towards this end, consider

$$\begin{aligned}\Psi(x;b) &:= \int_{\mathbb{R}}\frac{1}{2b^2}\Big|\sum_{j=1}^m \theta_j(\mathfrak{f}(x, bu_j - s) - \mathfrak{f}(x,-s))\Big|^2 \mathrm{d}s = \Big(\int_{-\infty}^0 + \int_0^\infty\Big)\dots \\ &=: \Psi_1(x;b) + \Psi_2(x;b). \qquad (3.37)\end{aligned}$$

Then for any $x > 0$,

$$\Psi_1(x;b) = \frac{1}{4x^3}\Big|\sum_{j=1}^m \frac{\theta_j}{b}(1-\mathrm{e}^{-bxu_j})\Big|^2 \to \frac{1}{4x}\Big|\sum_{j=1}^m \theta_j u_j\Big|^2 =: \Psi(x), \quad b \to 0, \quad (3.38)$$

and

$$\begin{aligned}\Psi_2(x;b) &= \frac{1}{2x^2b^2}\int_0^{bu_m}\Big|\sum_{j=1}^m \theta_j(1-\mathrm{e}^{-x(bu_j-s)})\mathbf{1}(s<bu_j)\Big|^2 \mathrm{d}s \\ &\le \frac{C}{x^2b^2}\int_0^{bu_m}(xs)^2\mathrm{d}s \le Cb \to 0. \qquad (3.39)\end{aligned}$$

Hence, $\Psi(x;b) \to \Psi(x)$. From the inequality $1 - \mathrm{e}^{-x} \le x$, $x \ge 0$, it easily follows the dominating bound $0 \le \Psi(x;b) \le C \min(1, 1/x)$, $\forall b, x > 0$. The convergence in (3.36), or $\int_0^\infty (1 - \mathrm{e}^{-\Psi(x;b)}) x^\beta \mathrm{d}x \to \int_0^\infty (1 - \mathrm{e}^{-\Psi(x)}) x^\beta \mathrm{d}x$ now easily follows by the dominating convergence theorem.

(iii) As in (3.36), it suffices to show that

$$I(b) := \psi_1 \int_0^\infty (1 - \mathrm{e}^{-\Psi(x,b)/\log(1/b)}) \mathrm{d}x \to \frac{\psi_1}{4} \Big| \sum_{j=1}^m \theta_j u_j \Big|^2, \tag{3.40}$$

where $\Psi(x,b)$ is defined in (3.37). Split the integral

$$I(b) = \psi_1 \Big(\int_0^{1/b} + \int_{1/b}^\infty \Big) (1 - \mathrm{e}^{-\Psi(x,b)/\log(1/b)}) \mathrm{d}x =: \psi_1 (I_1(b) + I_2(b)).$$

Then using (3.38) and (3.39) we infer that

$$\begin{aligned} I_1(b) &\sim \int_0^{1/b} \Big(1 - \exp\Big\{ - \frac{1}{4x\log(1/b)} \Big| \sum_{j=1}^m \theta_j u_j \Big|^2 \Big\} \Big) \mathrm{d}x \\ &\sim O\Big(\frac{1}{\log(1/b)} \Big) + \frac{|\sum_{j=1}^m \theta_j u_j|^2}{4\log(1/b)} \int_{1/\log(1/b)}^{1/b} \frac{\mathrm{d}x}{x} \\ &\sim \frac{1}{4} \Big| \sum_{j=1}^m \theta_j u_j \Big|^2 . \end{aligned}$$

On the other hand, using $|\Psi(x;b)| \le C/(b^2x^3 + bx^2)$, see (3.38), (3.39), with C independent of $x, b > 0$ we obtain that

$$I_2(b) \le C \int_{1/b}^\infty \frac{\mathrm{d}x}{(b^2x^3 + bx^2)\log(1/b)} = O\Big(\frac{1}{\log(1/b)} \Big) = o(1),$$

proving (3.40) and part (iii).

(iv) Similarly as above, it suffices to prove that

$$\begin{aligned} &\int_0^\infty \Big(1 - \exp\Big\{ - \frac{1}{2b} \int_{\mathbb{R}} \Big| \sum_{j=1}^m \theta_j (\mathfrak{f}(x, bu_j - s) - \mathfrak{f}(x, bu_{j-1} - s)) \Big|^2 \mathrm{d}s \Big\} \Big) \psi_1 x^\beta \mathrm{d}x \\ &\to \int_0^\infty \Big(1 - \exp\Big\{ - \frac{1}{2x^2} \sum_{j=1}^m \theta_j^2 (u_j - u_{j-1}) \Big\} \Big) \psi_1 x^\beta \mathrm{d}x \\ &= \frac{k_\beta}{2^{(1+\beta)/2}} \Big| \sum_{j=1}^m \theta_j^2 (u_j - u_{j-1}) \Big|^{(1+\beta)/2}, \end{aligned} \tag{3.41}$$

where $0 =: u_0 < u_1 < \cdots < u_m$. The l.h.s. of (3.41) can be rewritten as $\int_0^\infty (1 - \mathrm{e}^{-x^{-2} J(x;b)}) \psi_1 x^\beta \mathrm{d}x$, where for any $x > 0$,

$$\begin{aligned} J(x;b) &:= \frac{x^2}{2} \int_{\mathbb{R}} \Big| \sum_{j=1}^m \theta_j \big(\mathfrak{f}(x, b(u_j - s)) - \mathfrak{f}(x, b(u_{j-1} - s)) \big) \Big|^2 \mathrm{d}s \\ &\to J := \frac{1}{2} \sum_{j=1}^m \theta_j^2 (u_j - u_{j-1}) \end{aligned}$$

as $b \to \infty$ follows easily by substituting $\mathfrak{f}$ (3.9) into the integral above. The dominating bound $0 \le J(x;b) \le C\min(1,1/x)$ is elementary and allows to use the dominating convergence theorem, yielding the convergence in (3.41). Proposition 3.5 is proved. □

Proposition 3.5 entails the convergences

$$\mu^{1/2}\mathcal{Z}_\beta(\tau/\mu) \overset{\mathrm{fdd}}{\to} \mathcal{W}_\beta(\tau) \quad \text{and}$$

$$\begin{cases} \mu^{1-(\beta/2)}\mathcal{Z}_\beta(\tau/\mu) \overset{\mathrm{fdd}}{\to} B_{1-\beta/2}(\tau), & 0<\beta<1, \\ \mu\mathcal{Z}_\beta(\tau/\mu) \overset{\mathrm{fdd}}{\to} \tau V_\beta, & -1<\beta<0, \\ \mu(\log\mu)^{-1/2}\mathcal{Z}_\beta(\tau/\mu) \overset{\mathrm{fdd}}{\to} \tau V_0, & \beta=0, \end{cases} \tag{3.42}$$

as $\mu \to 0$ and $\mu \to \infty$, respectively. In other words, the 'intermediate' limit in Theorem 3.2(iii) plays the role of a 'bridge' between the limits in Cases (i) and (ii). Since $\mathcal{W}_\beta$, $B_{1-(\beta/2)}$ and τV_β are distinct processes, (3.42) imply that $\mathcal{Z}_\beta$ is not self-similar and not stable.

The definition of $\mathcal{Z}_\beta$ in (3.31) naturally extends to a two-parameter random field $\{\mathcal{Z}_\beta(\tau,x),\,(\tau,x)\in\mathbb{R}_+^2\}$ defined as a stochastic integral

$$\begin{aligned}\mathcal{Z}_\beta(\tau,x) \quad &:= \quad \int_{(0,x]\times(0,1)\times C(\mathbb{R})} z(\tau;v)\mathcal{M}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}B) \\ &\qquad + \int_{(0,x]\times[1,\infty)\times C(\mathbb{R})} z(\tau;v)\widetilde{\mathcal{M}}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}B), \quad \tau,x\ge 0,\end{aligned} \tag{3.43}$$

with respect to a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}_+\times C(\mathbb{R})$ with intensity $\mathrm{E}\mathcal{M}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}B) = \mathrm{d}y\mu(\mathrm{d}v,\mathrm{d}B) = \psi_1\mathrm{d}yv^\beta\mathrm{d}v\,\mathrm{P}_B(\mathrm{d}B(\cdot))$, see (3.29), where $\widetilde{\mathcal{M}}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}B) := \mathcal{M}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}B) - \mathrm{E}\mathcal{M}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}B)$. Then $\mathcal{Z}_\beta(\tau) = \mathcal{Z}_\beta(\tau,1)$. Note that for each $\tau>0$, $\{\mathcal{Z}_\beta(\tau,x),\,x\ge0\}$ is a homogeneous Lévy process with independent increments. The two-parameter process $\mathcal{Z}_\beta$ in (3.43) satisfies the following properties:
for any $(\tau_0,x_0)\in\mathbb{R}_+^2$,

$$\{\mathcal{Z}_\beta(\tau+\tau_0,x+x_0)-\mathcal{Z}_\beta(\tau+\tau_0,x_0)-\mathcal{Z}_\beta(\tau_0,x+x_0)+\mathcal{Z}_\beta(\tau_0,x_0)\} \overset{\mathrm{fdd}}{=} \{\mathcal{Z}_\beta(\tau,x)\}; \tag{3.44}$$

for any $c>0$,

$$\big\{\mathcal{Z}_\beta(c\tau,c^{1+\beta}x)\big\} \overset{\mathrm{fdd}}{=} \big\{c^{3/2}\mathcal{Z}_\beta(\tau,x)\big\}. \tag{3.45}$$

Property (3.44) is stationarity of increments and (3.45) is an anisotropic two-parameter scaling (self-similarity) property. Note that (3.45) implies

$$\begin{aligned}&\big\{(c\mu)^{-1}c^{-1/2}\mathcal{Z}_\beta(c\tau,(c\mu)^{1+\beta}),\,\tau\in\mathbb{R}_+\big\} \\ &\quad \overset{\mathrm{fdd}}{=} \big\{\mu^{1/2}\mathcal{Z}_\beta(\tau/\mu,1),\,\tau\in\mathbb{R}_+\big\} \quad \text{for all } c>0,\end{aligned} \tag{3.46}$$

which resembles the limit in Theorem 3.2(iii) for $N = (c\mu)^{1+\beta}, n = c$ growing as in (3.6). The two-parameter process $\{\mathcal{Y}_\beta(\tau,\mu) := \mu^{1/2}\mathcal{Z}_\beta(\tau/\mu,1)\}$ on the r.h.s. of (3.46) satisfies $\{\mathcal{Y}_\beta(c\tau,c\mu),\, \tau\in\mathbb{R}_+\} \overset{\mathrm{fdd}}{=} \{c^{1/2}\mathcal{Y}_\beta(\tau,\mu),\, \tau\in\mathbb{R}_+\}$. A related notion of self-similarity was introduced in Jørgensen et al. [51], who call a two-parameter process $Y = \{Y(\tau,\mu),\, (\tau,\mu)\in\mathbb{R}_+^2\}$ *self-similar with Hurst exponent* H *and rate parameter* μ if for all $c > 0$,

$$\{Y(c\tau, c^{H-1}\mu),\, \tau\in\mathbb{R}_+\} \overset{\mathrm{fdd}}{=} \{c^H Y(\tau,\mu),\, \tau\in\mathbb{R}_+\}. \tag{3.47}$$

Note that the two-parameter process $\{\tilde{\mathcal{Y}}_\beta(\tau,x) := \mathcal{Z}_\beta(\tau, x^{2(1+\beta)})\}$ satisfies the self-similarity property (3.47) with $H = 3/2$.

Another self-similarity property was introduced in Kaj [52]. Accordingly, a process $U = \{U(\tau),\, \tau\in\mathbb{R}_+\}$ is called *aggregate-similar with rigidity index* ρ if for any integer $m \ge 1$,

$$\Big\{\sum_{i=1}^m U^{(i)}(\tau),\, \tau\in\mathbb{R}_+\Big\} \overset{\mathrm{fdd}}{=} \{m^\rho U(\tau/m^\rho),\, \tau\in\mathbb{R}_+\}, \tag{3.48}$$

where $U^{(i)}$, $i\ge 1$, are independent copies of U. Let $\{U_\beta(\tau) := \mathcal{Z}_\beta(\tau^{2/3},1)\}$, then U_β satisfies (3.48) with $\rho := \frac{3}{2(1+\beta)}$, which again follows from (3.45) with $c = m^{1/(1+\beta)}$, $x = 1$ and the fact that $\{\sum_{i=1}^m \mathcal{Z}_\beta^{(i)}(\tau^{2/3},1)\} \overset{\mathrm{fdd}}{=} \{\mathcal{Z}_\beta(\tau^{2/3},m)\}$.

Remark 3.5. Kaj [52], Gaigalas [34] discussed the 'intermediate process'

$$Z_\beta(\tau) := \int_{\mathbb{R}_+\times\mathbb{R}} \Big\{\int_0^\tau \mathbf{1}(s-v<u<s)\mathrm{d}u\Big\}\widetilde{M}(\mathrm{d}v,\mathrm{d}s), \quad \tau\ge 0, \tag{3.49}$$

where $\widetilde{M}(\mathrm{d}v,\mathrm{d}s) = M(\mathrm{d}v,\mathrm{d}s) - \mathrm{E}M(\mathrm{d}v,\mathrm{d}s)$ is a centered Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ with mean $\mathrm{E}M(\mathrm{d}v,\mathrm{d}s) = Cv^{-\beta-2}\mathrm{d}v\mathrm{d}s$ and $0<\beta<1$ is a parameter. The process Z_β in (3.49) arises in the 'intermediate' aggregation regime of ON/OFF and infinite source Poisson models in network traffic. See also [54, 55, 68, 83]. Similarly to (3.43), Z_β extends to a two-parameter random field

$$Z_\beta(\tau,x) := \int_{(0,x]\times\mathbb{R}_+\times\mathbb{R}} \Big\{\int_0^\tau \mathbf{1}(s-v<u<s)\mathrm{d}u\Big\}\widetilde{M}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}s), \quad \tau,x\ge 0, \tag{3.50}$$

where $\widetilde{M}(\mathrm{d}y,\mathrm{d}v,\mathrm{d}s) = M(\mathrm{d}y,\mathrm{d}v,\mathrm{d}s) - \mathrm{d}y\mathrm{E}M(\mathrm{d}v,\mathrm{d}s)$ is a centered Poisson random measure on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}$ with mean $\mathrm{E}M(\mathrm{d}y,\mathrm{d}v,\mathrm{d}s) = \mathrm{d}y\mathrm{E}M(\mathrm{d}v,\mathrm{d}s)$. The random field in (3.50) satisfies the stationary increment property (3.44) and a scaling property similar to (3.45): $\{Z_\beta(c\tau, c^\beta x)\} \overset{\mathrm{fdd}}{=} \{cZ_\beta(\tau,x)\}$ for every $c>0$. These properties might be typical to random fields arising in the 'intermediate regime' of joint temporal and contemporaneous aggregation of independent copies of random processes with long-range dependence. We conjecture that (3.50) and (3.43) can be linked into a general class of Poisson stochastic integrals on the product space $\mathbb{R}_+ \times \mathcal{S}'(\mathbb{R})$, where $\mathcal{S}'(\mathbb{R})$ is the Schwartz space of tempered distributions equipped with a σ-finite shift and scaling invariant product measure, which includes the above mentioned 'intermediate' limits as particular cases and enjoys similar local and global scaling properties.

3.4 Proofs of Theorems 3.1–3.3

Proof of Theorem 3.1. Statement (3.13) follows from Theorems 2.1 and 3.1 in Puplinskaitė and Surgailis [88]. Next, consider (3.14). From [88, Proposition 2.3] we have that for any $-1 < \beta < 0$ and any $n \ge 1$ fixed,

$$N^{-1/2(1+\beta)} S_{N,n}(\tau) \stackrel{\mathrm{fdd}}{\to} [n\tau] V_\beta$$

as $N \to \infty$. Hence, (3.14) immediately follows. In a similar way, (3.15) is a consequence of $(N \log N)^{-1/2} S_{N,n}(\tau) \stackrel{\mathrm{fdd}}{\to} [n\tau] V_0$, or

$$(N \log N)^{-1/2} S_N(t) \stackrel{\mathrm{fdd}}{\to} V_0, \qquad S_N(t) := \sum_{i=1}^{N} X_i(t), \tag{3.51}$$

which is proved below.

Similarly to the rest of this chapter, we use the method of characteristic functions. We shall use the fact that the characteristic function of a standardized r.v. ε has the following representation in a neighborhood of the origin: there exists an $\epsilon > 0$ such that

$$\chi(\theta) := \mathrm{E}\mathrm{e}^{\mathrm{i}\theta\varepsilon} = \mathrm{e}^{-\theta^2 h(\theta)/2} \quad \text{for each } |\theta| < \epsilon, \tag{3.52}$$

where h is a positive function tending to 1 as $\theta \to 0$ (see, e.g., Ibragimov and Linnik [49, Theorem 2.6.5]). Fix $m \in \mathbb{N}$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, then $\sum_{t=1}^m \theta_t X(t) = \sum_{s\in\mathbb{Z}} \vartheta(s,a)\varepsilon(s)$, where $\vartheta(s,a) := \sum_{t=1}^m \theta_t a^{t-s} \mathbf{1}(s \le t)$. Using (3.52) similarly as in [88, pages 519, 521], the convergence (3.51) follows from

$$\begin{aligned} \Psi_N(\boldsymbol{\theta}) &:= N\mathrm{E}\Big[1 - \exp\Big\{ -\frac{1}{2N\log N} \sum_{s\in\mathbb{Z}} (\vartheta(s,a))^2 h\Big(\frac{\vartheta(s,a)}{(N\log N)^{1/2}}\Big)\Big\}\Big] \\ &\to \Psi(\boldsymbol{\theta}) := \frac{\psi_1}{4}\Big|\sum_{t=1}^m \theta_t\Big|^2. \end{aligned} \tag{3.53}$$

Arguing further as in [88, page 521], we reduce the proof of (3.53) to $\Psi_{N1}(\boldsymbol{\theta}) \to \Psi(\boldsymbol{\theta})$, where

$$\begin{aligned}
\Psi_{N1}(\boldsymbol{\theta}) &:= \psi_1 N \int_{1-\epsilon}^1 \Big(1-\exp\Big\{-\frac{1}{2N\log N}\sum_{s\le 0}(\vartheta(s,a))^2 h\Big(\frac{\vartheta(s,a)}{(N\log N)^{1/2}}\Big)\Big\}\Big)\mathrm{d}a \\
&\sim \psi_1 \int_0^{\epsilon N} \Big(1-\exp\Big\{-\frac{|\sum_{t=1}^m \theta_t|^2}{4y\log N}\Big\}\Big)\mathrm{d}y \\
&\sim \psi_1 \Big(O\Big(\frac{K}{\log N}\Big) + \int_{K/\log N}^{\epsilon N}\Big(1-\exp\Big\{-\frac{|\sum_{t=1}^m \theta_t|^2}{4y\log N}\Big\}\Big)\mathrm{d}y\Big) \\
&\sim \frac{\psi_1|\sum_{t=1}^m \theta_t|^2}{4\log N}\int_{K/\log N}^{\epsilon N}\frac{\mathrm{d}y}{y} \\
&\sim \frac{\psi_1|\sum_{t=1}^m \theta_t|^2}{4}
\end{aligned}$$

since $\frac{1}{\log N}\int_{K/\log N}^{\epsilon N} y^{-1}\mathrm{d}y \to 1$ when $\epsilon \to 0$ and $K \to \infty$ together with $N \to \infty$ but slowly enough (so that $\log(1/\epsilon) = o(\log N)$, $\log K = o(\log N)$). This proves (3.51) and (3.15).

It remains to prove (3.16). Let us first show that

$$n^{-1/2}S_{1,n}(\tau) = n^{-1/2}\sum_{t=1}^{[n\tau]} X(t) \stackrel{\mathrm{fdd}}{\to} (1-a)^{-1}B(\tau) \quad \text{as } n\to\infty, \tag{3.54}$$

where B is a Brownian motion as in (3.11) and $a \in [0,1)$ is independent of B and has the same (mixing) distribution in (3.5). Accordingly, it suffices to show that for any fixed $m \in \mathbb{N}$ and any $0 = \tau_0 < \tau_1 < \cdots < \tau_m$, $\boldsymbol{\theta} = (\theta_1,\dots,\theta_m) \in \mathbb{R}^m$,

$$\begin{aligned}
U_n(\boldsymbol{\theta}) &:= \mathrm{E}\exp\Big\{\mathrm{i}\, n^{-1/2}\sum_{j=1}^m \theta_j\,(S_{1,n}(\tau_j) - S_{1,n}(\tau_{j-1}))\Big\} \\
&\to \mathrm{E}\exp\Big\{\mathrm{i}\,(1-a)^{-1}\sum_{j=1}^m \theta_j\,(B(\tau_j) - B(\tau_{j-1}))\Big\} \\
&= \mathrm{E}\exp\Big\{-(1/2)(1-a)^{-2}\sum_{j=1}^m \theta_j^2(\tau_j - \tau_{j-1})\Big\} =: U(\boldsymbol{\theta})
\end{aligned}\tag{3.55}$$

as $n \to \infty$, where $S_{1,n}(0) := 0$. Denote

$$\tilde\vartheta_n(s,a) := \sum_{j=1}^m \theta_j \sum_{t=[n\tau_{j-1}]+1}^{[n\tau_j]} a^{t-s}\mathbf{1}(s\le t).$$

Then $\sum_{j=1}^m \theta_j(S_{1,n}(\tau_j) - S_{1,n}(\tau_{j-1})) = \sum_{s\le[n\tau_m]}\tilde\vartheta_n(s,a)\varepsilon(s)$. Let $A_n := \{a : 0 \le a < 1 - \log n/\sqrt{n}\}$, $A_n^c := [0,1)\setminus A_n$. Note that $\sup_{s\in\mathbb{Z},\,a\in A_n}|\tilde\vartheta_n(s,a)|/\sqrt{n} = O(1/\log n) \to 0$, implying $\sup_{s\in\mathbb{Z},\,a\in A_n}|h(\tilde\vartheta_n(s,a)/\sqrt{n}) - 1| = o(1)$. Using this and (3.52), for n large enough, split $U_n(\boldsymbol{\theta}) = U_{n1}(\boldsymbol{\theta}) + U_{n2}(\boldsymbol{\theta})$, where

$$U_{n1}(\boldsymbol{\theta}) \quad := \quad \mathrm{E}\Big[\exp\Big\{-\frac{1}{2n}\sum_{s\le[n\tau_m]}(\tilde{\vartheta}_n(s,a))^2 h\Big(\frac{\tilde{\vartheta}_n(s,a)}{n^{1/2}}\Big)\Big\}\mathbf{1}(a\in A_n)\Big] \tag{3.56}$$

and $U_{n2}(\boldsymbol{\theta}) := \mathrm{E}\big[\exp\{\mathrm{i}n^{-1/2}\sum_{j=1}^m \theta_j(S_{1,n}(\tau_j) - S_{1,n}(\tau_{j-1}))\}\mathbf{1}(a \in A_n^c)\big]$. Then (3.55) follows from $U_{n1}(\boldsymbol{\theta}) \to U(\boldsymbol{\theta})$ and $U_{n2}(\boldsymbol{\theta}) = o(1)$, where the last relation is immediate from $|U_{n2}(\boldsymbol{\theta})| \le \mathrm{P}(a \in A_n^c) = o(1)$. Using (3.56) and the argument above, the convergence $U_{n1}(\boldsymbol{\theta}) \to U(\boldsymbol{\theta})$ reduces to

$$\lim_{n\to\infty} n^{-1}\sum_{s\le[n\tau_m]}(\tilde{\vartheta}_n(s,a))^2 = (1-a)^{-2}\sum_{j=1}^m(\tau_j-\tau_{j-1})\,\theta_j^2 \tag{3.57}$$

for each $a\in[0,1)$. Relation (3.57) follows by splitting the sum on the l.h.s. of (3.57) as $\sum_{s\le[n\tau_m]} = \sum_{k=0}^m\sum_{[n\tau_{k-1}]<s\le[n\tau_k]}$, $[n\tau_{-1}] := -\infty$, and noting that

$$n^{-1}\sum_{[n\tau_{k-1}]<s\le[n\tau_k]}(\tilde{\vartheta}_n(s,a))^2 \to (1-a)^{-2}\theta_k^2(\tau_k-\tau_{k-1})$$

for $1\le k\le m$ and $n^{-1}\sum_{s\le 0}(\tilde{\vartheta}_n(s,a))^2 \le Cn^{-1}\sum_{s\le 0}(\sum_{t=1}^\infty a^{t-s})^2 \le Cn^{-1}(1-a^2)^{-1}(1-a)^{-2}\to 0$, for each $a\in[0,1)$. This proves (3.57) and (3.54), too.

Let $\mathcal{W} := \{(1-a)^{-1}B(\tau),\ \tau\ge 0\}$ and $\mathcal{W}_i$, $i=1,2,\dots$, be its independent copies. With (3.54) in mind, it remains to prove that

$$N^{-1/(1+\beta)}\sum_{i=1}^N \mathcal{W}_i(\tau) \overset{\mathrm{fdd}}{\to} \mathcal{W}_\beta(\tau). \tag{3.58}$$

For notational simplicity, we restrict the proof of (3.58) to two-dimensional convergence at $0\le\tau_1<\tau_2$, viz.,

$$\begin{aligned} U_N(\theta_1,\theta_2) \quad &:= \quad \mathrm{E}\exp\Big\{N^{-1/(1+\beta)}\Big(\mathrm{i}\,\theta_1\sum_{i=1}^N\mathcal{W}_i(\tau_1) + \mathrm{i}\,\theta_2\sum_{i=1}^N\big(\mathcal{W}_i(\tau_2)-\mathcal{W}_i(\tau_1)\big)\Big)\Big\} \\ &\to \quad \mathrm{E}\exp\big\{\mathrm{i}\theta_1\mathcal{W}_\beta(\tau_1) + \mathrm{i}\theta_2\big(\mathcal{W}_\beta(\tau_2)-\mathcal{W}_\beta(\tau_1)\big)\big\} \qquad (3.59) \\ &= \quad \mathrm{E}\exp\big\{-W_\beta\big(\tau_1\theta_1^2 + (\tau_2-\tau_1)\theta_2^2/2\big)\big\} \\ &= \quad \exp\big\{-k_\beta\big((\tau_1\theta_1^2+(\tau_2-\tau_1)\theta_2^2)/2\big)^{(1+\beta)/2}\big\}. \end{aligned}$$

We have $U_N(\theta_1,\theta_2) = (1-\frac{\Psi_N(\theta_1,\theta_2)}{N})^N$, where

$$\Psi_N(\theta_1,\theta_2) := N\int_0^1\Big(1-\exp\Big\{-\frac{\omega(\theta_1,\theta_2)}{N^{2/(1+\beta)}(1-a)^2}\Big\}\Big)\psi(a)(1-a)^\beta\mathrm{d}a,$$

where $\omega(\theta_1,\theta_2) := (1/2)(\tau_1\theta_1^2 + (\tau_2-\tau_1)\theta_2^2)\ge 0$. From the above expression and assumption (3.5), it easily follows that for any $\epsilon>0$,

$$\begin{aligned} \Psi_N(\theta_1,\theta_2) \quad &\sim \quad \psi_1 N\int_{1-\epsilon}^1\Big(1-\exp\Big\{-\frac{\omega(\theta_1,\theta_2)}{N^{2/(1+\beta)}(1-a)^2}\Big\}\Big)(1-a)^\beta\mathrm{d}a \\ &\to \quad k_\beta(\omega(\theta_1,\theta_2))^{(1+\beta)/2}, \end{aligned}$$

where

$$k_\beta := \frac{\psi_1}{2}\int_0^\infty (1-\mathrm{e}^{-y})\frac{\mathrm{d}y}{y^{(\beta+3)/2}} = \frac{\psi_1}{1+\beta}\Gamma\Big(\frac{1-\beta}{2}\Big). \tag{3.60}$$

This proves (3.59) and Theorem 3.1, too. □

Proof of Theorem 3.2. As in the proof of the previous theorem, we use the method of characteristic functions. For notational convenience we assume that $\varepsilon \overset{\mathrm{d}}{=} \mathcal{N}(0,1)$ or $h(\theta) \equiv 1$ in (3.52), and that $\psi(a) \equiv \psi_1 > 0$, $a \in [0,1)$, in the mixing density (3.5). For the general case, the proof of Theorem 3.2 does not require essential changes.

Case (i), $0 < \beta < 1$ (*proof of* (3.17)). It suffices to prove that for any fixed $m \in \mathbb{N}$, $0 < \tau_1 < \cdots < \tau_m$, and any $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$,

$$\begin{aligned} U_{N,n}(\boldsymbol{\theta}) \quad &:= \quad \mathrm{E}\exp\Big\{\mathrm{i}N^{-1/2}n^{-1+\beta/2}\sum_{j=1}^m \theta_j S_{N,n}(\tau_j)\Big\} \\ &\to \quad \mathrm{E}\exp\Big\{\mathrm{i}\sum_{j=1}^m \theta_j B_{1-(\beta/2)}(\tau_j)\Big\} =: U(\boldsymbol{\theta}), \end{aligned} \tag{3.61}$$

as $N, n, N/n^{1+\beta} \to \infty$. By definition,

$$\begin{aligned} U(\boldsymbol{\theta}) \quad &= \quad \exp\Big\{-\frac{\psi_1}{2}\int_0^\infty K(x)x^\beta \mathrm{d}x\Big\}, \quad \text{where} \\ K(x) \quad &:= \quad \int_{\mathbb{R}} \Big(\sum_{j=1}^m \theta_j\big(\mathfrak{f}(x,\tau_j - s) - \mathfrak{f}(x,-s)\big)\Big)^2 \mathrm{d}s \end{aligned} \tag{3.62}$$

and $\mathfrak{f}$ is given by (3.9). We also have

$$\begin{aligned} U_{N,n}(\boldsymbol{\theta}) \quad &= \quad \Big(\mathrm{E}\prod_{s\in\mathbb{Z}} \exp\Big\{-\frac{1}{2}\Big(\frac{\vartheta_n(s,a)}{N^{1/2}n^{1-\beta/2}}\Big)^2\Big\}\Big)^N \quad \text{with} \\ \vartheta_n(s,a) \quad &:= \quad \sum_{j=1}^m \theta_j \sum_{t=1}^{[n\tau_j]} a^{t-s}\mathbf{1}(s \le t). \end{aligned} \tag{3.63}$$

Then $U_{N,n}(\boldsymbol{\theta}) = (1 - \frac{\Psi_{N,n}(\boldsymbol{\theta})}{N})^N$, where

$$\Psi_{N,n}(\boldsymbol{\theta}) := N\Big[1 - \mathrm{E}\exp\Big\{-\frac{1}{2}\sum_{s\in\mathbb{Z}}\frac{(\vartheta_n(s,a))^2}{Nn^{2-\beta}}\Big\}\Big].$$

Thus (3.61) will be proved once we show that

$$\Psi_{N,n}(\boldsymbol{\theta}) \to \frac{\psi_1}{2}\int_0^\infty K(x)x^\beta \mathrm{d}x \quad \text{as } N, n, N/n^{1+\beta} \to \infty, \quad \text{for all } \boldsymbol{\theta} \in \mathbb{R}^m. \tag{3.64}$$

After a change of variable we obtain

$$\Psi_{N,n}(\boldsymbol{\theta}) = \frac{\psi_1 N}{n^{1+\beta}}\int_0^n \Big(1 - \exp\Big\{-\frac{n^{1+\beta}}{2N}K_{N,n}(x)\Big\}\Big)x^\beta \mathrm{d}x,$$

where

$$\begin{aligned} K_{N,n}(x) &:= \frac{1}{n^3}\sum_{s\in\mathbb{Z}}\Big(\vartheta_n\Big(s,1-\frac{x}{n}\Big)\Big)^2 \\ &= \int_{\mathbb{R}}\Big(\sum_{j=1}^m \theta_j \int_0^{\lceil n\tau_j\rceil/n}\Big(1-\frac{x}{n}\Big)^{\lceil nt\rceil-\lceil ns\rceil}\mathbf{1}(\lceil ns\rceil \le \lceil nt\rceil)\mathrm{d}t\Big)^2\mathrm{d}s \end{aligned}$$

is written as a Riemann sum. By the dominated convergence theorem it follows that

$$K_{N,n}(x) \to \int_{\mathbb{R}}\Big(\sum_{j=1}^m \theta_j \int_0^{\tau_j} \mathrm{e}^{-x(t-s)}\mathbf{1}(s\le t)\mathrm{d}t\Big)^2\mathrm{d}s = K(x) \quad \text{for each } x>0, \tag{3.65}$$

where $K(x)$ is the same as in (3.62). Moreover, the inequality $1-z\le \mathrm{e}^{-z}$, $z\ge 0$, yields $(1-x/n)^{\lceil nt\rceil-\lceil ns\rceil} \le \mathrm{e}^{-x(\lceil nt\rceil-\lceil ns\rceil)/n} \le C\mathrm{e}^{-x(t-s)}$, $x\ge 0$, $t\ge s$, and then

$$|K_{N,n}(x)| \le C\sum_{j=1}^m \int_{\mathbb{R}}\Big(\int_0^{\tau_j}\mathrm{e}^{-x(t-s)}\mathbf{1}(s\le t)\mathrm{d}t\Big)^2\mathrm{d}s =: \bar{K}(x) \tag{3.66}$$

with $\bar{K}(x)$ independent of N and n. We conclude by (3.65) and (3.66) that

$$J_{N,n}(x) := \frac{N}{n^{1+\beta}}\Big(1-\exp\Big\{-\frac{n^{1+\beta}}{2N}K_{N,n}(x)\Big\}\Big) \to \frac{K(x)}{2}$$

for each $x>0$, and that $|J_{N,n}(x)|$ is dominated by the function $\bar{K}(x)\ge 0$ satisfying $\int_0^\infty \bar{K}(x)x^\beta\mathrm{d}x<\infty$. Hence the dominated convergence theorem applies and leads to (3.64) and (3.61). This completes the proof of Case (i) of Theorem 3.2 for $0<\beta<1$.

Case (i), $-1<\beta<0$ (proof of (3.18)). Using the notation in (3.61) it suffices to show that

$$\begin{aligned} U_{N,n}(\boldsymbol{\theta}) &:= \mathrm{E}\exp\Big\{\mathrm{i}N^{-1/2(1+\beta)}n^{-1}\sum_{j=1}^m \theta_j S_{N,n}(\tau_j)\Big\} \\ &\to \mathrm{E}\exp\Big\{\mathrm{i}\Big(\sum_{j=1}^m\theta_j\tau_j\Big)V_\beta\Big\} =: U(\boldsymbol{\theta}) \quad \text{as } N,n,N/n^{1+\beta}\to\infty. \end{aligned} \tag{3.67}$$

Here, $U(\boldsymbol{\theta}) = \exp\{-K_\beta|\sum_{j=1}^m\theta_j\tau_j|^{2(1+\beta)}\}$ and $U_{N,n}(\boldsymbol{\theta}) = (1-\frac{\Psi_{N,n}(\boldsymbol{\theta})}{N})^N$, where

$$\Psi_{N,n}(\boldsymbol{\theta}) := \psi_1 N\int_0^1\Big(1-\exp\Big\{-\frac{1}{2N^{1/(1+\beta)}n^2}\sum_{s\in\mathbb{Z}}(\vartheta_n(s,a))^2\Big\}\Big)(1-a)^\beta\mathrm{d}a.$$

Hence to prove (3.67), it is enough to verify that for any $\boldsymbol{\theta}\in\mathbb{R}^m$

$$\Psi_{N,n}(\boldsymbol{\theta}) \to K_\beta\Big|\sum_{j=1}^m\theta_j\tau_j\Big|^{2(1+\beta)} \quad \text{as } N,n,N/n^{1+\beta}\to\infty. \tag{3.68}$$

We have $\sum_{s\in\mathbb{Z}}(\vartheta_n(s,a))^2 = R_0(a) + R_1(a)$, where

$$\begin{aligned} R_0(a) &:= (1-a^2)^{-1}\Big(\sum_{j=1}^m \theta_j \sum_{t=1}^{[n\tau_j]} a^t\Big)^2, \\ R_1(a) &:= \sum_{s=1}^{[n\tau_m]}\Big(\sum_{j=1}^m \theta_j \mathbf{1}(s\le [n\tau_j])\sum_{t=s}^{[n\tau_j]} a^{t-s}\Big)^2. \end{aligned} \tag{3.69}$$

Clearly, $R_1(a) \le C\min\{n^3, n/(1-a)^2\}$ for any $0\le a<1$. After a change of variable $1-a = N^{-1/(1+\beta)}x$, we get

$$\Psi_{N,n}(\boldsymbol{\theta}) = \psi_1 \int_0^{N^{1/(1+\beta)}} \Big(1-\exp\Big\{-\frac{1}{2}\big(\tilde R_0(x)+\tilde R_1(x)\big)\Big\}\Big)x^\beta \mathrm{d}x, \tag{3.70}$$

where

$$\begin{aligned} \tilde R_0(x) &:= \frac{1}{x(2-N^{-1/(1+\beta)}x)}\Big|\sum_{j=1}^m \theta_j\Big(\frac{1}{n}\sum_{t=1}^{[n\tau_j]}\big(1-N^{-1/(1+\beta)}x\big)^t\Big)\Big|^2 \\ &\to \frac{1}{2x}\Big|\sum_{j=1}^m \theta_j\tau_j\Big|^2 =: \tilde R(x) \end{aligned} \tag{3.71}$$

and

$$\tilde R_1(x) := \frac{R_1(1-N^{-1/(1+\beta)}x)}{N^{1/(1+\beta)}n^2} \le C\min\Big\{\frac{n}{N^{1/(1+\beta)}}, \frac{N^{1/(1+\beta)}}{x^2 n}\Big\} \to 0. \tag{3.72}$$

Write $\Psi_{N,n}(\boldsymbol{\theta}) = \sum_{i=1}^3 \tilde\Psi_i(\boldsymbol{\theta})$, where

$$\begin{aligned} \tilde\Psi_1(\boldsymbol{\theta}) &:= \psi_1\int_0^{N^{1/(1+\beta)}}\big(1-\mathrm{e}^{-\tilde R(x)/2}\big)x^\beta\mathrm{d}x, \\ \tilde\Psi_2(\boldsymbol{\theta}) &= \psi_1\int_0^{N^{1/(1+\beta)}}\big(\mathrm{e}^{-\tilde R(x)/2}-\mathrm{e}^{-\tilde R_0(x)/2}\big)x^\beta\mathrm{d}x, \\ \tilde\Psi_3(\boldsymbol{\theta}) &= \psi_1\int_0^{N^{1/(1+\beta)}}\big(\mathrm{e}^{-\tilde R_0(x)/2}-\mathrm{e}^{-(\tilde R_0(x)+\tilde R_1(x))/2}\big)x^\beta\mathrm{d}x. \end{aligned}$$

Now, relation $\tilde\Psi_1(\boldsymbol{\theta}) \to \psi_1\int_0^\infty(1-\mathrm{e}^{-\tilde R(x)/2})x^\beta\mathrm{d}x = K_\beta|\sum_{j=1}^m\theta_j\tau_j|^{2(1+\beta)}$ follows by the dominated convergence theorem. Relation $\tilde\Psi_2(\boldsymbol{\theta})\to 0$ follows in a similar way, since $H(x) := \mathrm{e}^{-\tilde R(x)/2}-\mathrm{e}^{-\tilde R_0(x)/2} \to 0$ (see (3.71)) and $|H(x)| \le |1-\mathrm{e}^{-\tilde R(x)/2}| + |1-\mathrm{e}^{-\tilde R_0(x)/2}| \le C(|\tilde R(x)|+|\tilde R_0(x)|) \le C\min(1,1/x) =: \bar H(x)$, with $\int_0^\infty \bar H(x)\, x^\beta\mathrm{d}x < \infty$. Finally, $\tilde\Psi_3(\boldsymbol{\theta})\to 0$ follows from the bound (3.72) since $|\tilde\Psi_3(\boldsymbol{\theta})| \le \psi_1\int_0^{N^{1/(1+\beta)}}|1-\mathrm{e}^{-\tilde R_1(x)/2}|x^\beta\mathrm{d}x \le C\int_0^{N^{1/(1+\beta)}}|\tilde R_1(x)|x^\beta\mathrm{d}x \le C\times(\int_0^{N^{1/(1+\beta)}/n}+\int_{N^{1/(1+\beta)}/n}^\infty)|\tilde R_1(x)|x^\beta\mathrm{d}x = O((N^{1/(1+\beta)}/n)^\beta) = o(1)$. This proves (3.68) and (3.18).

Case (i), $\beta=0$ (proof of (3.19)). Following (3.67), (3.68), it suffices to show that

$$\Psi_{N,n}(\boldsymbol{\theta}) \to \frac{\psi_1}{4}\Big(\sum_{j=1}^m\theta_j\tau_j\Big)^2 \quad \text{as } N,n,N/n\to\infty, \tag{3.73}$$

where

$$\Psi_{N,n}(\boldsymbol{\theta}) := \psi_1 N \int_0^1 \Big(1 - \exp\Big\{ - \frac{1}{2n^2 N \log(N/n)} \sum_{s\in\mathbb{Z}} (\vartheta_n(s,a))^2 \Big\}\Big) \mathrm{d}a,$$

$\vartheta_n(s,a)$ being defined in (3.63). By change of variable $1-a = x/N$ arguing as in the proof of (3.68), relation (3.73) follows from

$$\hat{\Psi}_0(\boldsymbol{\theta}) \;:=\; \int_0^N \Big(1 - \exp\Big\{ - \frac{\tilde{R}_0(x)}{2\log(N/n)} \Big\}\Big) \mathrm{d}x \to \frac{1}{4}\Big(\sum_{j=1}^m \theta_j \tau_j\Big)^2, \quad (3.74)$$

$$\hat{\Psi}_1(\boldsymbol{\theta}) \;:=\; \int_0^N \Big(1 - \exp\Big\{ - \frac{\tilde{R}_1(x)}{2\log(N/n)} \Big\}\Big) \mathrm{d}x \to 0, \quad (3.75)$$

where $\tilde{R}_0(x), \tilde{R}_1(x)$ are the same as in (3.70) with $\beta = 0$.

In order to simplify the exposition and notation, we restrict the subsequent proof of (3.74) to the one-dimensional case $m = \tau = 1$, $\boldsymbol{\theta} = \theta \in \mathbb{R}$. From definition in (3.71) we have $\tilde{R}_0(x) = Q_1(x) + Q_2(x)$, where

$$Q_1(x) \;:=\; \frac{\theta^2}{2x}\Big(\frac{1}{n}\sum_{t=1}^n \Big(1 - \frac{x}{N}\Big)^t\Big)^2,$$

$$Q_2(x) \;:=\; \frac{\theta^2}{2N(2-(x/N))}\Big(\frac{1}{n}\sum_{t=1}^n \Big(1 - \frac{x}{N}\Big)^t\Big)^2 \le \frac{C}{N}.$$

Since $\int_0^N (1 - \exp\{-Q_2(x)/(2\log(N/n))\})\mathrm{d}x \le \frac{C}{\log(N/n)} \int_0^N |Q_2(x)|\mathrm{d}x = O(\frac{1}{\log(N/n)})$ $= o(1)$, it suffices to show (3.74) with $\tilde{R}_0(x)$ replaced by $Q_1(x)$, viz.,

$$\Phi(\theta) := \int_0^N \Big(1 - \exp\Big\{ - \frac{\theta^2}{4x\log(N/n)} \Big(\frac{1}{n}\sum_{t=1}^n \Big(1 - \frac{x}{N}\Big)^t\Big)^2 \Big\}\Big) \mathrm{d}x \to \frac{\theta^2}{4}. \quad (3.76)$$

Rewrite $\Phi(\theta) = \frac{1}{\log(N/n)} \int_{(N\log(N/n))^{-1}}^\infty \Gamma_{N,n}(y) \frac{\mathrm{d}y}{y} = \sum_{i=1}^3 \Phi_i(\theta)$, where

$$\Gamma_{N,n}(y) := \frac{1}{y}\big(1 - \mathrm{e}^{-\theta^2 y \Lambda_{N,n}(y)/4}\big), \quad \Lambda_{N,n}(y) := \Big(\frac{1}{n}\sum_{t=1}^n \Big(1 - \frac{1}{yN\log(N/n)}\Big)^t\Big)^2$$

and $\Phi_1(\theta) := \frac{1}{\log(N/n)} \int_{1/(N\log(N/n))}^{n/N} \Gamma_{N,n}(y) \frac{\mathrm{d}y}{y}$, $\Phi_2(\theta) := \frac{1}{\log(N/n)} \int_{n/N}^1 \Gamma_{N,n}(y) \frac{\mathrm{d}y}{y}$, $\Phi_3(\theta) := \frac{1}{\log(N/n)} \int_1^\infty \Gamma_{N,n}(y) \frac{\mathrm{d}y}{y}$. We have

$$\begin{aligned}
\Phi_1(\theta) \;&\le\; \frac{C}{\log(N/n)} \int_{1/(N\log(N/n))}^{n/N} \Lambda_{N,n}(y) \frac{\mathrm{d}y}{y} \\
&=\; \frac{C}{\log(N/n)} \int_{1/(N\log(N/n))}^{n/N} \Big(\frac{1}{n}\sum_{t=1}^n \Big(1 - \frac{1}{yN\log(N/n)}\Big)^t\Big)^2 \frac{\mathrm{d}y}{y} \\
&=\; \frac{C}{n^2\log(N/n)} \sum_{t,s=1}^n \ell_{N,n}(t+s), \quad \text{with} \\
\ell_{N,n}(k) \;&:=\; \int_{1/(N\log(N/n))}^{n/N} \Big(1 - \frac{1}{yN\log(N/n)}\Big)^k \frac{\mathrm{d}y}{y}.
\end{aligned}$$

Using $1-x \le \mathrm{e}^{-x}$, $x \ge 0$, we obtain

$$\begin{aligned} \ell_{N,n}(k) &\le \int_{1/(N\log(N/n))}^{n/N} \exp\Big\{-\frac{k}{yN\log(N/n)}\Big\}\frac{\mathrm{d}y}{y} \\ &\le C\log\Big(\frac{n\log(N/n)}{k}\Big), \quad 1 \le k \le 2n. \end{aligned} \tag{3.77}$$

Indeed, by change of variable $z := \frac{k}{yN\log(N/n)}$, $z^{-1}\mathrm{d}z = -y^{-1}\mathrm{d}y$ the integral in (3.77) for $N\log(N/n) > k$ can be rewritten as $\int_{k/n\log(N/n)}^{k}\mathrm{e}^{-z}z^{-1}\mathrm{d}z = J_1 + J_2$, where $J_1 := \int_1^k \mathrm{e}^{-z}z^{-1}\mathrm{d}z \le C$, $J_2 := \int_{k/n\log(N/n)}^1 \mathrm{e}^{-z}z^{-1}\mathrm{d}z \le \int_{k/n\log(N/n)}^1 z^{-1}\mathrm{d}z = \log(\frac{n\log(N/n)}{k})$, proving (3.77). Using (3.77) we obtain

$$\begin{aligned} \Phi_1(\theta) &\le \frac{C}{n^2\log(N/n)}\sum_{t,s=1}^{n}\log\Big(\frac{n\log(N/n)}{t+s}\Big) \\ &\le \frac{C}{n^2\log(N/n)}\sum_{k=1}^{2n}k\log\Big(\frac{2n\log(N/n)}{k}\Big) \\ &= \frac{C}{n^2\log(N/n)}\sum_{k=1}^{2n}k\log\Big(\frac{2n}{k}\Big) + \frac{C\log\log(N/n)}{\log(N/n)}n^{-2}\sum_{k=1}^{2n}k \\ &= O\Big(\frac{\log\log(N/n)}{\log(N/n)}\Big) = o(1), \end{aligned}$$

since $\sum_{k=1}^{2n}k\log(\frac{2n}{k}) \le \int_1^{2n}x\log(\frac{2n}{x})\mathrm{d}x \le Cn^2$. Clearly, $\Gamma_{N,n}(y) \le y^{-1}$, implying $\Phi_3(\theta) = O(\frac{1}{\log(N/n)}) = o(1)$. Hence, (3.76) follows from $\Phi_2(\theta) \to \theta^2/4$. To show the last relation, split $\Phi_2(\theta) = \Phi_{21}(\theta) + \Phi_{22}(\theta)$, where

$$\Phi_{21}(\theta) := \frac{1}{\log(N/n)}\int_{n/N}^1 G(y)\frac{\mathrm{d}y}{y}, \qquad \Phi_{22}(\theta) := \frac{1}{\log(N/n)}\int_{n/N}^1[\Gamma_{N,n} - G(y)]\frac{\mathrm{d}y}{y}$$

and $G(y) := \frac{1}{y}(1-\mathrm{e}^{-(\theta^2/4)y})$. Using the facts that $G(n/N) - \theta^2/4 = o(1)$, $\int_{n/N}^1\frac{\mathrm{d}y}{y} = \log(N/n)$ and $\sup_{y\in(0,1]}|G'(y)| < C$, we obtain

$$\begin{aligned} |\Phi_{21}(\theta) - \theta^2/4| &\le |G(n/N) - \theta^2/4| + \frac{1}{\log(N/n)}\Big|\int_{n/N}^1(G(y) - G(n/N))\frac{\mathrm{d}y}{y}\Big| \\ &= o(1) + \frac{1}{\log(N/n)}\Big|\int_{n/N}^1 G'(y)(\log y)\mathrm{d}y\Big| = o(1). \end{aligned}$$

Next, consider $\Phi_{22}(\theta)$. Note that $\Lambda_{N,n}(y)$ is monotone in $y \in (0,1]$, hence $\Lambda_{N,n}(n/N) \le \Lambda_{N,n}(y) \le \Lambda_{N,n}(1) \le 1$, $n/N \le y \le 1$. Moreover, $\Lambda_{N,n}(n/N) = (\frac{1}{n}\sum_{t=1}^n(1-\frac{1}{n\log(N/n)})^t)^2 \to 1$ follows from $(1-\frac{1}{n\log(N/n)})^t \to 1$, $1 \le t \le n$. Using these facts, we obtain

$$\begin{aligned} \Phi_{22}(\theta) &= \frac{1}{\log(N/n)}\int_{n/N}^1\Big\{\frac{1}{y}\mathrm{e}^{-(\theta^2/4)y\Lambda_{N,n}(y)}\Big[1-\mathrm{e}^{-(\theta^2/4)y(1-\Lambda_{N,n}(y))}\Big]\Big\}\frac{\mathrm{d}y}{y} \\ &\le \frac{C}{\log(N/n)}\int_{n/N}^1(1-\Lambda_{N,n}(y))\frac{\mathrm{d}y}{y} \\ &\le C(1-\Lambda_{N,n}(n/N)) = o(1). \end{aligned}$$

It remains to show (3.75). Using $0 \le \tilde{R}_1(x) \le C\min(n/N, N/nx^2)$, see (3.72), it follows that $\hat{\Psi}_1(\boldsymbol{\theta}) \le \frac{C}{\log(N/n)}\int_0^N \tilde{R}_1(x)\mathrm{d}x$, where the last integral is bounded by a finite constant independent of N, n. This proves (3.75) and completes the proof of (3.19), too.

Case (ii), $-1 < \beta < 1$ (*proof of* (3.20)). Because of similarities with the proofs of (3.17) and (3.18), we restrict the proof to the one-dimensional convergence at $\tau = 1$. We have $\mathrm{E}\exp\{\mathrm{i}\theta N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(1)\} = (1 - \frac{\Psi_{N,n}(\theta)}{N})^N$, where

$$\Psi_{N,n}(\theta) := \psi_1 N\int_0^1 \Big(1 - \exp\Big\{-\frac{\theta^2(R_0(a)+R_1(a))}{2N^{2/(1+\beta)}n}\Big\}\Big)(1-a)^\beta\mathrm{d}a$$

and where $R_0(a) := \sum_{s\le 0}(\vartheta_n(s,a))^2 \le (1-a)^{-3}$, $R_1(a) := \sum_{s=1}^n(\vartheta_n(s,a))^2 = (1-a)^{-2}\sum_{k=1}^n(1-a^k)^2$. By the change of variables $1-a = N^{-1/(1+\beta)}x$, $\Psi_{N,n}(\theta)$ can be rewritten as

$$\Psi_{N,n}(\theta) = \psi_1\int_0^{N^{1/(1+\beta)}} \Big(1 - \exp\Big\{-\frac{\theta^2}{2}\Big(\tilde{R}_0(x) + \frac{\tilde{R}_1(x)}{x^2}\Big)\Big\}\Big)x^\beta\mathrm{d}x,$$

where

$$\begin{aligned}\tilde{R}_0(x) &:= \frac{1}{N^{2/(1+\beta)}n}R_0\Big(1 - \frac{x}{N^{1/(1+\beta)}}\Big) \le \frac{N^{3/(1+\beta)}}{x^3N^{2/(1+\beta)}n} \le \frac{N^{1/(1+\beta)}}{x^3 n} \to 0,\\ \tilde{R}_1(x) &:= \frac{1}{n}\sum_{k=1}^n\Big(1 - \Big(1 - \frac{x}{N^{1/(1+\beta)}}\Big)^k\Big)^2 \to 1.\end{aligned}$$

The above facts entail

$$\Psi_{N,n}(\theta) \to \psi_1\int_0^\infty (1 - \mathrm{e}^{-\theta^2/2x^2})x^\beta\mathrm{d}x = \frac{k_\beta|\theta|^{1+\beta}}{2^{(1+\beta)/2}} = -\log\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\mathcal{W}_\beta},$$

hence also the proof of (3.20).

Case (iii), $-1 < \beta < 1$ (*proof of* (3.21)). Similarly as above, it suffices to prove that for any $\boldsymbol{\theta} \in \mathbb{R}^m$,

$$\Psi_{N,n}(\boldsymbol{\theta}) \to \Psi_\mu(\boldsymbol{\theta}) \quad \text{as } N, n \to \infty,\ N^{1/(1+\beta)}/n \to \mu \in (0,\infty), \tag{3.78}$$

where

$$\Psi_\mu(\boldsymbol{\theta}) := -\log\mathrm{E}\exp\Big\{\mathrm{i}\sum_{j=1}^m \theta_j\mu^{1/2}\mathcal{Z}_\beta(\tau_j/\mu)\Big\}$$

$$= \psi_1\int_0^\infty\Big(1 - \exp\Big\{-\frac{1}{2\mu^2}\int_{\mathbb{R}}\Big(\sum_{j=1}^m\theta_j\big(\mathfrak{f}(x/\mu,\tau_j - s) - \mathfrak{f}(x/\mu,-s)\big)\Big)^2\mathrm{d}s\Big\}\Big)x^\beta\mathrm{d}x,$$

see (3.12), and

$$\Psi_{N,n}(\boldsymbol{\theta}) := \psi_1 N\int_0^1\Big(1 - \exp\Big\{-\frac{R_0(a)+R_1(a)}{2N^{2/(1+\beta)}n}\Big\}\Big)(1-a)^\beta\mathrm{d}a,$$

with $R_0(a)$, $R_1(a)$ defined in (3.69). By change of variable $1-a = N^{-1/(1+\beta)}x$ we obtain

$$\Psi_{N,n}(\boldsymbol{\theta}) = \psi_1 \int_0^{N^{1/(1+\beta)}} \Big(1 - \exp\Big\{ - (1/2)\big(\tilde{R}_0(x) + \tilde{R}_1(x)\big)\Big\}\Big) x^\beta \mathrm{d}x,$$

where

$$\begin{aligned}
\tilde{R}_0(x) &:= \frac{N^{1/(1+\beta)}(1-N^{-1/(1+\beta)}x)^2}{nx^3(2-N^{-1/(1+\beta)}x)} \Big(\sum_{j=1}^m \theta_j \Big(1 - \Big(1 - \frac{x}{N^{1/(1+\beta)}}\Big)^{[n\tau_j]}\Big)\Big)^2,\\
\tilde{R}_1(x) &:= \frac{1}{x^2 n} \sum_{s=1}^{[n\tau_m]} \Big(\sum_{j=1}^m \theta_j \Big(1 - \Big(1 - \frac{x}{N^{1/(1+\beta)}}\Big)^{[n\tau_j]-s+1}\Big) \mathbf{1}(s \le [n\tau_j])\Big)^2.
\end{aligned}$$

It is easy to verify that for each $x > 0$, $\tilde{R}_0(x) \to K_0(x)$, $\tilde{R}_1(x) \to K_1(x)$, where

$$\begin{aligned}
K_0(x) &:= \frac{\mu}{2x^3}\Big(\sum_{j=1}^m \theta_j (1 - \mathrm{e}^{-(x/\mu)\tau_j})\Big)^2\\
&= \mu^{-2} \int_{-\infty}^0 \Big(\sum_{j=1}^m \theta_j \big(\mathfrak{f}(x/\mu, \tau_j - s) - \mathfrak{f}(x/\mu, -s)\big)\Big)^2 \mathrm{d}s,\\
K_1(x) &:= x^{-2} \int_0^{\tau_m} \Big(\sum_{j=1}^m \theta_j (1 - \mathrm{e}^{-(x/\mu)(\tau_j - s)}) \mathbf{1}(s \le \tau_j)\Big)^2 \mathrm{d}s\\
&= \mu^{-2} \int_0^{\tau_m} \Big(\sum_{j=1}^m \theta_j \mathfrak{f}(x/\mu, \tau_j - s)\Big)^2 \mathrm{d}s.
\end{aligned}$$

Note $\Psi_\mu(\boldsymbol{\theta}) = \psi_1 \int_0^\infty [1 - \exp\{-(1/2)(K_0(x) + K_1(x))\}] x^\beta \mathrm{d}x$. The convergence (3.78) now follows by the dominated convergence theorem using a similar argument as in the proof of Theorem 3.1 in [88]. This proves (3.21) and thereby completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. The proof is analogous to that of the previous theorem. Let $S_{N,n} := S_{N,n}(1)$. We prove only one-dimensional convergence at $\tau = 1$, or

$$U_{N,n}(\theta) := \mathrm{E}\mathrm{e}^{\mathrm{i}\theta S_{N,n}/(Nn)^{1/2}} = \Big(1 - \frac{\Psi_{N,n}(\theta)}{N}\Big)^N \to \mathrm{e}^{-\theta^2\sigma^2/2}, \tag{3.79}$$

where

$$\begin{aligned}
\Psi_{N,n}(\theta) &:= N\mathrm{E}\Big[1 - \mathrm{e}^{\mathrm{i}\theta S_{1,n}/(Nn)^{1/2}}\Big] = N\mathrm{E}\Big[1 - \prod_{s \le n} \chi\Big(\theta \frac{\vartheta_n(s,a)}{(Nn)^{1/2}}\Big)\Big],\\
\vartheta_n(s,a) &:= \sum_{t=1}^n a^{t-s} \mathbf{1}(s \le t),
\end{aligned}$$

χ being the characteristic function of i.i.d. innovations $\{\varepsilon(s)\}$, see (3.52). Let $A_n := \{a : 0 \le a < 1 - \log n/\sqrt{n}\}$, $A_n^c := [0,1) \setminus A_n$ similarly to the proof of

Theorem 3.1. Accordingly, split $\Psi_{N,n}(\theta) = \Psi'_{N,n}(\theta) + \Psi''_{N,n}(\theta)$, where

$$\begin{aligned}
\Psi'_{N,n}(\theta) &:= N\mathrm{E}\left[1 - \mathrm{e}^{\mathrm{i}\theta S_{1,n}/(Nn)^{1/2}}\right]\mathbf{1}(a \in A_n),\\
\Psi''_{N,n}(\theta) &:= N\mathrm{E}\left[1 - \mathrm{e}^{\mathrm{i}\theta S_{1,n}/(Nn)^{1/2}}\right]\mathbf{1}(a \in A_n^c).
\end{aligned}$$

Since $|\Psi''_{N,n}(\theta)| = N|\mathrm{E}\mathrm{E}[1 - \mathrm{e}^{\mathrm{i}\theta S_{1,n}/(Nn)^{1/2}}|a]\mathbf{1}(a \in A_n^c)|$ and $\mathrm{E}[S_{1,n}|a] = 0$, $\mathrm{E}[S_{1,n}^2|a] = \sum_{s\le n}(\vartheta_n(s,a))^2$ satisfies $\sum_{s\le n}(\vartheta_n(s,a))^2 \le 2n/(1-a)^2$, we obtain

$$\begin{aligned}
\left|\Psi''_{N,n}(\theta)\right| &\le N(\theta^2/2)\mathrm{E}\Big[N^{-1}n^{-1}\sum_{s\le n}(\vartheta_n(s,a))^2\mathbf{1}(a \in A_n^c)\Big]\\
&\le \theta^2\mathrm{E}\left[(1-a)^{-2}\mathbf{1}(a \in A_n^c)\right] = O\left((\log n/\sqrt{n})^{\beta-1}\right) = o(1)
\end{aligned}$$

due to $\beta > 1$. Finally, (3.79) follows from

$$\begin{aligned}
\Psi'_{N,n}(\theta) &= N\mathrm{E}\Big[1 - \exp\Big\{-\frac{\theta^2}{2Nn}\sum_{s\le n}(\vartheta_n(s,a))^2 h\Big(\frac{\theta\vartheta_n(s,a)}{(Nn)^{1/2}}\Big)\Big\}\Big]\mathbf{1}(a \in A_N)\\
&\to \frac{\theta^2\sigma^2}{2},
\end{aligned}$$

by (3.55) and by Taylor expansion of the exponent in a standard way. This both proves (3.79) and Theorem 3.3. □

As a final remark, let us note that the above proof does not require (3.5) and the conclusion of Theorem 3.3 remains valid under the more general condition $\mathrm{E}(1-a)^{-2} < \infty$.

Chapter 4

Aggregation of AR(1) processes with common innovations

This chapter contains the article [80]. We discuss joint temporal and contemporaneous aggregation of N copies of stationary random-coefficient AR(1) processes with common i.i.d. standardized innovations, when N and time scale n increase at different rate. Assuming that the random coefficient a has a density, regularly varying at $a = 1$ with exponent $-1/2 < \beta < 0$, different joint limits of normalized aggregated partial sums are shown to exist when $N^{1/(1+\beta)}/n$ tends to (i) ∞, (ii) 0, (iii) $0 < \mu < \infty$. We extend the results of Chapter 3 from the case of idiosyncratic innovations to the case of common innovations.

4.1 Introduction

Let $X_i := \{X_i(t),\, t \in \mathbb{Z}\}$, $i = 1, \dots, N$, be stationary random-coefficient AR(1) processes

$$X_i(t) = a_i X_i(t-1) + \varepsilon(t), \quad t \in \mathbb{Z}, \tag{4.1}$$

with *common* standardized i.i.d. innovations $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ and i.i.d. random coefficients $a_i \in (-1, 1), i = 1, \dots, N$, independent of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$. Consider the double sum

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0, \tag{4.2}$$

representing joint temporal and contemporaneous aggregate of N individual AR(1) evolutions (4.1) at time scale n. We discuss the limit distribution of appropriately normalized double sums $S_{N,n}$ in (4.2) as N, n jointly increase to infinity, possibly at a *different rate*. Throughout this chapter we suppose that the distribution of generic coefficient $a \in (-1, 1)$ in (4.1), or the mixing distribution, satisfies the following two assumptions.

Assumption (A1). There exist $\beta > -1$ and $\epsilon \in (0, 1)$ such that $\mathrm{P}(a \le x)$ is differentiable on $(1-\epsilon, 1)$ with derivative

$$\mathrm{dP}(a \le x)/\mathrm{d}x = (1-x)^{\beta}\psi(x), \quad x \in (1-\epsilon, 1), \tag{4.3}$$

where ψ is bounded on $(1-\epsilon, 1)$ and continuous at $x = 1$ with $\psi_1 := \lim_{x\to 1}\psi(x) > 0$.

Assumption (A2). $\mathrm{E}(1+a)^{-1/2} < \infty$.

Assumptions (A1) and (A2) refer to the behavior of the mixing distribution in the vicinity of $a = 1$ and $a = -1$, respectively (the positive and negative unit roots of generic AR(1) process $X = X_i$ in (4.1)). Because of oscillation of the moving-average coefficients of X when $a < 0$, the behavior of the mixing distribution near $a = -1$ is generally less important for partial sums processes than its behavior near $a = 1$, the crucial role being played by the parameter β in (4.3). Assumption (A1) is similar to (3.5) on page 20 and [87, 88, 111], although the 'typical' range of β is different in the aggregation schemes with common and idiosyncratic innovations. The random-coefficient AR(1) process X has finite variance if and only if $\mathrm{E}X^2(t) = \mathrm{E}\sum_{s\le t} a^{2(t-s)} = \mathrm{E}(1-a^2)^{-1} < \infty$, which implies $\beta > 0$ in (4.3). It is well-known that under the condition (4.3) with $0 < \beta < 1$ (and $a \in [0, 1)$ a.s.), X has long memory in the sense that its covariance decays as $\mathrm{Cov}(X(0), X(t)) = O(t^{-\beta})$, $t \to \infty$, so that $\sum_{t=0}^{\infty} |\mathrm{Cov}(X(0), X(t))| = \infty$. Zaffaroni [111], Puplinskaitė and Surgailis [87] discussed the existence and long memory properties of the limit (in probability) $\mathcal{X}(t) := \lim_{N\to\infty} N^{-1}\sum_{i=1}^{N} X_i(t)$, $t \in \mathbb{Z}$, of aggregated AR(1) processes X_i in (4.1), written as a moving-average $\mathcal{X}(t) = \sum_{j=0}^{\infty} g(j)\varepsilon(t-j)$ with (deterministic) coefficients $g(j) := \mathrm{E}[a^j]$, $j \ge 0$. For $-1/2 < \beta < 0$ in (4.3) and under similar condition on the mixing distribution near $a = -1$, the coefficients $g(j) \sim \Gamma(1+\beta)j^{-\beta-1}$, $j \to \infty$, and the (normalized) partial sum process of $\mathcal{X}$ tends to a fractional Brownian motion with parameter $H = (1/2) - \beta \in (1/2, 1)$, see [87, Propositions 2 and 4]. We recall that Granger [40] proposed the scheme of contemporaneous aggregation of heterogeneous random-coefficient AR(1) processes as a possible explanation of the long memory phenomenon in macroeconomic time series. Subsequently, large-scale contemporaneous aggregation of linear and heteroscedastic heterogeneous time series models was studied in [19, 37, 39, 74, 79, 87, 88, 111, 112] and other papers.

Let us describe the main results of the present chapter. Assume that the mixing density satisfies Assumptions (A1) and (A2) with $-1/2 < \beta < 0$ and N, n

increase simultaneously so as

$$\frac{N^{1/(1+\beta)}}{n} \to \mu \in [0, \infty], \tag{4.4}$$

leading to the three cases (i)–(iii):

$$\text{Case (i): } \mu = \infty, \quad \text{Case (ii): } \mu = 0, \quad \text{Case (iii): } 0 < \mu < \infty. \tag{4.5}$$

Our main result is Theorem 4.3 of Section 4.2 which states that the 'simultaneous limit' of $S_{N,n}(\tau)$ exists in the sense of weak convergence of finite-dimensional distributions, and is different in all three Cases (i)–(iii), namely,

$$N^{-1} n^{\beta-(1/2)} S_{N,n}(\tau) \overset{\text{fdd}}{\to} \sigma_\beta B_{(1/2)-\beta}(\tau) \qquad \text{in Case (i),} \tag{4.6}$$

$$N^{-1/(1+\beta)} n^{-1/2} S_{N,n}(\tau) \overset{\text{fdd}}{\to} W_\beta B(\tau) \qquad \text{in Case (ii),} \tag{4.7}$$

$$N^{-1/(1+\beta)} n^{-1/2} S_{N,n}(\tau) \overset{\text{fdd}}{\to} \mu^{1/2} Z_\beta(\tau/\mu) \qquad \text{in Case (iii).} \tag{4.8}$$

Here, $B_{(1/2)-\beta}$ is a standard fractional Brownian motion with Hurst parameter $H = (1/2) - \beta$, σ_β is a constant defined in Proposition 4.2(ii), $W_\beta > 0$ is a $(1+\beta)$-stable r.v. independent of a standard Brownian motion B, and Z_β is an 'intermediate process' defined as the double stochastic integral

$$Z_\beta(\tau) := \int_{\mathbb{R}\times\mathbb{R}_+} \Big\{ \int_0^\tau \mathrm{e}^{-x(u-s)} \mathbf{1}(s \le u) \mathrm{d}u \Big\} \mathrm{d}B(s)\, N(\mathrm{d}x), \quad \tau \ge 0, \tag{4.9}$$

where $N = \{N(\mathrm{d}x),\, x \in \mathbb{R}_+\}$ is a Poisson random measure on $\mathbb{R}_+ := (0, \infty)$ with intensity $\nu(\mathrm{d}x) := \mathrm{E}N(\mathrm{d}x) := \psi_1 x^\beta \mathrm{d}x$, independent of standard Brownian motion B. The existence of the process Z_β in (4.9) and its properties are discussed in Section 4.2. In particular, we show that Z_β can be regarded as a 'bridge' between the limit processes in Cases (i) and (ii), in the sense that Z_β behaves as $B_{(1/2)-\beta}$ at 'small scales' and as $W_\beta B$ at 'large scales'. See Proposition 4.2 for rigorous formulation.

This chapter extends the previous one (based on [79]), where a similar problem was discussed for stationary random-coefficient AR(1) processes $Y_i = \{Y_i(t),\, t \in \mathbb{Z}\}$, $i = 1, \dots, N$, with *independent* (or idiosyncratic) innovations:

$$Y_i(t) = a_i Y_i(t-1) + \varepsilon_i(t), \quad t \in \mathbb{Z},$$

where $\{\varepsilon_i(t),\, t \in \mathbb{Z}\}$ are independent copies of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ in (4.1), independent of $a_i \in [0, 1)$, $i = 1, \dots, N$. Let $\mathcal{S}_{N,n}(\tau) := \sum_{i=1}^N \sum_{t=1}^{[n\tau]} Y_i(t)$, $\tau \ge 0$, be the analogue of $S_{N,n}(\tau)$ in (4.2). In Theorem 3.2 on page 24 under Assumption (A1) with $-1 < \beta < 1$ and N, n increasing as in (4.4), we obtained joint limits of $\mathcal{S}_{N,n}(\tau)$ in the respective Cases (i)–(iii) of (4.5). Namely, the limit process of $\mathcal{S}_{N,n}(\tau)$ in Case (i) is a fractional Brownian motion similarly to (4.6), but the limits of $\mathcal{S}_{N,n}(\tau)$ in Cases (ii) and (iii) differ from (4.7) and (4.8). In particular, the 'intermediate process' $\mathcal{Z}_\beta$ in Chapter 3 arising under Case (iii) is written as a 'Poisson mixture' of integrated Ornstein-Uhlenbeck (O-U) processes $\int_{-\infty}^{\tau}\{\int_0^{\tau} \mathrm{e}^{-x(u-s)}\mathbf{1}(s \le u)\mathrm{d}u\}\mathrm{d}B(s)$ on the product space $\mathbb{R}_+ \times C(\mathbb{R})$ equipped with the measure $\nu(\mathrm{d}x) \times \mathrm{P}_B$, with P_B being the Wiener measure on $C(\mathbb{R})$, while in the representation (4.9) of Z_β, these O-U processes are 'mixed' w.r.t. x only. These differences are due to different dependence structure between summands X_i and Y_i in the common and idiosyncratic aggregation schemes: the Y_i's are mutually independent processes while the X_i's are strongly interdependent due to common innovations.

The results of Chapter 3 and the present chapter are related to the study of joint limits of the aggregated input in network traffic models, see [27,34,35,55,70, 105] and references therein. See Chapter 3 for a discussion of the relation between AR(1) and network traffic aggregation schemes and their limit processes.

4.2 Main results

For $-1/2 < \beta < 0$, define a standard fractional Brownian motion $B_{(1/2)-\beta}$ with Hurst index $H = (1/2) - \beta \in (1/2, 1)$ as stochastic integral

$$B_{(1/2)-\beta}(\tau) := C_\beta^{-1} \int_{-\infty}^{\tau} \big((\tau - s)^{-\beta} - (-s)_+^{-\beta}\big)\mathrm{d}B(s), \quad \tau \ge 0, \tag{4.10}$$

w.r.t. a standard Brownian motion B, where $C_\beta^2 := -\beta \mathrm{B}(-\beta, 1+2\beta)/(1-2\beta) = \int_{-\infty}^{1}((1-s)^{-\beta} - (-s)_+^{-\beta})^2 \mathrm{d}s$. Note that $\mathrm{E}B^2_{(1/2)-\beta}(\tau) = \tau^{1-2\beta}$. See [36, page 545].

Next, let $W_\beta B := \{W_\beta B(\tau), \tau \ge 0\}$, $-1/2 < \beta < 0$, where $W_\beta > 0$ is a completely asymmetric $(1+\beta)$-stable r.v., independent of standard Brownian motion $B = \{B(\tau), \tau \ge 0\}$ and having the log-Laplace transform $\log \mathrm{E}\mathrm{e}^{-\theta W_\beta} = \psi_1 \int_0^\infty (\mathrm{e}^{-\theta/x} - 1)x^\beta \mathrm{d}x = -\psi_1(\Gamma(-\beta)/(1+\beta))\theta^{1+\beta}$, $\theta \ge 0$. Note, the process $W_\beta B$ has stationary increments and is self-similar with index 1/2.

Proposition 4.2 details the third limit process arising under (4.4)–(4.5). Before that, we discuss the double stochastic integral w.r.t. Gaussian and Poisson random measures.

Let $N = \{N(\mathrm{d}x), x \in \mathbb{R}_+\}$ be a Poisson random measure on $\mathbb{R}_+$ with intensity $\nu(\mathrm{d}x) := \mathrm{E}N(\mathrm{d}x) := \psi_1 x^\beta \mathrm{d}x$, $-1/2 < \beta < 0$, independent of a standard Brownian motion $B = \{B(s), s \in \mathbb{R}\}$. Let $\widetilde{N}(\mathrm{d}x) = N(\mathrm{d}x) - \nu(\mathrm{d}x)$ be the centered Poisson random measure. Let L_p ($p \ge 1$) be the space of all r.v.s ξ measurable w.r.t. the σ-field generated by N and B and such that $\mathrm{E}|\xi|^p < \infty$. Write $\mathrm{E} = \mathrm{E}_N \times \mathrm{E}_B$, where $\mathrm{E}_N, \mathrm{E}_B$ refer to expectation w.r.t. N, B only. For $1 \le p \le 2$, let $\mathcal{L}_p(\mathbb{R}_+ \times \mathbb{R})$ denote the Banach space of all measurable real-valued functions $h = h(x,s)$, $(x,s) \in \mathbb{R}_+ \times \mathbb{R}$ such that

$$\|h\|_{\mathcal{L}_p} := \Big(\int_{\mathbb{R}_+} \Big\{ \int_{\mathbb{R}} h^2(x,s)\mathrm{d}s \Big\}^{p/2} \nu(\mathrm{d}x) \Big)^{1/p} + \int_{\mathbb{R}_+} \Big\{ \int_{\mathbb{R}} h^2(x,s)\mathrm{d}s \Big\}^{1/2} \nu(\mathrm{d}x) < \infty.$$

Let $\mathcal{L}_0(\mathbb{R}_+ \times \mathbb{R})$ consist of all (step) functions $h = h(x,s)$ taking a finite number of non-zero values $h_n(k,j)$ on squares $(k/n,(k+1)/n] \times (j/n,(j+1)/n] \subset \mathbb{R}_+ \times \mathbb{R}$, $k = 0,1,\dots, j = 0, \pm 1, \pm 2, \dots$ for some $n = 1,2,\dots$. For such $h \in \mathcal{L}_0(\mathbb{R}_+ \times \mathbb{R})$, define the double stochastic integral $I(h) \equiv \int_{\mathbb{R}_+ \times \mathbb{R}} h(x,s)N(\mathrm{d}x)\mathrm{d}B(s)$ as a sum

$$I(h) := \sum_{k,j} h_n(k,j) N((k/n,(k+1)/n]) B((j/n,(j+1)/n]), \tag{4.11}$$

where $B((j/n,(j+1)/n]) := B((j+1)/n) - B(j/n)$.

Proposition 4.1. *For any $1 \le p \le 2$, the double stochastic integral $I(h) = \int_{\mathbb{R}_+ \times \mathbb{R}} h(x,s)N(\mathrm{d}x)\mathrm{d}B(s)$ in (4.11) extends to any $h \in \mathcal{L}_p(\mathbb{R}_+ \times \mathbb{R})$, by continuity in L_p, and satisfies the inequality*

$$\mathrm{E}|I(h)|^p \le C\|h\|^p_{\mathcal{L}_p} \tag{4.12}$$

with $C > 0$ independent of $h \in \mathcal{L}_p(\mathbb{R}_+ \times \mathbb{R})$. Moreover, for any $h \in \mathcal{L}_p(\mathbb{R}_+ \times \mathbb{R})$ and any $\theta \in \mathbb{R}$,

$$\begin{aligned} \mathrm{E}\mathrm{e}^{\mathrm{i}\theta I(h)} &= \mathrm{E}_B \exp\Big\{ \int_{\mathbb{R}_+} \Big(\exp\Big\{ \mathrm{i}\theta \int_{\mathbb{R}} h(x,s)\mathrm{d}B(s) \Big\} - 1 \Big) \nu(\mathrm{d}x) \Big\} \\ &= \mathrm{E}_N \exp\Big\{ -\frac{\theta^2}{2} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}_+} h(x,s)N(\mathrm{d}x) \Big)^2 \mathrm{d}s \Big\}. \end{aligned} \tag{4.13}$$

In particular, $I(h)$ has a mixed Gaussian distribution with 'random variance' $\int_{\mathbb{R}} (\int_{\mathbb{R}_+} h(x,s)N(\mathrm{d}x))^2 \mathrm{d}s$.

Proposition 4.2. *(i) The process $Z_\beta = \{Z_\beta(\tau), \tau \ge 0\}$ in (4.9) is well-defined for any $\beta \in (-1/2, 0)$, as a stochastic integral of Proposition 4.1, and satisfies $\mathrm{E}|Z_\beta(\tau)|^p < \infty$ for any $p \in [1, 2(1+\beta)) \subset [1,2)$. Moreover, Z_β has stationary increments and a.s. continuous paths on $\mathbb{R}_+$.*
(ii) (Asymptotic self-similarity.) For any $-1/2 < \beta < 0$,

$$b^{\beta-(1/2)} Z_\beta(b\tau) \overset{\mathrm{fdd}}{\to} \sigma_\beta B_{(1/2)-\beta}(\tau) \quad \text{as } b \to 0, \tag{4.14}$$

$$b^{-1/2} Z_\beta(b\tau) \overset{\mathrm{fdd}}{\to} W_\beta B(\tau) \quad \text{as } b \to \infty, \tag{4.15}$$

where $\sigma_\beta := -\psi_1 \Gamma(\beta) C_\beta$.

In Theorems 4.3 and 4.4, $S_{N,n}(\tau)$ is the aggregated sum (4.2), where X_i are stationary random-coefficient AR(1) processes

$$X_i(t) = \sum_{s=0}^{\infty} a_i^s \varepsilon(t-s), \quad t \in \mathbb{Z},\, i = 1, \ldots, N, \tag{4.16}$$

with common i.i.d. innovations $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ such that $\mathrm{E}\varepsilon(t) = 0$, $\mathrm{E}\varepsilon^2(t) = 1$, and i.i.d. random coefficients $a_i \in (-1, 1)$, $i = 1, \ldots, N$, independent of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$. Note that the series in (4.16) converges conditionally a.s. and in L_2 for any fixed $a_i \in (-1, 1)$.

Theorem 4.3. *Let Assumptions (A1) and (A2) be satisfied, where $-1/2 < \beta < 0$. Then the simultaneous limits of the normalized partial sums $S_{N,n}$ as $N, n \to \infty$ under* (4.4) *are given in* (4.6)–(4.8) *in respective Cases (i)–(iii) of* (4.5).

Theorem 4.4. *Let Assumptions (A1) and (A2) be satisfied, where $\beta > 0$. Then, as $N, n \to \infty$ in arbitrary way,*

$$N^{-1} n^{-1/2} S_{N,n}(\tau) \overset{\mathrm{fdd}}{\to} \sigma B(\tau), \tag{4.17}$$

where $\{B(\tau),\, \tau \geq 0\}$ is a standard Brownian motion and $\sigma := \mathrm{E}(1-a)^{-1}$.

4.3 Proofs

Proof of Proposition 4.1. Rewrite $I(h)$ in (4.11) as $I(h) = I_1(h) + I_2(h)$, where $I_1(h) := \sum_{k,j} h_n(k,j)\nu((k/n,(k+1)/n])B((j/n,(j+1)/n]), I_2(h) := \sum_{k,j} h_n(k,j) \times \widetilde{N}((k/n,(k+1)/n])B((j/n,(j+1)/n])$. By inequality (3.24) on page 26 for pth moment of Poisson stochastic integrals, it follows that for any $1 \leq p \leq 2$,

$$\begin{aligned}
\mathrm{E}|I_2(h)|^p &= \mathrm{E}\mathrm{E}\big[|I_2(h)|^p | B\big] \\
&\leq 2\mathrm{E}\sum_k \Big|\sum_j h_n(k,j)B((j/n,(j+1)/n])\Big|^p \nu((k/n,(k+1)/n]) \\
&\leq 2\sum_k \Big\{\mathrm{E}\Big|\sum_j h_n(k,j)B((j/n,(j+1)/n])\Big|^2\Big\}^{p/2} \nu((k/n,(k+1)/n]) \\
&= 2\sum_k \Big\{\sum_j h_n^2(k,j)(1/n)\Big\}^{p/2} \nu((k/n,(k+1)/n]) \\
&= 2\int_{\mathbb{R}_+} \Big\{\int_{\mathbb{R}} h^2(x,s)\mathrm{d}s\Big\}^{p/2} \nu(\mathrm{d}x),
\end{aligned}$$

while

$$\begin{aligned}
\mathrm{E}|I_1(h)|^p &\le \{\mathrm{E}|I_1(h)|^2\}^{p/2} \\
&= \Big\{\sum_j \Big(\sum_k h_n(k,j)\nu((k/n,(k+1)/n])\Big)^2 (1/n)\Big\}^{p/2} \\
&\le \Big\{\sum_k \Big(\Big|\sum_j h_n^2(k,j)(1/n)\Big|^2\Big)^{1/2} \nu((k/n,(k+1)/n])\Big\}^p \\
&= \Big\{\int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}} h^2(x,s)\mathrm{d}s\Big)^{1/2} \nu(\mathrm{d}x)\Big\}^p
\end{aligned}$$

by Minkowski's inequality. Hence, $I(h)$ in (4.11) satisfies (4.12). The set $\mathcal{L}_0(\mathbb{R}_+\times\mathbb{R})$ being dense in $\mathcal{L}_p(\mathbb{R}_+\times\mathbb{R})$, the linear map $I:\mathcal{L}_0(\mathbb{R}_+\times\mathbb{R})\to L_p$ in (4.11) extends by continuity in L_p to $\mathcal{L}_p(\mathbb{R}_+\times\mathbb{R})$ and satisfies (4.12). The second equality in (4.13) is obvious. Consider the first equality in (4.13), which obviously holds for $h\in\mathcal{L}_0(\mathbb{R}_+\times\mathbb{R})$. Note that $L(h) := \int_{\mathbb{R}_+}(\mathrm{e}^{\mathrm{i}\theta\int_{\mathbb{R}} h(x,s)\mathrm{d}B(s)}-1)\nu(\mathrm{d}x)$ is well-defined and satisfies $\mathrm{E}_B|L(h)| \le |\theta|\int_{\mathbb{R}_+}\mathrm{E}_B|\int_{\mathbb{R}} h(x,s)\mathrm{d}B(s)|\nu(\mathrm{d}x) \le |\theta|\int_{\mathbb{R}_+}\mathrm{E}_B^{1/2}|\int_{\mathbb{R}} h(x,s)\mathrm{d}B(s)|^2\nu(\mathrm{d}x) = |\theta|\int_{\mathbb{R}_+}\nu(\mathrm{d}x)\{\int_{\mathbb{R}_+} h^2(x,s)\mathrm{d}s\}^{1/2} \le |\theta|\|h\|_{\mathcal{L}_p}$ and $\mathrm{Re}(L(h))\le 0$. Due to these facts, the first equality in (4.13) easily extends to $h\in\mathcal{L}_p(\mathbb{R}_+\times\mathbb{R})$. Proposition 4.1 is proved. □

The proof of Proposition 4.2 uses Lemma 4.5. For $(x,t)\in\mathbb{R}_+\times\mathbb{R}$ define

$$f(x,t) := \begin{cases} (1-\mathrm{e}^{-xt})/x, & \text{if } x>0 \text{ and } t>0, \\ 0, & \text{otherwise.}\end{cases} \tag{4.18}$$

Lemma 4.5. *(i) Let* $L(\tau,x) := \int_{-\infty}^{\tau}(f(x,\tau-s)-f(x,-s))\mathrm{d}B(s)$*;* $\Lambda_b(\tau,x) := b^{-(1+\beta)}(\exp\{\mathrm{i}\theta b^{1+\beta}L(\tau,x)\}-1)$*;* $\Lambda_0(\tau,x) := \mathrm{i}\theta L(\tau,x)$ *for* $\tau>0$*,* $x>0$*,* $b>0$*,* $\theta\in\mathbb{R}$*. Then*

$$\int_0^\infty \Lambda_b(\tau,x)\nu(\mathrm{d}x) \xrightarrow{\mathrm{p}} \int_0^\infty \Lambda_0(\tau,x)\nu(\mathrm{d}x), \quad b\to 0. \tag{4.19}$$

(ii) Let $\mathcal{M}_b(\tau,x) := \exp\{\mathrm{i}\theta x^{-1}\int_0^\tau(1-\mathrm{e}^{-bx(\tau-s)})\mathrm{d}B(s)\}-1$*;* $\mathcal{M}_\infty(\tau,x) := \exp\{\mathrm{i}\theta x^{-1}\int_0^\tau \mathrm{d}B(s)\}-1$ *for* $\tau>0$*,* $x>0$*,* $b>0$*,* $\theta\in\mathbb{R}$*. Then*

$$\int_0^\infty \mathcal{M}_b(\tau,x)\nu(\mathrm{d}x) \xrightarrow{\mathrm{p}} \int_0^\infty \mathcal{M}_\infty(\tau,x)\nu(\mathrm{d}x), \quad b\to\infty. \tag{4.20}$$

Proof. (i) Let $1<p<2(1+\beta)<2$. Using $|\mathrm{e}^{\mathrm{i}x}-1-x|\le\min(2|x|,x^2/2)$, $x\in\mathbb{R}$, we obtain

$$\begin{aligned}
|\Lambda_b(\tau,x)-\Lambda_0(\tau,x)| &\le C\min\big(b^{1+\beta}L^2(\tau,x),|L(\tau,x)|\big) \\
&= C\big\{b^{1+\beta}|L(\tau,x)|\mathbf{1}(1\le b^{1+\beta}|L(\tau,x)|) \\
&\qquad +b^{1+\beta}|L(\tau,x)|^2\mathbf{1}(1\ge b^{1+\beta}|L(\tau,x)|)\big\} \\
&\le Cb^{(1+\beta)(p-1)}|L(\tau,x)|^p,
\end{aligned}$$

which tends to 0 with probability 1 as $b \to 0$ for any $x > 0$. Hence

$$\begin{aligned}\mathrm{E}_B\Big|\int_0^\infty (\Lambda_b(\tau,x) - \Lambda_0(\tau,x))\nu(\mathrm{d}x)\Big| &\le Cb^{(1+\beta)(p-1)}\int_0^\infty \mathrm{E}_B|L(\tau,x)|^p\nu(\mathrm{d}x)\\ &= O(b^{(1+\beta)(p-1)}) = o(1),\end{aligned}$$

since $\int_0^\infty \mathrm{E}_B|L(\tau,x)|^p\nu(\mathrm{d}x) \le C\int_0^\infty |\sigma(\tau,x)|^p\nu(\mathrm{d}x) < \infty$ by (4.22) with $\sigma^2(\tau,x) = \mathrm{E}_B L^2(\tau,x)$ evaluated in (4.21). This proves (4.19).

(ii) Since$|\mathcal{M}_b(\tau,x) - \mathcal{M}_\infty(\tau,x)| \le C\min(1, |\theta x^{-1}\int_0^\tau \mathrm{e}^{-bx(\tau-s)}\mathrm{d}B(s)|) =: L_b(\tau,x)$, we have

$$\begin{aligned}&\mathrm{E}_B\int_0^\infty |\mathcal{M}_b(\tau,x) - \mathcal{M}_\infty(\tau,x)|\nu(\mathrm{d}x)\\ &\quad\le \int_0^\epsilon \mathrm{E}_B|L_b(\tau,x)|\nu(\mathrm{d}x) + \int_\epsilon^\infty \mathrm{E}_B|L_b(\tau,x)|\nu(\mathrm{d}x) =: I_1 + I_2.\end{aligned}$$

Since $\beta > -1/2 > -1$ and L_b is bounded, the integral I_1 can be made arbitrary small by choosing $\epsilon > 0$ small enough. The proof is completed by showing that

$$\begin{aligned}I_2 &\le C\int_\epsilon^\infty \mathrm{E}_B\Big|x^{-1}\int_0^\tau \mathrm{e}^{-bx(\tau-s)}\mathrm{d}B(s)\Big|x^\beta\mathrm{d}x\\ &\le C\int_\epsilon^\infty \mathrm{E}_B^{1/2}\Big|\int_0^\tau \mathrm{e}^{-bx(\tau-s)}\mathrm{d}B(s)\Big|^2 x^{\beta-1}\mathrm{d}x\\ &\le C\int_\epsilon^\infty \Big|\int_0^\tau \mathrm{e}^{-2bxs}\mathrm{d}s\Big|^{1/2} x^{\beta-1}\mathrm{d}x\\ &\le C\int_\epsilon^\infty (xb)^{-1/2}x^{\beta-1}\mathrm{d}x \to 0, \quad b\to\infty.\end{aligned}$$

This proves (4.20) and the lemma, too. □

Proof of Proposition 4.2. (i) We have $h_\tau(x,s) := \int_0^\tau \mathrm{e}^{-x(u-s)}\mathbf{1}(s\le u)\mathrm{d}u = f(x,\tau-s) - f(x,-s)$, $(x,s)\in\mathbb{R}_+\times\mathbb{R}$, where $f(\tau,x)$ is defined in (4.18). By Proposition 4.1, $Z_\beta(\tau) = I(h_\tau)$ is well-defined provided $\|h_\tau\|_{\mathcal{L}_p} < \infty$ for some $1\le p\le 2$. From (3.33), (3.34) on page 28 we have

$$\sigma^2(\tau,x) := \int_{\mathbb{R}} h_\tau^2(x,s)\mathrm{d}s \le C\frac{\tau}{x^2}(1\wedge(\tau x)) \tag{4.21}$$

and hence

$$\begin{aligned}\int_{\mathbb{R}_+}\Big\{\int_{\mathbb{R}} h_\tau^2(x,s)\mathrm{d}s\Big\}^{p/2}\nu(\mathrm{d}x) &\le C\tau^{p/2}\int_0^\infty \Big\{\frac{1}{x^2}(1\wedge(\tau x))\Big\}^{p/2}x^\beta\mathrm{d}x\\ &\le C\tau^{(3p/2)-1-\beta} < \infty \qquad (4.22)\end{aligned}$$

for $1+\beta < 1 \le p < 2(1+\beta)$. Therefore

$$\mathrm{E}|Z_\beta(\tau)|^p \le C\|h_\tau\|^p_{\mathcal{L}_p} \le C(\tau^{(3p/2)-1-\beta} + \tau^{(1-2\beta)(p/2)}) < \infty \tag{4.23}$$

for $\tau > 0$, $1 \le p < 2(1+\beta)$. $Z_\beta(\tau) = \int_{\mathbb{R}_+\times\mathbb{R}} h_\tau(x,s)N(\mathrm{d}x)\mathrm{d}B(s)$ has stationary increments because the invariance properties $h_{\tau+u}(x,s) - h_u(x,s) = h_\tau(x, s-u)$, $\{\mathrm{d}B(s+u),\ s \in \mathbb{R}\} \overset{\mathrm{fdd}}{=} \{\mathrm{d}B(s),\ s \in \mathbb{R}\}$, $\tau, u \ge 0$, hold for the integrand and the white noise $\mathrm{d}B$. The fact that $Z_\beta(\tau)$ has a.s. continuous paths follows from (4.23) and stationarity of increments, and the Kolmogorov criterion [73, Theorem 2.2.3], by noting that both exponents of τ on the r.h.s. of (4.23) are strictly greater than 1 for $p < 2(1+\beta)$ sufficiently close to $2(1+\beta)$. This proves part (i).

(ii) We use the method of characteristic functions and restrict the proof to the one-dimensional convergence at fixed $\tau > 0$; the proof of finite-dimensional convergence follows similarly.

Proof of (4.14). Using (4.13) and the scaling property $f(x, bt) = bf(bx, t)$ we obtain

$$\begin{aligned}
U_b(\theta) \;&:=\; \mathrm{E}_B \exp\Big\{\int_0^\infty \Big(\exp\Big\{\mathrm{i}\theta b^{\beta-1/2}\int_{-\infty}^{b\tau}(f(x, b\tau - s) \\
&\qquad\qquad -f(x,-s))\mathrm{d}B(s)\Big\} - 1\Big)\nu(\mathrm{d}x)\Big\} \\
&= \mathrm{E}_B \exp\Big\{\int_0^\infty \Big(\exp\Big\{\mathrm{i}\theta b^{1+\beta}\int_{-\infty}^{\tau}(f(bx, \tau - s) \\
&\qquad\qquad -f(bx,-s))\mathrm{d}B(s)\Big\} - 1\Big)\nu(\mathrm{d}x)\Big\} \\
&= \mathrm{E}_B \exp\Big\{\int_0^\infty \Lambda_b(\tau, y)\nu(\mathrm{d}y)\Big\},
\end{aligned}$$

where we changed a variable to get the last equality with $\Lambda_b(\tau, y)$ defined in Lemma 4.5(i). Since $\mathrm{Re}\{\int_0^\infty \Lambda_b(\tau,y)\nu(\mathrm{d}y)\} \le 0$, Lemma 4.5(i) implies the convergence $U_b(\theta) \to U_0(\theta) := \mathrm{E}_B \exp\{\int_0^\infty \Lambda_0(\tau,y)\nu(\mathrm{d}y)\}$ for any $\theta \in \mathbb{R}$. It remains to show that $U_0(\theta) = \mathrm{E}\exp\{\mathrm{i}\theta\sigma_\beta B_{(1/2)-\beta}(\tau)\}$. Using the definitions of $B_{(1/2)-\beta}(\tau)$ in (4.10) and $f(x,\tau)$ in (4.18) and the identity $\int_0^\infty f(x,\tau)\nu(\mathrm{d}x) = -\psi_1\Gamma(\beta)\tau^{-\beta}$, $\beta \in (-1, 0)$, $\tau > 0$, we obtain

$$\begin{aligned}
\int_0^\infty \Lambda_0(\tau,x)\nu(\mathrm{d}x) \;&=\; \mathrm{i}\theta\int_{-\infty}^{\tau}\int_0^\infty (f(x,\tau-s) - f(x,-s))\nu(\mathrm{d}x)\mathrm{d}B(s) \\
&= \mathrm{i}\theta\sigma_\beta B_{(1/2)-\beta}(\tau),
\end{aligned} \tag{4.24}$$

where the interchange of the order of integration in the first equality of (4.24) can be justified by the stochastic Fubini theorem, see [85, Chapter 6, Theorem 65]. The proof of (4.14) is complete.

Proof of (4.15). It is well-known (see, e.g., [96, Theorem 3.12.2]) that the $(1+\beta)$-stable r.v. W_β in (4.15) can be written as stochastic integral w.r.t. Poisson random measure N: $W_\beta \overset{\mathrm{d}}{=} \int_0^\infty x^{-1} N(\mathrm{d}x)$. Let us prove that as $b \to \infty$,

$$\begin{aligned} V_b(\theta) &:= \mathrm{E}\exp\Big\{\mathrm{i}\theta b^{-1/2}\int_0^\infty\int_0^{b\tau} f(x, b\tau - s)N(\mathrm{d}x)\mathrm{d}B(s)\Big\} \\ &\to \mathrm{E}\exp\Big\{\mathrm{i}\theta\int_0^\infty x^{-1}N(\mathrm{d}x)B(\tau)\Big\} = \mathrm{E}\mathrm{e}^{\mathrm{i}\theta W_\beta B(\tau)} =: V_\infty(\theta). \end{aligned}$$

Indeed, using (4.13) and scaling properties of $f(x,t)$ and B, we have $V_b(\theta) = \mathrm{E}_B \exp\{\int_0^\infty \mathcal{M}_b(\tau, x)\nu(\mathrm{d}x)\}$, where $\mathcal{M}_b(\tau, x)$ is defined in Lemma 4.5(ii). Since $\mathrm{Re}\{\int_0^\infty \mathcal{M}_b(\tau,x)\nu(\mathrm{d}x)\} \le 0$, relation $V_b(\theta) \to V_\infty(\theta) = \mathrm{E}_B\exp\{\int_0^\infty \mathcal{M}_\infty(\tau, x)$ $\nu(\mathrm{d}x)\}$, $b\to\infty$, follows from Lemma 4.5(ii). It remains to prove that

$$I(b,p) := b^{-p/2}\mathrm{E}\Big|\int_0^\infty\int_{-\infty}^0 \big(f(x,b\tau - s) - f(x,-s)\big)N(\mathrm{d}x)\mathrm{d}B(s)\Big|^p \to 0, \quad (4.25)$$

$b\to\infty$, for some $p>0$. Using $\int_{-\infty}^0 (f(x, b\tau - s) - f(x,-s))\mathrm{d}B(s) = x^{-1}(1 - \mathrm{e}^{-xb\tau})\int_0^\infty \mathrm{e}^{-xs}\mathrm{d}B(s)$ and the inequality in (3.24) on page 26 with $0 < p < 1+\beta < 1$, $3p/2 > 1+\beta$ we obtain

$$\begin{aligned} b^{p/2} I(b,p) &\le \int_0^\infty |(1-\mathrm{e}^{-xb\tau})/x|^p \mathrm{E}_B\Big|\int_0^\infty \mathrm{e}^{-xs}\mathrm{d}B(s)\Big|^p \nu(\mathrm{d}x) \\ &\le C\int_0^\infty |(1-\mathrm{e}^{-xb\tau})/x|^p\Big(\int_0^\infty \mathrm{e}^{-2xs}\mathrm{d}s\Big)^{p/2}\nu(\mathrm{d}x) \\ &\le C\int_0^\infty |(1-\mathrm{e}^{-xb\tau})/x|^p x^{-p/2}\nu(\mathrm{d}x) = O(b^{(p/2)+p-1-\beta}) = o(b^{p/2}), \end{aligned}$$

which yields (4.25) and completes the proof of (4.15). Proposition 4.2 is proved. □

To prove Theorem 4.3 we need the following lemma.

Lemma 4.6. *Let* $\eta_n(a,s) := \sum_{t=1}^{[n\tau]} a^{t-s}\mathbf{1}(s\le t)$, $s\in\mathbb{Z}$, $a\in(-1,1)$. *Then as* $N, n\to\infty$ *and* $N^{1/(1+\beta)}/n \to \mu \in \{1,\infty\}$,

$$N^{-2}n^{2\beta-1}\sum_{s\in\mathbb{Z}}\Big(\sum_{i=1}^N \eta_n(a_i,s)\Big)^2 \xrightarrow{\mathrm{d}} \begin{cases} \int_{\mathbb{R}}\Big(\int_0^\infty\int_0^\tau \mathrm{e}^{-x(t-s)}\mathbf{1}(s\le t)\mathrm{d}tN(\mathrm{d}x)\Big)^2\mathrm{d}s, & \mu = 1, \\ (-\psi_1\Gamma(\beta))^2\int_{\mathbb{R}}((\tau-s)_+^{-\beta} - (-s)_+^{-\beta})^2\mathrm{d}s, & \mu=\infty. \end{cases} \quad (4.26)$$

Proof. We use the criterion in Cremers and Kadelka [23]. Rewrite (4.26) as $I_{N,n} \xrightarrow{\mathrm{d}} I$, where $I_{N,n} := \int_{\mathbb{R}} A_{N,n}^2(s)\mathrm{d}s$, $I := \int_{\mathbb{R}} A^2(s)\mathrm{d}s$ and

$$\begin{aligned} A_{N,n}(s) &:= \frac{n^\beta}{N}\sum_{i=1}^N \eta_n(a_i, \lceil ns\rceil), \\ A(s) &:= \begin{cases} \int_0^\infty\int_0^\tau \mathrm{e}^{-x(t-s)}\mathbf{1}(s\le t)\mathrm{d}tN(\mathrm{d}x), & \mu=1, \\ \kappa((\tau-s)_+^{-\beta} - (-s)_+^{-\beta}), & \mu = \infty, \end{cases} \end{aligned}$$

with $\kappa := -\psi_1\Gamma(\beta)$. Accordingly (see [23], the second Corollory to Theorem 3), it suffices to verify two conditions:

$$A_{N,n}(s) \overset{\text{fdd}}{\to} A(s) \tag{4.27}$$

and

$$\mathrm{E}\Big[\int_{\mathbb{R}} A_{N,n}^2(s)\mathrm{d}s\Big]^{1/2} < C. \tag{4.28}$$

Relation (4.27) follows from the convergence of the joint characteristic functions:

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j A_{N,n}(s_j)} = \Big(1 + \frac{\Theta_{N,n}}{N}\Big)^N \to \mathrm{e}^{\Theta} = \mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j A(s_j)} \tag{4.29}$$

for any $(\theta_1,\dots,\theta_m) \in \mathbb{R}^m$, $-\infty < s_1 < \dots < s_m < \infty$, $m \in \mathbb{N}$, where

$$\Theta_{N,n} := N\mathrm{E}\Big[\exp\Big\{\mathrm{i}\frac{n^\beta}{N}\sum_{j=1}^m \theta_j \eta_n(a, \lceil ns_j\rceil)\Big\} - 1\Big]$$

and

$$\Theta := \begin{cases} \psi_1 \int_0^\infty \Big(\exp\Big\{\mathrm{i}\sum_{j=1}^m \theta_j \int_0^\tau \mathrm{e}^{-x(t-s_j)}\mathbf{1}(s_j \le t)\mathrm{d}t\Big\} - 1\Big)x^\beta \mathrm{d}x, & \mu = 1, \\ \mathrm{i}\sum_{j=1}^m \theta_j A(s_j) = \mathrm{i}\kappa \sum_{j=1}^m \theta_j((\tau - s_j)_+^{-\beta} - (-s_j)_+^{-\beta}), & \mu = \infty. \end{cases}$$

Observe that $A(s) = \psi_1\int_0^\infty(\int_0^\tau \mathrm{e}^{-x(t-s)}\mathbf{1}(s \le t)\mathrm{d}t)x^\beta\mathrm{d}x$ if $\mu = \infty$. Split $\Theta_{N,n} = \Theta_{N,n,1} + \Theta_{N,n,2}$, where

$$\begin{aligned} \Theta_{N,n,1} &:= N\mathrm{E}\Big[\exp\Big\{\mathrm{i}\frac{n^\beta}{N}\sum_{j=1}^m \theta_j\eta_n(a,\lceil ns_j\rceil)\Big\} - 1\Big]\mathbf{1}(1-\epsilon < a < 1), \\ \Theta_{N,n,2} &:= N\mathrm{E}\Big[\exp\Big\{\mathrm{i}\frac{n^\beta}{N}\sum_{j=1}^m \theta_j\eta_n(a,\lceil ns_j\rceil)\Big\} - 1\Big]\mathbf{1}(-1 < a \le 1-\epsilon), \end{aligned}$$

with the same $\epsilon > 0$ as in Assumption (A1). From $\eta_n(a,s) \le 2/(1-a)$ and $|\mathrm{e}^{\mathrm{i}z} - 1| \le |z|$ $(z \in \mathbb{R})$, we obtain $|\Theta_{N,n,2}| \le Cn^\beta\mathrm{E}[(1-a)^{-1}\mathbf{1}(-1 < a < 1-\epsilon)] = o(1)$ as $\beta < 0$. Next, with $h_n(x,s) := \int_0^{\lceil n\tau\rceil/n}(1-\frac{x}{n})^{\lceil nt\rceil - \lceil ns\rceil}\mathbf{1}(\lceil ns\rceil \le \lceil nt\rceil)\mathrm{d}t$ by change of variable $a = 1 - x/n$ we obtain

$$\begin{aligned} \Theta_{N,n,1} &= N\int_{1-\epsilon}^1 \Big(\exp\Big\{\mathrm{i}\frac{n^{1+\beta}}{N}\sum_{j=1}^m \theta_j \int_0^{\lceil n\tau\rceil/n} a^{\lceil nt\rceil - \lceil ns_j\rceil}\mathbf{1}(\lceil ns_j\rceil \le \lceil nt\rceil)\mathrm{d}t\Big\} - 1\Big) \\ &\qquad \times\psi(a)(1-a)^\beta\mathrm{d}a \\ &= \frac{N}{n^{1+\beta}}\int_0^{\epsilon n}\Big(\exp\Big\{\mathrm{i}\frac{n^{1+\beta}}{N}\sum_{j=1}^m \theta_j h_n(x,s_j)\Big\} - 1\Big)\psi\Big(1 - \frac{x}{n}\Big)x^\beta\mathrm{d}x \\ &\to \Theta \end{aligned} \tag{4.30}$$

in both cases $\mu = 1$ and $\mu = \infty$. This follows from the pointwise convergence $h_n(x,s) \to \int_0^\tau \mathrm{e}^{-x(t-s)}\mathbf{1}(s \le t)\mathrm{d}t$, $(x,s) \in \mathbb{R}_+ \times \mathbb{R}$, and the dominating bound

$$\frac{N}{n^{1+\beta}}\Big|\exp\Big\{\mathrm{i}\frac{n^{1+\beta}}{N}\sum_{j=1}^m \theta_j h_n(x,s_j)\Big\} - 1\Big|\mathbf{1}(0 < x < \epsilon n) \le C\min(1,(1/x)),$$

which is a consequence of the inequalities $|\mathrm{e}^{\mathrm{i}z} - 1| \le |z|$ $(z \in \mathbb{R})$, $|1-u| \le \mathrm{e}^{-u}$ $(u \in [0,1])$. This proves (4.29) and (4.27).

Consider (4.28). Write $J_{N,n}$ for the l.h.s. of (4.28). By Minkowski's inequality,

$$\begin{aligned} J_{N,n} &= n^{\beta-(1/2)}N^{-1}\mathrm{E}\Big[\sum_{s\in\mathbb{Z}}\Big(\sum_{i=1}^N \eta_n(a_i,s)\Big)^2\Big]^{1/2} \\ &\le n^{\beta-(1/2)}\mathrm{E}\Big[\sum_{s\in\mathbb{Z}}\eta_n^2(a,s)\Big]^{1/2} \\ &= n^{\beta-(1/2)}\Big\{\mathrm{E}\Big[\sum_{s\in\mathbb{Z}}\eta_n^2(a,s)\Big]^{1/2}\mathbf{1}(1-\epsilon < a < 1) \\ &\qquad +\mathrm{E}\Big[\sum_{s\in\mathbb{Z}}\eta_n^2(a,s)\Big]^{1/2}\mathbf{1}(-1 < a \le 1-\epsilon)\Big\} \\ &=: J_{N,n,1} + J_{N,n,2} \end{aligned}$$

for the same $\epsilon > 0$ as in Assumption (A1). Since $\sum_{s\le 0}\eta_n^2(a,s) + \sum_{s=1}^{[n\tau]}\eta_n^2(a,s) \le C((1+a)^{-1}+n)$ for $-1 < a \le 1-\epsilon$, we therefore get $J_{N,n,2} = O(n^\beta) = o(1)$ under Assumption (A2). Next, similarly to (4.30), by change $a = 1 - x/n$ of variable and with the same $h_n(x,s)$ as in (4.30), we obtain

$$J_{N,n,1} \le C\int_0^{\epsilon n} x^\beta \mathrm{d}x\Big[\int_{-\infty}^\tau h_n^2(x,s)\mathrm{d}s\Big]^{1/2} < C.$$

This proves (4.28) and completes the proof of Lemma 4.6. □

Proof of Theorem 4.3. We use the method of characteristic functions as in Chapter 3. For notational convenience, we restrict the proof to one-dimensional convergence at $\tau > 0$. The case of general finite-dimensional distributions does not require essential changes.

Case (iii) (proof of (4.8)). Let $\mu = 1$. As in the proof of Theorem 3.2, we first assume ε to be a standard normal r.v., i.e. $\varepsilon \overset{\mathrm{d}}{=} \mathcal{N}(0,1)$. It suffices to show that for each $\theta \in \mathbb{R}$,

$$\begin{aligned} &\mathrm{E}\exp\{\mathrm{i}\theta N^{-1/(1+\beta)}n^{-1/2}S_{N,n}(\tau)\} &(4.31)\\ &\quad\to \mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z_\beta(\tau)} = \mathrm{E}_N\exp\Big\{-\frac{\theta^2}{2}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}_+}\int_0^\tau \mathrm{e}^{-x(t-s)}\mathbf{1}(s\le t)\mathrm{d}tN(\mathrm{d}x)\Big)^2\mathrm{d}s\Big\} \end{aligned}$$

as $N, n \to \infty$, $N^{1/(1+\beta)}/n \to 1$, where the characteristic function of $Z_\beta(\tau)$ follows from (4.9), (4.13). Use $X_i(t) = \sum_{s \le t} a_i^{t-s} \varepsilon(s)$ and $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\varepsilon} = \mathrm{e}^{-\theta^2/2}$ to write the l.h.s. of (4.31) as

$$\begin{aligned} &\mathrm{E}\mathrm{E}\Big[\exp\Big\{\mathrm{i}\theta N^{-1/(1+\beta)} n^{-1/2} \sum_{s\in\mathbb{Z}} \Big(\sum_{i=1}^N \eta_n(a_i, s)\Big)\varepsilon(s)\Big\}\Big| a_1, \dots, a_N\Big] \\ &\quad = \mathrm{E}\exp\Big\{-(\theta^2/2) N^{-2/(1+\beta)} n^{-1} \sum_{s\in\mathbb{Z}} \Big(\sum_{i=1}^N \eta_n(a_i, s)\Big)^2\Big\}, \qquad (4.32) \end{aligned}$$

with the same $\eta_n(a, s)$ as in Lemma 4.6. Whence, (4.31) immediately follows from the above-mentioned lemma.

In a general case of ε, the above argument needs some modification, see the proof of Theorem 3.1. Namely, we use the fact (see, e.g., Ibragimov and Linnik [49, Theorem 2.6.5]) that the characteristic function of ε has the following representation in a neighborhood of the origin: there exists $\delta > 0$ such that

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\varepsilon} := \mathrm{e}^{-(1/2)\theta^2 h(\theta)} \quad \text{for each } |\theta| < \delta, \qquad (4.33)$$

where $h(\theta)$ is a positive function tending to 1 as $\theta \to 0$. For $0 < p < 1/2$ consider the set $\Omega_{N,n} := \{\boldsymbol{a} = (a_1, a_2, \dots) \in [0,1)^{\mathbb{N}} : \sum_{i=1}^N (1-a_i)^{-1} < N^{1/(1+\beta)} n^p\}$. Then $\sup_{\boldsymbol{a}\in\Omega_{N,n},\, s\in\mathbb{Z}} |\sum_{i=1}^N \eta_n(a_i, s)| \le 2N^{1/(1+\beta)} n^p$ implies

$$\sup_{\boldsymbol{a}\in\Omega_{N,n},\, s\in\mathbb{Z}} \Big| h\Big(\theta N^{-1/(1+\beta)} n^{-1/2} \sum_{i=1}^N \eta_n(a_i, s)\Big) - 1\Big| = o(1). \qquad (4.34)$$

For N and n large enough, split $U_{N,n}(\theta) := \mathrm{E}\exp\{\mathrm{i}\theta N^{-1/(1+\beta)} n^{-1/2} S_{N,n}(\tau)\} = U_{N,n,1}(\theta) + U_{N,n,2}(\theta)$, where $U_{N,n,1}(\theta) := \mathrm{E}[\exp\{\mathrm{i}\theta N^{-1/(1+\beta)} n^{-1/2} S_{N,n}(\tau)\}\mathbf{1}(\boldsymbol{a} \in \Omega_{N,n})]$,
$U_{N,n,2}(\theta) := \mathrm{E}[\exp\{\mathrm{i}\theta N^{-1/(1+\beta)} n^{-1/2} S_{N,n}(\tau)\}\mathbf{1}(\boldsymbol{a} \notin \Omega_{N,n})]$. By Markov's inequality, for $(1+\beta)/(1+p) < q < 1+\beta < 1$, we get

$$\begin{aligned} |U_{N,n,2}(\theta)| &\le \mathrm{P}\Big(\sum_{i=1}^N (1-a_i)^{-1} \ge N^{1/(1+\beta)} n^p\Big) \le \frac{\mathrm{E}\big(\sum_{i=1}^N (1-a_i)^{-1}\big)^q}{\big(N^{1/(1+\beta)} n^p\big)^q} \\ &\le \mathrm{E}\Big[\frac{1}{(1-a)^q}\Big]\Big(\frac{N^{1/(1+\beta)}}{n}\Big)^{pq} N^{1-q(1+p)/(1+\beta)} \to 0. \end{aligned}$$

Using (4.33), (4.34), $\mathrm{P}(\boldsymbol{a} \notin \Omega_{N,n}) \to 0$ and (4.31), we obtain

$$\begin{aligned} U_{N,n,1}(\theta) &= \mathrm{E}\Big[\exp\Big\{-(\theta^2/2) N^{-2/(1+\beta)} n^{-1} \sum_{s\le[n\tau]} \Big(\sum_{i=1}^N \eta_n(a_i, s)\Big)^2 \\ &\qquad \times h\Big(\theta N^{-1/(1+\beta)} n^{-1/2} \sum_{i=1}^N \eta_n(a_i, s)\Big)\Big\}\mathbf{1}(\boldsymbol{a} \in \Omega_{N,n})\Big] \to \mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z_\beta(\tau)}, \end{aligned}$$

and finish the proof of $U_{N,n}(\theta) \to \mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z_\beta(\tau)}$ in the general case of ε. Finally, the general case of $0 < \mu < \infty$ reduces to $\mu = 1$, since $n^{-1/2} S_{N,n}(\tau) = \mu^{1/2} \tilde{n}^{-1/2} S_{N,\tilde{n}}(\tau/\mu)$ with $\tilde{n} = n\mu$ satisfying $N^{1/(1+\beta)}/\tilde{n} \to 1$.

Case (i) (proof of (4.6)). Follows similarly to Case (iii) by using (4.32) and Lemma 4.6 with $\mu = \infty$.

Case (ii) (proof of (4.7)). Split

$$S_{N,n}(\tau) = \Sigma_{N,n,1}(\tau) - \Sigma_{N,n,2}(\tau) + \Sigma_{N,n,3}(\tau), \tag{4.35}$$

where

$$\begin{aligned}
\Sigma_{N,n,1}(\tau) &:= \sum_{s=1}^{[n\tau]} \varepsilon(s) \sum_{i=1}^{N} \frac{1}{1-a_i}, \\
\Sigma_{N,n,2}(\tau) &:= \sum_{s=1}^{[n\tau]} \Big(\sum_{i=1}^{N} \frac{a_i^{[n\tau]-s+1}}{1-a_i} \Big) \varepsilon(s), \\
\Sigma_{N,n,3}(\tau) &:= \sum_{s \le 0} \Big(\sum_{i=1}^{N} \frac{a_i^{1-s}(1-a_i^{[n\tau]})}{1-a_i} \Big) \varepsilon(s).
\end{aligned}$$

It suffices to prove that

$$\begin{aligned}
N^{-1/(1+\beta)} n^{-1/2} \Sigma_{N,n,1}(\tau) &\overset{\mathrm{fdd}}{\to} W_\beta B(\tau), \\
\Sigma_{N,n,i}(\tau) &= o_{\mathrm{p}}(N^{1/(1+\beta)} n^{1/2}), \quad i = 2, 3,
\end{aligned} \tag{4.36}$$

as $N, n \to \infty$, $N^{1/(1+\beta)}/n \to 0$. The first relation in (4.36) follows from

$$n^{-1/2} \sum_{s=1}^{[n\tau]} \varepsilon(s) \overset{\mathrm{d}}{\to} B(\tau) \quad \text{and} \quad N^{-1/(1+\beta)} \sum_{i=1}^{N} (1-a_i)^{-1} \overset{\mathrm{d}}{\to} W_\beta, \tag{4.37}$$

by independence of $\{\varepsilon(s)\}$ and $\{a_i\}$ and the continuous mapping theorem. In turn, the first relation in (4.37) follows by the classical central limit theorem for i.i.d. r.v.s with finite variance. A similar statement for sums of i.i.d. r.v.s in the domain of attraction of stable law (see [49, Theorem 2.6.7]) implies the second limit in (4.37), because the distribution of $(1-a)^{-1}$ belongs to the domain of attraction of the $(1+\beta)$-stable law W_β: $\mathrm{P}((1-a)^{-1} > x) = \mathrm{P}(a > 1 - x^{-1}) \sim (\psi_1/(1+\beta)) x^{-(1+\beta)}$, $x \to \infty$, according to (4.3). The remaining relations in (4.36) are established in Lemma 4.7. This proves (4.7) and completes the proof of Theorem 4.3. □

Lemma 4.7. *$\Sigma_{N,n,i}(\tau) = o_{\mathrm{p}}(n^{1/2} N^{1/(1+\beta)})$, $i = 2, 3$, as $N, n \to \infty$ and $N^{1/(1+\beta)}/n \to 0$.*

Proof. W.l.g., let $\tau = 1$ and $\Sigma_{N,n,i} := \Sigma_{N,n,i}(1)$, $i = 2, 3$. We shall prove that for $\frac{2}{3}(1+\beta) < p < 1+\beta$,

$$\mathrm{E}|\Sigma_{N,n,i}|^p = o(n^{p/2}N^{p/(1+\beta)}), \quad i = 2, 3. \tag{4.38}$$

We have $\mathrm{E}|\Sigma_{N,n,2}|^p \le \mathrm{E}V_2^p$, where

$$\begin{aligned} V_2 &:= \mathrm{E}^{1/2}\big[|\Sigma_{N,n,2}|^2\big|a_1, \dots, a_N\big] = \Big\{\sum_{s=1}^{n}\Big(\sum_{i=1}^{N}\frac{a_i^{n-s+1}}{1-a_i}\Big)^2\Big\}^{1/2} \\ &\le \sum_{i=1}^{N}\Big\{\sum_{s=1}^{n}\frac{a_i^{2s}}{(1-a_i)^2}\Big\}^{1/2} = \sum_{i=1}^{N}\frac{(1-a_i^{2n})^{1/2}}{(1-a_i)(1-a_i^2)^{1/2}} \end{aligned}$$

by Minkowski's inequality. Hence,

$$\mathrm{E}|\Sigma_{N,n,2}|^p \le N\mathrm{E}A_2^p, \quad \text{where } A_2 := \frac{(1-a^{2n})^{1/2}}{(1-a)(1-a^2)^{1/2}}, \tag{4.39}$$

as $p < 1$. Split $\mathrm{E}A_2^p = \mathrm{E}[A_2^p\mathbf{1}(a \le 1-\epsilon)] + \mathrm{E}[A_2^p\mathbf{1}(a > 1-\epsilon)] =: \Lambda_2' + \Lambda_2''$ for the same $\epsilon > 0$ as in Assumption (A1). Then $\Lambda_2' \le C\mathrm{E}(1+a)^{-p/2} < C$ under Assumption (A2). Next, by change of variable $1 - a = x/n$ we obtain

$$\begin{aligned} \Lambda_2'' &\le C\int_{1-\epsilon}^{1}\frac{(1-a^{2n})^{p/2}}{(1-a)^{3p/2}}(1-a)^\beta \mathrm{d}a \\ &= Cn^{(3p/2)-(1+\beta)}\int_0^{\epsilon n}\Big(1-\Big(1-\frac{x}{n}\Big)^{2n}\Big)^{p/2}x^{\beta-(3p/2)}\mathrm{d}x, \end{aligned}$$

where the last integral tends to $\int_0^\infty(1-\mathrm{e}^{-2x})^{p/2}x^{\beta-(3p/2)}\mathrm{d}x < \infty$ for $2(1+\beta)/3 < p < 1+\beta$ by the dominated convergence theorem. Therefore, $\mathrm{E}|\Sigma_{N,n,2}|^p \le CNn^{(3p/2)-(1+\beta)}$, proving (4.38) for $i = 2$.

The proof of (4.38) for $i = 3$ is similar. Namely, $\mathrm{E}|\Sigma_{N,n,3}|^p \le \mathrm{E}V_3^p$, where

$$\begin{aligned} V_3 &:= \mathrm{E}^{1/2}\big[|\Sigma_{N,n,2}|^2\big|a_1, \dots, a_N\big] = \Big\{\sum_{s\le 0}\Big(\sum_{i=1}^{N}\frac{a_i^{1-s}(1-a_i^n)}{1-a_i}\Big)^2\Big\}^{1/2} \\ &\le \sum_{i=1}^{N}\Big\{\sum_{s\le 0}^{n}\frac{a_i^{2(1-s)}(1-a_i^n)^2}{(1-a_i)^2}\Big\}^{1/2} \le \sum_{i=1}^{N}\frac{1-a_i^n}{(1-a_i)(1-a_i^2)^{1/2}}. \end{aligned}$$

Hence

$$\mathrm{E}|\Sigma_{N,n,3}|^p \le N\mathrm{E}A_3^p, \quad \text{where } A_3 := \frac{1-a^n}{(1-a)(1-a^2)^{1/2}}$$

similarly to (4.39). Next, $\mathrm{E}A_3^p = \mathrm{E}[A_3^p\mathbf{1}(a \le 1-\epsilon)] + \mathrm{E}[A_3^p\mathbf{1}(a > 1-\epsilon)] =: \Lambda_3' + \Lambda_3''$, where $\Lambda_3' \le C\mathrm{E}(1+a)^{-p/2} < C$ and

$$\begin{aligned} \Lambda_3'' &\le C\int_{1-\epsilon}^{1}\frac{(1-a^n)^p}{(1-a)^{3p/2}}(1-a)^\beta \mathrm{d}a \\ &= Cn^{(3p/2)-(1+\beta)}\int_0^{\epsilon n}\Big(1-\Big(1-\frac{x}{n}\Big)^{n}\Big)^{p}x^{\beta-(3p/2)}\mathrm{d}x, \end{aligned}$$

where the last integral tends to $\int_0^\infty (1-\mathrm{e}^{-x})^p x^{\beta-(3p/2)}\mathrm{d}x < \infty$. Lemma 4.7 is proved. $\square$

Proof of Theorem 4.4. We restrict the proof of (4.17) to one-dimensional convergence at $\tau > 0$. Split $S_{N,n}(\tau)$ as in (4.35). Then, by the central limit theorem and the law of large numbers, $n^{-1/2}N^{-1}\Sigma_{N,n,1}(\tau) \xrightarrow{\mathrm{d}} \sigma B(\tau)$ as $N, n \to \infty$ in an arbitrary way. It remains to show that $\Sigma_{N,n,i}(\tau) = o_{\mathrm{p}}(Nn^{1/2})$, $i = 2, 3$. W.l.g., let $\tau = 1$. According to (4.39), $\mathrm{E}|\Sigma_{N,n,2}(1)| \le \mathrm{E}\{\mathrm{E}[|\Sigma_{N,n,2}(1)|^2 \,|\, a_1, \dots, a_N]\}^{1/2} \le N\mathrm{E}A_2$. Split $\mathrm{E}A_2 = \mathrm{E}[A_2\mathbf{1}(a \le 1-\epsilon)] + \mathrm{E}[A_2\mathbf{1}(a > 1-\epsilon)]$ as in the proof of Lemma 4.7. Then $\mathrm{E}[A_2\mathbf{1}(a \le 1-\epsilon)] \le C\mathrm{E}(1+a)^{-1/2} < C$. Using $1 - u^n \le \min(1, n(1-u))$, $u \in (0,1)$, for $\max(0, 1/2-\beta) < q < 1/2$ we get $\mathrm{E}[A_2\mathbf{1}(a > 1-\epsilon)] \le Cn^q\mathrm{E}[(1-a)^{q-3/2}] < Cn^q$ and thus $\mathrm{E}|\Sigma_{N,n,2}(1)| \le CNn^q = o(Nn^{1/2})$. The proof of $\mathrm{E}|\Sigma_{N,n,3}(\tau)| = o(Nn^{1/2})$ is analogous. Theorem 4.4 is proved. $\square$

Chapter 5

Statistical inference from panel AR(1) data

This chapter contains the article [63]. We discuss nonparametric estimation of the distribution function G of the autoregressive coefficient $a \in (-1, 1)$ from a panel of N random-coefficient AR(1) series, each of length n, by the empirical distribution function of lag 1 sample autocorrelations of individual AR(1) processes. Consistency and asymptotic normality of the empirical distribution function and a class of kernel density estimators is established under some regularity conditions on G as N and n increase to infinity. The Kolmogorov–Smirnov goodness-of-fit test for simple and composite hypotheses of beta distributed a is discussed. A simulation study for goodness-of-fit testing compares the finite-sample performance of our nonparametric estimator to the performance of its parametric analogue discussed in [9].

5.1 Introduction

Panel data can describe a large population of heterogeneous units/agents which evolve over time, e.g., households, firms, industries, countries, stock market indices. In this chapter we consider a panel where each individual unit evolves over time according to order-one random coefficient autoregressive model (RCAR(1)). It is well known that aggregation of specific RCAR(1) models can explain long memory phenomenon, which is often empirically observed in economic time series (see [40] for instance). More precisely, consider a panel $\{X_i(t),\, t = 1, \dots, n,\, i = 1, \dots, N\}$, where each $X_i = \{X_i(t),\, t \in \mathbb{Z}\}$ is an RCAR(1) process with $(0, \sigma^2)$

noise and random coefficient $a_i \in (-1, 1)$, whose autocovariance

$$\mathrm{E}X_i(0)X_i(t) = \sigma^2 \int_{-1}^{1} \frac{x^{|t|}}{1-x^2} \,\mathrm{d}G(x)$$

is determined by the distribution function $G(x) = \mathrm{P}(a \le x)$, $x \in [-1, 1]$, of the AR coefficient. Granger [40] showed, for a specific beta-type distribution G, that the *contemporaneous* aggregation of independent processes $X_1, \dots, X_N$ results in a stationary Gaussian long memory process $\{\mathcal{X}(t),\ t \in \mathbb{Z}\}$, i.e.,

$$N^{-1/2} \sum_{i=1}^{N} X_i(t) \overset{\mathrm{fdd}}{\to} \mathcal{X}(t) \quad \text{as } N \to \infty, \tag{5.1}$$

where the autocovariance $\mathrm{E}\mathcal{X}(0)\mathcal{X}(t) = \mathrm{E}X_1(0)X_1(t)$ decays slowly as $t \to \infty$ so that $\sum_{t \in \mathbb{Z}} |\mathrm{E}\mathcal{X}(0)\mathcal{X}(t)| = \infty$.

A natural statistical problem is recovering the distribution G (the frequency of a across the population of individual AR(1) 'microagents') from the aggregated sample $\{\mathcal{X}(1), \dots, \mathcal{X}(n)\}$. This problem was treated in [20, 21, 61]. Some related results were obtained in [19, 48, 50]. Albeit nonparametric, the estimators in [20, 61] involve an expansion of the density $g = G'$ in an orthogonal polynomial basis and are sensitive to the choice of the tuning parameter (the number of polynomials), being limited in practice to very smooth densities g. The last difficulty in estimation of G from aggregated data is not surprising due to the fact that aggregation *per se* inflicts a considerable loss of information about the evolution of individual 'micro-agents'.

Clearly, if the available data comprises evolutions $\{X_i(1), \dots, X_i(n)\}$, $i = 1, \dots, N$, of all N individual 'micro-agents' (the panel data), we may expect a much more accurate estimate of G. Robinson [92] constructed an estimator for the moments of G using sample autocovariances of X_i and derived its asymptotic properties as $N \to \infty$, whereas the length n of each sample remains fixed. Beran et al. [9] discussed estimation of two-parameter beta densities g from panel RCAR(1) data using maximum likelihood estimators with unobservable a_i replaced by sample lag 1 autocorrelation coefficient of $X_i(1), \dots, X_i(n)$ (see Section 5.6), and derived the asymptotic normality together with some other properties of the estimators as N and n tend to infinity.

The present chapter studies nonparametric estimation of G from panel RCAR(1) data using the empirical distribution function:

$$\widehat{G}_{N,n}(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}(\widehat{a}_{i,n} \le x), \quad x \in [-1, 1], \tag{5.2}$$

where $\widehat{a}_{i,n}$ is the lag 1 sample autocorrelation coefficient of X_i, $i = 1, \dots, N$ (see (5.12)). We also discuss kernel estimation of the density $g = G'$ based on smoothed version of (5.2). We assume that individual AR(1) processes X_i are driven by identically distributed shocks containing both common and idiosyncratic (independent) components. Consistency and asymptotic normality as $N, n \to \infty$ of the above estimators are derived under some regularity conditions on G. Our results can be applied to test goodness-of-fit of the distribution G to a given hypothesized distribution (e.g., a beta distribution) using the Kolmogorov–Smirnov statistic, and to construct confidence intervals for $G(x)$ or $g(x)$.

The chapter is organized as follows. In Section 5.2 we obtain the rate of convergence of the lag 1 sample autocorrelation coefficient $\widehat{a}_{i,n}$ to a_i in probability, the result of independent interest. In Section 5.3 we prove the weak convergence of the empirical process in (5.2) to a generalized Brownian bridge. Section 5.4 treats the Kolmogorov–Smirnov goodness-of-fit test for simple and composite hypotheses of beta distributed a. In Section 5.5 we show that kernel density estimators of $g(x)$ are asymptotically normally distributed and their mean integrated squared error tends to zero. In Section 5.6 a simulation study compares the empirical performance of (5.2) and the parametric estimator of [9] when testing the equality of G to a given beta distribution. The proofs of auxiliary statements can be found in Section 5.7.

5.2 Estimation of random AR coefficient

Consider an RCAR(1) process

$$X(t) = aX(t-1) + \zeta(t), \quad t \in \mathbb{Z}, \tag{5.3}$$

where innovations $\{\zeta(t)\}$ admit the following decomposition:

$$\zeta(t) = b\eta(t) + c\xi(t), \quad t \in \mathbb{Z}, \tag{5.4}$$

where random sequences $\{\eta(t)\}$, $\{\xi(t)\}$ and random coefficients a, b, c satisfy the following conditions:

Assumption (A1). $\{\eta(t)\}$ are i.i.d. r.v.s with $\mathrm{E}\eta(0) = 0$, $\mathrm{E}\eta^2(0) = 1$, $\mathrm{E}|\eta(0)|^{2p} < \infty$ for some $p > 1$.

Assumption (A2). $\{\xi(t)\}$ are i.i.d. r.v.s with $\mathrm{E}\xi(0) = 0$, $\mathrm{E}\xi^2(0) = 1$, $\mathrm{E}|\xi(0)|^{2p} < \infty$ for the same p as in (A1).

Assumption (A3). b and c are possibly dependent r.v.s such that $\mathrm{P}(b^2 + c^2 > 0) = 1$ and $\mathrm{E}b^2 < \infty$, $\mathrm{E}c^2 < \infty$.

Assumption (A4). $a \in (-1,1)$ is a r.v. with a distribution function (d.f.) $G(x) := \mathrm{P}(a \le x)$ supported on $[-1,1]$ and satisfying

$$\mathrm{E}\Big(\frac{1}{1-|a|}\Big) = \int_{-1}^{1} \frac{\mathrm{d}G(x)}{1-|x|} < \infty. \tag{5.5}$$

Assumption (A5). a, $\{\eta(t)\}$, $\{\xi(t)\}$ and the vector $(b,c)^\top$ are mutually independent.

Remark 5.1. In the context of panel observations (see (5.10)), $\{\eta(t)\}$ is the common component and $\{\xi(t)\}$ is the idiosyncratic component of shocks. The innovation process $\{\zeta(t)\}$ in (5.4) is i.i.d. if the coefficients b and c are nonrandom. In the general case $\{\zeta(t)\}$ is a dependent and uncorrelated stationary process with $\mathrm{E}\zeta(0) = 0$, $\mathrm{E}\zeta^2(0) = \mathrm{E}b^2 + \mathrm{E}c^2$, $\mathrm{E}\zeta(0)\zeta(t) = 0$, $t \ne 0$.

Under conditions (A1)–(A5), a unique strictly stationary solution of (5.3) with finite variance exists and is written as

$$X(t) = \sum_{s \le t} a^{t-s}\zeta(s), \quad t \in \mathbb{Z}. \tag{5.6}$$

Clearly, $\mathrm{E}X(t) = 0$ and $\mathrm{E}X^2(t) = \mathrm{E}\zeta^2(0)\mathrm{E}(1-a^2)^{-1} < \infty$. Note that (5.5) is equivalent to

$$\mathrm{E}\Big(\frac{1}{1-|a|^p}\Big) < \infty, \quad 1 < p \le 2,$$

since $1-|a| \le 1-|a|^p \le 2(1-|a|)$ for $a \in (-1,1)$.

For an observed sample $X(1), \dots, X(n)$ from the stationary process in (5.6), define the sample mean $\bar{X}_n := n^{-1}\sum_{t=1}^n X(t)$ and the sample lag 1 autocorrelation coefficient

$$\widehat{a}_n := \frac{\sum_{t=1}^{n-1}(X(t)-\bar{X}_n)(X(t+1)-\bar{X}_n)}{\sum_{t=1}^{n}(X(t)-\bar{X}_n)^2}. \tag{5.7}$$

Note the estimator $\widehat{a}_n$ in (5.7) does not exceed 1 a.s. in absolute value by the Cauchy–Schwarz inequality. Moreover, it is invariant to shift and scale transformations of $\{X(t)\}$ in (5.3), i.e., we can replace $\{X(t)\}$ by $\{\rho X(t) + \mu\}$ with some (unknown) $\mu \in \mathbb{R}$ and $\rho > 0$.

Proposition 5.1. *Under Assumptions (A1)–(A5), for any $0 < \gamma < 1$ and $n \ge 1$, it holds*

$$\mathrm{P}(|\widehat{a}_n - a| > \gamma) \le C(n^{-(\frac{p}{2}\wedge(p-1))}\gamma^{-p} + n^{-1}),$$

with $C > 0$ independent of n, γ.

Proof. See Section 5.7. □

Assume now that the d.f. G of a satisfies the following Hölder condition:

Assumption (A6). There exist constants $L_G > 0$ and $\varrho \in (0,1]$ such that

$$|G(x) - G(y)| \le L_G |x-y|^{\varrho}, \quad x, y \in [-1,1]. \tag{5.8}$$

Consider the d.f. of $\widehat{a}_n$:

$$G_n(x) := \mathrm{P}(\widehat{a}_n \le x), \quad x \in [-1,1]. \tag{5.9}$$

Corollary 5.2. *Let Assumptions (A1)–(A6) hold. Then, as $n \to \infty$,*

$$\sup_{x\in[-1,1]} |G_n(x) - G(x)| = O(n^{-\frac{\varrho}{\varrho+p}(\frac{p}{2}\wedge(p-1))}).$$

Proof. Denote $\delta_n := \widehat{a}_n - a$. For any (nonrandom) $\gamma > 0$ from (5.8) we have

$$\begin{aligned}
\sup_{x\in[-1,1]} |G_n(x) - G(x)| &= \sup_{x\in[-1,1]} |\mathrm{P}(a+\delta_n \le x) - \mathrm{P}(a \le x)| \\
&\le L_G \gamma^{\varrho} + \mathrm{P}(|\delta_n| > \gamma),
\end{aligned}$$

implying

$$\sup_{x\in[-1,1]} |G_n(x) - G(x)| \le L_G \gamma^{\varrho} + C(n^{-1} + n^{-(\frac{p}{2}\wedge(p-1))}\gamma^{-p})$$

with $C > 0$ independent of n, γ. Then the corollary follows from Proposition 5.1 by taking $\gamma = \gamma_n = o(1)$ such that $\gamma_n^{\varrho} \sim n^{-(\frac{p}{2}\wedge(p-1))}\gamma_n^{-p}$ and noting that the exponent $\frac{\varrho}{\varrho+p}(\frac{p}{2} \wedge (p-1)) < 1$. □

5.3 Asymptotics of the empirical distribution function

Consider RCAR(1) processes $\{X_i(t)\}$, $i = 1, 2, \dots$, which are stationary solutions to

$$X_i(t) = a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z}, \tag{5.10}$$

with innovations $\{\zeta_i(t)\}$ having the same structure as in (5.4):

$$\zeta_i(t) = b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z}. \tag{5.11}$$

More precisely, we make the following assumption:

Assumption (B). $\{\eta(t)\}$ satisfies (A1); $\{\xi_i(t)\}$, $(b_i, c_i)^{\top}$, a_i, $i = 1, 2, \dots$, are independent copies of $\{\xi(t)\}$, $(b, c)^{\top}$, a, respectively, which satisfy Assumptions (A2)–(A6). (Note that we assume (A5) for every $i = 1, 2, \dots$.)

Remark 5.2. The individual processes $\{X_i(t)\}$ have covariance long memory if conditions (5.5) and $\int_{-1}^{1} |1-x^2|^{-2} \mathrm{d}G(x) = \infty$ hold, which is compatible with Assumption (B). The same is true about the limit aggregated process in (5.1) arising when the common component of shocks is absent (i.e. in case $\mathrm{P}(b = 0) = 1$). On the other hand, in the presence of the common component, the limit aggregated process has long memory if the individual processes have infinite variance and condition (5.5) fails, see [87].

Define the sample mean $\bar{X}_{i,n} := n^{-1} \sum_{t=1}^{n} X_i(t)$, the corresponding lag 1 sample autocorrelation coefficient

$$\widehat{a}_{i,n} := \frac{\sum_{t=1}^{n-1} (X_i(t) - \bar{X}_{i,n})(X_i(t+1) - \bar{X}_{i,n})}{\sum_{t=1}^{n} (X_i(t) - \bar{X}_{i,n})^2}, \quad 1 \le i \le N, \tag{5.12}$$

and the empirical d.f.

$$\widehat{G}_{N,n}(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}(\widehat{a}_{i,n} \le x), \quad x \in [-1,1]. \tag{5.13}$$

Recall that (5.13) is a nonparametric estimate of the d.f. $G(x) = \mathrm{P}(a_i \le x)$ from the observed panel data $\{X_i(t),\, t = 1, \dots, n,\, i = 1, \dots, N\}$. In the following theorem we show that $\widehat{G}_{N,n}(x)$ is an asymptotically unbiased estimator of $G(x)$ as n and N both tend to infinity, and prove the weak convergence of the corresponding empirical process.

Theorem 5.3. *Let the panel data model in* (5.10)–(5.11) *satisfy Assumption (B). Then, as $N, n \to \infty$,*

$$\sup_{x \in [-1,1]} |\mathrm{E}\widehat{G}_{N,n}(x) - G(x)| = O(n^{-\frac{\varrho}{\varrho+p}(\frac{p}{2} \wedge (p-1))}). \tag{5.14}$$

If, in addition,

$$N = o(n^{\frac{2\varrho}{\varrho+p}(\frac{p}{2} \wedge (p-1))}), \tag{5.15}$$

then

$$N^{1/2}(\widehat{G}_{N,n}(x) - G(x)) \Rightarrow W(x), \quad x \in [-1,1], \tag{5.16}$$

where $\{W(x),\, x \in [-1,1]\}$ is a continuous Gaussian process with zero mean and $\mathrm{Cov}(W(x), W(y)) = G(x \wedge y) - G(x)G(y)$, *$x, y \in [-1,1]$, and $\Rightarrow$ denotes the weak convergence in the space $D[-1,1]$ with the supremum (uniform) metric.*

Proof. Note $\widehat{a}_{1,n}, \dots, \widehat{a}_{N,n}$ are identically distributed, in particular, $\mathrm{E}\widehat{G}_{N,n}(x) = G_n(x)$ with $G_n(x)$ defined in (5.9). Hence, (5.14) follows immediately from Corollary 5.2.

To prove the second statement of the theorem, we approximate $\widehat{G}_{N,n}(x)$ by the empirical d.f.

$$\widehat{G}_N(x) := \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}(a_i \le x), \quad x \in [-1,1],$$

of i.i.d. r.v.s $a_1, \dots, a_N$. We have $N^{1/2}(\widehat{G}_{N,n}(x) - G(x)) = N^{1/2}(\widehat{G}_N(x) - G(x)) + D_{N,n}(x)$ with $D_{N,n}(x) := N^{1/2}(\widehat{G}_{N,n}(x) - \widehat{G}_N(x))$. Since (A6) guarantees the continuity of G, it holds

$$N^{1/2}(\widehat{G}_N(x) - G(x)) \Rightarrow W(x), \quad x \in [-1,1],$$

by the classical Donsker theorem. Then (5.16) follows once we prove

$$\sup_{x\in[-1,1]} |D_{N,n}(x)| \xrightarrow{\mathrm{p}} 0.$$

By definition,

$$D_{N,n}(x) = N^{-1/2}\sum_{i=1}^{N}(\mathbf{1}(a_i + \delta_{i,n} \le x) - \mathbf{1}(a_i \le x)) = D'_{N,n}(x) - D''_{N,n}(x),$$

where $\delta_{i,n} := \widehat{a}_{i,n} - a_i$, $i = 1, \dots, N$, and

$$\begin{aligned} D'_{N,n}(x) &:= N^{-1/2}\sum_{i=1}^{N} \mathbf{1}(x < a_i \le x - \delta_{i,n},\, \delta_{i,n} \le 0), \\ D''_{N,n}(x) &:= N^{-1/2}\sum_{i=1}^{N} \mathbf{1}(x - \delta_{i,n} < a_i \le x,\, \delta_{i,n} > 0). \end{aligned}$$

For $\gamma > 0$ we have

$$\begin{aligned} D'_{N,n}(x) &\le N^{-1/2}\sum_{i=1}^{N}\mathbf{1}(x < a_i \le x + \gamma) + N^{-1/2}\sum_{i=1}^{N}\mathbf{1}(|\delta_{i,n}| > \gamma) \\ &=: V'_N(x) + V''_{N,n}. \end{aligned}$$

(Note that $V''_{N,n}$ does not depend on x.) By Proposition 5.1, we obtain

$$\mathrm{E}V''_{N,n} = N^{-1/2}\sum_{i=1}^{N}\mathrm{P}(|\delta_{i,n}| > \gamma) \le CN^{1/2}(n^{-((p/2)\wedge(p-1))}\gamma^{-p} + n^{-1}),$$

which tends to 0 when γ is chosen as $\gamma^{\varrho+p} = n^{-((p/2)\wedge(p-1))} \to 0$. Next,

$$\begin{aligned} V'_N(x) &= N^{1/2}(\widehat{G}_N(x+\gamma) - \widehat{G}_N(x)) \\ &= N^{1/2}(G(x+\gamma) - G(x)) + U_N(x, x+\gamma], \\ U_N(x, x+\gamma] &:= N^{1/2}(\widehat{G}_N(x+\gamma) - G(x+\gamma)) - N^{1/2}(\widehat{G}_N(x) - G(x)). \end{aligned}$$

The above choice of $\gamma^{\varrho+p} = n^{-((p/2)\wedge(p-1))}$ implies $\sup_{x\in[-1,1]} N^{1/2}|G(x+\gamma) - G(x)| = O(N^{1/2}\gamma^{\varrho}) = o(1)$, whereas $U_N(x, x+\gamma]$ vanishes in the uniform metric in probability (see Lemma 5.11 in Section 5.7). Since $D''_{N,n}(x)$ is analogous to $D'_{N,n}(x)$, this proves the theorem. □

Remark 5.3. (5.15) implies that $n \gg N^{(\varrho+p)/\varrho p}$ asymptotically for $p \geq 2$. Note that $(\varrho+p)/\varrho p > 1$ and $\lim_{p\to\infty}(\varrho+p)/\varrho p = 1/\varrho$ for any $\varrho \in (0,1]$. We may conclude that Theorem 5.3 as well as other results of this chapter apply to *long* panels with n increasing much faster than N, except maybe for the limiting case $p = \infty$ for $\varrho = 1$. The main reason for this conclusion is that a_i need to be accurately estimated by (5.12) in order that $\widehat{G}_{N,n}$ behaves similarly to the empirical d.f. $\widehat{G}_N$ based on unobserved autocorrelation coefficients $a_1, \ldots, a_N$.

5.4 Goodness-of-fit testing

Theorem 5.3 can be used for testing goodness-of-fit. In the case of *simple* hypothesis, we test the null $H_0\colon G = G_0$ vs. $H_1\colon G \neq G_0$ with G_0 being a certain hypothetical distribution satisfying the Hölder condition in (5.8). Accordingly, the corresponding Kolmogorov–Smirnov (KS) test rejecting H_0 whenever

$$N^{1/2} \sup_{x\in[-1,1]} |\widehat{G}_{N,n}(x) - G_0(x)| > c(\omega) \tag{5.17}$$

has asymptotic size $\omega \in (0,1)$ provided N, n, G_0 satisfy the assumptions for (5.16) in Theorem 5.3. (Here, $c(\omega)$ is the upper ω-quantile of the Kolmogorov distribution.) However, the goodness-of-fit test in (5.17) requires the knowledge of parameters of the model considered, which is not typically a very realistic situation. Below, we consider testing *composite* hypothesis using the Kolmogov–Smirnov statistic with estimated parameters. The parameters will be estimated by the method of moments.

Write $\mu = (\mu^{(1)}, \ldots, \mu^{(m)})^\top$ and $\widehat{\mu}_{N,n} = (\widehat{\mu}^{(1)}_{N,n}, \ldots, \widehat{\mu}^{(m)}_{N,n})^\top$, where

$$\mu^{(u)} := \mathrm{E}a^u = \int_{-1}^{1} x^u \mathrm{d}G(x), \quad \widehat{\mu}^{(u)}_{N,n} := \frac{1}{N}\sum_{i=1}^{N} (\widehat{a}_{i,n})^u, \quad 1 \leq u \leq m.$$

Proposition 5.4. *Let the panel data model in* (5.10)–(5.11) *satisfy Assumption (B) with exception of Assumption (A6). If* $N = o(n^{\frac{2}{1+p}(\frac{p}{2}\wedge(p-1))})$ *as* $N, n \to \infty$*, then*

$$N^{1/2}(\widehat{\mu}_{N,n} - \mu) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \Sigma), \quad \textit{where } \Sigma := \big(\mathrm{Cov}(a^u, a^v)\big)_{1\leq u,v\leq m}. \tag{5.18}$$

Proof. Write

$$N^{1/2}(\widehat{\mu}_{N,n} - \mu) = N^{1/2}(\widehat{\mu}_{N,n} - \widehat{\mu}_N) + N^{1/2}(\widehat{\mu}_N - \mu),$$

where $\widehat{\mu}_N := \frac{1}{N}\sum_{i=1}^N (a_i, \dots, a_i^m)^\top$. We have $N^{1/2}(\widehat{\mu}_N - \mu) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \Sigma)$ as $N \to \infty$ by the multivariate central limit theorem. On the other hand, $N^{1/2}(\widehat{\mu}_{N,n} - \widehat{\mu}_N) \xrightarrow{\mathrm{p}} 0$ follows from $\mathrm{E}|\widehat{a}_n^u - a^u| \le C\mathrm{E}|\widehat{a}_n - a| \le C(\gamma + \mathrm{P}(|\widehat{a}_n - a| > \gamma))$ and Proposition 5.1 with $\gamma^{1+p} = n^{-((p/2)\wedge(p-1))}$, proving the proposition. □

Remark 5.4. Robinson [92, Theorem 7] discussed a different estimator of μ and proved it to be asymptotically normally distributed for fixed n as $N \to \infty$ in contrast to ours. However, his result holds in the case of idiosyncratic innovations only and under stronger assumption on G than in Proposition 5.4, which does not allow for long memory.

Consider testing the composite null hypothesis that G belongs to the family $\mathcal{G} = \{G_\theta,\ \theta = (\alpha, \beta)^\top \in (1, \infty)^2\}$ of beta d.f.s versus an alternative $G \notin \mathcal{G}$, where

$$G_\theta(x) = \frac{1}{\mathrm{B}(\alpha, \beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t, \quad x \in [0, 1], \tag{5.19}$$

and $\mathrm{B}(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the beta function. The uth moment of G_θ is given by

$$\mu^{(u)} = \int_0^1 x^u \mathrm{d}G_\theta(x) = \prod_{r=0}^{u-1} \frac{\alpha + r}{\alpha + \beta + r}.$$

Parameters α, β can be found from the first two moments $\mu = (\mu^{(1)}, \mu^{(2)})^\top$ as

$$\alpha = \frac{\mu^{(1)}(\mu^{(1)} - \mu^{(2)})}{\mu^{(2)} - (\mu^{(1)})^2}, \quad \beta = \frac{(1 - \mu^{(1)})(\mu^{(1)} - \mu^{(2)})}{\mu^{(2)} - (\mu^{(1)})^2}. \tag{5.20}$$

The moment-based estimator $\widehat{\theta}_{N,n} := (\widehat{\alpha}_{N,n}, \widehat{\beta}_{N,n})^\top$ of $\theta = (\alpha, \beta)^\top$ is obtained by replacing μ in (5.20) by its estimator $\widehat{\mu}_{N,n}$. The consistency and asymptotic normality of this estimator follows by the delta method from Proposition 5.4, see Corollary 5.5, where we need condition $\alpha > 1, \beta > 1$ to satisfy Assumptions (A4) and (A6).

Corollary 5.5. *Let the panel data model in* (5.10)–(5.11) *satisfy Assumption (B). Let $G = G_\theta$, $\theta = (\alpha, \beta)^\top$, be a beta d.f. in* (5.19)*, where $\alpha > 1$, $\beta > 1$. Let N, n increase as in* (5.15) *where $\varrho = 1$. Then*

$$N^{1/2}(\widehat{\theta}_{N,n} - \theta) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \Lambda_\theta), \quad \Lambda_\theta := \Delta^{-1}\Sigma(\Delta^{-1})', \tag{5.21}$$

where Σ is the 2×2 matrix in (5.18) *and*

$$\Delta := \partial\mu/\partial\theta = \begin{pmatrix} \partial\mu^{(1)}/\partial\alpha & \partial\mu^{(1)}/\partial\beta \\ \partial\mu^{(2)}/\partial\alpha & \partial\mu^{(2)}/\partial\beta \end{pmatrix}.$$

Moreover, $\widehat{\theta}_{N,n}$ *is asymptotically linear:*

$$
\begin{aligned}
N^{1/2}(\widehat{\theta}_{N,n} - \theta) &= N^{-1/2}\sum_{i=1}^{N} l_\theta(a_i) + o_{\mathrm{p}}(1), \\
l_\theta(x) &:= \Delta^{-1}(x - \mu^{(1)}, x^2 - \mu^{(2)})^\top,
\end{aligned}
\tag{5.22}
$$

where $\mathrm{E} l_\theta(a) = \int_0^1 l_\theta(x)\mathrm{d}G_\theta(x) = 0$ *and* $\mathrm{E} l_\theta(a) l_\theta(a)^\top = \int_0^1 l_\theta(x) l_\theta(x)^\top \mathrm{d}G_\theta(x) = \Lambda_\theta$.

Corollary 5.6. *Let assumptions of Corollary 5.5 hold. Then*

$$N^{1/2}(\widehat{G}_{N,n}(x) - G_{\widehat{\theta}_{N,n}}(x)) \Rightarrow V_\theta(x), \quad x \in [0,1],$$

where $\{V_\theta(x),\, x \in [0,1]\}$ *is a continuous Gaussian process with zero mean and covariance*

$$
\begin{aligned}
\mathrm{Cov}(V_\theta(x), V_\theta(y)) &= G_\theta(x \wedge y) - G_\theta(x)G_\theta(y) + \partial_\theta G_\theta(x)^\top \Lambda_\theta \partial_\theta G_\theta(y) \\
&\quad - \int_0^x l_\theta(u)^\top \mathrm{d}G_\theta(u)\partial_\theta G_\theta(y) - \int_0^y l_\theta(u)^\top \mathrm{d}G_\theta(u)\partial_\theta G_\theta(x),
\end{aligned}
$$

where $\partial_\theta G_\theta(x) := \partial G_\theta(x)/\partial\theta = (\partial G_\theta(x)/\partial\alpha, \partial G_\theta(x)/\partial\beta)^\top$, $x, y \in [0,1]$, *and* Λ_θ *is defined in* (5.21).

Proof. The d.f. G_θ with $\alpha > 1, \beta > 1$ satisfies Assumptions (A4) and (A6) with $\varrho = 1$. Recall $\widehat{G}_N(x) := N^{-1}\sum_{i=1}^N \mathbf{1}(a_i \le x)$, $x \in [0,1]$. Since condition (5.15) is satisfied, so $N^{1/2}\sup_{x\in[0,1]} |\widehat{G}_{N,n}(x) - \widehat{G}_N(x)|$ vanishes in probability by Theorem 5.3, whereas the convergence $N^{1/2}(\widehat{G}_N(x) - G_{\widehat{\theta}_{N,n}}(x)) \Rightarrow V_\theta(x)$, $x \in [0,1]$, follows from (5.22) using the fact that $\partial_\theta G_\theta(x)$, $x \in [0,1]$, is continuous in θ, see [30] or [106, Theorem 19.23]. □

With Corollary 5.6 in mind, the Kolmogorov–Smirnov test for the composite hypothesis $G \in \mathcal{G}$ can be defined as

$$\sup_{x\in[0,1]} N^{1/2}|\widehat{G}_{N,n}(x) - G_{\widehat{\theta}_{N,n}}(x)| > c_{\widehat{\theta}_{N,n}}(\omega), \tag{5.23}$$

where $c_\theta(\omega)$ is the upper ω-quantile of the distribution of $\sup_{x\in[0,1]} |V_\theta(x)|$:

$$\mathrm{P}\Big(\sup_{x\in[0,1]} |V_\theta(x)| > c_\theta(\omega)\Big) = \omega, \quad \omega \in (0,1).$$

The test in (5.23) has correct asymptotic size for any $\omega \in (0,1)$, which follows from Corollary 5.6 and the continuity of the quantile function $c_\theta(\omega)$ in θ, see [102, page 69], [106]. By writing $N^{1/2}(\widehat{G}_{N,n}(x) - G_{\widehat{\theta}_{N,n}}(x)) = N^{1/2}(\widehat{G}_{N,n}(x) - G(x)) + N^{1/2}(G(x) - G_{\widehat{\theta}_{N,n}}(x))$, it follows that the Kolmogorov–Smirnov statistic on the l.h.s. of (5.23) tends to infinity (in probability) under any fixed alternative $G \notin \mathcal{G}$ which cannot be approximated by a beta d.f. G_θ in the uniform metric, i.e., such that $\inf_\theta \sup_{x\in[0,1]} |G(x) - G_\theta(x)| > 0$. Moreover, even under the alternative, we preserve the consistency of $\widehat{\mu}_{N,n}$, hence $c_{\widehat{\theta}_{N,n}}(\omega)$ being a continuous function of sample moments, converges in probability to some finite limit. Therefore the test (5.23) is consistent.

In practice, the evaluation of $c_\theta(\omega)$ requires Monte Carlo approximation which is time-consuming. Alternatively, [98, 102] discussed parametric bootstrap procedures to produce asymptotically correct critical values. We note that the assumptions of [102, Theorem 1] are valid for the family of beta d.f.s and the moment-based estimator of θ in Corollary 5.6. The consistency of the test when using bootstrap critical values follows by a similar argument as in (5.23).

5.5 Kernel density estimation

In this section we assume G has a bounded probability density function $g(x) = G'(x)$, $x \in [-1, 1]$, implying Assumption (A6) with Hölder exponent $\varrho = 1$ in (5.8). It is of interest to estimate g in a nonparametric way from $\widehat{a}_{1,n}, \dots, \widehat{a}_{N,n}$ (5.12).

Consider the kernel density estimator

$$\widehat{g}_{N,n}(x) := \frac{1}{Nh} \sum_{i=1}^{N} K\Big(\frac{x - \widehat{a}_{i,n}}{h}\Big), \quad x \in \mathbb{R},$$

where K is a kernel, satisfying Assumption (A7) and $h = h_{N,n}$ is a bandwidth which tends to zero as N and n tend to infinity.

Assumption (A7). $K : [-1, 1] \to \mathbb{R}$ is a continuous function of bounded variation that satisfies $\int_{-1}^{1} K(x)\mathrm{d}x = 1$. Set $\|K\|_2^2 := \int_{-1}^{1} K(x)^2 \mathrm{d}x$ and $\mu_2(K) := \int_{-1}^{1} x^2 K(x)\mathrm{d}x$ and $K(x) := 0$, $x \in \mathbb{R} \setminus [-1, 1]$.

We consider two cases separately.

Case (i): $\mathrm{P}(b_1 = 0) = 1$, meaning that the coefficient $b_i = 0$ for the common shock in (5.11) is zero and that the individual processes $\{X_i(t)\}$, $i = 1, 2, \dots$, are independent and satisfy

$$X_i(t) = a_i X_i(t-1) + c_i \xi_i(t), \quad t \in \mathbb{Z}.$$

Case (ii): $\mathrm{P}(b_1 \neq 0) > 0$, meaning that $\{X_i(t)\}$, $i = 1, 2, \dots$, are mutually dependent processes.

Proposition 5.7. *Let the panel data model in* (5.10)–(5.11) *satisfy Assumption (B) and let Assumption (A7) hold. If* $n^{(p/2)\wedge(p-1)}h^{1+p} \to \infty$*, then*

$$\mathrm{E}\widehat{g}_{N,n}(x) \to g(x) \tag{5.24}$$

at every continuity point $x \in \mathbb{R}$ *of* g*. Moreover, if*

$$\begin{cases} n^{(p/2)\wedge(p-1)}h^{1+p} \to \infty & \text{in Case (i)}, \\ n^{(p/2)\wedge(p-1)}(h/N)^{1+p} \to \infty & \text{in Case (ii)}, \end{cases} \tag{5.25}$$

then

$$Nh\,\mathrm{Cov}(\widehat{g}_{N,n}(x_1), \widehat{g}_{N,n}(x_2)) \to \begin{cases} g(x_1)\|K\|_2^2 & \text{if } x_1 = x_2, \\ 0 & \text{if } x_1 \neq x_2 \end{cases} \tag{5.26}$$

at any continuity points $x_1, x_2 \in \mathbb{R}$ *of* g*. If* $Nh \to \infty$ *holds in addition to* (5.25)*, then the estimator* $\widehat{g}_{N,n}(x)$ *is consistent at every continuity point* $x \in \mathbb{R}$*:*

$$\mathrm{E}|\widehat{g}_{N,n}(x) - g(x)|^2 \to 0. \tag{5.27}$$

Proof. Throughout the proof, let $K_h(x) := K(x/h)$, $x \in \mathbb{R}$. Consider (5.24). Note $\mathrm{E}\widehat{g}_{N,n}(x) = h^{-1}\mathrm{E}K_h(x - \widehat{a}_n)$, because $\widehat{a}_{1,n}, \dots, \widehat{a}_{N,n}$ are identically distributed. Let us approximate $\widehat{g}_{N,n}(x)$ by

$$\widehat{g}_N(x) := \frac{1}{Nh}\sum_{i=1}^{N} K_h(x - a_i), \quad x \in \mathbb{R}, \tag{5.28}$$

which satisfies $\mathrm{E}\widehat{g}_N(x) = h^{-1}\mathrm{E}K_h(x - a) \to g(x)$ as $h \to 0$ at a continuity point x of g, see [76]. Integration by parts and Corollary 5.2 yield

$$\begin{aligned} h|\mathrm{E}\widehat{g}_{N,n}(x) - \mathrm{E}\widehat{g}_N(x)| &= \Big|\int_{\mathbb{R}} (G_n(y) - G(y))\mathrm{d}K_h(x - y)\Big| \\ &\le V(K) \sup_{y\in[-1,1]} |G_n(y) - G(y)| \\ &= O(n^{-((p/2)\wedge(p-1))/(1+p)}), \end{aligned} \tag{5.29}$$

uniformly in $x \in \mathbb{R}$, where $V(K)$ denotes the total variation of K and $V(K) = V(K_h)$. This proves (5.24).

Next, let us prove (5.26). We have

$$\begin{aligned} Nh\,\mathrm{Cov}(\widehat{g}_N(x_1), \widehat{g}_N(x_2)) &= \frac{1}{h}\mathrm{E}K_h(x_1 - a)K_h(x_2 - a) \\ &\to \begin{cases} g(x_1)\|K\|_2^2 & \text{if } x_1 = x_2, \\ 0 & \text{if } x_1 \neq x_2, \end{cases} \end{aligned}$$

as $h \to 0$ at any points x_1, x_2 of continuity of g, see [76]. Split $Nh\{\mathrm{Cov}(\widehat{g}_{N,n}(x_1), \widehat{g}_{N,n}(x_2)) - \mathrm{Cov}(\widehat{g}_N(x_1), \widehat{g}_N(x_2))\} = \sum_{i=1}^3 Q_i(x_1, x_2)$, where

$$\begin{aligned}
Q_1(x_1,x_2) &:= h^{-1}\{\mathrm{E}K_h(x_1-\widehat{a}_n)K_h(x_2-\widehat{a}_n) - \mathrm{E}K_h(x_1-a)K_h(x_2-a)\},\\
Q_2(x_1,x_2) &:= h^{-1}\{\mathrm{E}K_h(x_1-\widehat{a}_n)\mathrm{E}K_h(x_2-\widehat{a}_n) - \mathrm{E}K_h(x_1-a)\mathrm{E}K_h(x_2-a)\},\\
Q_3(x_1,x_2) &:= (N-1)h^{-1}\,\mathrm{Cov}(K_h(x_1-\widehat{a}_{1,n}), K_h(x_2-\widehat{a}_{2,n})).
\end{aligned}$$

Note $Q_3(x_1,x_2) = 0$ in Case (i). Similarly to (5.29),

$$\begin{aligned}
|Q_1(x_1,x_2)| &= h^{-1}\Big|\int_{\mathbb{R}} (G_n(y)-G(y))\mathrm{d}K_h(x_1-y)K_h(x_2-y)\Big|\\
&\le Ch^{-1}n^{-((p/2)\wedge(p-1))/(1+p)} \to 0,
\end{aligned}$$

since $V(K_h(x_1-\cdot)K_h(x_2-\cdot)) \le C$ and $|Q_2(x_1,x_2)| \le Ch^{-1}n^{-((p/2)\wedge(p-1))/(1+p)} \to 0$ uniformly in x_1, x_2. Finally, by Lemma 5.12,

$$\begin{aligned}
|Q_3(x_1,x_2)| &= \frac{N-1}{h}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} (\mathrm{P}(\widehat{a}_{1,n}\le y_1, \widehat{a}_{2,n}\le y_2)\\
&\qquad -\mathrm{P}(\widehat{a}_{1,n}\le y_1)\mathrm{P}(\widehat{a}_{2,n}\le y_2))\mathrm{d}K_h(x_1-y_1)\mathrm{d}K_h(x_2-y_2)\Big|\\
&\le \frac{CN}{h}\sup_{y_1,y_2\in[-1,1]} |\mathrm{P}(\widehat{a}_{1,n}\le y_1, \widehat{a}_{2,n}\le y_2) - \mathrm{P}(\widehat{a}_{1,n}\le y_1)\mathrm{P}(\widehat{a}_{2,n}\le y_2)|\\
&= O(Nh^{-1}n^{-((p/2)\wedge(p-1))/(1+p)}) = o(1),
\end{aligned}$$

proving (5.26) and the proposition. □

Remark 5.5. It follows from the proof of the above proposition that in the case of a (uniformly) continuous density $g(x)$, $x \in [-1,1]$, relations (5.24), (5.27) and the first relation in (5.26) hold uniformly in $x \in \mathbb{R}$, implying the convergence of the mean integrated squared error:

$$\int_{-\infty}^{\infty} \mathrm{E}|\widehat{g}_{N,n}(x) - g(x)|^2 \mathrm{d}x \to 0.$$

Proposition 5.8. (Asymptotic normality) *Let the panel data model in* (5.10)–(5.11) *satisfy Assumption (B) and let Assumption (A7) hold. Moreover, let K be a Lipschitz function in Case* (ii) *and assume $Nh \to \infty$ in addition to* (5.25). *Then*

$$\frac{\widehat{g}_{N,n}(x) - \mathrm{E}\widehat{g}_{N,n}(x)}{\sqrt{\mathrm{Var}(\widehat{g}_{N,n}(x))}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1) \tag{5.30}$$

at every continuity point $x \in (-1,1)$ of g such that $g(x) \ne 0$.

Proof. First, consider Case (i). Since $\widehat{g}_{N,n}(x) = (Nh)^{-1}\sum_{i=1}^N V_{i,N}$ is a (normalized) sum of i.i.d. r.v.s $V_{i,N} := K_h(x - \widehat{a}_{i,n})$ with common distribution $V_N := V_{1,N}$, it suffices to verify Lyapunov's condition

$$\frac{\mathrm{E}|V_N - \mathrm{E}V_N|^{2+\delta}}{N^{\delta/2}\{\mathrm{Var}(V_N)\}^{(2+\delta)/2}} \to 0, \tag{5.31}$$

for some $\delta > 0$. This follows by the same arguments as in [76]. Analogously to Proposition 5.7, we have $\mathrm{E}|V_N|^{2+\delta} = \mathrm{E}|K_h(x-\widehat{a}_n)|^{2+\delta} \sim hg(x)\int_{-1}^1 |K(y)|^{2+\delta}\mathrm{d}y = O(h)$ while $\mathrm{Var}(V_N) = Nh^2\,\mathrm{Var}(\widehat{g}_{N,n}(x)) \sim hg(x)\|K\|_2^2$ according to (5.26). Hence the l.h.s. of (5.31) is $O((Nh)^{-\delta/2}) = o(1)$, proving (5.30) in Case (i).

Let us turn to Case (ii). It suffices to prove that $\sqrt{Nh}(\widehat{g}_{N,n}(x) - \widehat{g}_N(x)) \xrightarrow{\mathrm{p}} 0$, for $\widehat{g}_N(x)$ given in (5.28). By $|K(x) - K(y)| \le L_K|x-y|$, $x, y \in \mathbb{R}$, for $\epsilon > 0$,

$$
\begin{aligned}
\mathrm{P}\Big(\sqrt{Nh}|\widehat{g}_{N,n}(x) - \widehat{g}_N(x)| > \epsilon\Big) &\le \mathrm{P}\Big(\frac{L_K}{\sqrt{Nh}}\sum_{i=1}^N \frac{|\widehat{a}_{i,n} - a_i|}{h} > \epsilon\Big) \\
&\le N\mathrm{P}\Big(|\widehat{a}_n - a| > \sqrt{Nh}\Big(\frac{h}{N}\Big)\frac{\epsilon}{L_K}\Big) \\
&\le C\Big(h(Nh)^{-p/2}\Big(\frac{N}{h}\Big)^{1+p} n^{-((p/2)\wedge(p-1))} + \frac{N}{n}\Big) \\
&= o(1)
\end{aligned}
$$

from Proposition 5.1 and (5.25) with $Nh \to \infty$. $\square$

Corollary 5.9. *Let assumptions of Proposition 5.8 hold with $h \sim cN^{-1/5}$ for some $c > 0$, i.e.,*

$$
N = \begin{cases} o(n^{\frac{5}{3}\frac{1}{1+p}(\frac{p}{2}\wedge(p-1))}) & \textit{in Case } \mathrm{(i)}, \\ o(n^{\frac{5}{6}\frac{1}{1+p}(\frac{p}{2}\wedge(p-1))}) & \textit{in Case } \mathrm{(ii)}. \end{cases}
$$

Moreover, let $g \in C^2[-1,1]$ and $\int_{-1}^1 yK(y)\mathrm{d}y = 0$. Then

$$
N^{2/5}(\widehat{g}_{N,n}(x) - g(x)) \xrightarrow{\mathrm{d}} \mathcal{N}(\mu(x), \sigma^2(x)),
$$

where $\mu(x) := (c^2/2)g''(x)\mu_2(K)$ and $\sigma^2(x) := (1/c)g(x)\|K\|_2^2$.

Proof. This follows from Proposition 5.8, by noting that $\mathrm{E}\widehat{g}_N(x) - g(x) \sim h^2 g''(x)\,\mu_2(K)/2$ as $h \to 0$ and $\mathrm{E}\widehat{g}_{N,n}(x) - \mathrm{E}\widehat{g}_N(x) = O(h^{-1}n^{-((p/2)\wedge(p-1))/(1+p)})$ by (5.29). $\square$

5.6 Simulations

In this section we compare our nonparametric goodness-of-fit test in (5.17) for testing the null hypothesis $G = G_0$ with its parametric analogue studied in [9]. In accordance with the last paper, we assume $\{X_i(t)\}$ in (5.10) to be independent RCAR(1) processes with standard normal i.i.d. innovations $\{\zeta_i(t)\}$, $\zeta(0) \stackrel{\mathrm{d}}{=} \mathcal{N}(0,1)$ and the random AR coefficient $a_i \in (0,1)$ having a beta-type density g with unknown parameters $\theta := (\alpha, \beta)^\top$:

$$
g(x) = \frac{2}{\mathrm{B}(\alpha,\beta)}x^{2\alpha-1}(1-x^2)^{\beta-1}, \quad x \in (0,1),\ \alpha > 1, \beta > 1. \tag{5.32}
$$

Note that $\beta \in (1,2)$ implies the long memory property in $\{X_i(t)\}$. Beran et al. [9] discuss a maximum likelihood estimator $\widehat{\theta}_{N,n,\kappa} = (\widehat{\alpha}_{N,n,\kappa}, \widehat{\beta}_{N,n,\kappa})^\top$ of $\theta = (\alpha, \beta)^\top$ when each unobservable coefficient a_i is replaced by its estimate $\widehat{a}_{i,n,\kappa} := \min\{\max\{\widehat{a}_{i,n}, \kappa\}, 1-\kappa\}$ with $\widehat{a}_{i,n}$ given in (5.12) and $0 < \kappa = \kappa_{N,n} \to 0$ is a truncation parameter. Under certain conditions on $N, n \to \infty$ and $\kappa \to 0$, Beran et al. [9, Theorem 2] showed that

$$N^{1/2}(\widehat{\theta}_{N,n,\kappa} - \theta_0) \xrightarrow{\mathrm{d}} \mathcal{N}(0, A^{-1}(\theta_0)), \tag{5.33}$$

where θ_0 is the true parameter vector,

$$A(\theta) := \begin{pmatrix} \psi_1(\alpha) - \psi_1(\alpha+\beta) & -\psi_1(\alpha+\beta) \\ -\psi_1(\alpha+\beta) & \psi_1(\beta) - \psi_1(\alpha+\beta) \end{pmatrix},$$

and $\psi_1(x) := \mathrm{d}^2 \ln \Gamma(x)/\mathrm{d}x^2$ is the trigamma function. Based on (5.17) and (5.33), we consider testing both ways (nonparametrically and parametrically) the hypothesis that the unobserved AR coefficients $a_1, \dots, a_N$ are drawn from the reference distribution G_0 having density function in (5.32) with a specific θ_0, i.e., the null $G = G_0$ vs. the alternative $G \neq G_0$. The respective test statistics are

$$T_1 := N^{1/2} \sup_{x \in [0,1]} |\widehat{G}_{N,n}(x) - G_0(x)|; \ T_2 := N(\widehat{\theta}_{N,n,\kappa} - \theta_0)^\top A(\theta_0)(\widehat{\theta}_{N,n,\kappa} - \theta_0). \tag{5.34}$$

Under the null hypothesis, the distributions of statistics T_1 and T_2 converge to the Kolmogorov distribution and the chi-square distribution with 2 degrees of freedom, respectively, see (5.17), (5.33).

To compare the performance of the above testing procedures, we compute the empirical d.f.s of the p-values of T_1 and T_2 under null and alternative hypotheses. The p-value of observed T_i is defined as $p(T_i) = 1 - \mathcal{K}_i(T_i)$, $i = 1,2$, where $\mathcal{K}_i(y)$, $i = 1,2$, denote the limit d.f.s of (5.34). Recall that when the significance level of the test is correct, the (asymptotic) distribution of the p-value is uniform on $[0,1]$. The simulation procedure to compare the performance of T_1 and T_2 is the following:

Step (S0). We fix the parameter under the null hypothesis $H_0 : \theta = \theta_0$ with $\theta_0 = (2, 1.4)^\top$.

Step (S1). We simulate 5000 panels with $N = 250$, $n = 817$ for five chosen values $\theta = (2,1.2)^\top, (2,1.3)^\top, (2,1.4)^\top, (2,1.5)^\top, (2,1.6)^\top$ of beta parameters.

Step (S2). For each simulated panel we compute the p-value of statistics T_1 and T_2.

Step (S3). The empirical d.f.s of computed p-values of statistics T_1 and T_2 are plotted.

The values of beta parameters $\theta_0 = (2, 1.4)^\top$, N, n were chosen in accordance with the simulation study in [9].

Figure 5.1 presents the simulation results under the true hypothesis $\theta = \theta_0$ with zoom-in on small p-values. We see that both d.f.s in the left graph are approximately linear. Somewhat surprisingly, it appears that the empirical size of T_1 (the nonparametric test) is better than the size of T_2 (the parametric test). Particularly, for significance levels 0.05 and 0.1 we provide the empirical size values in Table 5.1.

Figure 5.2 gives the graphs of the empirical d.f.s of p-values of T_1 and T_2 for several alternatives $\theta \neq \theta_0$. It appears that for $\beta > \beta_0 = 1.4$ the parametric test T_2 is more powerful than the nonparametric test T_1 but for $\beta < \beta_0$ the power differences are less significant. Table 5.1 illustrates the empirical power for the significance levels 0.05, 0.1.

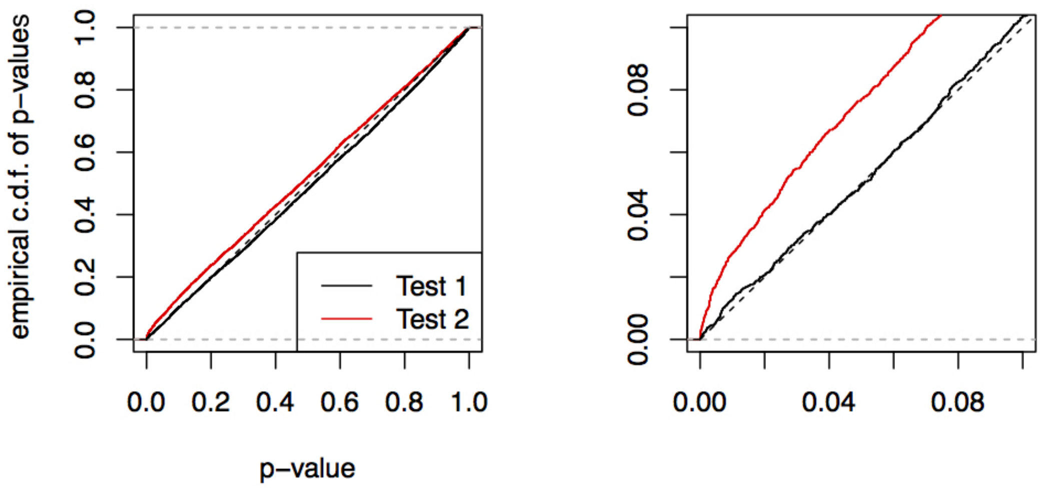

Figure 5.1: [left] Empirical d.f. of p-values of T_1 and T_2 under $H_0 : \theta = (2, 1.4)^\top$; 5000 replications with $N = 250$, $n = 817$. [right] Zoom-in on the region of interest: p-values smaller than 0.1.

The above simulations (Figures 5.1 and 5.2, Table 5.1) refer to the case of independent individual processes $\{X_i(t)\}$. There are no theoretical results for the parametric test T_2, when RCAR(1) series are dependent. Although the nonparametric test T_1 is valid for the latter case, one may expect that the presence of the common shock component in the panel data in (5.11) has a negative effect on the test performance for short series. To illustrate this effect, we simulate 5000 panels

ω	5%					10%				
β	1.2	1.3	1.4	1.5	1.6	1.2	1.3	1.4	1.5	1.6
T_1	.532	.137	.049	.208	.576	.653	.223	.103	.315	.702
T_2	.500	.104	.077	.313	.735	.634	.184	.134	.421	.827

Table 5.1: Numerical results of the comparison for testing procedure $H_0\colon \theta = (2, 1.4)^\top$ at the significance level $\omega = 5\%$ and $\omega = 10\%$. The column for $\beta = 1.4$ provides the empirical size.

with RCAR(1) processes $\{X_i(t)\}$ driven by dependent shocks in (5.11) with $b_i = b$, $c_i = (1 - b^2)^{1/2}$. As previously, we choose $\theta_0 = (2, 1.4)^\top$, $N = 250$, $n = 817$ and we fix $\theta = (2, 1.4)^\top$ to evaluate the empirical size of T_1. Figure 5.3[left] presents the graphs of the empirical d.f.s of the p-values of T_1 for $b = 1$, $b = 0.6$ and $b = 0$, the latter corresponding to independent individual processes as in Figure 5.1. We see that the size of the test worsens as b increases, particularly for $b = 1$ when $\{X_i(t)\}$ are all driven by the same shocks. To overcome the last effect, the sample length n of each series in the panel may be increased as in Figure 5.3[right], where the choice of $n = 5500$ and $b = 1$ shows a much better performance of T_1 under the null hypothesis $\theta = \theta_0 = (2, 1.4)^\top$ and the alternative ($\theta = (2, 1.5)^\top$ and $\theta = (2, 1.6)^\top$) scenarios.

In conclusion,

1. We do not observe an important loss of the power for the nonparametric KS test T_1 compared to the parametric approach.

2. The KS test T_1 does not require to choose any tuning parameter contrary to the test T_2.

3. One can use the KS test T_1 under weaker assumptions on RCAR(1) innovations. We only impose moment conditions. The dependence between the series is allowed by (5.11).

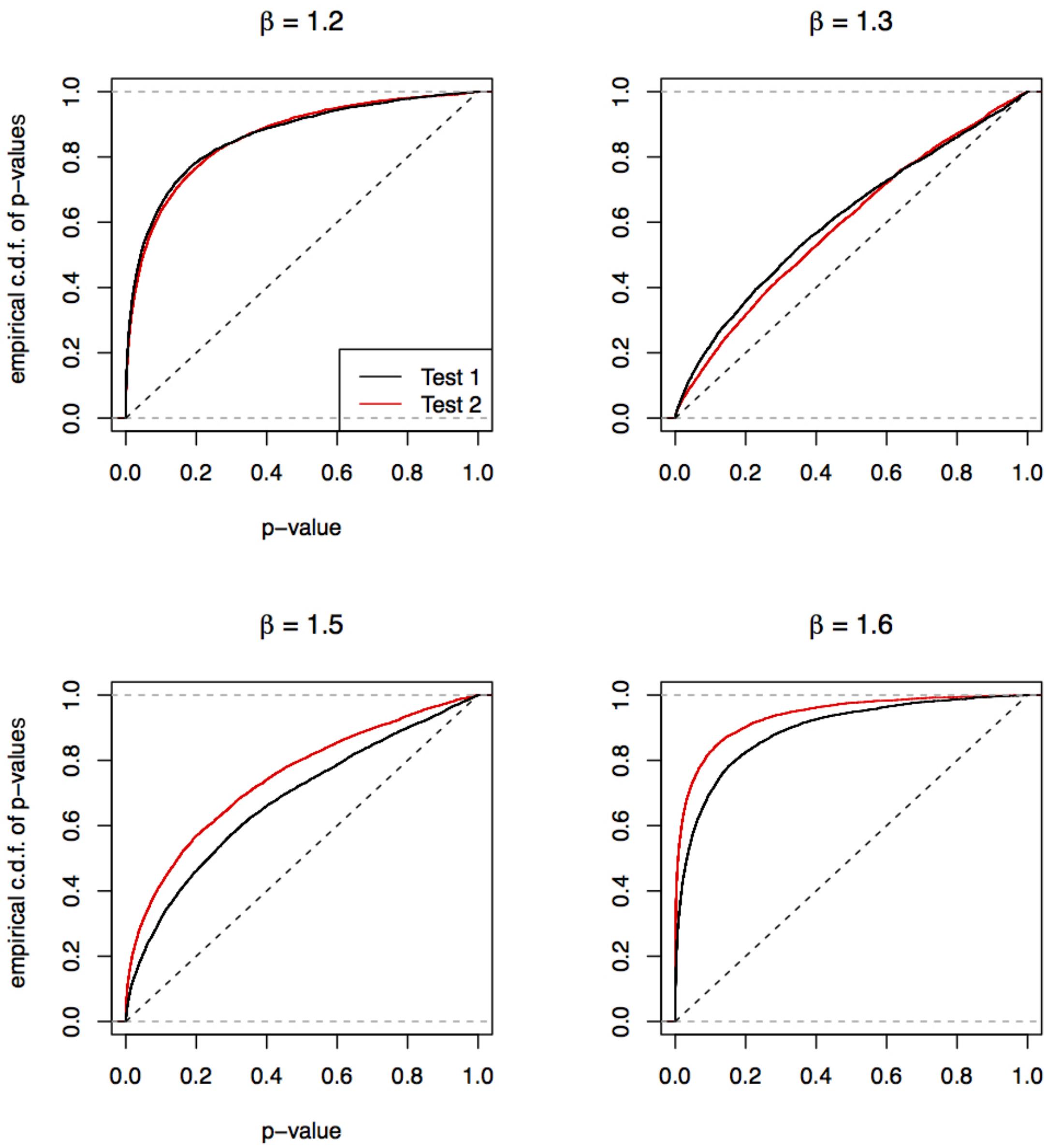

Figure 5.2: Empirical d.f. of p-values of T_1 and T_2 for testing $H_0 : \theta = (2, 1.4)^\top$ under several alternatives of the form $\theta = (2, \beta)^\top$; 5000 replications with $N = 250$, $n = 817$.

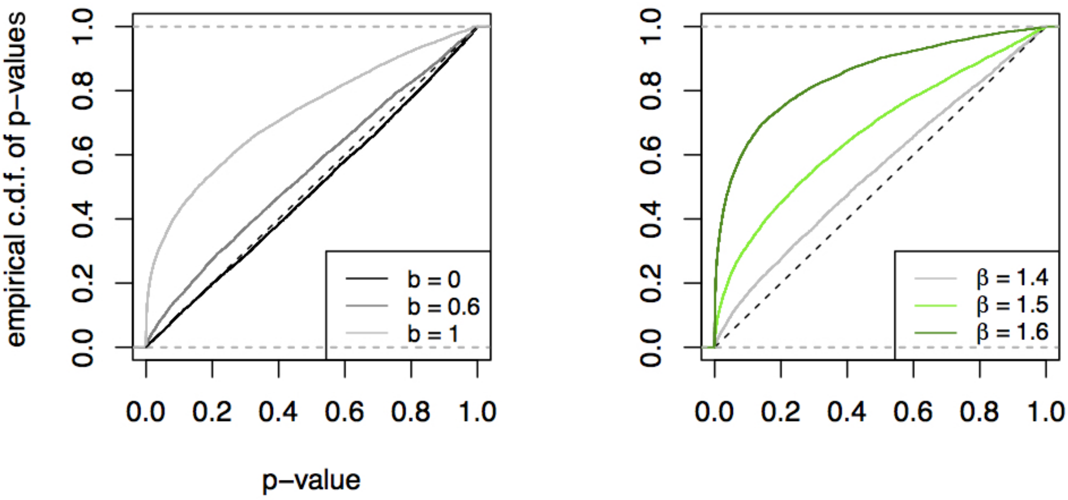

Figure 5.3: [left] Empirical d.f. of p-values of T_1 under $H_0\colon \theta = (2, 1.4)^\top$ for different dependence structure between RCAR(1) series : $b_i = b$ and $c_i = (1-b^2)^{1/2}$ and $N = 250$, $n = 817$. [right] Empirical d.f. of p-values of T_1 for testing $H_0 : \theta = (2, 1.4)^\top$. RCAR(1) series are driven by common innovations, i.e., $b_i = 1$, $c_i = 0$, for $\theta = (2, \beta)^\top$; 5000 replications with $N = 250$, $n = 5500$.

5.7 Some proofs and auxiliary lemmas

We use the following martingale moment inequality.

Lemma 5.10. *Let $p > 1$ and $\{\xi_j, j \ge 1\}$ be a martingale difference sequence: $\mathrm{E}[\xi_j|\xi_1,\dots,\xi_{j-1}] = 0$, $j = 2,3,\dots$, with $\mathrm{E}|\xi_j|^p < \infty$. Then there exists a constant $C_p < \infty$ depending only on p and such that*

$$\mathrm{E}\Big|\sum_{j=1}^{\infty}\xi_j\Big|^p \le C_p \begin{cases} \sum_{j=1}^{\infty}\mathrm{E}|\xi_j|^p, & 1 < p \le 2, \\ \big(\sum_{j=1}^{\infty}(\mathrm{E}|\xi_j|^p)^{2/p}\big)^{p/2}, & p > 2. \end{cases} \tag{5.35}$$

For $1 < p \le 2$, inequality (5.35) is known as von Bahr and Esséen inequality, see [107], and for $p > 2$, it is a consequence of the Burkholder and Rosenthal inequality ([18, 93], see also [36, Lemma 2.5.2]).

Proof of Proposition 5.1. Since $\widehat{a}_n$ in (5.7) is invariant w.r.t. a scale factor of innovations $\{\zeta(t)\}$, w.l.g. we can assume $b^2 + c^2 = 1$ and $\mathrm{E}\zeta^2(0) = 1$, $\mathrm{E}|\zeta(0)|^{2p} < \infty$. Then $\widehat{a}_n - a = \sum_{i=1}^{3}\delta_{ni}$, where

$$\begin{aligned} \delta_{n1} &:= -\frac{aX^2(n)}{\sum_{t=1}^{n}X^2(t) - n(\bar X_n)^2}, \quad \delta_{n2} := \frac{\sum_{t=1}^{n-1}X(t)\zeta(t+1)}{\sum_{t=1}^{n}X^2(t) - n(\bar X_n)^2}, \\ \delta_{n3} &:= \frac{\bar X_n(X(1)+X(n)) - (\bar X_n)^2(1+n(1-a))}{\sum_{t=1}^{n}X^2(t) - n(\bar X_n)^2}. \end{aligned}$$

The statement of the proposition follows from

$$\mathrm{P}(|\delta_{ni}| > \gamma) \le C(n^{-1} + n^{-((p/2)\wedge(p-1))}\gamma^{-p}) \quad (0 < \gamma < 1,\ i = 1,2,3). \tag{5.36}$$

To show (5.36) for $i = 1$, note that $\delta_{n1} = L_n/(n + D_n)$, where $L_n := -a(1-a^2)X^2(n)$ and $D_n = D_{n1} - D_{n2}$, $D_{n1} := \sum_{t=1}^{n}((1-a^2)X^2(t) - 1)$, $D_{n2} := n(1-a^2)(\bar X_n)^2$. We have $\mathrm{P}(|\delta_{n1}| > \gamma) \le \mathrm{P}(|D_n| > n/2) + \mathrm{P}(|L_n| > n\gamma/2)$. Thus, (5.36) for $i = 1$ follows from

$$\mathrm{E}|D_{n1}|^{p\wedge 2} \le Cn, \quad \mathrm{E}|D_{n2}| \le C \quad \text{and} \quad \mathrm{E}|L_n|^p \le C. \tag{5.37}$$

Consider the first relation in (5.37). Clearly, it suffices to prove it for $1 < p \le 2$ only. We have $D_{n1} = 2D'_{n1} + D''_{n1}$, where

$$\begin{aligned} D'_{n1} &:= (1-a^2)\sum_{s_2 < s_1 \le n}\ \sum_{t=1\vee s_1}^{n} a^{2(t-s_1)}a^{s_1-s_2}\zeta(s_1)\zeta(s_2), \\ D''_{n1} &:= (1-a^2)\sum_{s \le n}\ \sum_{t=1\vee s}^{n} a^{2(t-s)}(\zeta^2(s) - 1). \end{aligned}$$

We will use the following elementary inequality: for any $-1 \le a \le 1$, $n \ge 1$, $s \le n$,

$$\begin{aligned} \alpha_n(s) := (1-a^2) \sum_{t=1 \vee s}^{n} a^{2(t-s)} &= \begin{cases} a^{2(1-s)}(1-a^{2n}), & s \le 0, \\ 1-a^{2(n+1-s)}, & 1 \le s \le n \end{cases} \\ &\le C \begin{cases} a^{-2s} \min(1, 2n(1-|a|)), & s \le 0, \\ 1, & 1 \le s \le n. \end{cases} \end{aligned} \tag{5.38}$$

Using the independence of $\{\zeta(s)\}$ and a and inequality (5.35) (twice) for $1 < p \le 2$ we obtain

$$\begin{aligned} \mathrm{E}|D'_{n1}|^p &= \mathrm{E}\Big| \sum_{s_1 \le n} \alpha_n(s_1)\zeta(s_1) \sum_{s_2 < s_1} a^{s_1-s_2}\zeta(s_2)\Big|^p \\ &\le C\mathrm{E} \sum_{s_1 \le n} \Big|\alpha_n(s_1)\zeta(s_1) \sum_{s_2 < s_1} a^{s_1-s_2}\zeta(s_2)\Big|^p \\ &\le C\mathrm{E} \sum_{s_1 \le n} |\alpha_n(s_1)|^p \sum_{s_2 < s_1} |a|^{p(s_1-s_2)} \\ &\le C\mathrm{E}(1-|a|)^{-1} \sum_{s \le n} |\alpha_n(s)|^p \le Cn \end{aligned}$$

since $\mathrm{E}(1-|a|)^{-1} < \infty$ (see (5.5)) and $\sum_{s \le n} |\alpha_n(s)|^p \le Cn$ follows from (5.38). Similarly, since $\{\zeta^2(s) - 1,\ s \le n\}$ form a martingale difference sequence,

$$\mathrm{E}|D''_{n1}|^p \le C\mathrm{E} \sum_{s \le n} |\alpha_n(s)|^p \le Cn,$$

proving the first inequality (5.37). The second inequality in (5.37) follows by noting that $n\bar{X}_n = \sum_{s \le n}(\sum_{t=1\vee s}^{n} a^{t-s})\zeta(s)$ and

$$(1-a^2)\mathrm{E}[(n\bar{X}_n)^2|a] = a^2\Big(\frac{1-a^n}{1-a}\Big)^2 + (1-a^2)\sum_{s=1}^{n}\Big(\frac{1-a^s}{1-a}\Big)^2 \le \frac{Cn}{1-a}.$$

Consider the last inequality in (5.37). We have $|L_n| \le |2L'_n + L''_n + 1|$, where

$$\begin{aligned} L'_n &:= (1-a^2) \sum_{s_2 < s_1 \le n} a^{2(n-s_1)} a^{s_1-s_2} \zeta(s_1)\zeta(s_2), \\ L''_n &:= (1-a^2) \sum_{s \le n} a^{2(n-s)}(\zeta^2(s) - 1). \end{aligned}$$

We use Lemma 5.10, as above. Let $1 \le p \le 2$. Then $\mathrm{E}|L''_n|^p \le C\mathrm{E}\sum_{s\le n}\{(1-a^2)a^{2(n-s)}\}^p \le C$ and $\mathrm{E}|L'_n|^p \le C\mathrm{E}\sum_{s_2<s_1\le n}\{(1-a^2)|a|^{2(n-s_1)}|a|^{s_1-s_2}\}^p \le C\mathrm{E}(1-|a|)^{p-2} \le C$. Next, let $p \ge 2$. Then $\mathrm{E}|L''_n|^p \le C\mathrm{E}\{\sum_{s\le n}|(1-a^2)a^{2(n-s)}|^2\}^{p/2} \le C$ and $\mathrm{E}|L'_n|^p \le C\mathrm{E}(1-a^2)^p\{\sum_{s_2<s_1\le n} a^{4(n-s_1)}a^{2(s_1-s_2)}\}^{p/2} \le C$, proving (5.37) and hence (5.36) for $i = 1$.

Consider (5.36) for $i = 2$. We have $\delta_{n2} = R_n/(n + D_n)$, where $R_n := (1 - a^2)\sum_{t=1}^{n-1} X(t)\zeta(t+1)$ and D_n is the same as in (5.37). Then $\mathrm{P}(|\delta_{n2}| > \gamma) \le \mathrm{P}(|R_n| > n\gamma/2) + \mathrm{P}(|D_n| > n/2)$, where

$$\begin{aligned} \mathrm{P}(|D_n| > n/2) &\le (n/4)^{-(p\wedge 2)}\mathrm{E}|D_{n1}|^{p\wedge 2} + (n/4)^{-1}\mathrm{E}|D_{n2}| \\ &\le C\begin{cases} n^{-(p-1)}, & 1 < p \le 2, \\ n^{-1}, & p > 2, \end{cases} \end{aligned} \tag{5.39}$$

according to (5.37). Therefore (5.36) for $i = 2$ follows from

$$\mathrm{E}|R_n|^p \le C\begin{cases} n, & 1 < p \le 2, \\ n^{p/2}, & p > 2. \end{cases} \tag{5.40}$$

Since $R_n = (1 - a^2)\sum_{s\le n-1}\zeta(s)\sum_{t=1\vee s}^{n-1} a^{t-s}\zeta(t+1)$ is a sum of martingale differences, by inequality (5.35) with $1 < p \le 2$ we obtain

$$\begin{aligned} \mathrm{E}|R_n|^p &\le C\mathrm{E}\sum_{s\le n-1}\Big|(1-a^2)\zeta(s)\sum_{t=1\vee s}^{n-1} a^{t-s}\zeta(t+1)\Big|^p \\ &\le C\mathrm{E}|1-a^2|^p\sum_{s\le n-1}\sum_{t=1\vee s}^{n-1}|a|^{p(t-s)} \\ &\le C\mathrm{E}|1-a^2|^p\Big(\sum_{s\le 0}|a|^{-ps}\sum_{t=1}^{n-1}|a|^{pt} + \sum_{s=1}^{n-1}\sum_{t=s}^{n-1}|a|^{p(t-s)}\Big) \\ &\le C\mathrm{E}|1-a^2|^p\{(1-|a|^p)^{-2} + n(1-|a|^p)^{-1}\} \le Cn, \end{aligned}$$

proving (5.40) for $p \le 2$. Similarly, using (5.35) with $p > 2$ we get

$$\begin{aligned} \mathrm{E}|R_n|^p &= \mathrm{E}\Big[|1-a^2|^p\mathrm{E}\Big[\Big|\sum_{s\le n-1}\zeta(s)\sum_{t=1\vee s}^{n-1} a^{t-s}\zeta(t+1)|^p\Big|a\Big]\Big] \\ &\le C\mathrm{E}\Big[|1-a^2|^p\Big\{\sum_{s\le n-1}\Big(\mathrm{E}\Big[\Big|\zeta(s)\sum_{t=1\vee s}^{n-1} a^{t-s}\zeta(t+1)|^p\Big|a\Big]\Big)^{2/p}\Big\}^{p/2}\Big] \\ &\le C\mathrm{E}|1-a^2|^p\Big\{\sum_{s\le n-1}\sum_{t=1\vee s}^{n-1} a^{2(t-s)}\Big\}^{p/2} \\ &\le C\mathrm{E}|1-a^2|^p\Big\{\sum_{s\le 0} a^{-2s}\sum_{t=1}^{n-1} a^{2t} + \sum_{s=1}^{n-1}\sum_{t=s}^{n-1} a^{2(t-s)}\Big\}^{p/2} \\ &\le C\mathrm{E}|1-a^2|^p\{(1-a^2)^{-2} + n(1-a^2)^{-1}\}^{p/2} \le Cn^{p/2}, \end{aligned}$$

proving (5.40) and (5.36) for $i = 2$.

It remains to prove (5.36) for $i = 3$. Similarly as above, $\mathrm{P}(|\delta_{n3}| > \gamma) \le \mathrm{P}(|Q_n| > n\gamma/2) + \mathrm{P}(|D_n| > n/2)$, where $Q_n := (1 - a^2)\{\bar{X}_n(X(1) + X(n)) -$

$(\bar{X}_n)^2(1+n(1-a))\}$ and D_n is evaluated in (5.39). Thus, (5.36) for $i=3$ follows from (5.39) and

$$\begin{aligned} \mathrm{E}|Q_n|^p &\le C\{\mathrm{E}|(1-a^2)X^2(n)|^p + \mathrm{E}|(1-a^2)(\bar{X}_n)^2|^p \\ &\quad + n^p\mathrm{E}|(1-a)(1-a^2)(\bar{X}_n)^2|^p\} \le C. \end{aligned} \tag{5.41}$$

Since $n\bar{X}_n = \sum_{s\le n}(\sum_{t=1\vee s}^n a^{t-s})\zeta(s)$, an application of the second inequality of (5.35) yields

$$\mathrm{E}[|n\bar{X}_n|^{2p}|a] \le C\Big(\frac{(1-a^n)^2}{(1-a^2)(1-a)^2} + \sum_{s=1}^n \Big(\frac{1-a^s}{1-a}\Big)^2\Big)^p.$$

Using $1-a^n \le 1\wedge(n(1-a))$ we obtain $\mathrm{E}|(1-a)(1-a^2)(\bar{X}_n)^2|^p \le Cn^{-p}$ and $\mathrm{E}|(1-a^2)(\bar{X}_n)^2|^p \le C\mathrm{E}(1-a)^{-1}n^{-1}$. Finally, $\mathrm{E}|(1-a^2)X^2(n)|^p \le C$ follows by the same arguments as $\mathrm{E}|L_n|^p \le C$ (see (5.37)). This proves (5.41), thereby completing the proof of (5.36) and of the proposition, too. □

Let $a, a_1, \ldots, a_N$ be i.i.d. r.v.s with d.f. $G(x) = \mathrm{P}(a \le x)$ supported on $[-1,1]$. Define $\widehat{G}_N(x) := N^{-1}\sum_{i=1}^N \mathbf{1}(a_i \le x)$, $U_N(x) := N^{1/2}(\widehat{G}_N(x) - G(x))$, $x \in [-1,1]$, and $\omega_N(\delta)$ (= the modulus of continuity of U_N) by

$$\omega_N(\delta) := \sup_{0\le y-x\le\delta} |U_N(y) - U_N(x)|, \quad \delta > 0.$$

Lemma 5.11. *Assume that G satisfies Assumption (A6). Then for all $\epsilon > 0$,*

$$\epsilon^4\mathrm{P}(\omega_N(\delta) > 6\epsilon) \le (3+3C)L_G\delta^{\varrho} + N^{-1},$$

where C is a constant independent of ϵ, δ, N.

Proof. As in [13, page 106, equation (13.17)] we have that

$$\begin{aligned} \mathrm{E}|U_N(y) - U_N(x)|^2|U_N(z) - U_N(y)|^2 &\le 3\mathrm{P}(a \in (x,y])\mathrm{P}(a \in (y,z]), \\ \mathrm{E}|U_N(y) - U_N(x)|^4 &\le 3\mathrm{P}(a \in (x,y])^2 + N^{-1}\mathrm{P}(a \in (x,y]) \end{aligned}$$

for $-1 \le x \le y \le z \le 1$, where the second inequality treats the 4th central moment of a binomial variable. Now fix $\delta > 0$ and split $[-1,1] = \cup_i \Delta_i$, where $\Delta_i = [-1+i\delta, -1+(i+1)\delta]$, $i = 0,1,\ldots,\lfloor 2/\delta\rfloor - 1$, $\Delta_{\lfloor 2/\delta\rfloor} = [-1+\lfloor 2/\delta\rfloor\delta, 1]$. According to [109, page 49, Lemma 1], for all $\epsilon > 0$,

$$\epsilon^4\mathrm{P}(\omega_N(\delta) > 6\epsilon) \le (3+3C)\max_i \mathrm{P}(a \in \Delta_i) + N^{-1},$$

where C is a constant independent of ϵ, δ, N. Lemma follows from Assumption (A6) on the d.f. G of the r.v. a. □

Note that if we take $\delta = \delta_N = o(1)$, we then get $\mathrm{P}(\omega_N(\delta) > \epsilon) \to 0$ as $N \to \infty$.

Lemma 5.12. *Let $\widehat{a}_{1,n}$, $\widehat{a}_{2,n}$ be given in* (5.12) *under Assumptions (A1)–(A6) with $\varrho = 1$. Then for all $\gamma \in (0,1)$ and $n \geq 1$, it holds*

$$\sup_{x,y\in[-1,1]} |\mathrm{P}(\widehat{a}_{1,n} \leq x,\, \widehat{a}_{2,n} \leq y) - \mathrm{P}(\widehat{a}_{1,n} \leq x)\mathrm{P}(\widehat{a}_{2,n} \leq y)| = O(n^{-((p/2)\wedge(p-1))/(1+p)}).$$

Proof. Define $\delta_{i,n} := \widehat{a}_{i,n} - a_i$, $i = 1, 2$. For $\gamma \in (0,1)$, we have

$$\begin{aligned}\mathrm{P}(|\delta_{1,n}| > \gamma \,\text{or}\, |\delta_{2,n}| > \gamma) &\leq \mathrm{P}(|\delta_{1,n}| > \gamma) + \mathrm{P}(|\delta_{2,n}| > \gamma)\\ &\leq C(n^{-((p/2)\wedge(p-1))}\gamma^{-p} + n^{-1})\end{aligned}$$

by Proposition (5.1). Consider now

$$\begin{aligned}\mathrm{P}(\widehat{a}_{1,n} \leq x,\, \widehat{a}_{2,n} \leq y) &= \mathrm{P}(a_1 + \delta_{1,n} \leq x,\, a_2 + \delta_{2,n} \leq y)\\ &\leq \mathrm{P}(a_1 + \delta_{1,n} \leq x,\, a_2 + \delta_{2,n} \leq y,\, |\delta_{1,n}| \leq \gamma,\, |\delta_{2,n}| \leq \gamma)\\ &\qquad + \mathrm{P}(|\delta_{1,n}| > \gamma \,\text{or}\, |\delta_{2,n}| > \gamma).\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{P}(a_1 + \delta_{1,n} \leq x,\, a_2 + \delta_{2,n} \leq y,\, |\delta_{1,n}| \leq \gamma,\, |\delta_{2,n}| \leq \gamma)\\ &\quad \leq \mathrm{P}(a_1 \leq x + \gamma,\, a_2 \leq y + \gamma,\, |\delta_{1,n}| \leq \gamma,\, |\delta_{2,n}| \leq \gamma)\\ &\quad \leq G(x+\gamma)G(y+\gamma)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{P}(a_1 + \delta_{1,n} \leq x,\, a_2 + \delta_{2,n} \leq y,\, |\delta_{1,n}| \leq \gamma,\, |\delta_{2,n}| \leq \gamma)\\ &\quad \geq \mathrm{P}(a_1 \leq x - \gamma,\, a_2 \leq y - \gamma, |\delta_{1,n}| \leq \gamma,\, |\delta_{2,n}| \leq \gamma)\\ &\quad \geq G(x-\gamma)G(y-\gamma) - \mathrm{P}(|\delta_{1,n}| > \gamma \,\text{or}\, |\delta_{2,n}| > \gamma).\end{aligned}$$

From (5.8) we obtain

$$\begin{aligned}&|G(x \pm \gamma)G(y \pm \gamma) - G(x)G(y)|\\ &\quad = |(G(x) + O(\gamma))(G(y) + O(\gamma)) - G(x)G(y)| \leq C\gamma.\end{aligned}$$

Hence,

$$|\mathrm{P}(a_1 \leq x,\, a_2 \leq y) - G(x)G(y)| \leq C(\gamma + n^{-1} + n^{-((p/2)\wedge(p-1))}\gamma^{-p}). \tag{5.42}$$

In a similar way,

$$|\mathrm{P}(a_1 \leq x)\mathrm{P}(a_2 \leq y) - G(x)G(y)| \leq C(\gamma + n^{-1} + n^{-((p/2)\wedge(p-1))}\gamma^{-p}). \tag{5.43}$$

By (5.42), (5.43), the proof of the lemma is complete with $\gamma = \gamma_n = o(1)$, which satisfies $\gamma_n \sim n^{-((p/2)\wedge(p-1))}\gamma_n^{-p}$. □

Chapter 6

Scaling transition for nonlinear random fields

This chapter contains the article [82]. We obtain a complete description of anisotropic scaling limits and the existence of scaling transition for nonlinear functions (Appell polynomials) of stationary linear random fields on $\mathbb{Z}^2$ with moving average coefficients decaying at possibly different rate in the horizontal and vertical direction. This chapter extends recent results on scaling transition for linear random fields in [89, 90].

6.1 Introduction

[90] introduced the notion of scaling transition for stationary random field (RF) $X = \{X(t,s),\, (t,s) \in \mathbb{Z}^2\}$ in terms of partial sums limits

$$D_{\lambda,\gamma}^{-1} \sum_{(t,s)\in K_{[\lambda x, \lambda^\gamma y]}} X(t,s) \overset{\mathrm{fdd}}{\to} V_\gamma(x,y), \quad (x,y) \in \mathbb{R}_+^2, \quad \lambda \to \infty, \quad \gamma > 0, \tag{6.1}$$

where $D_{\lambda,\gamma} \to \infty$ is normalization and $K_{[\lambda x, \lambda^\gamma y]} := \{(t,s) \in \mathbb{Z}^2 : 1 \le t \le \lambda x, 1 \le s \le \lambda^\gamma y\}$ is a family of rectangles whose sides grow at possibly different rate $O(\lambda)$ and $O(\lambda^\gamma)$ and $\gamma > 0$ is *arbitrary*. RF X is said to exhibit *scaling transition* at $\gamma_0 > 0$ if the limit RFs $V_\gamma \equiv V_\gamma^X$ in (6.1) do not depend on γ for $\gamma > \gamma_0$ and $\gamma < \gamma_0$ and are different up to a multiplicative constant, viz.,

$$V_\gamma^X \overset{\mathrm{fdd}}{=} V_+^X \ (\forall \gamma > \gamma_0), \quad V_\gamma^X \overset{\mathrm{fdd}}{=} V_-^X \ (\forall \gamma < \gamma_0), \quad V_+^X \overset{\mathrm{fdd}}{\neq} aV_-^X \ (\forall a > 0).$$

In such case, RF $V_{\gamma_0}^X$ is called the *well-balanced* while RFs V_+^X and V_-^X the *unbalanced* scaling limits of X.

It appears that scaling transition is a new and general feature of spatial dependence which occurs for many isotropic and anisotropic RFs on $\mathbb{Z}^2$ with long-range dependence (LRD). It was established for a class of aggregated α-stable autoregressive models [90], a class of Gaussian LRD RFs [89], and some RFs arising by aggregation of network traffic and random-coefficient time series models in telecommunications and economics; see [35, 70, 79, 80], also [90, Remark 2.3]. The unbalanced limits $V_{\pm}^{X}$ in these studies have a very special dependence structure (either independent or invariant rectangular increments along one of the coordinate axes) and coincide in the Gaussian case with a fractional Brownian sheet (FBS) B_{H_1,H_2} with one of the two parameters $H_1, H_2 \in (0,1]$ equal to 1/2 or 1.

The above mentioned works deal with linear RF models written as sums (stochastic integrals) w.r.t. i.i.d. 'noise'. It is well-known that nonlinear RFs can display quite complicated nongaussian scaling behavior. See Dobrushin and Major [26], also [2, 4, 36, 38, 45, 59, 66, 100, 104] and the references therein.

The present chapter establishes the existence of scaling transition for a class of nonlinear subordinated RFs:

$$X(t,s) = G(Y(t,s)), \quad (t,s) \in \mathbb{Z}^2, \tag{6.2}$$

where $Y = \{Y(t,s), (t,s) \in \mathbb{Z}^2\}$ is a stationary linear LRD RF in (6.3) and $G(x) = A_k(x)$, $x \in \mathbb{R}$, is the Appell polynomial of degree $k \geq 1$ (see Section 6.2 for the definition) with $\mathrm{E}G(Y(0,0))^2 < \infty$, $\mathrm{E}G(Y(0,0)) = 0$. The (underlying) RF Y is written as a moving-average

$$Y(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a(t-u, s-v)\varepsilon(u,v), \quad (t,s) \in \mathbb{Z}^2, \tag{6.3}$$

in a standardized i.i.d. sequence $\{\varepsilon(u,v), (u,v) \in \mathbb{Z}^2\}$ with deterministic moving-average coefficients such that

$$a(t,s) \sim \mathrm{const}(|t|^2 + |s|^{2q_2/q_1})^{-q_1/2}, \quad |t| + |s| \to \infty, \tag{6.4}$$

where parameters $q_1, q_2 > 0$ satisfy

$$1 < Q := \frac{1}{q_1} + \frac{1}{q_2} < 2. \tag{6.5}$$

In Theorems 6.1–6.5, the moving-average coefficients $a(t,s)$ may take a more general form in (6.10) including an 'angular function'. Condition $Q < 2$ guarantees that $\sum_{(t,s)\in\mathbb{Z}} a(t,s)^2 < \infty$ or Y in (6.3) is well-defined, while $Q > 1$ implies that $\sum_{(t,s)\in\mathbb{Z}} |a(t,s)| = \infty$ (in other words, that RF Y is LRD). Note $a(t,0) = O(|t|^{-q_1})$, $a(0,s) = O(|s|^{-q_2})$ decay at a different rate when $q_1 \neq q_2$ in which case Y exhibits strong anisotropy. The form of moving-average coefficients in (6.4) implies a similar behavior of the covariance function $r_Y(t,s) := \mathrm{E}Y(0,0)Y(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a(u,v)a(t+u,s+v)$, namely,

$$C_1(|t|^2+|s|^{2p_2/p_1})^{-p_1/2} \le r_Y(t,s) \le C_2(|t|^2+|s|^{2p_2/p_1})^{-p_1/2}, \quad |t|+|s|\to\infty, \tag{6.6}$$

for some positive constants $C_1, C_2 > 0$, where

$$p_i := q_i(2-Q), \quad i=1,2. \tag{6.7}$$

Note $p_1/p_2 = q_1/q_2$ and the 1-1 correspondence between (q_1,q_2) and (p_1,p_2):

$$q_i := \frac{p_i}{2}(1+P), \quad i=1,2, \quad \text{where } P := \frac{1}{p_1}+\frac{1}{p_2}. \tag{6.8}$$

(6.6) implies that for any integer $k \ge 1$ and $P \notin \mathbb{N}$,

$$\sum_{(t,s)\in\mathbb{Z}^2} |r_Y(t,s)|^k = \infty \Longleftrightarrow 1 \le k < P. \tag{6.9}$$

See Propositions 6.8 and 6.10. In the case when Y in (6.3) is a (standardized) Gaussian RF, $r_Y(t,s)^k k!$ coincides with the covariance of the kth Hermite polynomial $H_k(Y(t,s))$ of Y and the (nonlinear) subordinated RF $X = H_k(Y)$ is LRD if condition (6.9) holds. A similar result is true for nongaussian moving-average RF Y in (6.3) and Hermite polynomial H_k replaced by Appell polynomial A_k.

The following summary describes the main results of this chapter.

(R1) Subordinated RFs $X = A_k(Y)$, $1 \le k < P$, exhibit scaling transition at the same point $\gamma_0 := p_1/p_2 = q_1/q_2$ independent of k.

(R2) The well-balanced scaling limit $V^X_{\gamma_0}$ of $X = A_k(Y)$ is nongaussian unless $k=1$ and is given by a k-tuple Itô–Wiener integral.

(R3) Unbalanced scaling limits $V^X_+ = V^X_\gamma$, $\gamma > \gamma_0$, of $X = A_k(Y)$ agree with FBS $B_{H^+_{1k},1/2}$ with Hurst parameter $H^+_{1k} \in (1/2,1)$ if $kp_2 > 1$, and with a 'generalized Hermite slide' $V^X_+(x,y) = xZ^+_k(y)$ if $kp_2 < 1$, where Z^+_k is a self-similar process written as a k-tuple Itô–Wiener integral. A similar fact holds for unbalanced limits $V^X_- = V^X_\gamma$, $\gamma < \gamma_0$.

(R4) For $k > P$, RF $X = A_k(Y)$ does not exhibit scaling transition and all scaling limits V^X_γ, $\gamma > 0$, agree with Brownian sheet $B_{1/2,1/2}$.

(R5) In the case of Gaussian underlying RF Y in (6.3), the above conclusions hold for general nonlinear function G in (6.2) and k equal to the Hermite rank of G.

The above list contains several new noncentral and central limit results. (R2), (R4) and (R5) are new in the 'anisotropic' case $p_1 \neq p_2$ while (R3) is new even for linear RF $X = A_1(Y) = Y$ (see Remark 6.1 concerning the terminology in (R3)). Similarly as in the case of linear models (see [89, 90]), unbalanced limits in (R3) have either independent or completely dependent increments along one of the coordinate axes. According to (R3), the sample mean of nonlinear LRD RF $X = A_k(Y)$, $1 < k < P$, on rectangles $K_{[\lambda,\lambda^\gamma]}$, $\gamma \neq \gamma_0$, may have Gaussian or nongaussian limit distribution depending on k, γ and parameters p_1, p_2, moreover, in both cases the variance of the sum $\sum_{(t,s)\in K_{[\lambda,\lambda^\gamma]}} X(t,s)$ grows faster than $\lambda^{1+\gamma}$, or the number of summands. The dichotomy of the limit distribution in (R3) is related to the presence or absence of the vertical/horizontal LRD property of X, see Remark 6.4. We also note that our proofs of the central limit results in (R3) and (R4) use rather simple approximation by m-dependent r.v.s and do not require a combinatorial argument or Malliavin's calculus as in [16, 72] and other papers.

It is well-known that Appell polynomials play a similar role to Hermite polynomials in limit theorems for nonlinear functions of linear nongaussian LRD processes, except that they lack the orthogonality property of the latter and therefore expansions in Appell polynomials are of limited use. See [4, 36, 101]. Particularly, the results in (R1)–(R4) hold for arbitrary polynomial $G(x) = \sum_{j=k}^{m} c_j A_j(x)$ with $c_k \neq 0$ under suitable moment assumptions on the innovations. However, except for the Gaussian case, dealing with non-polynomial functions of LRD processes requires different techniques, see e.g. [45, 101], and is much harder in the case of noncausal spatial models, c.f. [28].

The results of this chapter have direct relevance for statistics of strongly dependent spatial data by showing that the (asymptotic) shape of the spatial region may have a drastic effect on the limit distribution of linear and nonlinear statistics. See Section 6.8 (Final comment) at the end of the chapter.

The rest of the chapter is organized as follows. Section 6.2 provides the precise assumptions on RFs Y and X and some known properties of Appell polynomials. Section 6.3 contains formulations of the main results (Theorems 6.1–6.5) as described in (R1)–(R5) above. Section 6.4 provides two examples of linear fractionally integrated RFs satisfying the assumptions in Section 6.2. Section 6.5 discusses some properties of generalized homogeneous functions and their convolutions used to prove the results. Section 6.6 discusses the asymptotic form of the covariance function and the asymptotics of the variance of anisotropic partial sums of subordinated RF $X = A_k(Y)$. All proofs are collected in Section 6.7.

In this chapter, we denote $\mathbb{R}_+ := (0,\infty)$, $\mathbb{R}^2_+ := (0,\infty)^2$, $\mathbb{R}^2_0 := \mathbb{R}^2 \setminus \{(0,0)\}$, $\mathbb{Z}_+ := \{0,1,\dots\}$, $\mathbb{N} := \{1,2,\dots\}$, $\mathbb{Z}^{\bullet 2k} := \{((u_1,v_1),\dots,(u_k,v_k)) \in \mathbb{Z}^{2k} : (u_i,v_i) \neq (u_j,v_j), 1 \le i < j \le k\}$, $k \in \mathbb{N}$.

6.2 Assumptions and preliminaries

Assumption (A1). $\{\varepsilon, \varepsilon(t,s), (t,s) \in \mathbb{Z}^2\}$ is an i.i.d. sequence with $\mathrm{E}\varepsilon = 0$, $\mathrm{E}\varepsilon^2 = 1$.

Assumption (A2). $Y = \{Y(t,s), (t,s) \in \mathbb{Z}^2\}$ is a moving-average RF in (6.3) with coefficients

$$a(t,s) = \frac{1}{(|t|^2 + |s|^{2q_2/q_1})^{q_1/2}}\Big(L_0\Big(\frac{t}{(|t|^2+|s|^{2q_2/q_1})^{1/2}}\Big) + o(1)\Big), \quad |t|+|s| \to \infty, \tag{6.10}$$

$(t,s) \neq (0,0)$, where $q_i > 0$, $i = 1,2$, satisfy $Q = \sum_{i=1}^2 q_i^{-1} \in (1,2)$ (see (6.5)) and $L_0(u) \ge 0$, $u \in [-1,1]$, is a bounded piece-wise continuous function on $[-1,1]$.

We refer to L_0 in (6.10) as *angular function.* Assumptions (A1)–(A2) imply $\mathrm{E}Y(0,0)^2 = \sum_{(t,s)\in\mathbb{Z}^2} a(t,s)^2 < \infty$ and hence RF Y in (6.3) is well-defined and stationary, with zero mean $\mathrm{E}Y(0,0) = 0$. Moreover, if $\mathrm{E}|\varepsilon|^\alpha < \infty$ for some $\alpha > 2$ then $\mathrm{E}|Y(0,0)|^\alpha < \infty$ follows by Rosenthal's inequality; see e.g. [36, Corollary 2.5.1].

Given a r.v. ξ with $\mathrm{E}|\xi|^k < \infty$, $k \in \mathbb{Z}_+$, the kth Appell polynomial $A_k(x)$ relative to the distribution of ξ is defined by $A_k(x) := (-\mathrm{i})^k \mathrm{d}^k(\mathrm{e}^{\mathrm{i}ux}/\mathrm{E}\mathrm{e}^{\mathrm{i}u\xi})/\mathrm{d}u^k\big|_{u=0}$. See [4, 36] for various properties of Appell polynomials. In what follows, $A_k(\xi)$ stands for the r.v. obtained by substituting $x = \xi$ in the Appell polynomial $A_k(x)$ relative to the distribution of ξ. Particularly, if $\mathrm{E}\xi = 0$ then $A_1(\xi) = \xi$, $A_2(\xi) = \xi^2 - \mathrm{E}\xi^2$, $A_3(\xi) = \xi^3 - 3\xi\mathrm{E}\xi^2 - \mathrm{E}\xi^3$ etc. For standard normal $\xi \overset{\mathrm{d}}{=} \mathcal{N}(0,1)$ the Appell polynomials $A_k(\xi) = H_k(\xi) = (-\mathrm{i})^k \mathrm{d}^k \mathrm{e}^{\mathrm{i}u\xi + u^2/2}/\mathrm{d}u^k\big|_{u=0}$ agree with the Hermite polynomials.

Assumption (A3)$_k$. For $k \in \mathbb{N}$, $\mathrm{E}|\varepsilon|^{2k} < \infty$ and

$$X = \{X(t,s) := A_k(Y(t,s)), (t,s) \in \mathbb{Z}^2\}, \tag{6.11}$$

where A_k is the kth Appell polynomial relative to the (marginal) distribution of $Y(t,s)$ in (6.3).

We also use the representation of (6.11) via *Wick products* of noise variables (see [36, Chapter 14]):

$$A_k(Y(t,s)) = \sum_{(u,v)_k \in \mathbb{Z}^{2k}} a(t-u_1, s-v_1)\cdots a(t-u_k, s-v_k) :\varepsilon(u_1,v_1)\cdots\varepsilon(u_k,v_k): . \tag{6.12}$$

By definition, for *mutually distinct* points $(u_j, v_j) \neq (u_{j'}, v_{j'})$ $(j \neq j', 1 \leq j, j' \leq i)$ the Wick product $:\varepsilon(u_1, v_1)^{k_1} \cdots \varepsilon(u_i, v_i)^{k_i} := \prod_{j=1}^{i} A_{k_j}(\varepsilon(u_j, v_j))$ equals the product of *independent* r.v.s $A_{k_j}(\varepsilon(u_j, v_j))$, $1 \leq j \leq i$. (6.12) leads to the decomposition of (6.11) into the 'off-diagonal' and 'diagonal' parts:

$$A_k(Y(t,s)) = Y^{\bullet k}(t,s) + \mathcal{Z}(t,s), \tag{6.13}$$

where

$$Y^{\bullet k}(t,s) := \sum_{(u,v)_k}^{\bullet} a(t-u_1, s-v_1) \cdots a(t-u_k, s-v_k)\varepsilon(u_1,v_1) \cdots \varepsilon(u_k,v_k) \tag{6.14}$$

and the sum $\sum^{\bullet}_{(u,v)_k}$ is taken over all $(u,v)_k = ((u_1,v_1), \cdots, (u_k,v_k)) \in \mathbb{Z}^{2k}$ such that $(u_i, v_i) \neq (u_j, v_j)$ $(i \neq j)$ (the set of such $(u,v)_k \in \mathbb{Z}^{2k}$ will be denoted by $\mathbb{Z}^{\bullet 2k}$). By definition, the 'diagonal' part $\mathcal{Z}(t,s)$ in (6.13) is given by the r.h.s. of (6.12) with $(u,v)_k \in \mathbb{Z}^{2k}$ replaced by $(u,v)_k \in \mathbb{Z}^{2k} \setminus \mathbb{Z}^{\bullet 2k}$. In most of our limit results, $\mathcal{Z}(t,s)$ is negligible and $Y^{\bullet k}(t,s)$ is the main term which is easier to handle compared to $A_k(Y(t,s))$ in (6.13). We also note that limit distributions of partial sums of 'off-diagonal' polynomial forms in i.i.d. r.v.s were studied in [5, 36, 100] and other works.

Assumption (A4)$_k$. $\varepsilon(0,0) \overset{\mathrm{d}}{=} Z$ and $Y(0,0) \overset{\mathrm{d}}{=} Z$ have standard normal distribution $Z \overset{\mathrm{d}}{=} \mathcal{N}(0,1)$ and $X(t,s) = G(Y(t,s))$, where $G = G(x)$, $x \in \mathbb{R}$, is a measurable function with $\mathrm{E}G(Z)^2 < \infty$, $\mathrm{E}G(Z) = 0$ and Hermite rank $k \geq 1$.

Assumptions (A1), (A2) and (A4)$_k$ imply that Y in (6.3) is a Gaussian RF. As noted above, under Assumption (A4)$_k$ Appell polynomials $A_k(x)$ coincide with Hermite polynomials $H_k(x)$. Recall that the Hermite rank of a measurable function $G : \mathbb{R} \to \mathbb{R}$ with $\mathrm{E}G(Z)^2 < \infty$ is defined as the index k of the lowest nonzero coefficient c_j in the Hermite expansion of G, viz., $G(x) = \sum_{j=k}^{\infty} c_j H_j(x)/j!$ where $c_k \neq 0$.

Let $L^2(\mathbb{R}^{2k})$ denote the Hilbert space of real-valued functions $h = h((u,v)_k)$, $(u,v)_k = (u_1, v_1, \ldots, u_k, v_k) \in \mathbb{R}^{2k}$ with finite norm $\|h\|_k := \{\int_{\mathbb{R}^{2k}} h((u,v)_k)^2 \mathrm{d}(u,v)_k\}^{1/2}$, $\mathrm{d}(u,v)_k = \mathrm{d}u_1\mathrm{d}v_1 \cdots \mathrm{d}u_k\mathrm{d}v_k$. Let $W = \{W(\mathrm{d}u, \mathrm{d}v),\ (u,v) \in \mathbb{R}^2\}$ denote a real-valued Gaussian white noise with zero mean and variance $\mathrm{E}W(\mathrm{d}u,\mathrm{d}v)^2 = \mathrm{d}u\mathrm{d}v$. For any $h \in L^2(\mathbb{R}^{2k})$ the k-tuple Itô–Wiener integral $\int_{\mathbb{R}^{2k}} h((u,v)_k)\mathrm{d}^k W = \int_{\mathbb{R}^{2k}} h(u_1, v_1, \ldots, u_k, v_k) W(\mathrm{d}u_1, \mathrm{d}v_1) \cdots W(\mathrm{d}u_k, \mathrm{d}v_k)$ is well-defined and satisfies $\mathrm{E}\int_{\mathbb{R}^{2k}} h((u,v)_k)\mathrm{d}^k W = 0$, $\mathrm{E}(\int_{\mathbb{R}^{2k}} h((u,v)_k)\mathrm{d}^k W)^2 \leq k!\|h\|_k^2$; see e.g. [36].

6.3 Main results

Recall the definitions of p_i, P in (6.7), (6.8); $\gamma_0 = q_1/q_2 = p_1/p_2$. Denote

$$S^X_{\lambda,\gamma}(x,y) := \sum_{(t,s)\in K_{[\lambda x, \lambda^\gamma y]}} X(t,s), \quad S^X_{\lambda,\gamma} := S^X_{\lambda,\gamma}(1,1).$$

Consider a RF

$$V^X_{k,\gamma_0}(x,y) := \int_{\mathbb{R}^{2k}} h(x,y;(u,v)_k)\mathrm{d}^k W, \quad (x,y)\in\mathbb{R}^2_+, \tag{6.15}$$

where

$$\begin{aligned} h(x,y;(u,v)_k) &:= \int_{(0,x]\times(0,y]} \prod_{i=1}^k a_\infty(t-u_i, s-v_i)\mathrm{d}t\mathrm{d}s, \qquad (6.16)\\ a_\infty(t,s) &:= (|t|^2 + |s|^{2q_2/q_1})^{-q_1/2} L_0\big(t/(|t|^2+|s|^{2q_2/q_1})^{1/2}\big), \quad (t,s)\in\mathbb{R}^2. \end{aligned}$$

Theorem 6.1. *(i) The RF V^X_{k,γ_0} in (6.15)–(6.16) is well-defined for $1\le k<P$ as an Itô–Wiener stochastic integral and has zero mean $\mathrm{E}V^X_{k,\gamma_0}(x,y)=0$ and finite variance $\mathrm{E}V^X_{k,\gamma_0}(x,y)^2 = k!\|h(x,y;\cdot)\|^2_k$. Moreover, RF V^X_{k,γ_0} has stationary rectangular increments and satisfies the operator-scaling property (see [12])*

$$\{V^X_{k,\gamma_0}(\lambda x, \lambda^{\gamma_0} y), (x,y)\in\mathbb{R}^2_+\} \overset{\mathrm{fdd}}{=} \{\lambda^{H(\gamma_0)} V^X_{k,\gamma_0}(x,y), (x,y)\in\mathbb{R}^2_+\}, \quad \forall\lambda>0, \tag{6.17}$$

where $H(\gamma_0) := 1+\gamma_0 - kp_1/2$.
(ii) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1\le k<P$. Then as $\lambda\to\infty$,

$$\mathrm{Var}(S^X_{\lambda,\gamma_0}) \sim c(\gamma_0)\lambda^{2H(\gamma_0)}, \quad c(\gamma_0) := k!\|h(1,1;\cdot)\|^2_k \tag{6.18}$$

and

$$\lambda^{-H(\gamma_0)} S^X_{\lambda,\gamma_0}(x,y) \overset{\mathrm{fdd}}{\to} V^X_{k,\gamma_0}(x,y). \tag{6.19}$$

Next, we discuss the case $1\le k<P$, $\gamma\ne\gamma_0$. This case is split into four subcases: (c1): $\gamma>\gamma_0$, $k>1/p_2$, (c2): $\gamma>\gamma_0$, $k<1/p_2$, (c3): $\gamma<\gamma_0$, $k>1/p_1$, and (c4): $\gamma<\gamma_0$, $k<1/p_1$ (the 'boundary' cases $k=1/p_i$, $i=1,2$, are more delicate and omitted, see Remark 6.2). Cases (c3) and (c4) are symmetric to (c1) and (c2) and essentially follow by exchanging the coordinates t and s. Introduce random processes Z^+_k and Z^-_k with one-dimensional time:

$$Z^+_k(y) := \int_{\mathbb{R}^{2k}} h_+(y;(u,v)_k)\mathrm{d}^k W, \quad Z^-_k(x) := \int_{\mathbb{R}^{2k}} h_-(x;(u,v)_k)\mathrm{d}^k W, \quad x,y\in\mathbb{R}_+, \tag{6.20}$$

where

$$h_+(y;(u,v)_k) := \int_0^y \prod_{i=1}^k a_\infty(u_i, s-v_i)\mathrm{d}s, \quad h_-(x;(u,v)_k) := \int_0^x \prod_{i=1}^k a_\infty(t-u_i, v_i)\mathrm{d}t, \tag{6.21}$$

and $a_\infty(t,s)$ is defined in (6.16). In Theorem 6.3, $\star$ stands for convolution of functions indexed by $\mathbb{R}^2$ (see Section 6.5 for definition).

Theorem 6.2. *(i) Processes Z_k^+ and Z_k^- in (6.20) are well-defined for $1 \le k < 1/p_2$ and $1 \le k < 1/p_1$, respectively, as Itô–Wiener stochastic integrals. They have zero mean, finite variance, stationary increments and are self-similar with respective indices $H_{2k}^+ := 1 - kp_2/2 \in (1/2, 1)$ and $H_{1k}^- := 1 - kp_1/2 \in (1/2, 1)$.*
(ii) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1 \le k < 1/p_2$. Then for any $\gamma > \gamma_0$, as $\lambda \to \infty$,

$$\mathrm{Var}(S_{\lambda,\gamma}^X) \sim c(\gamma)\lambda^{2H(\gamma)}, \tag{6.22}$$

where $H(\gamma) := 1 + \gamma H_{2k}^+$ and $c(\gamma) := k!\|h_+(1;\cdot)\|_k^2 > 0$. Moreover,

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}^X(x,y) \overset{\mathrm{fdd}}{\to} x Z_k^+(y), \quad \lambda \to \infty. \tag{6.23}$$

(iii) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1 \le k < 1/p_1$. Then for any $\gamma < \gamma_0$, as $\lambda \to \infty$,

$$\mathrm{Var}(S_{\lambda,\gamma}^X) \sim c(\gamma)\lambda^{2H(\gamma)},$$

where $H(\gamma) := \gamma + H_{1k}^-$ and $c(\gamma) := k!\|h_-(1;\cdot)\|_k^2 > 0$. Moreover,

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}^X(x,y) \overset{\mathrm{fdd}}{\to} y Z_k^-(x), \qquad \lambda \to \infty.$$

Remark 6.1. Processes $Z_k^\pm$ in (6.20) have a similar structure and properties to generalized Hermite processes discussed in [5] except that (6.20) are defined as k-tuple Itô–Wiener integrals with respect to white noise in $\mathbb{R}^2$ and not in $\mathbb{R}$ as in [5]. Following the terminology in [81], RFs $xZ_k^+(y)$ and $yZ_k^-(x)$ may be called a *generalized Hermite slide* since they represent a random surface 'sliding linearly to 0' along one of the coordinate on the plane from a generalized Hermite process indexed by the other coordinate. In the Gaussian case $k = 1$, a generalized Hermite slide agrees with a FBS B_{H_1,H_2} where one of the two parameters H_1, H_2 equals 1. Recall that a fractional Brownian sheet (FBS) $B_{H_1,H_2} = \{B_{H_1,H_2}(x,y), (x,y) \in \mathbb{R}_+^2\}$ with parameters $0 < H_1, H_2 \le 1$ is a Gaussian process with zero mean and covariance function

$$\begin{aligned} \mathrm{E}B_{H_1,H_2}(x_1,y_1)B_{H_1,H_2}(x_2,y_2) &= (1/4)(x_1^{2H_1} + x_2^{2H_1} - |x_1 - x_2|^{2H_1}) \\ &\quad \times (y_1^{2H_2} + y_2^{2H_2} - |y_1 - y_2|^{2H_2}). \end{aligned} \tag{6.24}$$

Theorem 6.3. *(i) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1/p_2 < k < P$. Then for any $\gamma > \gamma_0$, as $\lambda \to \infty$,*

$$\mathrm{Var}(S^X_{\lambda,\gamma}) \sim c(\gamma)\lambda^{2H(\gamma)}, \tag{6.25}$$

where $H(\gamma) := H^+_{1k} + \gamma/2$, $H^+_{1k} := 1 + \gamma_0/2 - kp_1/2 \in (1/2, 1)$ and $c(\gamma) := k! \int_{(0,1]^2\times\mathbb{R}}((a_\infty \star a_\infty)(t_1 - t_2, s))^k \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s > 0$. Moreover,

$$\lambda^{-H(\gamma)} S^X_{\lambda,\gamma}(x, y) \overset{\mathrm{fdd}}{\to} c(\gamma)^{1/2} B_{H^+_{1k},1/2}(x, y), \quad \lambda \to \infty. \tag{6.26}$$

(ii) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1/p_1 < k < P$. Then for any $\gamma < \gamma_0$, as $\lambda \to \infty$,

$$\mathrm{Var}(S^X_{\lambda,\gamma}) \sim c(\gamma)\lambda^{2H(\gamma)}, \tag{6.27}$$

where $H(\gamma) := \gamma H^-_{2k} + 1/2$, $H^-_{2k} := 1 + 1/(2\gamma_0) - kp_2/2 \in (1/2, 1)$ and $c(\gamma) := k! \int_{\mathbb{R}\times(0,1]^2}((a_\infty \star a_\infty)(t, s_1 - s_2))^k \mathrm{d}t\mathrm{d}s_1\mathrm{d}s_2 > 0$. Moreover,

$$\lambda^{-H(\gamma)} S^X_{\lambda,\gamma}(x, y) \overset{\mathrm{fdd}}{\to} c(\gamma)^{1/2} B_{1/2,H^-_{2k}}(x, y), \quad \lambda \to \infty. \tag{6.28}$$

Remark 6.2. Note $H^+_{1k} = 1$ ($kp_2 = 1$) and $H^-_{2k} = 1$ ($kp_1 = 1$). We expect that the convergences (6.26) and (6.28) remain true (modulo a logarithmic correction of normalization) in the 'boundary' cases $kp_2 = 1$ and $kp_1 = 1$ of Theorem 6.3(i) and (ii) and the limit RFs in these cases agree with FBS $B_{1,1/2}$ or $B_{1/2,1}$, respectively, having both parameters equal to 1 or 1/2.

The next theorem discusses the case $k > P$.

Theorem 6.4. *Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$ and $k > P$. Then for any $\gamma > 0$, as $\lambda \to \infty$,*

$$\lambda^{-(1+\gamma)}\,\mathrm{Var}(S^X_{\lambda,\gamma}) \to \sigma^2_X,$$

where $\sigma^2_X := \sum_{(t,s)\in\mathbb{Z}^2} \mathrm{Cov}(X(0, 0), X(t, s)) \in [0, \infty)$. Moreover, if $\sigma^2_X > 0$ then

$$\lambda^{-(1+\gamma)/2} S^X_{\lambda,\gamma}(x, y) \overset{\mathrm{fdd}}{\to} \sigma_X B_{1/2,1/2}(x, y), \quad \lambda \to \infty.$$

Our last theorem extends the above results to general function G having Hermite rank k and Gaussian underlying RF Y.

Theorem 6.5. *Let $X = G(Y)$ satisfy Assumptions (A1), (A2) and (A4)$_k$. Assume w.l.g. that G has Hermite expansion $G(x) = H_k(x) + \sum_{j=k+1}^{\infty} c_j H_j(x)/j!$.*
(i) Let $1 \le k < P$. Then RF X satisfies all statements of Theorems 6.1–6.3.
(ii) Let $k > P$. Then RF X satisfies the statements of Theorem 6.4.

According to Theorems 6.2–6.3, the unbalanced scaling limits $V_\pm^X$ of RF $X = A_k(Y)$ satisfying Assumptions (A1)–(A3)$_k$ are given by

$$\begin{aligned} V_+^X(x,y) &= \begin{cases} xZ_k^+(y), & kp_2 < 1, \\ c_+^{1/2} B_{H_{1k}^+,1/2}(x,y), & kp_2 > 1, \end{cases} \\ V_-^X(x,y) &= \begin{cases} yZ_k^-(x), & kp_1 < 1, \\ c_-^{1/2} B_{1/2,H_{2k}^-}(x,y), & kp_1 > 1, \end{cases} \end{aligned} \tag{6.29}$$

where $c_\pm \equiv c(\gamma) > 0$ are constants given in Theorem 6.3. The covariance functions of RFs $V_\pm^X$ in (6.29) agree (modulo a constant) with the covariance of FBS B_{H_1,H_2} where at least one of the two parameters H_1, H_2 equals 1 or 1/2, namely $(H_1,H_2) = (1, H_{2k}^+)$ if $kp_2 < 1$, $= (H_{1k}^+, 1/2)$ if $kp_2 > 1$ in the case of V_+^X, and $(H_1,H_2) = (H_{1k}^-, 1)$ if $kp_1 < 1$, $= (1/2, H_{2k}^-)$ if $kp_1 > 1$ in the case of V_-^X. These facts and the explicit form of the covariance of FBS, see (6.24), imply that $V_+^X \overset{\mathrm{fdd}}{\neq} aV_-^X$ $(\forall a > 0)$, for any k, p_1, p_2 in Theorems 6.2–6.3, yielding the following corollary.

Corollary 6.6. *Let RF $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1 \le k < P$, $kp_i \neq 1$, $i = 1,2$. Then X exhibits scaling transition at $\gamma_0 = p_1/p_2$.*

6.4 Examples: fractionally integrated RFs

In this section we present two examples of linear fractionally integrated RFs Y on $\mathbb{Z}^2$ satisfying Assumptions (A1) and (A2).

Example 1. Isotropic fractionally integrated random field. Introduce the (discrete) Laplace operator $\Delta Y(t,s) := (1/4)\sum_{|u|+|v|=1}(Y(t+u,s+v) - Y(t,s))$ and a lattice isotropic fractionally integrated RF $\{Y(t,s), (t,s) \in \mathbb{Z}^2\}$ satisfying the equation:

$$(-\Delta)^d Y(t,s) = \varepsilon(t,s), \tag{6.30}$$

where $\{\varepsilon(t,s), (t,s) \in \mathbb{Z}^2\}$ are standard i.i.d. r.v.s, $0 < d < 1/2$ is the order of fractional integration, $(1-z)^d = \sum_{j=0}^\infty \psi_j(d) z^j$, $\psi_j(d) := \Gamma(j-d)/\Gamma(j+1)\Gamma(-d)$. More explicitly,

$$(-\Delta)^d Y(t,s) = \sum_{j=0}^\infty \psi_j(d)(1+\Delta)^j Y(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} b(u,v) Y(t-u, s-v), \tag{6.31}$$

where $b(u,v) := \sum_{j=0}^\infty \psi_j(d) p_j(u,v)$ and $p_j(u,v)$ are j-step transition probabilities of a symmetric nearest-neighbor random walk $\{W_j, j = 0,1,\dots\}$ on $\mathbb{Z}^2$ with equal 1-step probabilities $\mathrm{P}(W_1 = (u,v)|W_0 = (0,0)) = 1/4$, $|u|+|v| = 1$. Note $\sum_{(u,v)\in\mathbb{Z}^2} |b(u,v)| = \sum_{j=0}^{\infty} |\psi_j(d)| < \infty$, $d > 0$, and therefore the l.h.s. of (6.31) is well-defined for any stationary RF Y with $\mathrm{E}|Y(0,0)| < \infty$. As shown in [56], for $0 < d < 1/2$ a stationary solution of (6.30) with zero-mean and finite variance can be defined as a moving-average RF:

$$Y(t,s) = (-\Delta)^{-d}\varepsilon(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a(u,v)\varepsilon(t-u, s-v), \tag{6.32}$$

with coefficients

$$a(u,v) = \sum_{j=0}^{\infty} \psi_j(-d) p_j(u,v) \tag{6.33}$$

satisfying $\sum_{(u,v)\in\mathbb{Z}^2} a(u,v)^2 < \infty$. Moreover, RF Y in (6.32) has an explicit spectral density $f(x,y) = (2\pi)^{-2}2^{-2d}|(1-\cos x) + (1-\cos y)|^{-2d}$, $(x,y) \in [-\pi,\pi]^2$, which behaves as $\mathrm{const}(x^2+y^2)^{-2d}$ as $x^2+y^2 \to 0$. According to [56, Proposition 5.1], the moving-average coefficients in (6.33) satisfy the isotropic asymptotics:

$$a(t,s) = (A + o(1))(t^2+s^2)^{-(1-d)}, \quad t^2+s^2 \to \infty,$$

where $A := \pi^{-1}\Gamma(1-d)/\Gamma(d)$ and hence Assumption (A2) with $q_1 = q_2 = 2(1-d) \in (1,2)$, $Q = 1/(1-d) \in (1,2)$ and a constant angular function $L_0(z) = A$, $z \in [-1,1]$.

Example 2. Anisotropic fractionally integrated random field. Consider the 'discrete heat operator' $\Delta_{1,2}Y(t,s) = Y(t,s) - \theta Y(t-1,s) - \frac{1-\theta}{2}(Y(t-1,s+1) + Y(t-1,s-1))$, $0 < \theta < 1$, and a fractionally integrated RF satisfying

$$\Delta_{1,2}^{d} Y(t,s) = \varepsilon(t,s), \tag{6.34}$$

where $\{\varepsilon(t,s)\}$ are as in (6.30). Similarly to (6.32), a stationary solution of (6.34) can be written as a moving-average RF:

$$Y(t,s) = \Delta_{1,2}^{-d}\varepsilon(t,s) = \sum_{(u,v)\in\mathbb{Z}_+\times\mathbb{Z}} a(u,v)\varepsilon(t-u, s-v), \tag{6.35}$$

with coefficients

$$a(u,v) = \psi_u(-d) q_u(v), \tag{6.36}$$

where $q_u(v)$ are u-step transition probabilities of a random walk $\{W_u,\, u = 0,1,\dots\}$ on $\mathbb{Z}$ with 1-step probabilities $\mathrm{P}(W_1 = v|W_0 = 0) = \theta$ if $v = 0$, $= (1-\theta)/2$ if $v = \pm 1$. As shown in [60], $\sum_{(u,v)\in\mathbb{Z}^2} a(u,v)^2 < \infty$ and the RF in (6.35) is well-defined for any $0 < d < 3/4$, $\theta \in [0,1)$; moreover, the spectral density $f(x,y)$ of (6.35) is singular at the origin: $f(x,y) \sim \mathrm{const}(x^2 + (1-\theta)^2 y^4/4)^{-d}$, $(x,y) \to (0,0)$.

Proposition 6.7. *For any* $0 < d < 3/4$, $0 < \theta < 1$, *the coefficients in* (6.36) *satisfy Assumption (A2) with* $q_1 = 3/2 - d$, $q_2 = 2q_1$ *and a continuous angular function* $L_0(z)$, $z \in [-1, 1]$, *given by*

$$L_0(z) = \begin{cases} \dfrac{z^{d-3/2}}{\Gamma(d)\sqrt{2\pi(1-\theta)}} \exp\Big\{ - \dfrac{\sqrt{(1/z)^2 - 1}}{2(1-\theta)} \Big\}, & 0 < z \le 1, \\ 0, & -1 \le z \le 0. \end{cases} \tag{6.37}$$

Remark 6.3. [14,41] discussed fractionally integrated RFs satisfying the equation

$$\Delta_1^{d_1} \Delta_2^{d_2} Y(t,s) = \varepsilon(t,s), \tag{6.38}$$

where $\Delta_1 Y(t,s) := Y(t,s) - Y(t-1,s)$, $\Delta_2 Y(t,s) := Y(t,s) - Y(t,s-1)$ are difference operators and $0 < d_1, d_2 < 1/2$ are parameters. Stationary solution of (6.38) is a moving-average RF $Y(t,s) = \sum_{(u,v)\in\mathbb{Z}^2_+} a(u,v)\varepsilon(t-u, v-s)$ with coefficients $a(u,v) := \psi_u(-d_1)\psi_v(-d_2)$. Following the proof of Theorem 6.1 one can show that for *any* $\gamma > 0$ the (normalized) partial sums process of RF Y in (6.38) tends to a FBS depending on d_1, d_2 only, viz., $\lambda^{-H_1-\gamma H_2} S^Y_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} c(d_1)c(d_2)B_{H_1,H_2}(x,y)$, where $H_i = d_i + 1/2$ and $c(d_i) > 0$ are some constants. See [89, Proposition 3.2] for related result. We conclude that the fractionally integrated RF in (6.38) featuring a 'separation of LRD along coordinate axes' does not exhibit scaling transition in contrast to models in (6.30) and (6.34).

6.5 Properties of convolutions of generalized homogeneous functions

For a given $\varpi > 0$ denote

$$\rho(t,s) := (|t|^2 + |s|^{2/\varpi})^{1/2}, \quad \rho_+(t,s) := 1 \vee \rho(t,s), \quad (t,s) \in \mathbb{R}^2.$$

Let $f(t,s) = \rho(t,s)^{-h} L(t/\rho(t,s))$, where $h \in \mathbb{R}$ and $L = L(z)$, $z \in [-1,1]$, is an arbitrary measurable function, then $f(t,s)$ satisfies the scaling property: $f(\lambda t, \lambda^{\varpi} s) = \lambda^{-h} f(t,s)$, $(t,s) \in \mathbb{R}^2_0$, for each $\lambda > 0$. Such functions are called generalized homogeneous functions (see [42]).

We use the notation $(a_1 \star a_2)(t,s) = \int_{\mathbb{R}^2} a_1(u,v) a_2(t+u, s+v) \mathrm{d}u \mathrm{d}v$ for convolution of functions a_i, $i = 1,2$, defined on $\mathbb{R}^2$. Similarly, we write $[a_1 \star a_2](t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a_1(u,v) a_2(t+u, s+v)$ for 'discrete' convolution of sequences a_i, $i = 1,2$, defined on $\mathbb{Z}^2$. Note the symmetry $a_i(t,s) = a_i(t,-s)$, $i = 1,2$, implies the symmetry $[a_1 \star a_2](t,s) = [a_1 \star a_2](t,-s)$, $(a_1 \star a_2)(t,s) = (a_1 \star a_2)(t,-s)$ of convolutions.

Let $B_\delta(t,s) := \{(u,v)\in\mathbb{R}^2 : |t-u|+|s-v|\le\delta\}$, $B^c_\delta(t,s) := \mathbb{R}^2\setminus B_\delta(t,s)$.

Proposition 6.8. *(i) For any $\delta>0$, $h>0$,*

$$\int_{B_\delta(0,0)}\rho(t,s)^{-h}\mathrm{d}t\mathrm{d}s<\infty \Longleftrightarrow h<1+\varpi \tag{6.39}$$

and

$$\left.\begin{array}{l}\displaystyle\int_{B^c_\delta(0,0)}\rho(t,s)^{-h}\mathrm{d}t\mathrm{d}s<\infty\\ \displaystyle\sum_{(t,s)\in\mathbb{Z}^2}\rho_+(t,s)^{-h}<\infty\end{array}\right\}\Longleftrightarrow h>1+\varpi. \tag{6.40}$$

(ii) Let $h_i>0$, $i=1,2$, $h_1+h_2>1+\varpi$. Then there exists $C>0$ such that for any $(t,s)\in\mathbb{R}^2_0$,

$$\begin{aligned}(\rho^{-h_1}\star\rho^{-h_2})(t,s) &\le C\rho(t,s)^{1+\varpi-h_1-h_2}, && h_i<1+\varpi, i=1,2, &(6.41)\\ (\rho_+^{-h_1}\star\rho^{-h_2})(t,s) &\le C\rho_+(t,s)^{-h_2}, && h_2<1+\varpi<h_1, &(6.42)\\ (\rho_+^{-h_1}\star\rho_+^{-h_2})(t,s) &\le C\rho_+(t,s)^{-h_1\wedge h_2}, && h_i>1+\varpi, i=1,2. &(6.43)\end{aligned}$$

Moreover, inequalities (6.41)–(6.43) *are also valid for 'discrete' convolution $[\rho_+^{-h_1}\star\rho_+^{-h_2}](t,s)$, $(t,s)\in\mathbb{Z}^2$ with $\rho(t,s)$ on the r.h.s. of* (6.41) *replaced by $\rho_+(t,s)$.*
(iii) Let $a_i=a_i(t,s)$, $(t,s)\in\mathbb{Z}^2$, satisfy $a_i(t,s)=\rho_+(t,s)^{-h_i}(L_i(t/\rho_+(t,s))+o(1))$, $|t|+|s|\to\infty$, where $0<h_i<1+\varpi<h_1+h_2$, and $L_i(u)\not\equiv 0$, $u\in[-1,1]$, are bounded piecewise continuous functions, $i=1,2$. Let $a_{i\infty}(t,s):=\rho(t,s)^{-h_i}L_i(t/\rho(t,s))$, $(t,s)\in\mathbb{R}^2$, $i=1,2$. Then

$$(a_{1\infty}\star a_{2\infty})(t,s)=\rho(t,s)^{1+\varpi-h_1-h_2}L_{12}(t/\rho(t,s)),\quad (t,s)\in\mathbb{R}^2, \tag{6.44}$$

and

$$[a_1\star a_2](t,s)=\rho_+(t,s)^{1+\varpi-h_1-h_2}\big(L_{12}(t/\rho_+(t,s))+o(1)\big),\quad |t|+|s|\to\infty, \tag{6.45}$$

where

$$L_{12}(z):=(a_{1\infty}\star a_{2\infty})(z,(1-z^2)^{\varpi/2})=\int_{\mathbb{R}^2}a_{1\infty}(u,v)a_{2\infty}(u+z,v+(1-z^2)^{\varpi/2})\mathrm{d}u\mathrm{d}v \tag{6.46}$$

is a bounded continuous function on the interval $z\in[-1,1]$. Moreover, if $L_1(z)=L_2(z)\ge 0$ then $L_{12}(z)$ in (6.46) *is strictly positive on $[-1,1]$.*

Proposition 6.9. *Let* $b(t,s) := \rho_+(t,s)^{-h}(L(t/\rho_+(t,s)) + o(1))$, $|t|+|s| \to \infty$, $(t,s) \in \mathbb{Z}^2$, $b_\infty(t,s) := \rho(t,s)^{-h}L(t/\rho(t,s))$, $(t,s) \in \mathbb{R}^2$, *where* $0 < h < 1+\varpi$ *and* $L(u) \ge 0$, $u \in [-1,1]$, *is a continuous function. Then for any* $\gamma > 0$,

$$B_\lambda(\gamma) := \sum_{(t_i,s_i)\in K_{[\lambda,\lambda^\gamma]}, i=1,2} b(t_1-t_2, s_1-s_2) \sim \mathcal{C}(\gamma)\lambda^{2\mathcal{H}(\gamma)}, \quad \lambda \to \infty, \tag{6.47}$$

where

$$\mathcal{H}(\gamma) := \begin{cases} 1+\varpi-\frac{h}{2}, & \text{(I)} \\ 1+\gamma-\frac{\gamma h}{2\varpi}, & \text{(II)} \\ 1+\frac{\gamma}{2}-\frac{h-\varpi}{2}, & \text{(III)} \\ 1+\gamma-\frac{h}{2}, & \text{(IV)} \\ \frac{1}{2}+\gamma-\frac{\gamma(h-1)}{2\varpi}, & \text{(V)} \end{cases}$$

$$\mathcal{C}(\gamma) := \begin{cases} \int_{(0,1]^4} b_\infty(t_1-t_2, s_1-s_2)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2, & \text{(I)} \\ \int_{(0,1]^2} b_\infty(0, s_1-s_2)\mathrm{d}s_1\mathrm{d}s_2, & \text{(II)} \\ \int_{(0,1]^2\times\mathbb{R}} b_\infty(t_1-t_2, s)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s, & \text{(III)} \\ \int_{(0,1]^2} b_\infty(t_1-t_2, 0)\mathrm{d}t_1\mathrm{d}t_2, & \text{(IV)} \\ \int_{\mathbb{R}\times(0,1]^2} b_\infty(t, s_1-s_2)\mathrm{d}t\mathrm{d}s_1\mathrm{d}s_2, & \text{(V)} \end{cases} \tag{6.48}$$

in respective cases (I)*:* $\gamma = \varpi$, (II)*:* $\gamma > \varpi$, $h < \varpi$, (III)*:* $\gamma > \varpi$, $h > \varpi$, (IV)*:* $\gamma < \varpi$, $h < 1$ *and* (V)*:* $\gamma < \varpi$, $h > 1$.

6.6 Covariance structure of subordinated anisotropic RFs

In this section from Propositions 6.8 and 6.9 with $\varpi = \gamma_0$, $\rho(t,s) = (|t|^2 + |s|^{2/\gamma_0})^{1/2}$ we obtain the asymptotic form of the covariance function of $r_X(t,s) := \mathrm{E}X(0,0)X(t,s)$ and the asymptotics of the variance of anisotropic partial sums $S^X_{\lambda,\gamma}$ of subordinated RF $X = A_k(Y)$.

Proposition 6.10. *Let RF* $X = A_k(Y)$ *satisfy assumptions (A1), (A2) and (A3)*$_k$.

(i) Let $k \ge 1$. *Then* $X(t,s) = Y^{\bullet k}(t,s) + \mathcal{Z}(t,s)$, *where* $\mathcal{Z}(t,s)$ *is defined in* (6.14) *and*

$$r_{\mathcal{Z}}(t,s) = O(\rho(t,s)^{-2q_1}), \quad |t|+|s| \to \infty. \tag{6.49}$$

(ii) Let $1 \le k < P$. *Then*

$$r_X(t,s) = k!\rho(t,s)^{-kp_1}\big(L_X(t/\rho(t,s)) + o(1)\big), \quad |t|+|s| \to \infty, \tag{6.50}$$

where $L_X(z) := ((a_\infty \star a_\infty)(z,(1-z^2)^{\gamma_0/2}))^k$, $z \in [-1,1]$, *is a strictly positive continuous function and* a_∞ *is defined in* (6.16). *Moreover,* $r_{\mathcal{Z}}(t,s) = o(\rho(t,s)^{-kp_1})$, $|t|+|s| \to \infty$.
(iii) Let $k > P$. *Then*

$$r_X(t,s) = O(\rho(t,s)^{-(kp_1)\wedge(2q_1)}), \quad |t|+|s| \to \infty. \tag{6.51}$$

Clearly, (6.50) implies $C_1\rho(t,s)^{-kp_1} \le r_X(t,s) \le C_2\rho(t,s)^{-kp_1}$ for all $|t|+|s| > C_3$ and some $0 < C_i < \infty$, $i = 1,2,3$. The last fact together with Proposition 6.8(i) implies the following corollary.

Corollary 6.11. *Let* $X = A_k(Y)$, $k \ge 1$, *be the subordinated RF defined in Proposition 6.10 and satisfying the conditions therein.*
(i) Let $1 \le k < P$. *Then* $\sum_{(t,s)\in\mathbb{Z}^2} |r_X(t,s)| = \infty$. *Moreover,* $\sum_{s\in\mathbb{Z}} |r_X(0,s)| = \infty \iff kp_2 \le 1$ *and* $\sum_{t\in\mathbb{Z}} |r_X(t,0)| = \infty \iff kp_1 \le 1$.
(ii) Let $k > P$. *Then* $\sum_{(t,s)\in\mathbb{Z}^2} |r_X(t,s)| < \infty$.

Remark 6.4. Following the terminology in [81], we say that a stationary RF $X = \{X(t,s),\,(t,s) \in \mathbb{Z}^2\}$ with finite variance has *vertical LRD property* (respectively, *horizontal LRD property*) if $\sum_{s\in\mathbb{Z}} |r_X(0,s)| = \infty$ (respectively, $\sum_{t\in\mathbb{Z}} |r_X(t,0)| = \infty$). From Corollary 6.11 we see the dichotomy of the limit distribution in Theorems 6.2–6.3 at points $kp_2 = 1$ and $kp_1 = 1$ is related to the change of vertical and horizontal LRD properties of the subordinated RF $X = A_k(Y)$.

Corollary 6.12. *Let* $X(t,s) = A_k(Y(t,s)) = Y^{\bullet k}(t,s) + \mathcal{Z}(t,s)$, $1 \le k < P$, $kp_i \ne 1$, $i = 1,2$, *be the subordinated RF defined in Proposition 6.10 and satisfying the conditions therein. Then for any* $\gamma > 0$, *as* $\lambda \to \infty$,

$$\mathrm{Var}(S^X_{\lambda,\gamma}) \sim \mathrm{Var}(S^{Y^{\bullet k}}_{\lambda,\gamma}) \sim c(\gamma)\lambda^{2H(\gamma)} \tag{6.52}$$

and

$$\mathrm{Var}(S^{\mathcal{Z}}_{\lambda,\gamma}) = O(\lambda^{1+\gamma}), \tag{6.53}$$

where $H(\gamma) \in ((1+\gamma)/2, 1+\gamma)$ *and* $c(\gamma)$ *are defined in Theorems 6.1–6.3.*

6.7 Proofs

6.7.1 Proofs of Propositions 6.8–6.10 and Corollary 6.12

Proof of Proposition 6.8. With the notation $\varrho := \rho(t,s)$ we have that $(t,s) \mapsto (\varrho, t/\varrho)$ is a 1-1 mapping from $\mathbb{R} \times [0,\infty)$ to $[0,\infty) \times [-1,1]$. Particularly, if $\varpi = 1$ then $(\varrho, \arccos(t/\varrho))$ are the polar coordinates of $(t,s) \in \mathbb{R} \times [0,\infty)$. We use the inequality:

$$\rho(t_1 + t_2, s_1 + s_2) \le C_\varpi \sum_{i=1}^{2} \rho(t_i, s_i), \tag{6.54}$$

with $C_\varpi := 1 \vee 2^{1/\varpi - 1}$, which follows from

$$\rho(t_1 + t_2, s_1 + s_2)^{1\wedge\varpi} \le \sum_{i=1}^{2} \rho(t_i, s_i)^{1\wedge\varpi}, \quad (t_i, s_i) \in \mathbb{R}^2,\ i = 1,2. \tag{6.55}$$

(i) W.l.g., let $\delta = 1$. Then $\int_{B_1(0,0)} \rho(t,s)^{-h} \mathrm{d}t\mathrm{d}s \le 4 \int_0^1 t^{\varpi - h} \mathrm{d}t \int_0^{1/t^\varpi} (1 + u^{2/\varpi})^{-h/2} \mathrm{d}u$, where the inner integral $= O(1)$ if $h > \varpi$, $= O(t^{h-\varpi})$ if $h < \varpi$, $= O(|\log t|)$ if $h = \varpi$, as $u \to 0$. This proves (6.39) and (6.40) follows analogously.

(ii) After the change of variables: $u \to \varrho u$, $v \to \varrho^\varpi v$, $\varrho := \rho(t,s)$, we get

$$\begin{aligned}(\rho^{-h_1} \star \rho^{-h_2})(t,s) &= \varrho^{1+\varpi-h_1-h_2} \int_{\mathbb{R}^2} \rho(u,v)^{-h_1} \rho((t/\varrho) + u, (s/\varrho^\varpi) + v)^{-h_2} \mathrm{d}u\mathrm{d}v, \\ &= \varrho^{1+\varpi-h_1-h_2} (I_1 + I_2 + I_{12}), \end{aligned} \tag{6.56}$$

where

$$\begin{aligned} I_1 &:= \int_{B_\delta(0,0)} \rho(u,v)^{-h_1} \rho((t/\varrho) + u, (s/\varrho^\varpi) + v)^{-h_2} \mathrm{d}u\mathrm{d}v, \\ I_2 &:= \int_{B_\delta(-t/\varrho, -s/\varrho^\varpi)} \ldots \mathrm{d}u\mathrm{d}v, \qquad I_{12} := \int_{B^c_\delta(0,0) \cap B^c_\delta(-t/\varrho, -s/\varrho^\varpi)} \ldots \mathrm{d}u\mathrm{d}v \end{aligned}$$

with $\delta > 0$ such that $B_\delta(0,0) \cap B_\delta(-t/\varrho, -s/\varrho^\varpi) = \emptyset$ for any $(t,s) \ne (0,0)$. The integrals $I_i \le C$, $i = 1,2$ by (6.39) and $0 < h_i < 1 + \varpi$, $i = 1,2$. Next, by Hölder's inequality with $h := h_1 + h_2$,

$$I_{12} \le \int_{B^c_\delta(0,0)} \rho(u,v)^{-h} \mathrm{d}u\mathrm{d}v \le C,$$

in view of (6.40) and

$$\int_{B^c_\delta(-t/\varrho, -s/\varrho^\varpi)} \rho((t/\varrho) + u, (s/\varrho^\varpi) + v)^{-h} \mathrm{d}u\mathrm{d}v = \int_{B^c_\delta(0,0)} \rho(u,v)^{-h} \mathrm{d}u\mathrm{d}v.$$

This proves (6.41).

Next, consider (6.42), or the case $0 < h_2 < 1 + \varpi < h_1$. By changing the variables as in (6.56), we get $(\rho_+^{-h_1} \star \rho^{-h_2})(t,s) \le \varrho^{1+\varpi-h_1-h_2}(I'_1 + I_2 + I_{12})$, where $I_2 < C$, $I_{12} < C$ are the same as in (6.56), whereas

$$I'_1 := \int_{B_\delta(0,0)} (\varrho^{-1} \vee \rho(u,v))^{-h_1} \rho((t/\varrho) + u, (s/\varrho^\varpi) + v)^{-h_2} \mathrm{d}u\mathrm{d}v.$$

Note that if given small enough $\delta > 0$, then (6.55) implies $\rho((t/\varrho) + u, (s/\varrho^\varpi) + v)^{1\wedge\varpi} \ge 1 - \rho(u,v)^{1\wedge\varpi} \ge 1/2$ for all $(u,v) \in B_\delta(0,0)$, and hence $I'_1 \le C\varrho^{h_1-1-\varpi} \int_{\mathbb{R}^2} \rho_+(u,v)^{-h_1}\mathrm{d}u\mathrm{d}v \le C\varrho^{h_1-1-\varpi}$ according to (6.40). Since $\rho(t,s)^{1+\varpi-h_1-h_2} = o(\rho(t,s)^{-h_2})$ as $|t| + |s| \to \infty$, the proof of (6.42) is complete.

Finally, consider (6.43). We follow the proof of (6.42) and get $(\rho_+^{-h_1} \star \rho_+^{-h_2})(t,s) \le \varrho^{1+\varpi-h_1-h_2}(I'_1 + I'_2 + I_{12})$ with the same $I'_1 < C\varrho^{h_1-1-\varpi}$, $I_{12} < C$, whereas

$$I'_2 := \int_{B_\delta(-t/\varrho, -s/\varrho^\varpi)} \rho(u,v)^{-h_1} (\varrho^{-1} \vee \rho((t/\varrho) + u, (s/\varrho^\varpi) + v))^{-h_2} \mathrm{d}u\mathrm{d}v.$$

For small enough $\delta > 0$, we have $\rho(u,v)^{1\wedge\varpi} \ge 1 - \rho((t/\varrho)+u, (s/\varrho^\varpi)+v)^{1\wedge\varpi} \ge 1/2$ for all $(u,v) \in B_\delta(-t/\varrho, -s/\varrho^\varpi)$, and hence $I'_2 \le C\varrho^{h_2-1-\varpi} \int_{\mathbb{R}^2} \rho_+(t+u, s+v)^{-h_2}\mathrm{d}u\mathrm{d}v \le C\varrho^{h_2-1-\varpi}$ by (6.40). Using $\rho(t,s)^{1+\varpi-h_1-h_2} = o(\rho(t,s)^{-h_1\wedge h_2})$ as $|t| + |s| \to \infty$, we conclude (6.43). Extension of (6.41)–(6.43) to 'discrete' convolution $[\rho_+^{-h_1} \star \rho_+^{-h_2}](t,s)$ requires minor changes and we omit the details. This proves part (ii).

(iii) It suffices to show (6.45) for $(t,s) \in \mathbb{Z} \times \mathbb{Z}_+$, $(t,s) \neq (0,0)$, in which case $\rho_+(t,s) = \rho(t,s)$. We have $[a_1 \star a_2](t,s) = \sum_{i,j=0}^1 [a_1^i \star a_2^j](t,s)$, where $a_i^0(t,s) := \rho_+(t,s)^{-h_i} L_i(t/\rho_+(t,s))$, $a_i^1(t,s) := a_i(t,s) - a_i^0(t,s) = o(\rho_+(t,s)^{-h_i})$, $i = 1,2$. Clearly, (6.45) follows from

$$\lim_{|t|+|s|\to\infty} \big|\rho(t,s)^{h_1+h_2-1-\varpi}[a_1^0 \star a_2^0](t,s) - L_{12}(t/\rho(t,s))\big| = 0 \tag{6.57}$$

and

$$[a_1^i \star a_2^j](t,s) = o(\rho(t,s)^{1+\varpi-h_1-h_2}), \quad (i,j) \neq (0,0),\ i,j = 0,1, \quad |t| + |s| \to \infty. \tag{6.58}$$

The proof of (6.58) mimics the proof of (6.57) and is omitted. To prove (6.57), write $[a_1^0 \star a_2^0](t,s)$ as the integral: $[a_1^0 \star a_2^0](t,s) = \int_{\mathbb{R}^2} a_1^0([u],[v]) a_2^0([u]+t, [v]+s)\mathrm{d}u\mathrm{d}v$. After the same change of variables $u \to \varrho u$, $v \to \varrho^\varpi v$, $\varrho := \rho(t,s)$ as in the proof of (ii) we obtain $[a_1^0 \star a_2^0](t,s) = \varrho^{1+\varpi-h_1-h_2} L_\varrho(t/\varrho)$, where

$$L_\varrho(z) := \int_{\mathbb{R}^2} g_\varrho(u,v;z)\mathrm{d}u\mathrm{d}v, \quad z \in [-1,1]$$

and where

$$g_\varrho(u,v;z) := a_{1\varrho}\big(\tilde u, \tilde v\big) a_{2\varrho}\big(\tilde u + z, \tilde v + (1-z^2)^{\varpi/2}\big),$$

with $\tilde{u} := [\varrho u]/\varrho$, $\tilde{v} := [\varrho^{\varpi} v]/\varrho^{\varpi}$ and

$$a_{i\varrho}(u,v) := \big(\varrho^{-1} \vee \rho(u,v)\big)^{-h_i} L_i\big(u/\big(\varrho^{-1} \vee \rho(u,v)\big)\big), \quad i = 1,2, \tag{6.59}$$

since $s/\varrho^{\varpi} = (1-z^2)^{\varpi/2}$ for $z = t/\varrho \in [-1,1]$, $s \ge 0$. Then with $a_{i\infty}(u,v)$, $i = 1,2$, defined by the statement of Proposition 6.8(iii) we get that

$$g_\varrho(u,v;z) \to g_\infty(u,v;z) := a_{1\infty}(u,v)a_{2\infty}(u+z, v+(1-z^2)^{\varpi/2}) \tag{6.60}$$

as $\varrho = \rho(t,s) \to \infty$ ($|t|+|s| \to \infty$) for any fixed $(u,v;z) \in \mathbb{R}^2 \times [-1,1]$ such that $(u,v) \notin \{(0,0), (-z, -(1-z^2)^{\varpi/2})\}$ and $u/\rho(u,v)$, $(u+z)/\rho(u+z, v+(1-z^2)^{\varpi/2})$ being continuity points of L_1 and L_2 respectively. Let us prove that

$$L_\varrho(z) \to L_{12}(z) \quad \text{as } \varrho \to \infty \tag{6.61}$$

uniformly in $z \in [-1,1]$, which implies (6.57), viz., $|L_\varrho(t/\varrho) - L_{12}(t/\varrho)| \le \sup_{z\in[-1,1]} |L_\varrho(z) - L_{12}(z)| = o(1)$ as $\varrho \to \infty$. The uniform convergence in (6.61) follows if $\lim_{\varrho\to\infty} L_\varrho(z_\varrho) = L_{12}(z)$ holds for any $z \in [-1,1]$ and every sequence $\{z_\varrho\} \subset [-1,1]$ tending to z: $\lim_{\varrho\to\infty} z_\varrho = z$. Choose $\delta > 0$ and split the difference $L_\varrho(z_\varrho) - L_{12}(z) = I_1 + I_2 + I_{12}$, where

$$\begin{aligned} I_1 &:= \int_{B_\delta(0,0)} (g_\varrho(u,v;z_\varrho) - g_\infty(u,v;z))\mathrm{d}u\mathrm{d}v, \\ I_2 &:= \int_{B_\delta(-z,-z')} \ldots \mathrm{d}u\mathrm{d}v, \qquad I_{12} := \int_{B^c_\delta(0,0)\cap B^c_\delta(-z,-z')} \ldots \mathrm{d}u\mathrm{d}v \end{aligned}$$

with the notation $z' := (1-z^2)^{\varpi/2}$. Note that $\rho(z,z') = 1$ and $\delta > 0$ is chosen small enough so that $B_\delta(0,0) \cap B_\delta(-z,-z') = \emptyset$. Let us first check that $|I_i|$, $i = 1,2$, can be made arbitrary small by taking sufficiently small δ. Towards this end, we need the bound

$$|a_{i\varrho}(\tilde{u},\tilde{v})| \le C\rho(u,v)^{-h_i}, \quad (u,v) \in \mathbb{R}^2, \quad i = 1,2. \tag{6.62}$$

Indeed, by (6.54), $\rho(u,v) \le C_\varpi(\rho(\tilde{u},\tilde{v}) + \rho(u-\tilde{u}, v-\tilde{v}))$, where $|u-\tilde{u}| \le \varrho^{-1}$, $|v-\tilde{v}| \le \varrho^{-\varpi}$ and hence $\rho(u-\tilde{u}, v-\tilde{v}) \le \sqrt{2}\varrho^{-1}$, with $C_\varpi > 0$ dependent only on $\varpi > 0$. Therefore, $\rho(u,v) \le \sqrt{2}C_\varpi(\rho(\tilde{u},\tilde{v}) + \varrho^{-1}) \le 2\sqrt{2}C_\varpi(\rho(\tilde{u},\tilde{v}) \vee \varrho^{-1})$ implying

$$\rho(\tilde{u},\tilde{v}) \vee \varrho^{-1} \ge (2\sqrt{2}C_\varpi)^{-1}\rho(u,v), \tag{6.63}$$

or (6.62) in view of the definition of $a_{i\varrho}$ in (6.59). Using (6.62) it follows that

$$|g_\varrho(u,v;z_\varrho) - g_\infty(u,v;z)| \le C\rho(u,v)^{-h_1}\big(\rho(u+z_\varrho, v+z'_\varrho)^{-h_2} + \rho(u+z, v+z')^{-h_2}\big). \tag{6.64}$$

From (6.64) we obtain $|I_1| \le C\int_{B_\delta(0,0)} \rho(u,v)^{-h_1}\mathrm{d}u\mathrm{d}v \le C\delta^{1+\varpi-h_1} = o(1)$ and similarly, $|I_2| \le C\delta^{1+\varpi-h_2} = o(1)$. Hence it suffices to show that $I_{12} \to 0$ ($z_\varrho \to z$), viz., that for each $\delta > 0$

$$\int_{B^c_\delta(0,0)\cap B^c_\delta(-z,-z')} |g_\varrho(u,v;z_\varrho) - g_\infty(u,v;z)|\mathrm{d}u\mathrm{d}v \to 0 \quad \text{as } \varrho \to \infty. \tag{6.65}$$

From (6.55), $\rho(u+z_\varrho, v+z'_\varrho)^{1\wedge\varpi} \ge \rho(u+z,v+z')^{1\wedge\varpi} - (\delta/2)^{1\wedge\varpi}/2 \ge (1/2)\rho(u+z,v+z')^{1\wedge\varpi}$ for all $(u,v) \in B^c_\delta(-z,-z')$ and ϱ large enough that $\rho(z-z_\varrho, z'-z'_\varrho)^{1\wedge\varpi} \le (\delta/2)^{1\wedge\varpi}/2$ (in view of $z_\varrho \to z$). Hence and from (6.64) we obtain that the integrand in (6.65) is dominated on $B^c_\delta(0,0)\cap B^c_\delta(-z,-z')$ by an integrable function independent of ϱ, viz., $|g_\varrho(u,v;z_\varrho)-g_\infty(u,v;z)| \le C\rho(u,v)^{-h_1}\rho(u+z,v+z')^{-h_2}$. Since this integrand vanishes a.e. on $B^c_\delta(0,0)\cap B^c_\delta(-z,-z')$ as $\varrho \to \infty$, see (6.60), relation (6.65) follows by the dominated convergence theorem, proving (6.61). The continuity of L_{12} (6.46) follows similarly by the dominated convergence theorem.

It remains to prove the strict positivity of L_{12} in the case where $L_1(z) \equiv L_2(z) =: L(z) \ge 0$. Under assumption of piecewise continuity of L and $L \not\equiv 0$ a.e., we can find $0 < |z_0| < 1$ and $\delta > 0$ such that $L(z) > \delta$ for any $|z - z_0| < \delta$. We also have $|u/\rho(u,v) - (u+z)/\rho(u+z,v+z')| \le \rho(u,v)^{-1} + |1-\rho(u+z,v+z')/\rho(u,v)| = O(\rho(u,v)^{-1\wedge\varpi})$ uniformly in $z \in [-1,1]$ for $\rho(u,v) \ge 1$. Indeed, this follows from $|1-(\rho(u+z,v+z')/\rho(u,v))^{1\wedge\varpi}| \le \rho(u,v)^{-1\wedge\varpi}$ by (6.55), when combined with $1-x \le \varpi^{-1}(1-x^\varpi)$, $0<x<1$, if $0<\varpi<1$ and $\rho(u+z,v+z')/\rho(u,v) \le 2C_\varpi$ for $\rho(u,v) \ge 1$ by (6.54). Hence, given K large enough $|u/\rho(u,v)-(u+z)/\rho(u+z,v+z')| < \delta/2$ for all $(u,v) \in B^c_K(0,0)$. Next, we choose the interior point (u_0,v_0) of $B^c_K(0,0)$ such that $u_0/\rho(u_0,v_0) = z_0$. In view of continuity of $u/\rho(u,v)$, there exists $\varepsilon > 0$ such that $|z_0 - u/\rho(u,v)| < \delta/2$ holds for all $(u,v) \in B_\varepsilon(u_0,v_0) \subset B^c_K(0,0)$. Consequently, $L(u/\rho(u,v))L((u+z)/\rho(u+z,v+z')) > \delta^2 > 0$ for any $z \in [-1,1]$ and all $(u,v) \in B_\varepsilon(u_0,v_0)$. Finally, $L_{12}(z) > \delta^2(2C_\varpi)^{-h_2}\int_{B_\varepsilon(u_0,v_0)} \rho(u,v)^{-h_1-h_2}\mathrm{d}u\mathrm{d}v > 0$, proving $L_{12}(z) > 0, z \in [-1,1]$, and part (iii). Proposition 6.8 is proved. □

Proof of Proposition 6.9. Rewrite the l.h.s. of (6.47) as

$$B_\lambda(\gamma) = \int_{\widetilde{K}^2_{[\lambda,\lambda^\gamma]}} b([t_1]-[t_2],[s_1]-[s_2])\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2, \tag{6.66}$$

where $\widetilde{K}_{[\lambda,\lambda^\gamma]} := \{(t,s) \in \mathbb{R}^2 : ([t],[s]) \in K_{[\lambda,\lambda^\gamma]}\}$.

Case (I): $\gamma = \varpi$. By changing the variables in (6.66) as $t_i \to \lambda t_i$, $s_i \to \lambda^\varpi s_i$,

$i = 1, 2$, we obtain $\lambda^{-2\mathcal{H}(\varpi)} B_\lambda(\varpi) = \int_{\mathbb{R}^4} \widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}s_1 \mathrm{d}s_2$, where

$$\begin{aligned} \widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \quad &:= \quad b_\lambda(([\lambda t_1] - [\lambda t_2])/\lambda, ([\lambda^\varpi s_1] - [\lambda^\varpi s_2])/\lambda^\varpi) \\ &\quad\quad \times \mathbf{1}(([\lambda t_i], [\lambda^\varpi s_i]) \in (0, \lambda] \times (0, \lambda^\varpi], \ i = 1, 2) \end{aligned} \tag{6.67}$$

with $b_\lambda(t, s) := (\lambda^{-1} \vee \rho(t, s))^{-h}(L(t/(\lambda^{-1} \vee \rho(t, s))) + o(1))$ as $\lambda \to \infty$. Then

$$\widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \to b_\infty(t_1 - t_2, s_1 - s_2)\mathbf{1}((t_i, s_i) \in (0, 1]^2, \ i = 1, 2), \quad \lambda \to \infty,$$

point-wise for any $(t_1, t_2, s_1, s_2) \in \mathbb{R}^4$, $(t_1, s_1) \neq (t_2, s_2)$ fixed. The dominating bound

$$\lambda^{-1} \vee \rho\big(([\lambda t_1] - [\lambda t_2])/\lambda, ([\lambda^\varpi s_1] - [\lambda^\varpi s_2])/\lambda^\varpi\big) \geq C\rho(t_1 - t_2, s_1 - s_2),$$

follows by the same arguments as (6.63). These facts and the dominated convergence theorem justify the limit $\lim_{\lambda\to\infty} \lambda^{-2\mathcal{H}(\varpi)} B_\lambda(\varpi) = \mathcal{C}(\varpi)$ since the integral $\mathcal{C}(\varpi) \leq C \int_{(-1,1]^2} \rho(t, s)^{-h} \mathrm{d}t\mathrm{d}s < \infty$ in (6.48) converges by Proposition 6.8(i).

Case (II): $\gamma > \varpi$, $h < \varpi$. By changing the variables in (6.66) as $t_i \to \lambda t_i$, $s_i \to \lambda^\gamma s_i$, $i = 1, 2$, we obtain $\lambda^{-2\mathcal{H}(\gamma)} B_\lambda(\gamma) = \int_{\mathbb{R}^4} \widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}s_1 \mathrm{d}s_2$, where

$$\begin{aligned} \widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \quad &:= \quad b_\lambda(([\lambda t_1] - [\lambda t_2])/\lambda^{\gamma/\varpi}, ([\lambda^\gamma s_1] - [\lambda^\gamma s_2])/\lambda^\gamma) \\ &\quad\quad \times \mathbf{1}(([\lambda t_i], [\lambda^\gamma s_i]) \in (0, \lambda] \times (0, \lambda^\gamma], \ i = 1, 2) \end{aligned}$$

with $b_\lambda(t, s) := (\lambda^{-\gamma/\varpi} \vee \rho(t, s))^{-h}(L(t/(\lambda^{-\gamma/\varpi} \vee \rho(t, s))) + o(1))$ as $\lambda \to \infty$. Hence since $\gamma/\varpi > 1$ it follows that

$$\widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \to b_\infty(0, s_1 - s_2)\mathbf{1}((t_i, s_i) \in (0, 1]^2, i = 1, 2), \quad \lambda \to \infty,$$

point-wise for any $(t_1, t_2, s_1, s_2) \in \mathbb{R}^4$, $s_1 \neq s_2$ fixed. Note $b_\infty(0, s) = L(0)|s|^{-h/\varpi}$ is integrable on $[-1, 1]$ due to $h < \varpi$. The limit $\lim_{\lambda\to\infty} \lambda^{-2\mathcal{H}(\varpi)} B_\lambda(\varpi) = \mathcal{C}(\varpi)$ can be justified by the dominated convergence theorem using the bound

$$\begin{aligned} &\lambda^{-\gamma/\varpi} \vee \rho\big(([\lambda t_1] - [\lambda t_2])/\lambda^{\gamma/\varpi}, ([\lambda^\gamma s_1] - [\lambda^\gamma s_2])/\lambda^\gamma\big) \\ &\quad \geq \lambda^{-\gamma/\varpi} \vee \rho(0, ([\lambda^\gamma s_1] - [\lambda^\gamma s_2])/\lambda^\gamma) \\ &\quad \geq C\rho(0, s_1 - s_2), \end{aligned}$$

which follows by the same arguments as (6.63).

Case (III): $\gamma > \varpi$, $h > \varpi$. By changing the variables in (6.66) as $t_i \to \lambda t_i$, $i = 1, 2$, $s_1 - s_2 \to \lambda^\varpi s_1$, $s_2 \to \lambda^\gamma s_2$, we obtain $\lambda^{-2\mathcal{H}(\gamma)} B_\lambda(\gamma) = \int_{\mathbb{R}^4} \widetilde{b}_\lambda(t_1, t_2, s_1, s_2) \mathrm{d}t_1 \mathrm{d}t_2$

$\mathrm{d}s_1\mathrm{d}s_2$, where

$$\begin{aligned}\widetilde{b}_\lambda(t_1,t_2,s_1,s_2) \ &:= \ b_\lambda(([\lambda t_1]-[\lambda t_2])/\lambda, ([\lambda^\varpi s_1+\lambda^\gamma s_2]-[\lambda^\gamma s_2])/\lambda^\varpi) \\ &\quad \times \mathbf{1}([\lambda t_i]\in(0,\lambda],\ i=1,2, \\ &\qquad [\lambda^\varpi s_1+\lambda^\gamma s_2]\in(0,\lambda^\gamma], [\lambda^\gamma s_2]\in(0,\lambda^\gamma])\end{aligned}$$

with $b_\lambda(t,s) := (\lambda^{-1}\vee\rho(t,s))^{-h}(L(t/(\lambda^{-1}\vee\rho(t,s))+o(1))$ as $\lambda\to\infty$. Then

$$\widetilde{b}_\lambda(t_1,t_2,s_1,s_2)\to b_\infty(t_1-t_2,s_1)\mathbf{1}((t_1,t_2,s_2)\in(0,1]^3), \quad \lambda\to\infty,$$

for any $t_1\neq t_2$, $s_1\in\mathbb{R}\setminus\{0\}$, $s_2\in\mathbb{R}\setminus\{0,1\}$ fixed since $\gamma>\varpi$ implies $\mathbf{1}(0<[\lambda^\varpi s_1+\lambda^\gamma s_2]\le\lambda^\gamma)\to\mathbf{1}(0<s_2<1)$. The dominating bound

$$\lambda^{-1}\vee\rho\big(([\lambda t_1]-[\lambda t_2])/\lambda, ([\lambda^\varpi s_1+\lambda^\gamma s_2]-[\lambda^\gamma s_2])/\lambda^\varpi\big)\ge C\rho(t_1-t_2,s_1)$$

follows in the same way as (6.63), because $|([\lambda^\varpi s_1+\lambda^\gamma s_2]-[\lambda^\gamma s_2])/\lambda^\varpi - s_1|\le 2\lambda^{-\varpi}$. Then the dominated convergence in (6.47) is proved in view of $\mathcal{C}(\gamma)\le C\int_{-1}^1\int_{\mathbb{R}}\rho(t,s)^{-h}\mathrm{d}t\mathrm{d}s<\infty$.
Cases (IV) and (V) can be treated similarly to Cases (II) and (III) and we omit the details. Proposition 6.9 is proved. □

In the rest of the chapter, we apply Propositions 6.8 and 6.9 with $\varpi=\gamma_0$ and use the notation $\rho(t,s)=(|t|^2+|s|^{2/\gamma_0})^{1/2}$, $(t,s)\in\mathbb{R}^2$.

Proof of Proposition 6.10. (i) Since $\mathcal{Z}(t,s)\equiv 0$ for $k=1$, let $k\ge 2$ in what follows. According to (6.12),

$$\begin{aligned}\mathcal{Z}(t,s) \ &= \ \sum_{i=1}^{k-1}\sum_{(D)_i}\sum_{(u,v)_i}^{\bullet} a(t-u_1,s-v_1)^{|D_1|}\cdots a(t-u_i,s-v_i)^{|D_i|} \\ &\qquad \times A_{|D_1|}(\varepsilon(u_1,v_1))\cdots A_{|D_i|}(\varepsilon(u_i,v_i)),\end{aligned} \tag{6.68}$$

where the sum $\sum_{(D)_i}$ is taken over all partitions of $\{1,2,\cdots,k\}$ into i nonempty sets $D_1,\dots,D_i$ having cardinality $|D_1|\ge 1,\dots,|D_i|\ge 1$, $|D_1|+\cdots+|D_i|=k$. Thus, (6.68) is a decomposition of $\mathcal{Z}(t,s)=A_k(Y(t,s))-Y^{\bullet k}(t,s)$ into a sum of stationary 'off-diagonal' polynomial forms of order $i<k$ in i.i.d. r.v. $A_{|D_\ell|}(\varepsilon(u_\ell,v_\ell))$, $1\le\ell\le i$, with $\max(|D_1|,\cdots,|D_i|)\ge 2$. From (6.68) it follows that

$$|\mathrm{E}\mathcal{Z}(0,0)\mathcal{Z}(t,s)|\le C\sum_{i=1}^{k-1}\sum_{(d)_i,(d')_i}\prod_{\ell=1}^{i}[|a|^{d_\ell}\star|a|^{d'_\ell}](t,s), \tag{6.69}$$

where the second sum is taken over all collections $(d)_i=(d_1,\dots,d_i)$, $(d')_i=(d'_1,\dots,d'_i)$ of integers $d_\ell\ge 1$, $d'_\ell\ge 1$ with $\sum_{\ell=1}^i d_\ell=\sum_{\ell=1}^i d'_\ell=k$. See [36], proof of Theorem 14.2.1. Then $a(t,s)^{d_\ell} \le C\rho(t,s)^{-\beta_\ell}$, $a(t,s)^{d'_\ell} \le C\rho(t,s)^{-\beta'_\ell}$, where $\beta_\ell := d_\ell q_1$, $\beta'_\ell := d'_\ell q_1$. By Proposition 6.8(ii),

$$|\mathrm{E}\mathcal{Z}(0,0)\mathcal{Z}(t,s)| \le C\sum_{i=1}^{k-1}\sum_{(d)_i,(d')_i}\prod_{\ell=1}^{i}\rho(t,s)^{-w_\ell}, \tag{6.70}$$

where

$$w_\ell := \begin{cases} 2q_1 - 1 - \gamma_0 = p_1, & \text{if } d_\ell = d'_\ell = 1, \\ q_1, & \text{if } d_\ell \ge 2,\ d'_\ell = 1 \text{ or } d_\ell = 1,\ d'_\ell \ge 2, \\ 2q_1, & \text{if } d_\ell \ge 2,\ d'_\ell \ge 2. \end{cases} \tag{6.71}$$

Relations (6.71) and $\max_{1\le\ell\le i} d_\ell \ge 2$, $\max_{1\le\ell\le i} d'_\ell \ge 2$ imply $\sum_{\ell=1}^{i} w_\ell \ge 2q_1$ and hence (6.49).

(ii) Since RFs $\{Y^{\bullet k}(t,s)\}$ and $\{\mathcal{Z}(t,s)\}$ are uncorrelated: $\mathrm{Cov}(Y^{\bullet k}(t,s),\mathcal{Z}(u,v)) = 0$ for any (t,s), $(u,v) \in \mathbb{Z}^2$, relation (6.50) follows from (6.49) and

$$\mathrm{Cov}(Y^{\bullet k}(t,s), Y^{\bullet k}(0,0)) = k! r_Y(t,s)^k(1+o(1)), \quad |t|+|s| \to \infty. \tag{6.72}$$

To show (6.72), note that the difference $|r_Y(t,s)^k k! - \mathrm{Cov}(Y^{\bullet k}(t,s), Y^{\bullet k}(0,0))| = |(\left[a \star a\right](t,s))^k - \sum^{\bullet}_{(u,v)_k}\prod_{i=1}^{k} a(t+u_i, s+v_i)a(u_i,v_i)|k!$ satisfies the same bound as in (6.70) and therefore this difference is $O(\rho(t,s)^{-2q_1}) = o(r_Y(t,s)^k)$ according to (6.49). This proves (6.72) and part (ii).

(iii) follows similarly to (ii) using (6.49) and $|\mathrm{Cov}(Y^{\bullet k}(t,s), Y^{\bullet k}(0,0))| \le k!(\left[|a| \star |a|\right](t,s))^k \le C\rho_+(t,s)^{-kp_1}$. Proposition 6.10 is proved. $\square$

Proof of Corollary 6.12. Relation (6.53) follows from (6.49) and Proposition 6.8(i) since the l.h.s. of (6.53) does not exceed $\sum_{(t_1,s_1),(t_2,s_2)\in K_{[\lambda,\lambda^\gamma]}} |r_{\mathcal{Z}}(t_1-t_2, s_1-s_2)| \le \lambda^{1+\gamma}\sum_{(t,s)\in\mathbb{Z}^2}|r_{\mathcal{Z}}(t,s)| \le C\lambda^{1+\gamma}\sum_{(t,s)\in\mathbb{Z}^2}\rho_+(t,s)^{-2q_1}$ and the last sum converges by Proposition 6.8(i) due to $2q_1 > 1+\gamma_0$.

Relations (6.52) follow from (6.53), the orthogonality of $\{Y^{\bullet k}(t,s)\}$ and $\{\mathcal{Z}(t,s)\}$ and

$$\mathcal{V}^{\bullet}_{\lambda,\gamma} := \mathrm{Var}\Big(\sum\nolimits_{(t,s)\in K_{[\lambda,\lambda^\gamma]}} Y^{\bullet k}(t,s)\Big) \sim c(\gamma)\lambda^{2H(\gamma)}. \tag{6.73}$$

In turn, (6.73) follows from

$$\mathcal{V}_{\lambda,\gamma} := k!\sum_{(t_1,s_1),(t_2,s_2)\in K_{[\lambda,\lambda^\gamma]}} r_Y(t_1-t_2, s_1-s_2)^k \sim c(\gamma)\lambda^{2H(\gamma)} \tag{6.74}$$

and

$$\mathcal{V}^{\bullet}_{\lambda,\gamma} - \mathcal{V}_{\lambda,\gamma} = o(\lambda^{2H(\gamma)}). \tag{6.75}$$

Relation (6.74) follows from $r_Y(t,s) = [a \star a](t,s)$, Propositions 6.8(iii), 6.9 and the fact that the asymptotic constants $\mathcal{C}(\gamma)$ in (6.47) coincide with $c(\gamma)$ in Theorems 6.1–6.3. (The last fact follows by exchanging the order of integration in these integrals, e.g. $c(\gamma_0)$ in (6.18) writes as $c(\gamma_0) = k! \int_{\mathbb{R}^{2k}} \big(\int_{(0,1]^2} \prod_{i=1}^k a_\infty(t-u_i, s-v_i)\mathrm{d}t\mathrm{d}s\big)^2 \prod_{i=1}^k \mathrm{d}u_i \mathrm{d}v_i = k! \int_{(0,1]^4} b_\infty(t_1-t_2, s_1-s_2)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2 = \mathcal{C}(\gamma_0)$, where $b_\infty(t,s) = ((a_\infty \star a_\infty)(t,s))^k$, see (6.44).) Finally, the difference in (6.75) can be estimated as in (6.69)–(6.70) and therefore this difference is $O(\lambda^{1+\gamma}) = o(\lambda^{2H(\gamma)})$ as shown in (6.53). This proves (6.73) and the proposition. $\square$

6.7.2 Proofs of Theorems 6.1–6.5 and Proposition 6.7

We use the criterion in Proposition 6.13 for the convergence in distribution of off-diagonal polygonal forms towards Itô–Wiener integral which is a straightforward extension of [36, Proposition 14.3.2].

Let $L^2(\mathbb{Z}^{2k})$ be the class of all real functions $g = g((u,v)_k)$, $(u,v)_k \in \mathbb{Z}^{2k}$, with $\sum_{(u,v)_k \in \mathbb{Z}^{2k}} g((u,v)_k)^2 < \infty$ and $Q_k(g) := \sum^{\bullet}_{(u,v)_k} g((u,v)_k)\varepsilon(u_1,v_1)\cdots\varepsilon(u_k,v_k)$, $g \in L^2(\mathbb{Z}^{2k})$ be a k-tuple off-diagonal form in i.i.d. r.v.s $\{\varepsilon(u,v)\}$ satisfying Assumption (A1). For $g_{\lambda,\gamma} \in L^2(\mathbb{Z}^{2k})$ $(\lambda > 0, \gamma > 0)$ define a step function $\widetilde{g}_{\lambda,\gamma} \in L^2(\mathbb{R}^{2k})$ by

$$\widetilde{g}_{\lambda,\gamma}((u,v)_k) \quad := \quad \lambda^{k\gamma(1+\gamma_0^{-1})/2} g_{\lambda,\gamma}([\lambda^{\gamma/\gamma_0}u_1], [\lambda^\gamma v_1], \dots, [\lambda^{\gamma/\gamma_0}u_k], [\lambda^\gamma v_k]), \qquad (u,v)_k \in \mathbb{R}^{2k}. \tag{6.76}$$

Proposition 6.13. *Assume there exists $h_\gamma \in L^2(\mathbb{R}^{2k})$ such that $\lim_{\lambda\to\infty} \|\widetilde{g}_{\lambda,\gamma} - h_\gamma\|_k \to 0$. Then $Q_k(g_{\lambda,\gamma}) \xrightarrow{\mathrm{d}} \int_{\mathbb{R}^{2k}} h_\gamma((u,v)_k)\mathrm{d}^k W$ $(\lambda\to\infty)$.*

Proof of Theorem 6.1. (i) Let us show that the stochastic integral $V^X_{k,\gamma_0}(x,y)$ is well-defined or $\|h(x,y;\cdot)\|_k < \infty$, where $h(x,y;(u,v)_k)$ is defined in (6.16). It suffices to consider the case $x = y = 1$. By (6.41), (6.39) of Proposition 6.8, $\|h(1,1;\cdot)\|_k^2 = \int_{(0,1]^4}((a_\infty \star a_\infty)(t_1-t_2, s_1-s_2))^k \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2 \le C\int_{(0,1]^4} \rho(t_1-t_2, s_1-s_2)^{-kp_1}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2 < \infty$ since $kp_1 < 1+\gamma_0 = 1 + p_1/p_2$ or $k < P$ holds. The self-similarity property in (6.17) follows by scaling properties $a_\infty(\lambda t, \lambda^{\gamma_0}s) = \lambda^{-q_1}a_\infty(t,s)$, $\{W(\mathrm{d}\lambda u, \mathrm{d}\lambda^{\gamma_0}v)\} \overset{\mathrm{fdd}}{=} \{\lambda^{(1+\gamma_0)/2}W(\mathrm{d}u,\mathrm{d}v)\}$ of the integrand and the white noise, and the change of variables rules for multiple Itô–Wiener integral, see [25], also [36, Proposition 14.3.5].

(ii) Relation (6.18) is proved in Corollary 6.12. Let us prove (6.19). Recall the decomposition $X(t,s) = Y^{\bullet k}(t,s) + \mathcal{Z}(t,s)$ in (6.13). Using $\mathrm{Var}(S^{\mathcal{Z}}_{\lambda,\gamma_0}) = O(\lambda^{1+\gamma_0}) =$

$o(\lambda^{2H(\gamma_0)})$, see (6.53), relation (6.19) follows from

$$Q_k(g_{\lambda,\gamma_0}(x,y;\cdot)) = \lambda^{-H(\gamma_0)} \sum_{(t,s)\in K_{[\lambda x,\lambda^{\gamma_0} y]}} Y^{\bullet k}(t,s) \overset{\mathrm{fdd}}{\to} V^X_{k,\gamma_0}(x,y), \tag{6.77}$$

where

$$g_{\lambda,\gamma_0}(x,y;(u,v)_k) \ := \ \lambda^{-H(\gamma_0)} \sum_{(t,s)\in K_{[\lambda x,\lambda^{\gamma_0} y]}} a(t-u_1,s-v_1)\cdots a(t-u_k,s-v_k), \quad (u,v)_k \in \mathbb{Z}^{2k}. \tag{6.78}$$

Using Proposition 6.13 and Cramér–Wold device, relation (6.77) follows from

$$\lim_{\lambda\to\infty} \Big\| \sum\nolimits_{i=1}^m \theta_i(\widetilde{g}_{\lambda,\gamma_0}(x_i,y_i;\cdot) - h(x_i,y_i;\cdot)) \Big\|_k = 0, \tag{6.79}$$

for any $m \ge 1$ and any $\theta_i \in \mathbb{R}$, $(x_i,y_i) \in \mathbb{R}^2_+$, $1 \le i \le m$, where the limit function $h(x,y;(u,v)_k)$ is given in (6.16). We restrict the subsequent proof of (6.79) to the case $m = \theta_1 = 1$, $(x_1,y_1) = (x,y)$ since the general case of (6.79) follows analogously. Using (6.10), (6.78), (6.76) and notation $a_\lambda(t,s) := (\lambda^{-1} \vee \rho(t,s))^{-q_1}(L_0(t/(\lambda^{-1}\vee\rho(t,s))) + o(1))$, $\lambda\to\infty$, and $\lambda' := \lambda^{\gamma_0}$ similarly to (6.67) we get

$$\begin{aligned}
\widetilde{g}_{\lambda,\gamma_0}(x,y;(u,v)_k) \ &= \ \int_{\mathbb{R}^2} \prod_{i=1}^k a_\lambda\big(\tfrac{[\lambda t]-[\lambda u_i]}{\lambda}, \tfrac{[\lambda' s]-[\lambda' v_i]}{\lambda'}\big) \\
&\qquad\qquad \times \mathbf{1}\big(([\lambda t],[\lambda' s]) \in (0,\lambda x]\times(0,\lambda' y]\big)\mathrm{d}t\mathrm{d}s \\
&\to \ h(x,y;(u,v)_k)
\end{aligned} \tag{6.80}$$

point-wise for any $(u,v)_k \in \mathbb{R}^{2k}$, $(u_i,v_i) \ne (u_j,v_j)$ $(i \ne j)$ fixed. A similar inequality to (6.63), viz.,

$$\tfrac{1}{\lambda} \vee \rho\big(\tfrac{[\lambda t]-[\lambda u]}{\lambda}, \tfrac{[\lambda' s]-[\lambda' v]}{\lambda'}\big) \ge c\rho(t-u,s-v), \quad \forall t,u,s,v\in\mathbb{R}, \tag{6.81}$$

holds with some constant $c > 0$ independent of $t,u,s,v \in \mathbb{R}$, implying the dominating bound

$$|\widetilde{g}_{\lambda,\gamma_0}(x,y;(u,v)_k)| \le C \int_{(0,2x]\times(0,2y]} \prod_{i=1}^k \rho(t-u_i,s-v_i)^{-q_1}\mathrm{d}t\mathrm{d}s =: \bar{g}(x,y:(u,v)_k),$$

where $\|\bar{g}(x,y;\cdot)\|_k < \infty$ by (6.41), (6.39) of Proposition 6.8, so that (6.79) follows by the dominated convergence theorem in view of (6.80). Theorem 6.1 is proved. □

Proof of Theorem 6.2. As noted in Section 6.3, part (iii) follows by the same argument as part (ii) by exchanging the coordinates t and s and we omit the details.

(i) Let us show that the stochastic integral in (6.20) is well-defined or $\|h_+(y;\cdot)\|_k < \infty$, where $h_+(y;(u,v)_k)$ is defined in (6.21). Indeed by (6.41) of Proposition 6.8 $\|h_+(y;\cdot)\|_k^2 = \int_{(0,1]^2}((a_\infty \star a_\infty)(0,s_1-s_2))^k \mathrm{d}s_1\mathrm{d}s_2 \le C\int_{(0,1]^2}\rho(0,s_1-s_2)^{-kp_1}\mathrm{d}s_1\mathrm{d}s_2 \le C\int_{[-1,1]}|s|^{-kp_2}\mathrm{d}s < \infty$ since $kp_2 < 1$. The remaining facts in (i) follow similarly as in the proof of Theorem 6.1(i).

(ii) Relation (6.22) is proved in Corollary 6.12. Similarly to the proof of (6.19), the weak convergence in (6.23) follows from

$$Q_k(g_{\lambda,\gamma}(x,y;\cdot)) = \lambda^{-H(\gamma)} \sum_{(t,s)\in K_{[\lambda x,\lambda^\gamma y]}} Y^{\bullet k}(t,s) \stackrel{\mathrm{fdd}}{\to} xZ_k^+(y), \tag{6.82}$$

where

$$g_{\lambda,\gamma}(x,y;(u,v)_k) \;:=\; \lambda^{-H(\gamma)} \sum_{(t,s)\in K_{[\lambda x,\lambda^\gamma y]}} a(t-u_1,s-v_1)\cdots a(t-u_k,s-v_k), \quad (u,v)_k \in \mathbb{Z}^{2k}.$$

Again, we restrict the proof of (6.82) to one-dimensional convergence at $(x,y)\in\mathbb{R}_+^2$. By Proposition 6.13 this follows from

$$\lim_{\lambda\to\infty} \|\widetilde{g}_{\lambda,\gamma}(x,y;\cdot) - xh_+(y;\cdot)\|_k = 0, \tag{6.83}$$

where, with $\lambda' := \lambda^\gamma$, $\lambda'' := \lambda^{\gamma/\gamma_0}$, $\lambda = o(\lambda'')$, $a_{\lambda''}(t,s) := ((\lambda'')^{-1}\vee\rho(t,s))^{-q_1}(L_0(t/((\lambda'')^{-1}\vee\rho(t,s))) + o(1))$,

$$\begin{aligned}\widetilde{g}_{\lambda,\gamma}(x,y;(u,v)_k) &= \int_{\mathbb{R}^2}\prod_{i=1}^k a_{\lambda''}\big(\tfrac{[\lambda t]-[\lambda'' u_i]}{\lambda''}, \tfrac{[\lambda' s]-[\lambda' v_i]}{\lambda'}\big) \\ &\qquad \times \mathbf{1}\big(([\lambda t],[\lambda' s]) \in (0,\lambda x]\times(0,\lambda' y]\big)\mathrm{d}t\mathrm{d}s \\ &\to xh_+(y;(-u,v)_k) \end{aligned} \tag{6.84}$$

point-wise for any $(u,v)_k\in\mathbb{R}^{2k}$, $(u_i,v_i)\ne(u_j,v_j)$ $(i\ne j)$ fixed.

The dominating convergence argument to prove (6.83) from (6.84) uses Pratt's lemma [84], as follows. Similarly to (6.81) note that

$$\tfrac{1}{\lambda''}\vee\rho\big(\tfrac{[\lambda t]-[\lambda'' u]}{\lambda''}, \tfrac{[\lambda' s]-[\lambda' v]}{\lambda'}\big) \ge c\rho((\lambda t/\lambda'')-u, s-v),$$

with $c>0$ independent of $t,u,s,v\in\mathbb{R}$ and hence

$$\begin{aligned}|\widetilde{g}_{\lambda,\gamma}(x,y;(u,v)_k)| &\le C\int_{(0,2x]\times(0,2y]}\prod_{i=1}^k \rho((\lambda t/\lambda'')-u_i, s-v_i)^{-q_1}\mathrm{d}t\mathrm{d}s \\ &=: CG_\lambda((u,v)_k)\end{aligned}$$

with $C > 0$ independent of $\lambda > 0$, $(u,v)_k \in \mathbb{R}^{2k}$. Clearly, $\lim_{\lambda\to\infty} G_\lambda((u,v)_k) = G((u,v)_k) := 2x\int_{(0,2y]}\prod_{i=1}^k \rho(u_i, s-v_i)^{-q_1}\mathrm{d}s$ point-wise in $\mathbb{R}^{2k}$ and

$$\begin{aligned}\|G_\lambda\|_k^2 &= \int_{(0,2x]^2\times(0,2y]^2}\big((\rho^{-q_1}\star\rho^{-q_1})((\lambda/\lambda'')(t_1-t_2), s_1-s_2)\big)^k\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2\\ &\to \int_{(0,2x]^2\times(0,2y]^2}\big((\rho^{-q_1}\star\rho^{-q_1})(0, s_1-s_2)\big)^k\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2 = \|G\|_k^2 < \infty\end{aligned}$$

by (6.41) of Proposition 6.8 and condition $1 \le k < 1/p_2$, or $p_2 = q_2(2-Q) < 1/k$. Thus, application of [84] proves (6.83). Theorem 6.2 is proved. $\square$

To prove Theorem 6.3 we use approximation by m-dependent variables and the following CLT for triangular array of m-dependent r.v.s.

Lemma 6.14. *Let $\{\xi_{ni}, 1 \le i \le N_n\}$, $n \ge 1$, be a triangular array of m-dependent r.v.s with zero mean and finite variance. Assume that: (L1) ξ_{ni}, $1 \le i \le N_n$, are identically distributed for any $n \ge 1$, (L2) $\xi_n := \xi_{n1} \xrightarrow{\mathrm{d}} \xi$, $\mathrm{E}\xi_n^2 \to \mathrm{E}\xi^2 < \infty$ for some r.v. ξ and (L3) $\mathrm{Var}(\sum_{i=1}^{N_n}\xi_{ni}) \sim \sigma^2 N_n$, $\sigma^2 > 0$. Then $N_n^{-1/2}\sum_{i=1}^{N_n}\xi_{ni} \xrightarrow{\mathrm{d}} N(0,\sigma^2)$.*

Proof. W.l.g., we can assume $N_n = n$ in the subsequent proof. We use the CLT due to Orey [75]. Accordingly, let $\xi_{ni}^\tau := \xi_{ni}\mathbf{1}(|\xi_{ni}| \le \tau n^{1/2})$, $\alpha_{ni}^\tau := \mathrm{E}\xi_{ni}^\tau$, $\sigma_{nij}^\tau := \mathrm{Cov}(\xi_{ni}^\tau, \xi_{nj}^\tau)$. It suffices to show that for any $\tau > 0$ the following conditions in [75] are satisfied: (O1) $n^{-1/2}\sum_{i=1}^n \alpha_{ni}^\tau \to 0$, (O2) $n^{-1}\sum_{i,j=1}^n \sigma_{nij}^\tau \to \sigma^2$, (O3) $n^{-1}\sum_{i=1}^n \sigma_{nii}^\tau = O(1)$, and (O4) $\sum_{i=1}^n \mathrm{P}(|\xi_{ni}| > \tau n^{1/2}) \to 0$.
Consider (O1), or $n^{1/2}\alpha_n^\tau \to 0$, $\alpha_n^\tau := \alpha_{n1}^\tau$. We have $0 = n^{1/2}\mathrm{E}\xi_n = n^{1/2}\alpha_n^\tau + \kappa_n$, where $|\kappa_n| := n^{1/2}|\mathrm{E}\xi_n\mathbf{1}(|\xi_n| > \tau n^{1/2})| \le \tau^{-1}\mathrm{E}\xi_n^2\mathbf{1}(|\xi_n| > \tau n^{1/2})$. Therefore, (O1) follows from

$$\mathrm{E}\xi_n^2\mathbf{1}(|\xi_n| > \tau n^{1/2}) \to 0. \tag{6.85}$$

Using the Skorohod representation theorem [97] w.l.g. we can assume that r.v.s ξ, ξ_n, $n \ge 1$, are defined on the same probability space and $\xi_n \to \xi$ almost surely. The latter fact together with (L2) and Pratt's lemma [84] implies that $\mathrm{E}|\xi_n^2 - \xi^2| \to 0$ and hence (6.85) follows due to $\mathrm{P}(|\xi_n| > \tau n^{1/2}) \to 0$, see [71, Chapter 2, Proposition 5.3]. The above argument also implies (O4) since $\mathrm{P}(|\xi_n| > \tau n^{1/2}) \le \tau^{-2}n^{-1}\mathrm{E}\xi_n^2\mathbf{1}(|\xi_n| > \tau n^{1/2})$ by Markov's inequality. (O3) is immediate from (L1) and (L2). Finally, (O2) follows from (L3), (O1) and

$$n^{-1}\sum_{1\le i,j\le n, |i-j|\le m}\mathrm{E}(\xi_{ni}\xi_{nj} - \xi_{ni}^\tau\xi_{nj}^\tau) \to 0. \tag{6.86}$$

Let $\widetilde{\xi}_{ni}^\tau := \xi_{ni} - \xi_{ni}^\tau$. Since $|\mathrm{E}(\xi_{ni}\xi_{nj} - \xi_{ni}^\tau\xi_{nj}^\tau)| \le |\mathrm{E}(\widetilde{\xi}_{ni}^\tau\xi_{nj}^\tau + \xi_{ni}^\tau\widetilde{\xi}_{nj}^\tau + \widetilde{\xi}_{ni}^\tau\widetilde{\xi}_{nj}^\tau)| \le C\mathrm{E}^{1/2}\xi_n^2\mathbf{1}(|\xi_n| > \tau n^{1/2})$, relation (6.86) follows from (6.85). Lemma 6.14 is proved. $\square$

Proof of Theorem 6.3. Again, we prove part (i) only since part (ii) follows similarly by exchanging the coordinates t and s.
Relation (6.25) is proved in Corollary 6.12. Let us prove (6.26). Similarly as in the case of the previous theorems, we shall restrict ourselves with the proof of one-dimensional convergence at $(x,y) \in \mathbb{R}^2_+$. For $m \ge 1$, $\lambda > 0$, define stationary RFs

$$\begin{aligned} X_m(t,s) &:= A_k(Y_m(t,s)), \qquad \text{where} \\ Y_m(t,s) &:= \sum_{(u,v)\in\mathbb{Z}^2:|s-v|\le[\lambda^{\gamma_0}]m} a(t-u,s-v)\varepsilon(u,v), \end{aligned} \tag{6.87}$$

and where A_k stands for the Appell polynomial of degree k relative to the distribution of $Y_m(t,s)$. Note $X_m(t_1,s_1)$ and $X_m(t_2,s_2)$ are independent if $|s_1-s_2| > 2[\lambda^{\gamma_0}]m$. Then

$$S^{X_m}_{\lambda,\gamma}(x,y) := \sum_{(t,s)\in K_{[\lambda x,\lambda^\gamma y]}} X_m(t,s) = \sum_{i=1}^{N_\lambda+1} U_{\lambda,m}(i), \tag{6.88}$$

where $N_\lambda := [[\lambda^\gamma y]/[\lambda^{\gamma_0}]] = O(\lambda^{\gamma-\gamma_0})$ and

$$U_{\lambda,m}(i) := \sum_{1\le t\le[\lambda x]} \sum_{(i-1)[\lambda^{\gamma_0}]<s\le i[\lambda^{\gamma_0}]} X_m(t,s). \tag{6.89}$$

Note $U_{\lambda,m}(i)$ and $U_{\lambda,m}(j)$ are independent provided $|i-j| > 2m$ hence (6.88) is a sum of $2m$-dependent r.v.s. The one-dimensional convergence in (6.26) follows from standard Slutsky's argument (see e.g. [36, Lemma 4.2.1]) and the following lemma. Theorem 6.3 is proved. □

Lemma 6.15. *Under the conditions and notation of Theorem 6.3(i), for any $\gamma > \gamma_0$ and any $m = 1,2,\dots$,*

$$\begin{aligned} \mathrm{Var}(S^{X_m}_{\lambda,\gamma}(x,y)) &\sim \sigma^2_m(x,y)\lambda^{2H(\gamma)} \quad \textit{and} &(6.90)\\ \lambda^{-H(\gamma)}S^{X_m}_{\lambda,\gamma}(x,y) &\xrightarrow{\mathrm{d}} N(0,\sigma^2_m(x,y)) \quad \textit{as } \lambda\to\infty, &(6.91)\end{aligned}$$

where $\sigma^2_m(x,y)$ is defined in (6.93). *Moreover,*

$$\lim_{m\to\infty}\limsup_{\lambda\to\infty} \lambda^{-2H(\gamma)}\,\mathrm{Var}(S^X_{\lambda,\gamma}(x,y) - S^{X_m}_{\lambda,\gamma}(x,y)) = 0. \tag{6.92}$$

Proof. By adapting the argument in the proof of (6.25) and Proposition 6.9 Case (III), we can show the limits

$$\begin{aligned} \lambda^{-2H(\gamma)}\,\mathrm{Var}(S^{X_m}_{\lambda,\gamma}(x,y)) &\to k!y\int_{(0,x]^2\times\mathbb{R}} \big((a_{\infty,m}\star a_{\infty,m})(t_1-t_2,s)\big)^k \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s \\ &=: \sigma^2_m(x,y) \end{aligned} \tag{6.93}$$

and

$$
\begin{aligned}
&\lambda^{-2H(\gamma)}\operatorname{Var}(S^X_{\lambda,\gamma}(x,y)-S^{X_m}_{\lambda,\gamma}(x,y))\\
&\quad=\lambda^{-2H(\gamma)}\sum_{(t_i,s_i)\in K_{[\lambda x,\lambda^\gamma y]},i=1,2}\big\{\operatorname{Cov}(X(t_1,s_1),X(t_2,s_2))\\
&\qquad\qquad-\operatorname{Cov}(X(t_1,s_1),X_m(t_2,s_2))-\operatorname{Cov}(X_m(t_1,s_1),X(t_2,s_2))\\
&\qquad\qquad+\operatorname{Cov}(X_m(t_1,s_1),X_m(t_2,s_2))\big\}\\
&\quad\to k!y\int_{(0,x]^2\times\mathbb{R}}G_m(t_1-t_2,s)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s,\quad \lambda\to\infty,
\end{aligned}\tag{6.94}
$$

where $G_m(t,s):=((a_\infty\star a_\infty)(t,s))^k-((a_{\infty,m}\star a_\infty)(t,s))^k-((a_\infty\star a_{\infty,m})(t,s))^k+((a_{\infty,m}\star a_{\infty,m})(t,s))^k$ and

$$
a_{\infty,m}(t,s):=L_0(t/\rho(t,s))\rho(t,s)^{-q_1}\mathbf{1}(|s|\le m),\quad (t,s)\in\mathbb{R}^2,\tag{6.95}
$$

is a 'truncated' version of $a_\infty(t,s)$ in (6.15). Since $|G_m(t,s)|\le 4((a_\infty\star a_\infty)(t,s))^k$ and $G_m(t,s)$ vanishes with $m\to\infty$ for any fixed $(t,s)\ne(0,0)$, (6.92) follows from (6.94) by the dominated convergence theorem.

The proof of (6.91) uses Lemma 6.14. Accordingly, let $N_\lambda:=[[\lambda^\gamma y]/[\lambda^{\gamma_0}]]$ and $\xi_{\lambda i}:=\lambda^{-H(\gamma_0)}U_{\lambda,m}(i)$, where $H(\gamma_0)=1+\gamma_0-kp_1/2$ is the same as in Theorem 6.1 and $U_{\lambda,m}(i)$ are $2m$-dependent r.v.s defined in (6.89). Note $U_{\lambda,m}(i)$, $1\le i\le N_\lambda$, are identically distributed and $\lambda^{H(\gamma_0)}N_\lambda^{1/2}\sim\lambda^{H(\gamma)}y^{1/2}$. Thus, condition (L1) of Lemma 6.14 for $\xi_{\lambda i}$, $1\le i\le N_\lambda$, is satisfied and (L3) follows from $\operatorname{Var}(\sum_{i=1}^{N_\lambda}\xi_{\lambda i})\sim\lambda^{-2H(\gamma_0)}\operatorname{Var}(S^{X_m}_{\lambda,\gamma}(x,y))\sim\lambda^{\gamma-\gamma_0}\sigma_m^2(x,y)$, see (6.90). Finally, condition (L2), or

$$
\xi_{\lambda,1}=\lambda^{-H(\gamma_0)}U_{\lambda,m}(1)\xrightarrow{\mathrm{d}}\xi,\quad \mathrm{E}\xi^2_{\lambda,1}\to\mathrm{E}\xi^2
$$

follows similarly as in Theorem 6.1 with the limit r.v. ξ given by the k-tuple Itô–Wiener integral:

$$
\xi:=\int_{\mathbb{R}^{2k}}\Big\{\int_0^x\int_0^1\prod_{\ell=1}^k a_{\infty,m}(t-u_\ell,s-v_\ell)\,\mathrm{d}t\mathrm{d}s\Big\}\mathrm{d}^kW
$$

and $a_{\infty,m}(t,s)$ defined in (6.95). This proves (6.91) and Lemma 6.15, too. □

Proof of Theorem 6.4. The proof is an adaptation of the proof of CLT in [36, Theorem 4.8.1] for sums of 'off-diagonal' polynomial forms with one-dimensional 'time' parameter. Define

$$
\begin{aligned}
X_m(t,s)&:=A_k(Y_m(t,s)),\\
Y_m(t,s)&:=\sum_{(u,v)\in\mathbb{Z}^2:|t-u|+|s-v|\le m}a(t-u,s-v)\varepsilon(u,v),
\end{aligned}\tag{6.96}
$$

where A_k stands for the Appell polynomial of degree k relative to the distribution of $Y_m(t,s)$. Note the truncation level m in (6.96) does not depend on λ in contrast to the truncation level $m[\lambda^{\gamma_0}]$ in (6.87). Similarly to Lemma 6.15 it suffices to prove for any $\gamma > 0$, $m = 1, 2, \dots,$

$$\mathrm{Var}(S^{X_m}_{\lambda,\gamma}(x,y)) \sim xy\sigma^2_{X_m}\lambda^{1+\gamma}, \quad \lambda^{-(1+\gamma)/2}S^{X_m}_{\lambda,\gamma}(x,y) \xrightarrow{\mathrm{d}} N(0, xy\sigma^2_{X_m}) \tag{6.97}$$

$$\lim_{m\to\infty}\limsup_{\lambda\to\infty}\lambda^{-(1+\gamma)}\mathrm{Var}(S^X_{\lambda,\gamma}(x,y) - S^{X_m}_{\lambda,\gamma}(x,y)) = 0, \tag{6.98}$$

where $\sigma^2_{X_m} := \sum_{(t,s)\in\mathbb{Z}^2} r_{X_m}(t,s)$ and $r_{X_m}(t,s) := \mathrm{Cov}(X_m(0,0), X_m(t,s))$. Note $X_m(t_1,s_1)$ and $X_m(t_2,s_2)$ are independent if $|t_1 - t_2| + |s_1 - s_2| > 2m$. Therefore $\sum_{(t,s)\in\mathbb{Z}^2}|r_{X_m}(t,s)| < \infty$ and (6.97) follows from the CLT for m-dependent RFs, see [15]. Consider (6.98), where we can put $x = y = 1$ w.l.g. We have $\lambda^{-(1+\gamma)}\mathrm{Var}(S^X_{\lambda,\gamma} - S^{X_m}_{\lambda,\gamma}) \le \sum_{(t,s)\in\mathbb{Z}^2}|\phi_m(t,s)|$, where $\phi_m(t,s) := \mathrm{Cov}(X(0,0) - X_m(0,0), X(t,s) - X_m(t,s))$. From (6.69), (6.70) and (6.72) we conclude that

$$\begin{aligned}&|\mathrm{Cov}(X(0,0), X(t,s))| + |\mathrm{Cov}(X(0,0), X_m(t,s))| + |\mathrm{Cov}(X_m(0,0), X_m(t,s))| \\ &\quad \le C\rho_+(t,s)^{-(kp_1)\wedge(2q_1)}\end{aligned}$$

as in (6.51), with $C > 0$ independent of m. Hence $|\phi_m(t,s)| \le C\rho_+(t,s)^{-(kp_1)\wedge(2q_1)} =: \phi(t,s)$, where $\sum_{(t,s)\in\mathbb{Z}^2}\phi(t,s) < \infty$, see Proposition 6.8(i), also Corollary 6.11(ii). Thus, (6.98) follows by the dominated convergence theorem and the fact that $\lim_{m\to\infty}\phi_m(t,s) = 0$ for any $(t,s)\in\mathbb{Z}^2$. Theorem 6.4 is proved. □

Proof of Theorem 6.5. (i) Split $X = X_k + X'_k$, where $X'_k := \sum_{j=k+1}^\infty c_j X_j/j!$, $X_j(t,s) := H_j(Y(t,s))$. Since all statements of Theorems 6.1–6.3 hold for RF $X_k = H_k(Y)$ and $\mathrm{Cov}(X_k(t_1,s_1), X'_k(t_2,s_2)) = 0$, $\forall (t_i,s_i)\in\mathbb{Z}^2$, $i = 1,2$, it suffices to show that

$$\mathrm{Var}(S^{X'_k}_{\lambda,\gamma}) = o(\lambda^{2H(\gamma)}), \quad \lambda\to\infty, \tag{6.99}$$

for $H(\gamma)$ defined in Theorems 6.1–6.3. By well-known properties of Hermite polynomials, $\mathrm{Var}(S^{X'_k}_{\lambda,\gamma}) = \sum_{j=k+1}^\infty c_j^2\,\mathrm{Var}(S^{X_j}_{\lambda,\gamma})/(j!)^2$, $\mathrm{Var}(S^{X_j}_{\lambda,\gamma}) = j!\sum_{(t_i,s_i)\in K_{[\lambda,\lambda^\gamma]}, i=1,2} r_Y(t_1 - t_2, s_1 - s_2)^j \le j!\Sigma_{k+1}(\lambda)$, where $\Sigma_{k+1}(\lambda) := \sum_{(t_i,s_i)\in K_{[\lambda,\lambda^\gamma]}, i=1,2}|r_Y(t_1 - t_2, s_1 - s_2)|^{k+1}$ for $j \ge k+1$ since $|r_Y(t,s)| \le 1$ according to Assumption (A4)$_k$. Therefore, $\mathrm{Var}(S^{X'_k}_{\lambda,\gamma}) \le (\sum_{j=k+1}^\infty c_j^2/j!)\Sigma_{k+1}(\lambda) \le \mathrm{E}G(Y(0,0))^2\Sigma_{k+1}(\lambda)$, where $\Sigma_{k+1}(\lambda) = o(\lambda^{2H(\gamma)})$ follows by Proposition 6.9. This proves (6.99) and part (i).

(ii) For large $K\in\mathbb{N}$, $K > k$, split $X = \hat X_K + X'_K$, where $\hat X_K(t,s) := \sum_{j=k}^K c_j H_j(Y(t,s))/j!$ and $X'_K(t,s) := \sum_{j=K+1}^\infty c_j H_j(Y(t,s))/j!$. Then $\mathrm{Var}(S^{X'_K}_{\lambda,\gamma}) \le (\sum_{j=K+1}^\infty c_j^2/j!)\Sigma_{K+1}(\lambda)$ as in the proof of part (i), implying $\mathrm{Var}(S^{X'_K}_{\lambda,\gamma}) \le C\epsilon_K\lambda^{1+\gamma}$, where $\epsilon_K := \sum_{j=K+1}^\infty c_j^2/j!$ can be made arbitrary small by choosing K large enough. On the other hand, by Theorem 6.4, $\lambda^{-(1+\gamma)/2} S^{X_j}_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \sigma_{X_j} B_{1/2,1/2}(x, y)$ for any $j \ge k$ and the last result extends to finite sums of Hermite polynomials, viz., $\lambda^{-(1+\gamma)/2} S^{\hat{X}_K}_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \sigma_{\hat{X}_K} B_{1/2,1/2}(x,y)$, where $\sigma^2_{\hat{X}_K} = \sum_{(t,s)\in\mathbb{Z}^2} \mathrm{Cov}(\hat{X}_K(0, 0), \hat{X}_K(t,s)) \to \sigma^2_X$, $K \to \infty$. See e.g. [36, proof of Theorem 4.6.1]. The remaining details are easy. Theorem 6.4 is proved. □

Proof of Proposition 6.7. The transition probabilities $q_u(v)$ in (6.36) can be explicitly written in terms of binomial probabilities $\mathrm{bin}(j,k;p) := \binom{k}{j} p^j (1-p)^{k-j}$, $k = 0,1,\dots$, $j = 0,1,\dots,k$, $0 \le p \le 1$:

$$q_u(v) = \sum_{j=0}^{u} \mathrm{bin}(u-j, u; \theta)\, \mathrm{bin}((v+j)/2, j; 1/2), \quad u \in \mathbb{N},\ |v| \le u. \tag{6.100}$$

Similarly to [56, proof of Proposition 4.1], we shall use the following version of the Moivre–Laplace theorem (Feller [32, Chapter 7, §3, Theorem 1]): *There exists a constant C such when $j \to \infty$ and $k \to \infty$ vary in such a way that*

$$\frac{(j-kp)^3}{k^2} \to 0,$$

then

$$\Big| \frac{\mathrm{bin}(j,k;p)}{\frac{1}{\sqrt{2\pi kp(1-p)}} \exp\{-\frac{(j-kp)^2}{2kp(1-p)}\}} - 1 \Big| < \frac{C}{k} + \frac{C|j-kp|^3}{k^2}. \tag{6.101}$$

Let us first explain the idea of the proof. Using (6.100) and replacing the binomial probabilities by Gaussian densities according to (6.101) leads to

$$\begin{aligned}
a(u,v) &\sim \frac{1}{2}\sum_{j=0}^{u} \frac{1}{\Gamma(d)u^{1-d}} \frac{1}{\sqrt{2\pi\theta(1-\theta)u}} \exp\Big\{-\frac{(j-(1-\theta)u)^2}{2\theta(1-\theta)u}\Big\} \\
&\qquad \times \frac{1}{\sqrt{j\pi/2}} \exp\Big\{-\frac{v^2}{2j}\Big\} \\
&= \frac{u^{d-3/2}}{\Gamma(d)\sqrt{2\pi}} \sum_{j=0}^{u} \frac{1}{u\sqrt{2\pi\theta(1-\theta)/u}} \exp\Big\{-\frac{((j/u)-(1-\theta))^2}{2\theta(1-\theta)/u}\Big\} \\
&\qquad \times \frac{1}{\sqrt{j/u}} \exp\Big\{-\frac{v^2/u}{2j/u}\Big\} \\
&\sim \frac{u^{d-3/2}}{\Gamma(d)\sqrt{2\pi}} \int_0^1 \frac{1}{\sqrt{2\pi\theta(1-\theta)/u}} \exp\Big\{-\frac{(x-(1-\theta))^2}{2\theta(1-\theta)/u}\Big\} \\
&\qquad \times \frac{1}{\sqrt{x}} \exp\Big\{-\frac{v^2/u}{2x}\Big\} \mathrm{d}x \\
&\sim \frac{u^{d-3/2}}{\Gamma(d)\sqrt{2\pi(1-\theta)}} \exp\Big\{-\frac{v^2/u}{2(1-\theta)}\Big\} \\
&= \rho(u,v)^{d-3/2} L_0(u/\rho(u,v))
\end{aligned}$$

with $L_0(z)$, $z \in [-1,1]$, defined in (6.37). Here, factor 1/2 in front of the sum in the first line appears since $\mathrm{bin}((v+j)/2, j; 1/2) = 0$ whenever $v+j$ is odd, in other words, by using Gaussian approximation for all (even and odd) j we double the sum and therefore must divide it by 2. Note also that in the third line, the Gaussian kernel $\frac{1}{\sqrt{2\pi\theta(1-\theta)/u}} \exp\{-\frac{(x-(1-\theta))^2}{2\theta(1-\theta)/u}\}$ acts as a δ-function at $x = 1-\theta$ when $u \to \infty$.

Let us turn to a rigorous proof of the above asymptotics. For $(u,v) \in \mathbb{Z}^2$, $(u,v) \neq (0,0)$, denote $\varrho := (u^2+v^4)^{1/2}$, $z := u/\varrho \in [-1,1]$, then $u = z\varrho$, $v^2 = \varrho\sqrt{1-z^2}$. It suffices to prove

$$\varrho^{3/2-d} a(u,v) - L_0(z) \to 0 \quad \text{as } |u|+|v| \to \infty. \tag{6.102}$$

By definition (see (6.36), (6.37)), (6.102) holds for $u \geq 0$, $z \geq 0$ hence we can assume $u \geq 1$, $z > 0$ in what follows. Moreover, for any $\epsilon > 0$ there exists $K > 0$ such that

$$\varrho^{3/2-d} a(u,v) < \epsilon \quad \text{and} \quad L_0(z) < \epsilon \quad (\forall 1 \leq u < v^{9/5},\ \varrho > K). \tag{6.103}$$

The second relation in (6.103) is immediate by $\lim_{z\to 0} L_0(z) = L_0(0) = 0$ and $z = u/\varrho \leq \varrho^{9/10}/\varrho \to 0$ ($\varrho \to \infty$). To prove the first relation we use Hoeffding's inequality [46]. Let $\mathrm{bin}(j,k;p)$ be the binomial distribution. Then for any $\tau > 0$,

$$\sum_{0 \leq j \leq k: |j-kp| \geq \tau\sqrt{k}} \mathrm{bin}(j,k;p) \leq 2\mathrm{e}^{-2\tau^2}. \tag{6.104}$$

(6.104) implies $\mathrm{bin}((v+j)/2, j; 1/2) \leq 2\mathrm{e}^{-v^2/2j} \leq 2\mathrm{e}^{-v^2/2u}$ for any $|v| \leq u$, $0 \leq j \leq u$. Also note that $1 \leq u < v^{9/5}$ implies $2v^2 \geq u\varrho^{1/10}$. Using these facts and (6.100) with $\sum_{j=0}^{u} \mathrm{bin}(u-j, u; \theta) = 1$ for any $1 \leq u < v^{9/5}$ we obtain

$$\varrho^{3/2-d} a(u,v) \leq C\varrho^{3/2-d} q_u(v) \leq C\varrho^{3/2-d}\mathrm{e}^{-v^2/2u} \leq C\varrho^{3/2-d}\mathrm{e}^{-\varrho^{1/10}/4} \to 0, \quad \varrho \to \infty,$$

proving (6.103). Hence, it suffices to prove (6.102) for $u \to \infty$, $0 \leq v \leq u^{5/9}$. Next, we give the proof for v even, the proof for v odd being similar. Denote

$$\begin{aligned} \mathcal{D}^+(u,v) &:= \{0 \leq j \leq u/2 : |2j - u(1-\theta)| < u^{3/5} \text{ and } |v| < j^{3/5}\}, \\ \mathcal{D}^-(u,v) &:= \{0 \leq j \leq u/2 : |2j - u(1-\theta)| \geq u^{3/5} \text{ or } |v| \geq j^{3/5}\}. \end{aligned}$$

Split $a(u,v) = \psi_u(-d) \sum_{0 \leq j \leq u/2} \mathrm{bin}(u-2j, u; \theta)\, \mathrm{bin}(v/2+j, 2j; 1/2) = a^+(u,v) + a^-(u,v)$, where $a^\pm(u,v) := \psi_u(-d) \sum_{j \in \mathcal{D}^\pm(u,v)} \cdots$. It suffices to prove that

$$\varrho^{3/2-d} a^+(u,v) - L_0(z) \to 0 \quad \text{and} \quad \varrho^{3/2-d} a^-(u,v) \to 0 \tag{6.105}$$

as $u \to \infty$, $0 \le v \le u^{5/9}$. To show the first relation in (6.105), let $j_u^* := [u(1-\theta)/2]$ and

$$a^*(u,v) := \mathrm{bin}(v/2 + j_u^*, 2j_u^*; 1/2)\psi_u(-d) \sum_{j \in \mathcal{D}^+(u,v)} \mathrm{bin}(u-2j, u; \theta),$$

then

$$\begin{aligned} a^*(u,v) - a^+(u,v) &= \psi_u(-d) \sum_{j \in \mathcal{D}^+(u,v)} \mathrm{bin}(u-2j, u; \theta) \\ &\qquad \times \big(\mathrm{bin}(v/2 + j_u^*, 2j_u^*; 1/2) - \mathrm{bin}(v/2 + j, 2j; 1/2)\big). \end{aligned}$$

According to (6.101), for $j \in \mathcal{D}^+(u,v)$, $j_u^* \in \mathcal{D}^+(u,v)$,

$$\begin{aligned} \mathrm{bin}(v/2 + j, 2j; 1/2) &= \frac{1}{\sqrt{\pi j}} \mathrm{e}^{-v^2/4j}\big(1 + O(j^{-1/5})\big) \\ &= \frac{1}{\sqrt{\pi j}} \mathrm{e}^{-v^2/4j}\big(1 + O(u^{-1/5})\big), \\ \mathrm{bin}(v/2 + j_u^*, 2j_u^*; 1/2) &= \frac{1}{\sqrt{\pi j_u^*}} \mathrm{e}^{-v^2/4j_u^*}\big(1 + O(u^{-1/5})\big). \end{aligned}$$

Using $c_- u < j < c_+ u$, $j \in \mathcal{D}^+(u,v)$ for some $c_\pm > 0$, and elementary inequalities we obtain that $|\frac{1}{\sqrt{\pi j}}\mathrm{e}^{-v^2/4j} - \frac{1}{\sqrt{\pi j_u^*}}\mathrm{e}^{-v^2/4j_u^*}| \le Cu^{-7/10}\mathrm{e}^{-cv^2/u}$ for some $C, c > 0$ and hence the bound

$$|\,\mathrm{bin}(v/2 + j_u^*, 2j_u^*; 1/2) - \mathrm{bin}(v/2 + j, 2j; 1/2)| \le Cu^{-7/10}\mathrm{e}^{-cv^2/u}$$

for all $j \in \mathcal{D}^+(u,v)$ and all $u > 0$ large enough. Therefore since $\sum_{j \in \mathcal{D}^+(u,v)} \mathrm{bin}(u-2j, u; \theta) \le 1$ we obtain

$$\varrho^{3/2-d}|a^*(u,v) - a^+(u,v)| \le C\varrho^{3/2-d}u^{-7/10+d-1}\mathrm{e}^{-cv^2/u} = \varrho^{-1/5}L^*(z) \le C\varrho^{-1/5},$$

where $L^*(z) := Cz^{d-17/10}\mathrm{e}^{-c\sqrt{(1/z)^2-1}}$, $z \in (0,1]$, is a bounded function. As a consequence, it suffices to prove the first relation in (6.105) with $a^+(u,v)$ replaced by $a^*(u,v)$. This in turn follows from relations $\frac{1}{\sqrt{\pi j_u^*}}\mathrm{e}^{-v^2/4j_u^*} \sim \frac{1}{\sqrt{\pi u(1-\theta)/2}}\mathrm{e}^{-v^2/2u(1-\theta)}$, $\psi_u(-d) \sim \Gamma(d)^{-1}u^{d-1}$, and

$$\sum_{j \in \mathcal{D}^+(u,v)} \mathrm{bin}(u-2j, u; \theta) \to 1/2 \quad \text{as } u \to \infty, \tag{6.106}$$

each of which hold uniformly in $0 \le v \le u^{5/9}$. Let us check (6.106) for instance. Since $c_- u < j < c_+ u$, $j \in \mathcal{D}^+(u,v)$ for some $c_\pm > 0$, see above, so $u^{5/9} = o(j^{3/5})$ and (6.106) follows from

$$B'(u) \to 1/2 \quad \text{and} \quad B''(u) \to 0, \tag{6.107}$$

where $B'(u) := \sum_{j=0}^{u} \mathrm{bin}(u-j,u;\theta)\mathbf{1}(j \text{ is even})$, $B''(u) := \sum_{j=0}^{u} \mathrm{bin}(u-j,u;\theta)$ $\mathbf{1}(|j-u(1-\theta)| \ge u^{3/5})$. Here, the first relation in (6.107) is obvious by well-known properties of binomial coefficients while the second one follows from (6.104) according to which $B''(u) \le C\mathrm{e}^{-2u^{1/5}} \to 0$. This proves the first relation in (6.105).

The proof of the second relation in (6.105) uses Hoeffding's inequality in (6.104) in a similar way. We have $a^-(u,v) \le a_1^-(u,v) + a_2^-(u,v)$, where $a_1^-(u,v) := \psi_u(-d)\sum_{0\le j\le u:|j-u(1-\theta)|\ge u^{3/5}} \mathrm{bin}(u-j,u;\theta) \le Cu^{d-1}\mathrm{e}^{-2u^{1/5}}$ implying $\varrho^{3/2-d}a_1^-(u,v) \le Cu^{(10/9)(3/2-d)+(d-1)}\mathrm{e}^{-2u^{1/5}} \to 0$ $(u\to\infty)$ uniformly in $|v| \le u^{5/9}$. Finally,

$$\begin{aligned} a_2^-(u,v) &:= \psi_u(-d) \sum_{0\le j\le u:|j-u(1-\theta)|\le u^{3/5}, v\ge (j/2)^{3/5}} \mathrm{bin}(u-j,u;\theta) \\ &\qquad\qquad \times \mathrm{bin}((v+j)/2, j; 1/2) \\ &\le Cu^{d-1} \sum_{c_1 u \le j \le u, v \ge (j/2)^{3/5}} \mathrm{e}^{-v^2/2j} \le Cu^d \mathrm{e}^{-c_2 u^{1/5}} \end{aligned}$$

for some positive constants c_1, $c_2 > 0$, implying $\varrho^{3/2-d}a_2^-(u,v) \le Cu^{(10/9)(3/2-d)+d}$ $\mathrm{e}^{-c_2u^{1/5}} \to 0$ $(u \to \infty)$ uniformly in $|v| \le u^{5/9}$ as above. This proves (6.105) and Proposition 6.7, too. □

6.8 Final comment

Limit theorems for weakly dependent RFs usually assume very general shape of summation domains (spatial regions), the limit distribution being independent of the way in which these regions tend to infinity. Particularly, van Hove's condition (see e.g. [17]) roughly says that the volume (cardinality) of spatial region grows faster than that of its boundary. For rectangular domains, van Hove's condition means that all sides of rectangles grow to infinity in an arbitrary way.

The situation is very different for LRD RFs. We prove that for a class of nonlinear LRD RF X on $\mathbb{Z}^2$ and rectangular domains with sides increasing as $O(n)$ and $O(n^\gamma)$, the limit distribution of sums of X depends on γ in a crucial way. Specifically, there exists $\gamma_0 > 0$ such that the limit distribution is different whenever $\gamma < \gamma_0$, $\gamma = \gamma_0$ or $\gamma > \gamma_0$. For partial sums of Gaussian or stable LRD RFs, a similar trichotomy (termed scaling transition) was observed [90], [89].

The above facts have important implications for statistics of strongly dependent spatial data. The quantity $\gamma > 0$ can be broadly interpreted as the ratio of the vertical and horizontal dimensions of the sampling region (an 'external scale ratio') while $\gamma_0 = p_1/p_2$ can be defined as the ratio of the 'vertical and horizontal Hurst exponents' of the RF (an 'internal scale ratio'). Since the limit distribution of simple statistics such as the sample mean or the sample variance may depend on the relation between γ and γ_0, these quantities need to be estimated or decided in advance before applying the limit theorem. Particularly, deciding on the value of γ in a concrete situation might be difficult. For panel data, this is a question of dealing with either long, or short panel which is not easy to answer and then the natural limit theory leads to models where the limit is independent of how the numbers of horizontal (time series) and vertical (cross section) panel observations tend to infinity [78]. Nevertheless, for some panels with LRD, a 'scaling transition' occurs, see [79], and the above question must be answered in a practical situation.

Let us also mention some open problems related to the present chapter. It is of interest to extend our results for nonlocal functions or vector-valued RFs, particularly for covariance estimates, c.f. [3, 47]. Several works note that in many practical applications, sampling regions are non-rectangular, and possibly of a nonstandard shape, see [24, 57]. Extending scaling transition to such domains seems possible but is open at present. Using the terminology in [57], our results are limited to positively dependent RFs while the case of negatively dependent RFs is completely open. Finally, a complete description of anisotropic scaling limits of LRD RFs on $\mathbb{Z}^\nu$, $\nu \geq 3$, remains a challenging task, see [89].

Chapter 7

Anisotropic scaling of the random grain model

This is an extended version of the article [81]. We obtain a complete description of anisotropic scaling limits of random grain model on the plane with heavy tailed grain area distribution. The scaling limits have either independent or completely dependent increments along one or both coordinate axes and include stable, Gaussian and 'intermediate' infinitely divisible random fields. Asymptotic form of the covariance function of the random grain model is obtained. Application to superposed network traffic is included.

7.1 Introduction

It is well-known that many random fields (RFs) exhibit different scaling behavior in different directions. Important examples of RFs with such behavior is fractional Brownian sheet (FBS) and various classes of stochastic partial differential equations driven by FBS, see e.g. [2] and the references therein. For stationary RF $Y = \{Y(t,s),\, (t,s) \in \mathbb{R}^2\}$ the simplest form of anisotropic scaling is obtained by taking partial integrals $S_{\lambda,\gamma}(x,y) = \int_{(0,\lambda x]\times(0,\lambda^\gamma y]} Y(t,s)\mathrm{d}t\mathrm{d}s$ over rectangles $(0,\lambda x] \times (0,\lambda^\gamma y] \subset \mathbb{R}^2_+$ whose sides grow with $\lambda \to \infty$ at different rate $O(\lambda)$ and $O(\lambda^\gamma)$ (provided $\gamma \neq 1$). The (large-scale) behavior of Y is reflected in the scaling limit

$$a_{\lambda,\gamma}^{-1} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} V_\gamma(x,y) \quad \text{as } \lambda \to \infty, \tag{7.1}$$

where $a_{\lambda,\gamma} \to \infty$ is a normalization. Moreover, if $a_{\lambda,\gamma}$ is regularly varying at infinity with exponent $H(\gamma) > 0$, the limit RF V_γ in (7.1) has stationary rectangular increments and satisfies the self-similarity property:

$$\{V_\gamma(\lambda x, \lambda^\gamma y)\} \stackrel{\mathrm{fdd}}{=} \{\lambda^{H(\gamma)} V_\gamma(x,y)\} \quad \text{for each } \lambda > 0;$$

see [90], which is a particular case of the operator-scaling RF property introduced in Biermé et al. [12].

[90] observed that for many RFs Y on $\mathbb{Z}^2$ or $\mathbb{R}^2$, (nontrivial) scaling limits in (7.1) exist for any $\gamma > 0$, resulting in a one-dimensional family $\{V_\gamma, \gamma > 0\}$ of scaling limits termed the *scaling diagram of* Y below. Since scaling limits characterize the dependence structure and large-scale properties of the underlying random process, the scaling diagram provides a more complete 'large-scale summary of Y' compared to the (isotropic or anisotropic) scaling with fixed $\gamma > 0$ discussed in [1, 2, 16, 26, 59, 67, 100, 108] and elsewhere. Scaling diagrams of some classes of long-range dependent (LRD) Gaussian and aggregated nearest-neighbor autoregressive RFs on $\mathbb{Z}^2$ were identified in [89, 90]. It turned out that for these RFs, there exists a unique point $\gamma_0 > 0$ such that the scaling limits $V_\gamma \stackrel{\mathrm{fdd}}{=} V_\pm$ do not depend on γ for $\gamma < \gamma_0$ and $\gamma > \gamma_0$ and $V_+ \stackrel{\mathrm{fdd}}{\neq} V_-$. [90] termed this phenomenon *scaling transition* (at $\gamma = \gamma_0$). Scaling transition also arises under joint temporal and contemporaneous aggregation of independent LRD processes in telecommunication and economics, see [35, 55, 70, 79, 80], see also [90, Remark 2.3]. In this chapter we obtain a different kind of scaling diagram (see Figure 7.1) with *two* change-points $\gamma_- < \gamma_+$ of scaling limits which shows that this concept might be more complex and needs further studies.

The present chapter studies scaling limits (scaling diagram) of *random grain model*:

$$X(t,s) := \sum_i \mathbf{1}\big(((t-x_i)/R_i^p, (s-y_i)/R_i^{1-p}) \in B\big), \quad (t,s) \in \mathbb{R}^2, \tag{7.2}$$

where $B \subset \mathbb{R}^2$ ('generic grain') is a measurable bounded set of finite Lebesgue measure $\mathrm{leb}(B) < \infty$, $0 < p < 1$ is a shape parameter, $\{(x_i, y_i), R_i\}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity $\mathrm{d}x\mathrm{d}yF(\mathrm{d}r)$. We assume that F is a probability distribution on $\mathbb{R}_+$ having a density function f such that

$$f(r) \sim c_f r^{-1-\alpha} \quad \text{as } r \to \infty, \quad \text{for some } 1 < \alpha < 2,\ c_f > 0. \tag{7.3}$$

The sum in (7.2) counts the number of uniformly scattered and randomly dilated grains $(x_i, y_i) + R_i^P B$ containing (t,s), where $R^P B := \{(R^p x, R^{1-p} y) : (x,y) \in B\} \subset \mathbb{R}^2$ is the dilation of B by factors R^p and R^{1-p} in the horizontal and vertical directions, respectively. The case $p = 1/2$ corresponds to uniform or isotropic dilation. Note that the area $\mathrm{leb}(R^P B) = \mathrm{leb}(B)R$ of generic randomly dilated grain is proportional to R and does not depend on p and has a heavy-tailed distribution with finite mean $\mathrm{E}\,\mathrm{leb}(R^P B) < \infty$ and infinite second moment $\mathrm{E}\,\mathrm{leb}(R^P B)^2 = \infty$ according to (7.3). Condition (7.3) also guarantees that covariance of the random grain model is not integrable: $\int_{\mathbb{R}^2} |\operatorname{Cov}(X(0,0), X(t,s))| \mathrm{d}t\mathrm{d}s = \infty$, see Section 7.3, hence (7.2) is a LRD RF. Examples of the grain set B are the unit ball and the unit square, leading respectively to the *random ellipses model* $X(t,s) = \sum_i \mathbf{1}((t-x_i)^2/R_i^{2p} + (s-y_i)^2/R_i^{2(1-p)} \le 1)$ and the *random rectangles model*: $X(t,s) = \sum_i \mathbf{1}(x_i < t \le x_i + R_i^p,\, y_i < s \le y_i + R_i^{1-p})$. Note that for $p \ne 1/2$ the ratio $R^p/R^{1-p} = R^{2p-1}$ of sides of a generic rectangle tends to 0 or ∞ as $R \to \infty$ implying that large rectangles are 'elongated' or 'flat' and resulting in a strong anisotropy of the random rectangles model. A similar observation applies to the general random grain model in (7.2).

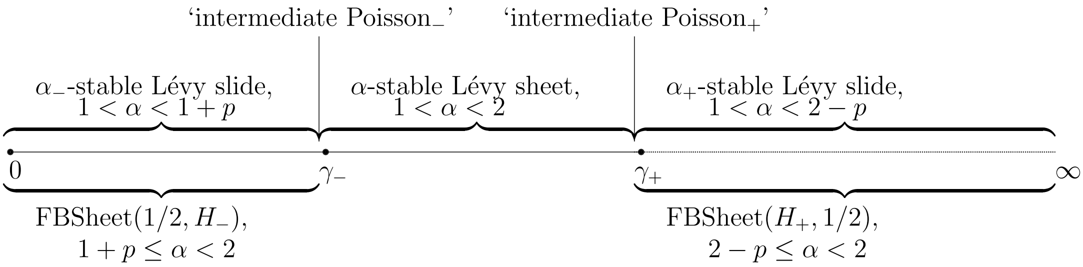

Figure 7.1: Scaling diagram of a random grain model.

Our main results are summarized in Figure 7.1 which shows a panorama of scaling limits V_γ in (7.1) as γ changes between 0 and ∞. Precise formulations pertaining to Figure 7.1 and the terminology therein are given in Section 7.2. Below we explain the most important facts about this diagram. First of all note that, due to the symmetry of the random grain model in (7.2), the scaling limits in (7.1) are symmetric under simultaneous exchange $x \leftrightarrow y$, $\gamma \leftrightarrow 1/\gamma$, $p \leftrightarrow 1-p$ and a reflection transformation of B. This symmetry is reflected in Figure 7.1, where the left region $0 < \gamma \le \gamma_-$ and the right region $\gamma_+ \le \gamma < \infty$ including the change points of the scaling limits

$$\gamma_- := \frac{1-p}{\alpha - (1-p)}, \quad \gamma_+ := \frac{\alpha}{p} - 1, \tag{7.4}$$

are symmetric with respect to the above transformations. The middle region $\gamma_- < \gamma < \gamma_+$ in Figure 7.1 corresponds to an *α-stable Lévy sheet* defined as a stochastic integral over $(0,x] \times (0,y]$ with respect to (w.r.t.) an α-stable random measure on $\mathbb{R}^2_+$. According to Figure 7.1, for $\gamma > \gamma_+$ the scaling limits in (7.1) exhibit a dichotomy depending on parameters α, p, featuring a Gaussian (fractional Brownian sheet) limit for $2-p \le \alpha < 2$, and an α_+-stable limit for $1 < \alpha < 2-p$ with stability parameter

$$\alpha_+ := \frac{\alpha - p}{1-p} > \alpha \tag{7.5}$$

larger than the parameter α. The terminology *$\alpha_\pm$-stable Lévy slide* refers to a RF of the form $xL_+(y)$ or $yL_-(x)$ 'sliding' linearly to zero along one of the coordinate axes, where $L_\pm$ are $\alpha_\pm$-stable Lévy processes (see Section 7.2 for definition). Finally, the 'intermediate Poisson' limits in Figure 7.1 at $\gamma = \gamma_\pm$ are not stable although infinitely divisible RFs given by stochastic integrals w.r.t. Poisson random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $c_f \mathrm{d}u\mathrm{d}v r^{-1-\alpha}\mathrm{d}r$.

The results of this chapter are related to those in, e.g. [11,27,35,53,55,70,79,80, 89,90] in which different scaling regimes occur for various classes of LRD models, in particular, heavy-tailed duration models. Isotropic scaling limits (case $\gamma = 1$) of random grain and random balls models in arbitrary dimension were discussed in Kaj et al. [53] and Biermé et al. [11]. The monograph [69] provides a nice discussion of limit behavior of heavy-tailed duration models whose spatial version is the random grain model in (7.2). From an application viewpoint, probably the most interesting is the study of different scaling regimes of superposed network traffic models [27,35,55,70]. In these studies, it is assumed that traffic is generated by independent sources and the problem concerns the limit distribution of the aggregated traffic as the time scale T and the number of sources M both tend to infinity, possibly at different rate. The present chapter extends the above-mentioned work, by considering the limit behavior of the aggregated workload process:

$$\begin{aligned} A_{M,K}(Tx) &:= \int_0^{Tx} W_{M,K}(t)\mathrm{d}t, \quad \text{where} \\ W_{M,K}(t) &:= \sum_i (R_i^{1-p} \wedge K)\mathbf{1}(x_i < t \le x_i + R_i^p,\, 0 < y_i < M), \quad t \ge 0, \end{aligned} \tag{7.6}$$

and where $\{(x_i, y_i), R_i\}$ is the same Poisson point process as in (7.2). The quantity $W_{M,K}(t)$ in (7.6) can be interpreted as the active workload at time t from sources arriving at x_i with $0 < y_i < M$ and transmitting at rate $R_i^{1-p} \wedge K$ during time interval $(x_i, x_i + R^p]$. Thus, the transmission rate in (7.6) is a (deterministic) function $(R^p)^{(1-p)/p} \wedge K$ of the transmission duration R^p depending on parameter $0 < p \le 1$, with $0 < K \le \infty$ playing the role of the maximal rate bound. The limiting case $p = 1$ in (7.6) corresponds to a constant rate workload from stationary M/G/∞ queue. Theorems 7.9–7.11 obtain the limit distributions of the centered and properly normalized process $\{A_{M,K}(Tx),\, x \ge 0\}$ with heavy-tailed distribution of R in (7.3) when the time scale T, the source intensity M and the maximal source rate K tend jointly to infinity so as $M = T^\gamma$, $K = T^\beta$ for some $0 < \gamma < \infty$, $0 < \beta \le \infty$. The results of Theorems 7.9 and 7.10 are summarized in Table 7.1. The workload process in (7.6) featuring a power-law dependence between transmission rate and duration is closely related to the random rectangles model with $B = (0,1]^2$, the last fact being reflected in Table 7.1, where most (but not all) of the limit processes can be linked to the scaling limits in Figure 7.1 and where γ_+, α_+ are the same as in (7.4), (7.5).

Parameter region		Limit process
$(1+\gamma)(1-p) < \alpha\beta \le \infty$	$1 < \alpha < 2$	α-stable Lévy process
$0 < \alpha\beta < (1+\gamma)(1-p)$	$1 < \alpha < 2p$	(α/p)-stable Lévy process
	$1 \vee 2p < \alpha < 2$	Brownian motion

a) Slow connection rate: $0 < \gamma < \gamma_+$.

Parameter region		Limit process
$0 < \alpha_+\beta < \gamma_+$	$1 < \alpha < 2p$	FBMotion, $H = (3 - (\alpha/p))/2$
	$1 \vee 2p < \alpha < 2$	Brownian motion
$\gamma_+ < \alpha_+\beta < \gamma$	$1 < \alpha < 2 - p$	Gaussian line
$\gamma < \alpha_+\beta \le \infty$		α_+-stable line
$\gamma_+ < \alpha_+\beta \le \infty$	$2 - p < \alpha < 2$	FBMotion, $H = (2 - \alpha + p)/2p$

b) Fast connection rate: $\gamma_+ < \gamma < \infty$.

Table 7.1: Limit distribution of the workload process in (7.6) with $M = T^\gamma$, $K = T^\beta$.

The rest of the chapter is organized as follows. Section 7.2 contains rigorous formulations (Theorems 7.1–7.6) of the asymptotic results pertaining to Figure 7.1. Section 7.3 discusses LRD properties and asymptotics of the covariance function of the random grain model. Section 7.4 obtains limit distributions of the aggregated workload process in (7.6). All proofs are relegated to Section 7.5.

7.2 Scaling limits of random grain model

We can rewrite the sum (7.2) as the stochastic integral

$$X(t,s) = \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s-v}{r^{1-p}}\Big) \in B\Big) N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r), \quad (t,s) \in \mathbb{R}^2, \tag{7.7}$$

w.r.t. a Poisson random measure $N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r)$ on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $\mathrm{E}N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) = \mathrm{d}u\mathrm{d}vF(\mathrm{d}r)$. The integral (7.7) is well-defined and follows a Poisson distribution with mean $\mathrm{E}X(t,s) = \mathrm{leb}(B)\int_0^\infty rF(\mathrm{d}r)$. The RF X in (7.7) is stationary with finite variance and the covariance function

$$\begin{aligned}
&\mathrm{Cov}(X(0,0), X(t,s)) \\
&\quad = \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathbf{1}\Big(\Big(\frac{u}{r^p}, \frac{v}{r^{1-p}}\Big) \in B, \Big(\frac{u-t}{r^p}, \frac{v-s}{r^{1-p}}\Big) \in B\Big) \mathrm{d}u\mathrm{d}vF(\mathrm{d}r).
\end{aligned} \tag{7.8}$$

Let

$$\begin{aligned}
S_{\lambda,\gamma}(x,y) &:= \int_0^{\lambda x}\int_0^{\lambda^\gamma y} (X(t,s) - \mathrm{E}X(t,s))\mathrm{d}t\mathrm{d}s \qquad (7.9)\\
&= \int_{\mathbb{R}^2\times\mathbb{R}_+} \Big\{\int_0^{\lambda x}\int_0^{\lambda^\gamma y} \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s-v}{r^{1-p}}\Big) \in B\Big)\mathrm{d}t\mathrm{d}s\Big\} \widetilde{N}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r)
\end{aligned}$$

for $(x,y) \in \mathbb{R}_+^2$, where $\widetilde{N}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) = N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) - \mathrm{E}N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r)$ is the centered Poisson random measure in (7.7). Recall the definition of $\gamma_\pm$:

$$\gamma_- := \frac{1-p}{\alpha - (1-p)}, \quad \gamma_+ := \frac{\alpha}{p} - 1.$$

In Theorems 7.1–7.6 we specify limit RFs V_γ and normalizations $a_{\lambda,\gamma}$ in (7.1) for all $\gamma > 0$ and $\alpha \in (1,2)$, $p \in (0,1)$ as in Figure 7.1. Throughout the chapter we assume that B is a bounded Borel set whose boundary ∂B has zero Lebesgue measure: $\mathrm{leb}(\partial B) = 0$.

7.2.1 Case $\gamma_- < \gamma < \gamma_+$

For $1 < \alpha < 2$, we introduce an α-stable Lévy sheet

$$L_\alpha(x,y) := Z_\alpha((0,x] \times (0,y]), \quad (x,y) \in \mathbb{R}_+^2, \tag{7.10}$$

as a stochastic integral w.r.t. an α-stable random measure $Z_\alpha(\mathrm{d}u, \mathrm{d}v)$ on $\mathbb{R}^2$ with control measure $\sigma^\alpha \mathrm{d}u\mathrm{d}v$ and skewness parameter 1, where the constant σ^α is given in (7.31). Thus, $\mathrm{E}\exp\{\mathrm{i}\theta Z_\alpha(A)\} = \exp\{-\mathrm{leb}(A)\sigma^\alpha|\theta|^\alpha(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan(\pi\alpha/2))\}$, $\theta \in \mathbb{R}$, for any Borel set $A \subset \mathbb{R}^2$ of finite Lebesgue measure $\mathrm{leb}(A) < \infty$. Note $\mathrm{E}Z_\alpha(A) = 0$.

Theorem 7.1. *Let $\gamma_- < \gamma < \gamma_+$, $1 < \alpha < 2$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} L_\alpha(x,y) \quad \text{as } \lambda \to \infty, \tag{7.11}$$

where $H(\gamma) := (1+\gamma)/\alpha$ and L_α is an α-stable Lévy sheet defined in (7.10).

7.2.2 Cases $\gamma > \gamma_+$, $1 < \alpha < 2 - p$ and $\gamma < \gamma_-$, $1 < \alpha < 1 + p$

For $1 < \alpha < 2 - p$ and $1 < \alpha < 1 + p$ introduce totally skewed stable Lévy processes $\{L_+(y),\, y \ge 0\}$ and $\{L_-(x),\, x \ge 0\}$ with respective stability indices $\alpha_\pm \in (1,2)$ defined as

$$\alpha_+ := \frac{\alpha - p}{1 - p}, \quad \alpha_- := \frac{\alpha - 1 + p}{p}$$

and characteristic functions

$$\mathrm{E}\exp\{\mathrm{i}\theta L_\pm(1)\} := \exp\{-\sigma^{\alpha_\pm}|\theta|^{\alpha_\pm}(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan(\pi\alpha_\pm/2))\}, \quad \theta \in \mathbb{R}, \tag{7.12}$$

where σ^{α_+} is given in (7.36) and σ^{α_-} can be found by symmetry, see (7.27).

Theorem 7.2. *(i) Let $\gamma > \gamma_+$, $1 < \alpha < 2 - p$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} x L_+(y) \quad \text{as } \lambda \to \infty, \tag{7.13}$$

where $H(\gamma) := 1 + \gamma/\alpha_+$ and L_+ is the α_+-stable Lévy process defined by (7.12).

(ii) Let $0 < \gamma < \gamma_-$, $1 < \alpha < 1 + p$. Then

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} y L_-(x) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma + 1/\alpha_-$ and L_- is the α_--stable Lévy process defined by (7.12).

7.2.3 Cases $\gamma > \gamma_+$, $2 - p \le \alpha < 2$ and $\gamma < \gamma_-$, $1 + p \le \alpha < 2$

A standard FBS B_{H_1,H_2} with Hurst indices $0 < H_1, H_2 \le 1$ is defined as a Gaussian process with zero mean and covariance

$$\begin{aligned} \mathrm{E}B_{H_1,H_2}(x_1,y_1)B_{H_1,H_2}(x_2,y_2) &= \frac{1}{4}(x_1^{2H_1} + x_2^{2H_1} - |x_1 - x_2|^{2H_1}) \\ &\quad \times (y_1^{2H_2} + y_2^{2H_2} - |y_1 - y_2|^{2H_2}), \end{aligned}$$

$(x_i, y_i) \in \mathbb{R}_+^2$, $i = 1, 2$. The constants σ_+ and $\widetilde{\sigma}_+$ appearing in Theorems 7.3(i) and 7.4(i) are defined in (7.40) and (7.42), respectively. The corresponding constants σ_- and $\widetilde{\sigma}_-$ in parts (ii) of these theorems can be found by symmetry (see (7.27)).

Theorem 7.3. *(i) Let $\gamma > \gamma_+$, $2 - p < \alpha < 2$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \sigma_+ B_{H_+,1/2}(x,y) \quad \text{as } \lambda \to \infty, \tag{7.14}$$

where $H(\gamma) := H_+ + \gamma/2$, $H_+ := 1/p - \gamma_+/2 = (2 - \alpha + p)/2p \in (1/2,1)$ and $B_{H_+,1/2}$ is an FBS with parameters $(H_+, 1/2)$.
(ii) Let $\gamma < \gamma_-$, $1 + p < \alpha < 2$. Then

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \sigma_- B_{1/2,H_-}(x,y) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma H_- + 1/2$, $H_- := 1/(1-p) + (1-p-\alpha)/2(1-p) \in (1/2,1)$ and $B_{1/2,H_-}$ is an FBS with parameters $(1/2, H_-)$.

Theorem 7.4. *(i) Let $\gamma > \gamma_+$, $\alpha = 2 - p$. Then*

$$\lambda^{-H(\gamma)} (\log \lambda)^{-1/2} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \widetilde{\sigma}_+ B_{1,1/2}(x,y) \quad \text{as } \lambda \to \infty, \tag{7.15}$$

where $H(\gamma) := 1 + \gamma/2$, $B_{1,1/2}$ is an FBS with parameters $(1, 1/2)$.
(ii) Let $\gamma < \gamma_-$, $\alpha = 1 + p$. Then

$$\lambda^{-H(\gamma)} (\log \lambda)^{-1/2} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \widetilde{\sigma}_- B_{1/2,1}(x,y) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma + 1/2$ and $B_{1/2,1}$ is an FBS with parameters $(1/2, 1)$.

7.2.4 Cases $\gamma = \gamma_\pm$

Define 'intermediate Poisson' RFs $I_\pm = \{I_\pm(x,y),\, (x,y) \in \mathbb{R}^2_+\}$ as stochastic integrals

$$\begin{aligned} I_+(x,y) &:= \int_{\mathbb{R}\times(0,y]\times\mathbb{R}_+} \int_{(0,x]\times\mathbb{R}} \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big) \mathrm{d}t\mathrm{d}s \widetilde{M}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r), \qquad (7.16) \\ I_-(x,y) &:= \int_{(0,x]\times\mathbb{R}\times\mathbb{R}_+} \int_{\mathbb{R}\times(0,y]} \mathbf{1}\Big(\Big(\frac{t}{r^p}, \frac{s-v}{r^{1-p}}\Big) \in B\Big) \mathrm{d}t\mathrm{d}s \widetilde{M}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) \end{aligned}$$

w.r.t. the centered Poisson random measure $\widetilde{M}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) = M(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) - \mathrm{E}M(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r)$ on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $\mathrm{E}M(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) = c_f \mathrm{d}u\mathrm{d}v\, r^{-(1+\alpha)}\mathrm{d}r$.

Proposition 7.5. *(i) The RF I_+ in (7.16) is well-defined for $1 < \alpha < 2$, $0 < p < 1$ and $\mathrm{E}|I_+(x,y)|^q < \infty$ for any $0 < q < \alpha_+ \wedge 2$. Moreover, if $2 - p < \alpha < 2$ then $\mathrm{E}|I_+(x,y)|^2 < \infty$ and*

$$\mathrm{E}I_+(x_1,y_1)I_+(x_2,y_2) = \sigma_+^2 \mathrm{E}B_{H_+,1/2}(x_1,y_1)B_{H_+,1/2}(x_2,y_2), \qquad (x_i,y_i) \in \mathbb{R}^2_+,\ i = 1,2, \tag{7.17}$$

where σ_+, H_+ are the same as in Theorem 7.3(i).

(ii) The RF I_- in (7.16) *is well-defined for $1 < \alpha < 2$, $0 < p < 1$ and $\mathrm{E}|I_-(x,y)|^q < \infty$ for any $0 < q < \alpha_- \wedge 2$. Moreover, if $1 + p < \alpha < 2$ then $\mathrm{E}|I_-(x,y)|^2 < \infty$ and*

$$\mathrm{E}I_-(x_1,y_1)I_-(x_2,y_2) = \sigma_-^2 \mathrm{E}B_{1/2,H_-}(x_1,y_1)B_{1/2,H_-}(x_2,y_2), \quad (x_i,y_i) \in \mathbb{R}_+^2,\ i = 1,2,$$

where σ_-, H_- are the same as in Theorem 7.3(ii).

Theorem 7.6. *(i) Let $\gamma = \gamma_+$, $1 < \alpha < 2$. Then*

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} I_+(x,y) \quad \text{as } \lambda \to \infty, \tag{7.18}$$

where $H(\gamma) := 1/p$ and RF I_+ is defined in (7.16)*.*
(ii) Let $\gamma = \gamma_-$, $1 < \alpha < 2$. Then

$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} I_-(x,y) \quad \text{as } \lambda \to \infty,$$

where $H(\gamma) := \gamma_-/(1-p)$ and RF I_- is defined in (7.16)*.*

Remark 7.1. The normalizing exponent $H(\gamma) \equiv H(\gamma,\alpha,p)$ in Theorems 7.1–7.6 is a jointly continuous (albeit non-analytic) function of $(\gamma,\alpha,p) \in (0,\infty)\times(1,2)\times(0,1)$.

Remark 7.2. Restriction $\alpha < 2$ is crucial for our results. Indeed, if $\alpha > 2$ then for *any* $\gamma > 0$, $p \in (0,1)$ the normalized integrals tend

$$\lambda^{-(1+\gamma)/2} S_{\lambda,\gamma}(x,y) \overset{\mathrm{fdd}}{\to} \sigma B_{1/2,1/2}(x,y) \quad \text{as } \lambda \to \infty,$$

to a classical Brownian sheet $B_{1/2,1/2}$ with variance $\sigma^2 = \mathrm{leb}(B)^2 \int_0^\infty r^2 F(\mathrm{d}r)$. We omit the proof of the last result which follows a general scheme of the proofs in Section 7.5.

7.3 LRD properties of random grain model

One of the most common definitions of LRD property pertains to stationary random processes with non-summable (non-integrable) autocovariance function. In the case of anisotropic RFs, the autocovariance function may decay at different rates in different directions, motivating a more detailed classification of LRD as in Definition 7.1. In this section we also verify these LRD properties for the random grain model in (7.2)–(7.3) and relate them to the change of the scaling limits or the dichotomies in Figure 7.1; see Remark 7.3.

Definition 7.1. Let $Y = \{Y(t,s),\, (t,s) \in \mathbb{R}^2\}$ be a stationary RF with finite variance and nonnegative covariance function $\rho_Y(t,s) := \mathrm{Cov}(Y(0,0), Y(t,s)) \ge 0$. We say that:
(i) Y has *short-range dependence (SRD)* property if $\int_{\mathbb{R}^2} \rho_Y(t,s)\mathrm{d}t\mathrm{d}s < \infty$; otherwise we say that Y has *long-range dependence (LRD)* property;
(ii) Y has *vertical SRD* property if $\int_{[-Q,Q]\times\mathbb{R}} \rho_Y(t,s)\mathrm{d}t\mathrm{d}s < \infty$ for any $0 < Q < \infty$; otherwise we say that Y has *vertical LRD* property;
(iii) Y has *horizontal SRD* property if $\int_{\mathbb{R}\times[-Q,Q]} \rho_Y(t,s)\mathrm{d}t\mathrm{d}s < \infty$ for any $0 < Q < \infty$; otherwise we say that Y has *horizontal LRD* property.

The main result of this section is Theorem 7.7 providing the asymptotics of the covariance function of the random grain model in (7.2)–(7.3) as $|t|+|s| \to \infty$ and enabling the verification of its integrability properties in Definition 7.1. Let

$$w := (|t|^{1/p} + |s|^{1/(1-p)})^p, \quad \text{for } (t,s) \in \mathbb{R}^2.$$

For $p = 1/2$, w is the Euclidean norm and $(w, \arccos(t/w))$ are the polar coordinates of $(t,s) \in \mathbb{R}^2$, $s \ge 0$. Introduce a function $b(z)$, $z \in [-1,1]$, by

$$b(z) := c_f \int_0^\infty \mathrm{leb}\Big(B \cap \Big(B + \big(z/r^p, (1-|z|^{1/p})^{1-p}/r^{1-p}\big)\Big)\Big) r^{-\alpha}\mathrm{d}r, \tag{7.19}$$

playing the role of the 'angular function' in the asymptotics (7.20). For the random balls model with $p = 1/2$ and $B = \{x^2 + y^2 \le 1\}$, $b(z)$ is a constant function independent on z.

Theorem 7.7. *Let* $1 < \alpha < 2$, $0 < p < 1$.
(i) The function $b(z)$ *in* (7.19) *is bounded, continuous and strictly positive on* $[-1,1]$.
(ii) The covariance function $\rho(t,s) := \mathrm{Cov}(X(0,0), X(t,s))$ *in* (7.8) *has the following asymptotics:*

$$\rho(t,s) \sim b(\mathrm{sgn}(s)t/w) w^{-(\alpha-1)/p} \quad \textit{as } |t|+|s| \to \infty. \tag{7.20}$$

Theorem 7.7 implies the following bound for covariance function $\rho(t,s) = \mathrm{Cov}(X(0,0), X(t,s))$ of the random grain model: there exist $Q > 0$ and strictly positive constants $0 < C_- < C_+ < \infty$ such that for any $|t|+|s| > Q$,

$$C_-(|t|^{1/p} + |s|^{1/(1-p)})^{1-\alpha} \le \rho(t,s) \le C_+(|t|^{1/p} + |s|^{1/(1-p)})^{1-\alpha}. \tag{7.21}$$

The bounds in (7.21) together with easy integrability properties of the function $(|t|^{1/p} + |s|^{1/(1-p)})^{1-\alpha}$ on $\{|t|+|s| > Q\}$ imply the following corollary.

Corollary 7.8. *The random grain model in* (7.2)–(7.3) *has:*
(i) LRD property for any $1 < \alpha < 2$, $0 < p < 1$*;*
(ii) vertical LRD property for $1 < \alpha \le 2 - p$ *and vertical SRD property for* $2 - p < \alpha < 2$ *and any* $0 < p < 1$*;*
(iii) horizontal LRD property for $1 < \alpha \le 1 + p$ *and horizontal SRD property for* $1 + p < \alpha < 2$ *and any* $0 < p < 1$.

Remark 7.3. The above corollary indicates that the dichotomy at $\alpha = 2 - p$ in Figure 7.1, region $\gamma > \gamma_+$ is related to the change from the vertical LRD to the vertical SRD property in the random grain model. Similarly, the dichotomy at $\alpha = 1+p$ in Figure 7.1, region $\gamma < \gamma_+$ is related to the change from the horizontal LRD to the horizontal SRD property.

[90] introduced Type I distributional LRD property for RF Y with two-dimensional 'time' in terms of dependence properties of rectangular increments of V_γ, $\gamma > 0$. The increment of a RF $V = \{V(x,y), (x,y) \in \mathbb{R}^2_+\}$ on rectangle $K = (u,x] \times (v,y] \subset \mathbb{R}^2_+$ is defined as the double difference $V(K) = V(x,y) - V(u,y) - V(x,v) + V(u,v)$. Let $\ell \subset \mathbb{R}^2$ be a line, $(0,0) \in \ell$. According to [90, Definition 2.2], a RF $V = \{V(x,y), (x,y) \in \mathbb{R}^2_+\}$ is said to have:

- *independent rectangular increments in direction* ℓ if $V(K)$ and $V(K')$ are independent for any two rectangles $K, K' \subset \mathbb{R}^2_+$ which are separated by an orthogonal line $\ell' \perp \ell$;
- *invariant rectangular increments in direction* ℓ if $V(K) = V(K')$ for any two rectangles K, K' such that $K' = (x,y) + K$ for some $(x,y) \in \ell$;
- *properly dependent rectangular increments* if V has neither independent nor invariant increments in arbitrary direction ℓ.

Further on, a stationary RF Y on $\mathbb{Z}^2$ is said to have *Type I distributional LRD* [90, Definition 2.4] if there exists a unique point $\gamma_0 > 0$ such that its scaling limit V_{γ_0} has properly dependent rectangular increments while all other scaling limits V_γ, $\gamma \ne \gamma_0$, have either independent or invariant rectangular increments in some direction $\ell = \ell(\gamma)$. The above definition trivially extends to RF Y on $\mathbb{R}^2$.

We end this section with the observation that *all scaling limits of the random grain model in* (7.2)–(7.3) *in Theorems 7.1–7.6 have either independent or invariant rectangular increments in direction of one or both coordinate axes.* The last fact is immediate from stochastic integral representations in (7.10), (7.16), the covariance function of FBS with Hurst indices H_1, H_2 equal to 1 or 1/2 (see also [90, Example 2.3]) and the limit RFs in (7.13). We conclude that the random grain model in (7.2)–(7.3) *does not have Type I distributional LRD* in contrast to Gaussian and other classes of LRD RFs discussed in [89, 90]. The last conclusion is not surprising since similar facts about scaling limits of heavy-tailed duration models with one-dimensional time are well-known; see e.g. [62].

7.4 Limit distributions of aggregated workload process

We rewrite the accumulated workload in (7.6) as the integral

$$\begin{aligned} &A_{M,K}(Tx) \\ &= \int_{\mathbb{R}\times(0,M]\times\mathbb{R}_+} \Big\{ (r^{1-p} \wedge K) \int_0^{Tx} \mathbf{1}(u < t \le u + r^p)\mathrm{d}t \Big\} N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r), \end{aligned} \tag{7.22}$$

where $N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r)$ is the same Poisson random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity $\mathrm{E}N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r) = \mathrm{d}u\mathrm{d}vF(\mathrm{d}r)$ as in (7.2). We assume that $F(\mathrm{d}r)$ has a density $f(r)$ satisfying (7.3) with $1 < \alpha < 2$ as in Section 7.2. We let $p \in (0, 1]$ in (7.22) and thus the parameter may take value $p = 1$ as well. We assume that K and M grow with T in such a way that

$$M = T^\gamma,\ K = T^\beta \quad \text{for some } 0 < \gamma < \infty,\ 0 < \beta \le \infty.$$

We are interested in the limit distribution

$$b_T^{-1}(A_{M,K}(Tx) - \mathrm{E}A_{M,K}(Tx)) \overset{\mathrm{fdd}}{\to} \mathcal{A}(x) \quad \text{as } T \to \infty, \tag{7.23}$$

where $b_T \equiv b_{T,\gamma,\beta} \to \infty$ is a normalization.

Recall from (7.4) and (7.5) the definitions

$$\gamma_+ = \frac{\alpha}{p} - 1, \quad \alpha_+ = \frac{\alpha - p}{1 - p}.$$

For $p = 1$, let $\alpha_+ := \infty$. By assumption (7.3), transmission durations R_i^p, $i \in \mathbb{Z}$, have a heavy-tailed distribution with tail parameter $\alpha/p > 1$. Following the terminology in [27, 35, 53, 70], the regions $\gamma < \gamma_+$, $\gamma > \gamma_+$ and $\gamma = \gamma_+$ will be respectively referred to as *slow connection rate*, *fast connection rate* and *intermediate connection rate*. For each of these ‘regimes’, Theorems 7.9, 7.10 and 7.11 detail the limit processes and normalizations in (7.23) depending on parameters β, α, p.

Apart from the classical Gaussian and stable processes listed in Table 7.1, some 'intermediate' infinitely divisible processes arise. Let us introduce

$$I(x) := \int_{\mathbb{R}\times\mathbb{R}_+} \Big\{ \int_0^x 1(u < t \le u + r^p)\mathrm{d}t \Big\} \widetilde{\mathcal{M}}(\mathrm{d}u, \mathrm{d}r), \quad x \ge 0, \tag{7.24}$$

where $\widetilde{M}(\mathrm{d}u, \mathrm{d}r)$ is a centered Poisson random measure with intensity measure $c_f \mathrm{d}u r^{-(1+\alpha)}\mathrm{d}r$. The process in (7.24) essentially depends on the ratio α/p only and is well-defined for $1 < \alpha < 2p$ and $1/2 < p \le 1$. Under the 'intermediate' regime this process arises for many heavy-tailed duration models (see e.g. [27, 35, 55]). It was studied in detail in [34]. We introduce a 'truncated' version of (7.24):

$$\widehat{I}(x) := \int_{\mathbb{R}\times\mathbb{R}_+} \Big\{ (r^{1-p} \wedge 1) \int_0^x \mathbf{1}(u < t \le u + r^p)\mathrm{d}t \Big\} \widetilde{\mathcal{M}}(\mathrm{d}u, \mathrm{d}r), \quad x \ge 0, \tag{7.25}$$

and its Gaussian counterpart

$$\widehat{Z}(x) := \int_{\mathbb{R}\times\mathbb{R}_+} \Big\{ (r^{1-p} \wedge 1) \int_0^x \mathbf{1}(u < t \le u + r^p)\mathrm{d}t \Big\} \mathcal{Z}(\mathrm{d}u, \mathrm{d}r), \quad x \ge 0, \tag{7.26}$$

where $\mathcal{Z}(\mathrm{d}u, \mathrm{d}r)$ is a Gaussian random measure on $\mathbb{R}\times\mathbb{R}_+$ with the same variance $c_f \mathrm{d}u r^{-(1+\alpha)}\mathrm{d}r$ as the centered Poisson random measure $\widetilde{\mathcal{M}}(\mathrm{d}u, \mathrm{d}r)$. The processes in (7.25) and (7.26) are well-defined for any $1 < \alpha < 2$, $0 < p \le 1$ and have the same covariance functions.

The RFs defined in Section 7.2 reappear in Theorems 7.9–7.11 for the certain grain set, namely the unit square $B = (0, 1]^2$. Recall that a homogeneous Lévy process $\{L(x),\, x \ge 0\}$ is completely specified by its characteristic function $\mathrm{E}e^{\mathrm{i}\theta L(1)}$, $\theta \in \mathbb{R}$. A standard fractional Brownian motion with Hurst parameter $H \in (0, 1]$ is a Gaussian process $\{B_H(x),\, x \ge 0\}$ with zero mean and covariance function $(1/2)(x^{2H} + y^{2H} - |x - y|^{2H})$, $x, y \ge 0$.

Theorem 7.9 (Slow connection rate). *Let $0 < \gamma < \gamma_+$. The convergence in* (7.23) *holds with the limit $\mathcal{A}$ and normalization $b_T = T^{\mathcal{H}}$ specified in (i)–(v) below.*
(i) Let $(1+\gamma)(1-p) < \alpha\beta \le \infty$. Then $\mathcal{H} := (1+\gamma)/\alpha$ and $\mathcal{A} := \{L_\alpha(x, 1),\, x \ge 0\}$ is an α-stable Lévy process defined by (7.10)*.*
(ii) Let $0 < \alpha\beta < (1 + \gamma)(1 - p)$ and $1 < \alpha < 2p$. Then $\mathcal{H} := \beta + (1 + \gamma)p/\alpha$ and $\mathcal{A} := \{L_{\alpha/p}(x),\, x \ge 0\}$ is an (α/p)-stable Lévy process with characteristic function given by (7.48)*.*
(iii) Let $0 < \alpha\beta < (1 + \gamma)(1 - p)$ and $1 \vee 2p < \alpha < 2$. Then $\mathcal{H} := (1/2)(1 + \gamma + \beta(2 - \alpha)/(1 - p))$ and $\mathcal{A} := \{\sigma_1 B(x),\, x \ge 0\}$ is a Brownian motion with variance σ_1^2 given by (7.49)*.*

(iv) Let $0 < \alpha\beta < (1+\gamma)(1-p)$ *and* $\alpha = 2p$. *Then* $b_T := T^{\mathcal{H}}(\log T)^{1/2}$ *with* $\mathcal{H} := \beta + (1+\gamma)/2$ *and* $\mathcal{A} := \{\widehat{\sigma}_1 B(x),\, x \ge 0\}$ *is a Brownian motion with variance* $\widehat{\sigma}_1^2$ *given by* (7.50).

(v) Let $\alpha\beta = (1+\gamma)(1-p)$. *Then* $\mathcal{H} := (1+\gamma)/\alpha$ *and* $\mathcal{A} := \{\widehat{L}(x),\, x \ge 0\}$ *is a Lévy process with characteristic function in* (7.51).

Theorem 7.10. (Fast connection rate.) *Let* $\gamma_+ < \gamma < \infty$. *The convergence in* (7.23) *holds with the limit* $\mathcal{A}$ *and normalization* $b_T := T^{\mathcal{H}}$ *specified in (i)–(ix) below.*

(i) Let $0 < \alpha_+\beta < \gamma_+$ *and* $1 < \alpha < 2p$. *Then* $\mathcal{H} := H + \beta + \gamma/2$ *and* $\mathcal{A} := \{\sigma_2 B_H(x),\, x \ge 0\}$ *is a fractional Brownian motion with* $H = (3 - \alpha/p)/2$ *and variance* σ_2^2 *given by* (7.52).

(ii) Let $0 < \alpha_+\beta < \gamma_+$ *and* $1 \vee 2p < \alpha < 2$. *Then* $\mathcal{H}$ *and* $\mathcal{A}$ *are the same as in Theorem 7.9(iii).*

(iii) Let $\gamma_+ < \alpha_+\beta < \gamma$ *and* $1 < \alpha < 2-p$. *Then* $\mathcal{H} := 1 + (1/2)(\gamma + \beta(2-\alpha-p)/(1-p))$ *and* $\mathcal{A} := \{xZ,\, x \ge 0\}$ *is a Gaussian line with random slope* $Z \sim \mathcal{N}(0, \sigma_3^2)$ *and* σ_3^2 *given in* (7.53).

(iv) Let $\gamma < \alpha_+\beta \le \infty$ *and* $1 < \alpha < 2-p$. *Then* $\mathcal{H} := 1 + \gamma/\alpha_+$ *and* $\mathcal{A} := \{xL_+(1),\, x \ge 0\}$ *is an* α_+*-stable line with random slope* $L_+(1)$ *having* α_+*-stable distribution defined by* (7.12).

(v) Let $\gamma_+ < \alpha_+\beta \le \infty$ *and* $2-p < \alpha < 2$. *Then* $\mathcal{H} := H_+ + \gamma/2$ *and* $\mathcal{A} := \{\sigma_+ B_{H_+,1/2}(x,1),\, x \ge 0\}$ *is a fractional Brownian motion with* $H = H_+ = (2-\alpha+p)/2p$ *and variance* σ_+^2 *given by* (7.40).

(vi) Let $0 < \alpha_+\beta < \gamma_+$ *and* $\alpha = 2p$. *Then* $b_T := T^{\mathcal{H}}(\log T)^{1/2}$ *with* $\mathcal{H} := \beta + (1+\gamma)/2$ *and* $\mathcal{A} := \{\widehat{\sigma}_2 B(x),\, x \ge 0\}$ *is a Brownian motion with variance* $\widehat{\sigma}_2^2$ *in* (7.54).

(vii) Let $\alpha_+\beta = \gamma_+$. *Then* $\mathcal{H} := (1/2)(1+\gamma+(2-\alpha)/p)$ *and* $\mathcal{A} := \{\widehat{Z}(x),\, x \ge 0\}$ *in an intermediate Gaussian process defined by* (7.26).

(viii) Let $\alpha_+\beta = \gamma$ *and* $1 < \alpha < 2-p$. *Then* $\mathcal{H} = 1 + \beta$ *and* $\mathcal{A} := \{x\widehat{Z},\, x \ge 0\}$, *where a slope* $\widehat{Z}$ *is a r.v. defined by* (7.55).

(ix) If $\gamma_+ < \alpha_+\beta \le \infty$ *and* $\alpha = 2-p$. *Then* $b_T := T^{\mathcal{H}}(\log T)^{1/2}$, $\mathcal{H} := 1 + \gamma/2$ *and* $\mathcal{A} := \{\widetilde{\sigma}_+ B_{1,1/2}(x,1),\, x \ge 0\} = \{x\widetilde{Z},\, x \ge 0\}$ *is a Gaussian line with random slope* $\widetilde{Z} \sim \mathcal{N}(0, \widetilde{\sigma}_+^2)$ *and* $\widetilde{\sigma}_+^2$ *given by* (7.42).

Theorem 7.11. (Intermediate connection rate.) *Let* $\gamma = \gamma_+$. *The convergence in* (7.23) *holds with the limit* $\mathcal{A}$ *and normalization* $b_T := T^{\mathcal{H}}$ *specified in (i)–(v) below.*

(i) Let $0 < \alpha_+\beta < \gamma_+$ *and* $1 < \alpha < 2p$*. Then* $\mathcal{H} := 1 + \beta$ *and* $\mathcal{A} := \{I(x),\, x \ge 0\}$ *is defined by* (7.24).
(ii) Let $0 < \alpha_+\beta < \gamma_+$ *and* $1 \vee 2p < \alpha < 2$*. Then* $\mathcal{H}$ *and* $\mathcal{A}$ *are the same as in Theorem 7.9(iii).*
(iii) Let $0 < \alpha_+\beta < \gamma_+$ *and* $\alpha = 2p$*. Then* $\mathcal{H}$ *and* $\mathcal{A}$ *are the same as in Theorem 7.9(iv).*
(iv) Let $\alpha_+\beta = \gamma_+$*. Then* $\mathcal{H} := 1/p$ *and* $\mathcal{A} := \{\widehat{I}(x),\, x \ge 0\}$ *is defined by* (7.25).
(v) Let $\gamma_+ < \alpha_+\beta \le \infty$*. Then* $\mathcal{H} := 1/p$ *and* $\mathcal{A} := \{I_+(x,1),\, x \ge 0\}$ *is defined by* (7.16).

Remark 7.4. Note that $p = 1$ implies $\gamma_+ = \alpha - 1$. In this case, Theorem 7.9 reduces to the α-stable limit in (i), whereas Theorem 7.10 reduces to the fractional Brownian motion limit in (v) discussed in [70] and other papers. A similar dichotomy appears for β close to zero and $1 < \alpha < 2p$ with the difference that α is now replaced by α/p. Intuitively, it can be explained as follows. For small $\beta > 0$, the workload process $W_{M,K}(t)$ in (7.6) behaves like a constant rate process $K \sum_i \mathbf{1}(x_i < t \le x_i + R_i^p,\, 0 < y_i < M)$ with transmission lengths R_i^p that are i.i.d. and follow the same distribution $\mathrm{P}(R_i^p > r) = \mathrm{P}(R_i > r^{1/p}) \sim (c_f/\alpha) r^{-(\alpha/p)}$, $r \to \infty$, with tail parameter $1 < \alpha/p < 2$. Therefore, for small β our results agree with [70], including the Gaussian limit in Theorems 7.9(iii) and 7.10(ii) arising when the R_i^p's have finite variance.

Remark 7.5. As it follows from the proof, the random line limits in Theorem 7.10(iv) and (iii) are caused by extremely long sessions starting in the past at times $x_i < 0$ and lasting $R_i^p = O(T^{\gamma/\gamma_+})$, $\gamma_+ < \gamma < \alpha_+\beta$ or $R_i^p = O(T^{\alpha_+\beta/\gamma_+})$, $\gamma_+ < \alpha_+\beta < \gamma$, respectively, so that typically these sessions end at times $x_i + R_i^p \gg T$.

7.5 Proofs

7.5.1 Proofs of Sections 7.2 and 7.3

Let

$$X^*(t,s) := \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathbf{1}\Big(\Big(\frac{t-u}{r^{1-p}}, \frac{s-v}{r^p}\Big) \in B^*\Big) N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r), \quad (t,s) \in \mathbb{R}^2,$$

be a 'reflected' version of (7.7), with B replaced by $B^* := \{(u,v) \in \mathbb{R}^2 : (v,u) \in B\}$, p replaced by $1-p$ and the same Poisson random measure $N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}r)$ as in (7.7). Let $S^*_{\lambda_*,\gamma_*}(y,x) := \int_0^{\lambda_* y} \int_0^{\lambda_*^{\gamma_*} x} (X^*(t,s) - \mathrm{E}X^*(t,s)) \mathrm{d}t \mathrm{d}s$, $(y,x) \in \mathbb{R}_+^2$, be the corresponding partial integral in (7.9). If λ_*, γ_* are related to λ, γ as $\lambda_* = \lambda^\gamma, \gamma_* = 1/\gamma$ then

$$S^*_{\lambda_*,\gamma_*}(y,x) \overset{\mathrm{fdd}}{=} S_{\lambda,\gamma}(x,y) \tag{7.27}$$

holds by symmetry property of the Poisson random measure. As noted in the Introduction, relation (7.27) allows to reduce the limits of $S_{\lambda,\gamma}(x,y)$ as $\lambda \to \infty$ and $\gamma \le \gamma_-$ to the limits of $S^*_{\lambda_*,\gamma_*}(y,x)$ as $\lambda_* \to \infty$ and $\gamma_* \ge \gamma_{*+} := \alpha/(1-p) - 1$. As a consequence, the proofs of parts (ii) of Theorems 7.2–7.6 can be omitted since they can be deduced from parts (i) of the corresponding statements.

The convergence of normalized partial integrals in (7.1) is equivalent to the convergence of characteristic functions:

$$\mathrm{E}\exp\Big\{\mathrm{i}a^{-1}_{\lambda,\gamma}\sum_{i=1}^m \theta_i S_{\lambda,\gamma}(x_i,y_i)\Big\} \to \mathrm{E}\exp\Big\{\mathrm{i}\sum_{i=1}^m \theta_i V_\gamma(x_i,y_i)\Big\} \quad \text{as } \lambda\to\infty, \tag{7.28}$$

for all $m = 1,2,\ldots$, $(x_i,y_i) \in \mathbb{R}^2_+$, $\theta_i \in \mathbb{R}$, $i = 1,\ldots,m$. We restrict the proof of (7.28) to one-dimensional convergence for $m = 1$, $(x,y) \in \mathbb{R}^2_+$ only. The general case of (7.28) follows analogously. We have

$$\begin{aligned} W_{\lambda,\gamma}(\theta) &:= \log \mathrm{E}\exp\{\mathrm{i}\theta a^{-1}_{\lambda,\gamma} S_{\lambda,\gamma}(x,y)\} \\ &= \int_{\mathbb{R}^2\times\mathbb{R}_+} \Psi\Big(\frac{\theta}{a_{\lambda,\gamma}}\int_0^{\lambda x}\int_0^{\lambda^\gamma y} \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s-v}{r^{1-p}}\Big) \in B\Big)\mathrm{d}t\mathrm{d}s\Big)\mathrm{d}u\mathrm{d}v f(r)\mathrm{d}r, \end{aligned} \tag{7.29}$$

where $\Psi(z) := \mathrm{e}^{\mathrm{i}z} - 1 - \mathrm{i}z$, $z \in \mathbb{R}$. We shall use the following inequality:

$$|\Psi(z)| \le \min(2|z|, z^2/2), \quad z \in \mathbb{R}. \tag{7.30}$$

Proof of Theorem 7.1. In the integrals on the r.h.s. of (7.29) we change the variables:

$$\frac{t-u}{r^p} \to t, \quad \frac{s-v}{r^{1-p}} \to s, \quad u \to \lambda u, \quad v \to \lambda^\gamma v, \quad r \to \lambda^{H(\gamma)} r.$$

This yields $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r)\mathrm{d}r$, where

$$f_\lambda(r) := \lambda^{(1+\alpha)H(\gamma)} f(\lambda^{H(\gamma)} r) \to c_f r^{-(1+\alpha)}, \quad \lambda \to \infty$$

according to (7.3), and

$$\begin{aligned} g_\lambda(r) &:= \int_{\mathbb{R}^2} \Psi(\theta h_\lambda(u,v,r))\mathrm{d}u\mathrm{d}v, \\ h_\lambda(u,v,r) &:= r\int_B \mathbf{1}(0 < u + \lambda^{-\delta_1} r^p t \le x,\ 0 < v + \lambda^{-\delta_2} r^{1-p} s \le y)\mathrm{d}t\mathrm{d}s, \end{aligned}$$

where the exponents $\delta_1 := 1 - H(\gamma)p = (\gamma_+ - \gamma)/(1+\gamma_+) > 0$, $\delta_2 := \gamma - H(\gamma)(1-p) = (\gamma - \gamma_-)/(1+\gamma_-) > 0$. Clearly,

$$h_\lambda(u,v,r) \to \mathrm{leb}(B) r\, \mathbf{1}(0 < u \le x, 0 < v \le y), \quad \lambda \to \infty,$$

for any fixed $(u,v,r) \in \mathbb{R}^2 \times \mathbb{R}_+$, $u \notin \{0,x\}$, $v \notin \{0,y\}$, implying

$$g_\lambda(r) \to xy\Psi(\theta\,\mathrm{leb}(B)r)$$

for any $r > 0$. Since $\int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v = xyr\,\mathrm{leb}(B)$ and $h_\lambda(u,v,r) \le Cr$, the dominating bound $|g_\lambda(r)| \le C\min(r,r^2)$ follows by (7.30). Whence and from Lemma 7.12 we conclude that

$$W_{\lambda,\gamma}(\theta) \to W_\gamma(\theta) := xyc_f \int_0^\infty (\mathrm{e}^{\mathrm{i}\theta\,\mathrm{leb}(B)r} - 1 - \mathrm{i}\theta\,\mathrm{leb}(B)r)r^{-(1+\alpha)}\mathrm{d}r.$$

It remains to verify that

$$W_\gamma(\theta) = -xy\sigma^\alpha|\theta|^\alpha(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan(\pi\alpha/2)) = \log \mathrm{E}\exp\{\mathrm{i}\theta L_\alpha(x,y)\},$$

where

$$\sigma^\alpha := c_f\,\mathrm{leb}(B)^\alpha\cos(\pi\alpha/2)\Gamma(2-\alpha)/\alpha(1-\alpha). \tag{7.31}$$

This proves the one-dimensional convergence in (7.11) and Theorem 7.1, too. □

Proof of Theorem 7.2. In (7.29), change the variables as follows:

$$\begin{aligned} &t \to \lambda t, \quad s - v \to \lambda^{(1-p)\gamma/(\alpha-p)}s, \\ &u \to \lambda^{p\gamma/(\alpha-p)}u, \quad v \to \lambda^\gamma v, \quad r \to \lambda^{\gamma/(\alpha-p)}r. \end{aligned} \tag{7.32}$$

This yields $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r)f_\lambda(r)\mathrm{d}r$, where

$$f_\lambda(r) := \lambda^{(1+\alpha)\gamma/(\alpha-p)}f(\lambda^{\gamma/(\alpha-p)}r) \to c_f r^{-(1+\alpha)}, \quad \lambda \to \infty, \tag{7.33}$$

and $g_\lambda(r) := \int_{\mathbb{R}^2}\Psi(\theta h_\lambda(u,v,r))\mathrm{d}u\mathrm{d}v$ with

$$h_\lambda(u,v,r) := \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}\Big(\Big(\frac{\lambda^{-\delta_1}t - u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big)\mathbf{1}(0 < v + \lambda^{-\delta_2}s < y)\mathrm{d}s, \tag{7.34}$$

where $\delta_1 := p\gamma/(\alpha-p) - 1 = (\gamma - \gamma_+)/\gamma_+ > 0$, $\delta_2 := \gamma(\alpha-1)/(\alpha-p) > 0$. Let $B(u) := \{v \in \mathbb{R} : (u,v) \in B\}$ and write $\mathrm{leb}_1(A)$ for the Lebesgue measure of a set $A \subset \mathbb{R}$. By the dominated convergence theorem,

$$\begin{aligned} h_\lambda(u,v,r) \to h(u,v,r) \ &:= \ x\mathbf{1}(0 < v < y)\int_{\mathbb{R}} \mathbf{1}\Big(\Big(\frac{-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big)\mathrm{d}s \\ &= \ x\mathbf{1}(0 < v < y)r^{1-p}\,\mathrm{leb}_1(B(-u/r^p)) \end{aligned} \tag{7.35}$$

for any $(u,v,r) \in \mathbb{R}^2 \times \mathbb{R}_+$, $v \notin \{0,y\}$, implying

$$g_\lambda(r) \to g(r) := \int_{\mathbb{R}^2}\Psi(\theta h(u,v,r))\mathrm{d}u\mathrm{d}v = yr^p\int_{\mathbb{R}}\Psi\big(\theta x r^{1-p}\,\mathrm{leb}_1(B(u))\big)\mathrm{d}u$$

for any $r > 0$. Indeed, since B is bounded, for fixed $r > 0$ the function $(u, v) \mapsto h_\lambda(u, v, r)$ has a bounded support uniformly in $\lambda \geq 1$. Therefore it is easy to verify domination criterion for the above convergence. Combining $h_\lambda(u, v, r) \leq C r^{1-p}$ with $\int_{\mathbb{R}^2} h_\lambda(u, v, r) \mathrm{d}u\mathrm{d}v = xyr\,\mathrm{leb}(B)$ gives $|g_\lambda(r)| \leq C \min(r, r^{2-p})$ by (7.30). Hence and by Lemma 7.12, $W_{\lambda,\gamma}(\theta) \to W_\gamma(\theta) := c_f \int_0^\infty g(r) r^{-(1+\alpha)} \mathrm{d}r$. By change of variable, the last integral can be rewritten as

$$
\begin{aligned}
W_\gamma(\theta) &= c_f\, y\, x^{\alpha_+} (1-p)^{-1} \int_{\mathbb{R}} \mathrm{leb}_1(B(u))^{\alpha_+} \mathrm{d}u \int_0^\infty (\mathrm{e}^{\mathrm{i}\theta w} - 1 - \mathrm{i}\theta w) w^{-(1+\alpha_+)} \mathrm{d}w \\
&= -(y\, x^{\alpha_+}) \sigma^{\alpha_+} |\theta|^{\alpha_+} (1 - \mathrm{i}\,\mathrm{sgn}(\theta) \tan(\pi\alpha_+/2)) = \log \mathrm{E} \exp\{\mathrm{i}\theta x L_+(y)\},
\end{aligned}
$$

where

$$
\sigma^{\alpha_+} := \frac{c_f \Gamma(2-\alpha_+) \cos(\pi\alpha_+/2)}{(1-p)\alpha_+(1-\alpha_+)} \int_{\mathbb{R}} \mathrm{leb}_1(B(u))^{\alpha_+} \mathrm{d}u, \tag{7.36}
$$

thus completing the proof of one-dimensional convergence in (7.13). Theorem 7.2 is proved. □

Proof of Theorem 7.3. In (7.29), change the variables as follows:

$$
t \to \lambda t, \quad s - v \to \lambda^{(1/p)-1} s, \quad u \to \lambda u, \quad v \to \lambda^\gamma v, \quad r \to \lambda^{1/p} r. \tag{7.37}
$$

We get $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r) \mathrm{d}r$, where

$$
\begin{aligned}
f_\lambda(r) &:= \lambda^{(1+\alpha)/p} f(\lambda^{1/p} r), \\
g_\lambda(r) &:= \int_{\mathbb{R}^2} \lambda^{2(H(\gamma)-1/p)} \Psi(\theta \lambda^{(1/p)-H(\gamma)} h_\lambda(u, v, r)) \mathrm{d}u\mathrm{d}v,
\end{aligned} \tag{7.38}
$$

with

$$
\begin{aligned}
h_\lambda(u, v, r) &:= \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}(0 < v + \lambda^{-\delta} s < y) \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big) \mathrm{d}s \\
&\to \mathbf{1}(0 < v < y) \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big) \mathrm{d}s \\
&= \mathbf{1}(0 < v < y)\, r^{1-p} \int_0^x \mathrm{leb}_1(B((t-u)/r^p)) \mathrm{d}t \\
&=: h(u, v, r)
\end{aligned} \tag{7.39}
$$

as $\lambda \to \infty$, for all $(u, v, r) \in \mathbb{R}^2 \times \mathbb{R}_+$, $v \notin \{0, y\}$, since $\delta := 1 + \gamma - (1/p) > 0$. Note that $2(H(\gamma) - 1/p) = \gamma - \gamma_+ > 0$ and hence

$$
\lambda^{2(H(\gamma)-1/p)} \Psi(\theta \lambda^{(1/p)-H(\gamma)} h_\lambda(u, v, r)) \to -(\theta^2/2) h^2(u, v, r), \quad \lambda \to \infty.
$$

Next, by the dominated convergence theorem

$$
g_\lambda(r) \to g(r) := -\frac{\theta^2}{2} \int_{\mathbb{R}^2} h^2(u, v, r) \mathrm{d}u\mathrm{d}v
$$

for any $r > 0$. Using $\int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v = xy\,\mathrm{leb}(B)r$ and $h_\lambda(u,v,r) \le C\min(r^{1-p}, r)$ similarly as in the proof of Theorem 7.2 we obtain $|g_\lambda(r)| \le C\int_{\mathbb{R}^2} h^2_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v \le C\min(r^{2-p}, r^2)$. Then by Lemma 7.12,

$$W_{\lambda,\gamma}(\theta) \to W_\gamma(\theta) := c_f \int_0^\infty g(r) r^{-(1+\alpha)}\mathrm{d}r = -(\theta^2/2)\sigma_+^2 x^{2H_+} y,$$

where

$$\sigma_+^2 := c_f \int_{\mathbb{R}} \mathrm{d}u \int_0^\infty \Big(\int_0^1 \mathrm{leb}_1(B((t-u)/r^p))\mathrm{d}t\Big)^2 r^{1-\alpha-2p}\mathrm{d}r, \tag{7.40}$$

where the last integral converges. (Indeed, since $u \mapsto \mathrm{leb}_1(B(u)) = \int \mathbf{1}((u,v) \in B)\mathrm{d}v$ is a bounded function with compact support, the inner integral in (7.40) does not exceed $C(1 \wedge r^p)\mathbf{1}(|u| < K(1+r^p))$ for some $C, K > 0$ implying $\sigma_+^2 \le C\int_0^\infty (1\wedge r^p)^2(1+r^p)r^{1-\alpha-2p}\mathrm{d}r < \infty$ since $2-p<\alpha<2$.) This ends the proof of one-dimensional convergence in (7.14). Theorem 7.3 is proved. $\square$

Proof of Theorem 7.4. After the same change of variables as in (7.32), viz.,

$$t \to \lambda t, \quad s - v \to \lambda^{\gamma/2}s, \quad u \to \lambda^{p\gamma/2(1-p)}u, \quad v \to \lambda^\gamma v, \quad r \to \lambda^{\gamma/2(1-p)}r,$$

we obtain $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r)\mathrm{d}r$ with $f_\lambda(r)$ as in (7.33) and $g_\lambda(r) := \int_{\mathbb{R}^2} \Psi(\theta(\log\lambda)^{-1/2} h_\lambda(u,v,r))\mathrm{d}u\mathrm{d}v$, where

$$h_\lambda(u,v,r) := \int_0^x \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}\Big(\Big(\frac{\lambda^{-\delta_1}t - u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big)\mathbf{1}(0 < v + \lambda^{-\delta_2}s < y)\mathrm{d}s,$$

$\delta_1 := p\gamma/2(1-p) - 1 = (\gamma - \gamma_+)/\gamma_+ > 0$, $\delta_2 := \gamma/2 > 0$ are the same as in (7.34) and

$$h_\lambda(u,v,r) \to h(u,v,r) := x\,\mathbf{1}(0 < v < y)\int_{\mathbb{R}} \mathbf{1}\Big(\Big(\frac{-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big)\mathrm{d}s$$

c.f. (7.35). Below we prove that the main contribution to the limit of $W_{\lambda,\gamma}(\theta)$ comes from the interval $\lambda^{-\delta_1/p} < r < 1$, namely, that $W_{\lambda,\gamma}(\theta) - W^0_{\lambda,\gamma}(\theta) \to 0$, where

$$\begin{aligned} W^0_{\lambda,\gamma}(\theta) &:= \int_{\lambda^{-\delta_1/p}}^1 g_\lambda(r) f_\lambda(r)\mathrm{d}r \\ &\sim -\frac{\theta^2}{2}\frac{c_f}{\log\lambda}\int_{\lambda^{-\delta_1/p}}^1 \frac{\mathrm{d}r}{r^{3-p}}\int_{\mathbb{R}^2} h^2(u,v,r)\mathrm{d}u\mathrm{d}v \\ &= -\frac{\theta^2}{2}x^2 y c_f \int_{\mathbb{R}} (\mathrm{leb}_1(B(u)))^2\mathrm{d}u \frac{1}{\log\lambda}\int_{\lambda^{-\delta_1/p}}^1 \frac{\mathrm{d}r}{r} \\ &= -\frac{\theta^2}{2}\widetilde{\sigma}_+^2 x^2 y =: W_\gamma(\theta), \end{aligned} \tag{7.41}$$

where

$$\widetilde{\sigma}_+^2 := \frac{c_f(\gamma - \gamma_+)}{2(1-p)} \int_{\mathbb{R}} \mathrm{leb}(B \cap (B + (0,u)))\mathrm{d}u \tag{7.42}$$

and where we used the fact that $\int_{\mathbb{R}^2} h^2(u,v,r)\mathrm{d}u\mathrm{d}v = x^2 y r^{2-p} \int_{\mathbb{R}} \mathrm{leb}_1(B(u))^2 \mathrm{d}u = x^2 y r^{2-p} \int_{\mathbb{R}} \mathrm{leb}(B \cap (B + (0,u)))\mathrm{d}u$.

Accordingly, write $W_{\lambda,\gamma}(\theta) = W^0_{\lambda,\gamma}(\theta) + W^-_{\lambda,\gamma}(\theta) + W^+_{\lambda,\gamma}(\theta)$, where $W^-_{\lambda,\gamma}(\theta) := \int_0^{\lambda^{-\delta_1/p}} g_\lambda(r) f_\lambda(r)\mathrm{d}r$ and $W^+_{\lambda,\gamma}(\theta) := \int_1^\infty g_\lambda(r) f_\lambda(r)\mathrm{d}r$ are remainder terms. Indeed, using (7.30) and

$$\int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v = xyr\,\mathrm{leb}(B), \quad h_\lambda(u,v,r) \le C(\lambda^{\delta_1} r) \wedge r^{1-p}. \tag{7.43}$$

it follows that

$$|W^+_{\lambda,\gamma}(\theta)| \le \frac{C}{(\log\lambda)^{1/2}} \int_1^\infty \frac{\mathrm{d}r}{r^{3-p}} \int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v = O((\log\lambda)^{-1/2}) = o(1).$$

Similarly,

$$\begin{aligned}
|W^-_{\lambda,\gamma}(\theta)| &\le \frac{C\lambda^{\delta_1}}{\log\lambda} \int_0^{\lambda^{-\delta_1/p}} r f_\lambda(r)\mathrm{d}r \int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v \\
&\le \frac{C\lambda^{\delta_1}}{\log\lambda} \int_0^{\lambda^{-\delta_1/p}} r^2 f_\lambda(r)\mathrm{d}r = \frac{C}{\lambda\log\lambda} \int_0^{\lambda^{1/p}} r^2 f(r)\mathrm{d}r \\
&= O((\log\lambda)^{-1}) = o(1),
\end{aligned}$$

since $\delta_1 = p\gamma/2(1-p) - 1$.

Consider the main term $W^0_{\lambda,\gamma}(\theta)$ in (7.41). Let $\widetilde{W}_{\lambda,\gamma}(\theta) := -\frac{\theta^2}{2\log\lambda} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r)\mathrm{d}r \int_{\mathbb{R}^2} h_\lambda^2(u,v,r)\mathrm{d}u\mathrm{d}v$. Then using (7.43) and $|\Psi(z) + z^2/2| \le |z|^3/6$ we obtain

$$\begin{aligned}
|W^0_{\lambda,\gamma}(\theta) - \widetilde{W}_{\lambda,\gamma}(\theta)| &\le \frac{C}{(\log\lambda)^{3/2}} \int_{\lambda^{-\delta_1/p}}^1 r^{2-2p} f_\lambda(r)\mathrm{d}r \int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v \\
&\le \frac{C}{(\log\lambda)^{3/2}} \int_{\lambda^{-\delta_1/p}}^1 r^{3-2p} f_\lambda(r)\mathrm{d}r \\
&\le \frac{C}{(\log\lambda)^{3/2}} \int_0^1 r^{-p}\mathrm{d}r = O((\log\lambda)^{-3/2}) = o(1).
\end{aligned}$$

Finally, it remains to estimate the difference $|\widetilde{W}_{\lambda,\gamma}(\theta) - W_\gamma(\theta)| \le C(J'_\lambda + J''_\lambda)$, where

$$\begin{aligned}
J'_\lambda &:= \frac{1}{\log\lambda} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r)\mathrm{d}r \int_{\mathbb{R}^2} |h_\lambda^2(u,v,r) - h^2(u,v,r)|\mathrm{d}u\mathrm{d}v, \\
J''_\lambda &:= \frac{1}{\log\lambda} \int_{\lambda^{-\delta_1/p}}^1 r^{2-p} |f_\lambda(r) - c_f r^{p-3}|\mathrm{d}r.
\end{aligned}$$

Let

$$\widetilde{h}_\lambda(u,v,r) := x \int_{\mathbb{R}} \mathbf{1}\Big(\Big(\frac{-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big) \mathbf{1}(0 < v + \lambda^{-\delta_2} s < y)\mathrm{d}s.$$

Then $J'_\lambda \le J'_{\lambda 1} + J'_{\lambda 2}$, where $J'_{\lambda 1} := (\log \lambda)^{-1} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r) \mathrm{d}r \int_{\mathbb{R}^2} |h^2_\lambda(u,v,r) - \widetilde{h}^2_\lambda(u,v,r)| \mathrm{d}u\mathrm{d}v$, $J'_{\lambda 2} := (\log \lambda)^{-1} \int_{\lambda^{-\delta_1/p}}^1 f_\lambda(r) \mathrm{d}r \int_{\mathbb{R}^2} |\widetilde{h}^2_\lambda(u,v,r) - h^2(u,v,r)| \mathrm{d}u\mathrm{d}v$. Using the fact that B is a bounded set with $\mathrm{leb}(\partial B) = 0$ we get that

$$\begin{aligned}\int_{\mathbb{R}^2} &|h_\lambda(u,v,r) - \widetilde{h}_\lambda(u,v,r)| \mathrm{d}u\mathrm{d}v \\ &\le yr \int_0^x \mathrm{d}t \int_{\mathbb{R}^2} \Big| \mathbf{1}\Big(\Big(\frac{\lambda^{-\delta_1} t}{r^p} - u, s\Big) \in B\Big) - \mathbf{1}((-u,s) \in B) \Big| \mathrm{d}u\mathrm{d}s \\ &\le r\epsilon(\lambda^{-\delta_1} r^{-p}),\end{aligned}$$

where $\epsilon(z)$, $z \ge 0$, is a bounded function with $\lim_{z\to 0} \epsilon(z) = 0$. We also have $h_\lambda(u,v,r) + \widetilde{h}_\lambda(u,v,r) \le Cr^{1-p}$ as in (7.43). Using these bounds together with $f_\lambda(r) \le Cr^{p-3}$, $r > \lambda^{-\delta_1/p}$ we obtain

$$J'_{\lambda 1} \log \lambda \le C \int_{\lambda^{-\delta_1/p}}^1 \epsilon(\lambda^{-\delta_1} r^{-p}) r^{-1} \mathrm{d}r = C \int_{\lambda^{-\delta_1}}^1 \epsilon(z) z^{-1} \mathrm{d}z = o(\log \lambda),$$

proving $J'_{\lambda 1} \to 0$ as $\lambda \to \infty$. In a similar way, using $\int_{\mathbb{R}^2} |\widetilde{h}_\lambda(u,v,r) - h(u,v,r)| \mathrm{d}u\mathrm{d}v \le xr \int_{\mathbb{R}^3} \mathbf{1}((-u,s) \in B) |\mathbf{1}(0 < v + \lambda^{-\delta_2} r^{1-p} s < y) - \mathbf{1}(0 < v < y)| \mathrm{d}u\mathrm{d}v\mathrm{d}s \le Cr^{2-p} \lambda^{-\delta_2}$ we obtain $J'_{\lambda 2} \log \lambda \le C\lambda^{-\delta_2} \int_0^1 r^{-p} \mathrm{d}r = O(\lambda^{-\delta_2})$, proving $J'_{\lambda 2} \to 0$ and hence $J'_\lambda \to 0$. Finally, $J''_\lambda = (\log \lambda)^{-1} \int_{\lambda^{1/p}}^\infty r^{2-p} |f(r) - c_f r^{p-3}| \mathrm{d}r \to 0$ follows from (7.3). This proves the limit $\lim_{\lambda\to\infty} W_{\lambda,\gamma}(\theta) = W_\gamma(\theta) = -(\theta^2/2)\widetilde{\sigma}^2_+ x^2 y$ for any $\theta \in \mathbb{R}$, or one-dimensional convergence in (7.15). Theorem 7.4 is proved. ☐

Proof of Proposition 7.5. We use well-known properties of Poisson stochastic integrals and inequality (3.3) in [79]. Accordingly, $I_+(x,y)$ is well-defined and satisfies $\mathrm{E}|I_+(x,y)|^q \le 2J_q(x,y)$ $(1 \le q \le 2)$ provided

$$\begin{aligned} J_q(x,y) &:= c_f \int_0^\infty r^{-(1+\alpha)} \mathrm{d}r \int_{\mathbb{R}\times(0,y]} \mathrm{d}u\mathrm{d}v \Big| \int_{(0,x]\times\mathbb{R}} \mathbf{1}\Big(\Big(\frac{t-u}{r^p}, \frac{s}{r^{1-p}}\Big) \in B\Big) \mathrm{d}t\mathrm{d}s \Big|^q \\ &= c_f y \int_0^\infty r^{q(1-p)-(1+\alpha)} \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}u \Big| \int_0^x \mathrm{leb}_1\Big(B\Big(\frac{t-u}{r^p}\Big)\Big) \mathrm{d}t \Big|^q < \infty. \end{aligned}$$

Split $J_q(x,y) = c_f y[\int_0^1 \mathrm{d}r + \int_1^\infty] \dots \mathrm{d}r =: c_f y[J' + J'']$. Then $J'' \le C \int_1^\infty r^{q(1-p)} r^{-(1+\alpha)} \mathrm{d}r \int \mathbf{1}(|u| \le Cr^p) \mathrm{d}u \le C \int_1^\infty r^{q(1-p)-(1+\alpha)+p} \mathrm{d}r < \infty$ provided $q < (\alpha - p)/(1-p)$. Similarly, $J' \le C \int_0^1 r^{q(1-p)-(1+\alpha)} \mathrm{d}r |\int \mathbf{1}(|t| \le Cr^p) \mathrm{d}t|^q \le C \int_0^1 r^{q(1-p)} r^{-(1+\alpha)+qp} \mathrm{d}r < \infty$ provided $\alpha < q$. Note that $\alpha < (\alpha-p)/(1-p) \le 2$ for $1 < \alpha \le 2-p$ and $(\alpha-p)/(1-p) > 2$ for $2-p < \alpha < 2$. Relation (7.17) follows from (7.14) and $J_2(x,y) = \sigma^2_+ y x^{2H_+}$ by a change of variables. This proves part (i). The proof of part (ii) is analogous. ☐

Proof of Theorem 7.6. Using the change of variables as in (7.37) we get $W_{\lambda,\gamma}(\theta) = \int_0^\infty g_\lambda(r) f_\lambda(r) \mathrm{d}r$ with the same $f_\lambda(r)$, $g_\lambda(r)$ as in (7.38) and $h_\lambda(u,v,r)$ satisfying

(7.39). (Note $H(\gamma) = H(\gamma_+) = 1/p$ hence $\lambda^{H(\gamma_+)-(1/p)} = 1$ in the definition of $g_\lambda(r)$ in (7.38).) Particularly, $\Psi(\theta h_\lambda(u,v,r)) \to \Psi(\theta h(u,v,r))$ for any $(u,v,r) \in \mathbb{R}^2 \times \mathbb{R}_+$, $v \notin \{0,y\}$. Then $g_\lambda(r) \to g(r) := \int_{\mathbb{R}^2} \Psi(\theta h(u,v,r))\mathrm{d}u\mathrm{d}v$ follows by the dominated convergence theorem. Using $\int_{\mathbb{R}^2} h_\lambda(u,v,r)\mathrm{d}u\mathrm{d}v = xyr\,\mathrm{leb}(B)$ and $h_\lambda(u,v,r) \le Cr$ we obtain $|g_\lambda(r)| \le C\min(r,r^2)$ and hence $W_{\lambda,\gamma}(\theta) \to \int_0^\infty g(r)r^{-(1+\alpha)}\mathrm{d}r = \log \mathrm{E}\exp\{\mathrm{i}\theta I_+(x,y)\}$, proving the one-dimensional convergence in (7.18). The proof of Theorem 7.6 is complete. □

Proof of Theorem 7.7. (i) Write $D_r(x,y) := \{(u,v) \in \mathbb{R}^2 : (u-x)^2+(v-y)^2 \le r^2\}$ for a ball in $\mathbb{R}^2$ centered at (x,y) and having radius r. Recall that B is bounded. Note that $\inf_{z\in[-1,1]}(|z|/r^p + (1-|z|^{1/(p-1)})^{1-p}/r^{1-p}) \ge c_0\min(r^{-p}, r^{-(1-p)})$ for some constant $c_0 > 0$. Therefore, there exists $r_0 > 0$ such that for all $0 < r < r_0$ the intersection $B_{z,r} := B \cap \big(B + \big(z/r^p, (1-|z|^{1/p})^{1-p}/r^{1-p}\big)\big) = \emptyset$ in (7.19). Hence $b(z) \le C < \infty$ uniformly in $z \in [-1,1]$.

Let $(x,y) \in B \setminus \partial B$. Then $D_{2r}(x,y) \subset B$ for all $r < r_0$ and some $r_0 > 0$. If we translate B by distance r_0 at most, the translated set still contains the ball $D_{r_0}(x,y)$. Since $\sup_{z\in[-1,1]}(|z|/r^p + (1-|z|^{1/p})^{1-p}/r^{1-p}) \le 2\max(r^{-p}, r^{-(1-p)})$, there exists $r_1 > 0$ for which $\inf_{r>r_1}\mathrm{leb}(B_{z,r}) \ge \pi r_0^2$, proving $\inf_{z\in[-1,1]} b(z) > 0$. The continuity of $b(z)$ follows from the above argument and the continuity of the mapping $z \mapsto \mathrm{leb}(B_{z,r}) : [-1,1] \to \mathbb{R}_+$, for each $r > 0$.

(ii) Let $s \ge 0$. In the integral (7.8) we change the variables: $u \to r^p u$, $v \to r^{1-p}v$, $r \to w^{1/p}r$. Then

$$\rho(t,s) = w^{-(\alpha-1)/p}\int_0^\infty \mathrm{leb}(B_{t/w,r})f_w(r)r\mathrm{d}r,$$

where $f_w(r) := w^{(1+\alpha)/p}f(w^{1/p}r) \to c_f\, r^{-(1+\alpha)}$, $w \to \infty$. Then (7.20) follows by Lemma 7.12 and the afore-mentioned properties of $\mathrm{leb}(B_{t/w,r})$. Theorem 7.7 is proved. □

In this chapter we often use the following lemma which is a version of Lemma 2 in [53] or Lemma 2.4 in [11].

Lemma 7.12. *Let F be a probability distribution that has a density function f satisfying (7.3). Set $f_\lambda(r) := \lambda^{1+\alpha}f(\lambda r)$ for $\lambda \ge 1$. Assume that g, g_λ are measurable functions on $\mathbb{R}_+$ such that $g_\lambda(r) \to g(r)$ as $\lambda \to \infty$ for all $r > 0$ and such that the inequality*

$$|g_\lambda(r)| \le C(r^{\beta_1} \wedge r^{\beta_2}) \tag{7.44}$$

holds for all $r > 0$ and some $0 < \beta_1 < \alpha < \beta_2$, where C does not depend on r, λ. Then

$$\int_0^\infty g_\lambda(r)f_\lambda(r)\mathrm{d}r \to c_f\int_0^\infty g(r)r^{-(1+\alpha)}\mathrm{d}r \quad \text{as } \lambda \to \infty.$$

Proof. Split $\int_0^\infty g_\lambda(r) f_\lambda(r)\mathrm{d}r = (\int_0^\epsilon + \int_\epsilon^\infty) g_\lambda(r) f_\lambda(r)\mathrm{d}r =: I_1(\lambda) + I_2(\lambda)$, where $\epsilon > 0$. It suffices to prove

$$\lim_{\lambda\to\infty} I_2(\lambda) = c_f \int_\epsilon^\infty g(r) r^{-(1+\alpha)}\mathrm{d}r \quad \text{and} \quad \lim_{\epsilon\to 0}\limsup_{\lambda\to\infty} I_1(\lambda) = 0. \tag{7.45}$$

The first relation in (7.45) follows by the dominated convergence theorem, using (7.44) and the bound $f_\lambda(r) \le Cr^{-(1+\alpha)}$ which holds for all $r > \rho/\lambda$ and a sufficiently large $\rho > 0$ by virtue of (7.3). The second relation in (7.45) follows from $|I_1(\lambda)| \le C\int_0^\epsilon r^{\beta_2} f_\lambda(r)\mathrm{d}r = C\lambda^{\alpha-\beta_2}\int_0^{\lambda\epsilon} x^{\beta_2} f(x)\mathrm{d}x \le C\lambda^{\alpha-\beta_2} + C\lambda^{\alpha-\beta_2}\int_1^{\lambda\epsilon} x^{\beta_2-(1+\alpha)}\mathrm{d}x \le C(\lambda^{\alpha-\beta_2} + \epsilon^{\beta_2-\alpha})$. □

7.5.2 Proofs of Section 7.4

Proof of Theorem 7.9. We have

$$\begin{aligned} W_{T,\gamma,\beta}(\theta) &:= \log \mathrm{E}\exp\big\{\mathrm{i}\theta b_T^{-1}\big(A_{M,K}(Tx) - \mathrm{E}A_{M,K}(Tx)\big)\big\} \qquad (7.46)\\ &= T^\gamma \int_{\mathbb{R}\times\mathbb{R}_+} \Psi\Big(\theta T^{-\mathcal{H}}(r^{1-p}\wedge T^\beta)\int_0^{Tx} \mathbf{1}(u<t<u+r^p)\mathrm{d}t\Big)\mathrm{d}u f(r)\mathrm{d}r, \end{aligned}$$

where $\Psi(z) = \mathrm{e}^{\mathrm{i}z} - 1 - \mathrm{i}z$, $z\in\mathbb{R}$, as in Section 7.5.1.

(i) Let $0 < p < 1, \delta_1 := \beta - (1+\gamma)(1-p)/\alpha > 0, \delta_2 := 1 - (1+\gamma)p/\alpha = (\gamma_+ - \gamma)p/\alpha > 0$. Using the change of variables $(t-u)/r^p \to t$, $u \to Tu$, $r \to T^{(1+\gamma)/\alpha} r$ in (7.46), we obtain

$$W_{T,\gamma,\beta}(\theta) = \int_0^\infty g_T(r) f_T(r)\mathrm{d}r, \tag{7.47}$$

where $f_T(r) := T^{(1+\alpha)(1+\gamma)/\alpha} f(T^{(1+\gamma)/\alpha} r)$ and

$$g_T(r) := \int_{\mathbb{R}} \Psi\big(\theta(r^{1-p}\wedge T^{\delta_1}) r^p h_T(u,r)\big)\mathrm{d}u$$

and where $h_T(u,r) := \int_0^1 \mathbf{1}(0 < u + T^{-\delta_2} r^p t < x)\mathrm{d}t \to \mathbf{1}(0<u<x)$ for fixed $(u,r)\in\mathbb{R}\times\mathbb{R}_+$, $u\notin\{0,x\}$. Hence $g_T(r)\to g(r) := x\Psi(\theta r)$ follows by the dominated convergence theorem. The bound $|g_T(r)| \le C\min(r, r^2)$ follows from (7.30) and $\int_{\mathbb{R}} h_T(u,r)\mathrm{d}u = x$ with $h_T(u,r)\le 1$. Finally, by Lemma 7.12, $W_{T,\gamma,\beta}(\theta) \to xc_f\int_0^\infty \Psi(\theta r) r^{-(1+\alpha)}\mathrm{d}r = \log\mathrm{E}\exp\{\mathrm{i}\theta L_\alpha(x,1)\}$, proving part (i) for $0<p<1$. The case $p=1$ follows similarly.

(ii) By the same change of variables as in part (i) we get $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where

$$g_T(r) := \int_{\mathbb{R}} \Psi\big(\theta((T^{-\delta_1} r^{1-p})\wedge 1) r^p h_T(u,r)\big)\mathrm{d}u,$$

where $\delta_1, f_T(r), h_T(u,r)$ are the same as in (7.47) except that now $\delta_1 < 0$. Next, $g_T(r) \to x\Psi(\theta r^p)$ by the dominated convergence theorem while $|g_T(r)| \le C\min(r^p, r^{2p})$ follows by (7.30) and $\int_{\mathbb{R}} \min(h_T(u,r), h_T^2(u,r))\mathrm{d}u \le C$. Then $W_{T,\gamma,\beta}(\theta) \to W_{\gamma,\beta}(\theta) := xc_f \int_0^\infty \Psi(\theta r^p) r^{-(1+\alpha)}\mathrm{d}r$ follows by Lemma 7.12. To finish the proof of part (ii) it suffices to check that

$$\begin{aligned} W_{\gamma,\beta}(\theta) &= -x\frac{c_f\Gamma(2-\alpha/p)}{\alpha(1-\alpha/p)}\cos\Big(\frac{\pi\alpha}{2p}\Big)|\theta|^{\alpha/p}\Big(1-\mathrm{i}\,\mathrm{sgn}(\theta)\tan\Big(\frac{\pi\alpha}{2p}\Big)\Big) \\ &=: \log \mathrm{E}\exp\{\mathrm{i}\theta L_{\alpha/p}(x)\}. \end{aligned} \tag{7.48}$$

(iii) Denote $\delta_1 := 1+\gamma-\alpha\beta/(1-p) > 0$, $\delta_2 := 1-p\beta/(1-p) > 0$. Then by change of variables: $(t-u)/r^p \to t$, $u \to Tu$, $r \to T^{\beta/(1-p)}r$ we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where $f_T(r) := T^{(1+\alpha)\beta/(1-p)} f(T^{\beta/(1-p)}r)$ and

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi\big(\theta T^{-\delta_1/2}(r^{1-p}\wedge 1)r^p h_T(u,r)\big)\mathrm{d}u$$

with $h_T(u,r) := \int_0^1 \mathbf{1}(0 < u + T^{-\delta_2}r^p t < x)\mathrm{d}t \to \mathbf{1}(0<u<x)$. Then $g_T(r) \to -(\theta^2/2)(r^{1-p}\wedge 1)^2 r^{2p} x$ by the dominated convergence theorem using the bounds $|\Psi(z)| \le z^2/2$, $z\in\mathbb{R}$ and $h_T(u,r) \le \mathbf{1}(-r^p < u < x)$. Moreover, $|g_T(r)| \le C\min(r^{2p}, r^2)$ holds in view of $\int_{\mathbb{R}} h_T^2(u,r)\mathrm{d}u \le C$. Using Lemma 7.12 we get $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)xc_f\int_0^\infty (r^{1-p}\wedge 1)^2 r^{2p-(1+\alpha)}\mathrm{d}r = -(\theta^2/2)\sigma_1^2 x$, where

$$\sigma_1^2 := \frac{2c_f(1-p)}{(2-\alpha)(\alpha-2p)} < \infty \tag{7.49}$$

since $\max(1,2p) < \alpha < 2$. This proves part (iii).

(iv) By the same change of variables as in part (iii), we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi\big(\theta T^{-\delta_1/2}(\log T)^{-1/2}(r^{1-p}\wedge 1)r^p h_T(u,r)\big)\mathrm{d}u$$

and $f_T(r)$ and δ_1, $\delta_2 > 0$ and $h_T(u,r) := \int_0^1 \mathbf{1}(0 < u + T^{-\delta_2}r^p t < x)\mathrm{d}t \to \mathbf{1}(0 < u < x)$ are the same as in (iii). We split $W_{T,\gamma,\beta}(\theta) = W^-_{T,\gamma,\beta}(\theta) + W^0_{T,\gamma,\beta}(\theta) + W^+_{T,\gamma,\beta}(\theta)$ and next prove that $W^-_{T,\gamma,\beta}(\theta) := \int_0^1 g_T(r)f_T(r)\mathrm{d}r$ and $W^+_{T,\gamma,\beta}(\theta) := \int_{T^{\delta_1/2p}}^\infty g_T(r)f_T(r)\mathrm{d}r$ are the remainder terms, whereas

$$\begin{aligned} W^0_{T,\gamma,\beta}(\theta) := \int_1^{T^{\delta_1/2p}} g_T(r)f_T(r)\mathrm{d}r &\sim -\frac{\theta^2}{2}\frac{xc_f}{\log T}\int_1^{T^{\delta_1/2p}} r^{2p-(1+2p)}\mathrm{d}r \\ &= -\frac{\theta^2}{2}\widehat{\sigma}_1^2 x =: W_{\gamma,\beta}(\theta), \end{aligned}$$

where

$$\widehat{\sigma}_1^2 := c_f\frac{\delta_1}{2p} = \frac{c_f}{2p(1-p)}((1+\gamma)(1-p)-2p\beta). \tag{7.50}$$

By (7.3), there exists $\rho > 0$ such that $f_T(r) \le Cr^{-(1+2p)}$ for all $r > \rho/T^{\beta/(1-p)}$. Using this bound along with $\int_{\mathbb{R}} h_T(u,r)\mathrm{d}u = x$, $h_T(u,r) \le 1$ and (7.30), we get

$$\begin{aligned}
|W^-_{T,\gamma,\beta}(\theta)| &\le \frac{C}{\log T}\int_0^1 r^2 f_T(r)\mathrm{d}r = O((\log T)^{-1}) = o(1),\\
|W^+_{T,\gamma,\beta}(\theta)| &\le C\frac{T^{\delta_1/2}}{(\log T)^{1/2}}\int_{T^{\delta_1/2p}}^{\infty} r^{p-(1+2p)}\mathrm{d}r = O((\log T)^{-1/2}) = o(1).
\end{aligned}$$

We now consider the main term $W^0_{T,\gamma,\beta}(\theta)$. Let $\widetilde{W}_{T,\gamma,\beta}(\theta) := -\frac{\theta^2}{2\log T}\int_1^{T^{\delta_1/2p}} r^{2p} f_T(r)\mathrm{d}r \int_{\mathbb{R}} h_T^2(u,r)\mathrm{d}u$. Then, by $|\Psi(z) + z^2/2| \le |z|^3/6$, $z \in \mathbb{R}$, it follows that

$$\begin{aligned}
|W^0_{T,\gamma,\beta}(\theta) - \widetilde{W}_{T,\gamma,\beta}(\theta)| &\le \frac{C}{(\log T)^{3/2}T^{\delta_1/2}}\int_1^{T^{\delta_1/2p}} r^{3p} f_T(r)\mathrm{d}r \int_{\mathbb{R}} h_T^3(u,r)\mathrm{d}u\\
&\le \frac{C}{(\log T)^{3/2}T^{\delta_1/2}}\int_1^{T^{\delta_1/2p}} r^{p-1}\mathrm{d}r\\
&= O((\log T)^{-3/2}) = o(1).
\end{aligned}$$

Finally, we estimate $|\widetilde{W}_{T,\gamma,\beta}(\theta) - W_{\gamma,\beta}(\theta)| \le C(J'_T + J''_T)$, where

$$\begin{aligned}
J'_T &:= \frac{1}{\log T}\int_1^{T^{\delta_1/2p}} r^{2p} f_T(r)\mathrm{d}r \int_{\mathbb{R}} |h_T^2(u,r) - \mathbf{1}(0<u<x)|\mathrm{d}u,\\
J''_T &:= \frac{1}{\log T}\int_1^{T^{\delta_1/2p}} r^{2p}|f_T(r) - c_f r^{-(1+2p)}|\mathrm{d}r.
\end{aligned}$$

Using

$$\begin{aligned}
&\int_{\mathbb{R}} |h_T^2(u,r) - \mathbf{1}(0<u<x)|\mathrm{d}u\\
&\qquad \le 2\int_0^1 \mathrm{d}t \int_{\mathbb{R}} |\mathbf{1}(0 < u + T^{-\delta_2}r^p t < x) - \mathbf{1}(0<u<x)|\mathrm{d}u \le Cr^pT^{-\delta_2},
\end{aligned}$$

we obtain $J'_T \le C(\log T)^{-1}T^{-\delta_2}\int_1^{T^{\delta_1/2p}} r^{p-1}\mathrm{d}r = o(1)$, since $\delta_1/2 \le \delta_2$ for $\gamma \le \gamma_+$. Then $J''_T = o(1)$ follows from (7.3), since $|f_T(r) - c_f r^{-(1+2p)}| \le \epsilon c_f r^{-(1+2p)}$ for all $r > \rho/T^{\beta/(1-p)}$ and some $\rho > 0$ if given any $\epsilon > 0$. This completes the proof of $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)\widehat{\sigma}_1^2 x = \log \mathrm{E}\exp\{\mathrm{i}\theta\widehat{\sigma}_1 B(x)\}$ as $T \to \infty$ for any $\theta \in \mathbb{R}$.

(v) After the same change of variables as in part (iii) we get $W_{T,\gamma,\beta}(\theta)$ in (7.47), where

$$g_T(r) := \int_{\mathbb{R}} \Psi\big(\theta(r^{1-p}\wedge 1)r^p h_T(u,r)\big)\mathrm{d}u$$

with the same $f_T(r)$ and $h_T(u,t) \to \mathbf{1}(0<u<x)$ as in (iii). By dominated convergence theorem, $g_T(r) \to x\Psi(\theta(r^{1-p}\wedge 1)r^p)$, where we justify its use by (7.30), and $h_T(u,r) \le \mathbf{1}(-r^p < u < x)$. The bound $|g_T(r)| \le C\min(r^p, r^2)$

follows from (7.30) and $\int_{\mathbb{R}} h_T(u,r)\mathrm{d}u = x$ with $h_T(u,r) \le 1$. Finally, by Lemma 7.12,

$$W_{T,\gamma,\beta}(\theta) \to x c_f \int_0^\infty \Psi\big(\theta(r^{1-p}\wedge 1)r^p\big) r^{-(1+\alpha)}\mathrm{d}r =: \log \mathrm{E}\exp\{\mathrm{i}\theta \widehat{L}(x)\}. \quad (7.51)$$

The proof of Theorem 7.9 is complete. □

Proof of Theorem 7.10. (i) Denote $\delta_1 := 1+\gamma-\alpha/p = \gamma-\gamma_+ > 0$ and $\delta_2 := (1-p)/p-\beta > 0$. By changing the variables in (7.29): $t \to Tt$, $u \to Tu$, $r \to T^{1/p}r$ we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where $f_T(r) := T^{(1+\alpha)/p} f(T^{1/p}r)$ and

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi\big(\theta T^{-\delta_1/2}((T^{\delta_2}r^{1-p})\wedge 1)h(u,r)\big)\mathrm{d}u$$

with $h(u,r) := \int_0^x \mathbf{1}(u<t<u+r^p)\mathrm{d}t$. The dominated convergence $g_T(r) \to g(r) := -(\theta^2/2)\int_{\mathbb{R}} h^2(u,r)\mathrm{d}u$ follows by (7.30). The latter combined with $\int_{\mathbb{R}} h^2(u,r)\mathrm{d}u \le C\min(1,r^p)\int_{\mathbb{R}} h(u,r)\mathrm{d}u \le C\min(r^p,r^{2p})$ gives the bound $|g_T(r)| \le C\min(r^p,r^{2p})$. Finally, by Lemma 7.12, $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)\sigma_2^2 x^{2H}$, where

$$\begin{aligned}\sigma_2^2 &:= c_f \int_{\mathbb{R}\times\mathbb{R}} \Big(\int_0^1 \mathbf{1}(u<t<u+r^p)\mathrm{d}t\Big)^2 \frac{\mathrm{d}u\mathrm{d}r}{r^{1+\alpha}} \\ &= \frac{2c_f}{\alpha(2-\alpha/p)(3-\alpha/p)(\alpha/p-1)}, \end{aligned}\quad (7.52)$$

proving part (i).

(ii) The proof is the same as that of Theorem 7.9(iii).

(iii) Let $\delta_1 := \gamma-\alpha_+\beta > 0$, $\delta_2 := \alpha_+\beta/\gamma_+ - 1 > 0$. By change of variables: $t \to Tt$, $u \to T^{\beta p/(1-p)}u$, $r \to T^{\beta/(1-p)}r$ we get (7.47) with $f_T(r) := T^{(1+\alpha)\beta/(1-p)} f(T^{\beta/(1-p)}r)$ and

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi(\theta T^{-\delta_1/2}(r^{1-p}\wedge 1)h_T(u,r))\mathrm{d}u,$$

with $h_T(u,r) := \int_0^x \mathbf{1}(0<(T^{-\delta_2}t-u)/r^p<1)\mathrm{d}t \to h(u,r) := x\mathbf{1}(-r^p<u<0)$. Then (7.30) and $h_T^2(u,r) \le x\mathbf{1}(-r^p<u<1)$ justify the dominated convergence $g_T(r) \to -(\theta^2/2)(r^{1-p}\wedge 1)^2 r^p x^2$. By (7.30) and $\int_{\mathbb{R}} h_T^2(u,r)\mathrm{d}u \le C\int_{\mathbb{R}} h_T(u,r)\mathrm{d}u \le Cr^p$, we have $|g_T(r)| \le C\min(r^p,r^{2-p})$. Finally, by Lemma 7.12 $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)x^2 c_f \int_0^\infty (r^{1-p}\wedge 1)^2 r^{p-(1+\alpha)}\mathrm{d}r = -(\theta^2/2)x^2\sigma_3^2$ with

$$\sigma_3^2 := \frac{2c_f(1-p)}{(2-p-\alpha)(\alpha-p)}, \quad (7.53)$$

proving part (iii).

(iv) Denote $\delta_1 := \beta - \gamma/\alpha_+ > 0, \delta_2 := \gamma/\gamma_+ - 1 > 0$. By the change of variables: $t \to Tt$, $u \to T^{\gamma/\gamma_+}u$, $r \to T^{\gamma/\gamma_+ p}r$ we get (7.47) with $f_T(r) := T^{(1+\alpha)\gamma/\gamma_+ p}f(T^{\gamma/\gamma_+ p}r)$ and

$$g_T(r) := \int_{\mathbb{R}} \Psi(\theta(r^{1-p} \wedge T^{\delta_1})h_T(u,r))\mathrm{d}u,$$

where $h_T(u,r) := \int_0^x \mathbf{1}(u < T^{-\delta_2}t < u + r^p)\mathrm{d}t \to h(u,r) := x\mathbf{1}(-r^p < u < 0)$. Then $g_T(r) \to g(r) := \int_{\mathbb{R}} \Psi(\theta x r^{1-p}\mathbf{1}(-r^p < u < 0))\mathrm{d}u$ and $W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\mathrm{d}r = \log \mathrm{E}\exp\{\mathrm{i}\theta x L_+(1)\}$ similarly to the proof of Theorem 7.2(ii).

(v) Set $\delta_1 := \gamma - \gamma_+ > 0$, $\delta_2 := \beta - (1-p)/p > 0$. After a change of variables: $t \to Tt$, $u \to Tu$, $r \to T^{1/p}r$, we get (7.47) with $f_T(r) := T^{(1+\alpha)/p}f(T^{1/p}r)$ and

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi(\theta T^{-\delta_1/2}(r^{1-p} \wedge T^{\delta_2})h(u,r))\mathrm{d}u,$$

where $h(u,r) := \int_0^x \mathbf{1}(u < t < u + r^p)\mathrm{d}t$. Then $g_T(r) \to g(r) := -(\theta^2/2)\int_{\mathbb{R}} r^{2(1-p)} h^2(u,r)\mathrm{d}u$ and $W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\mathrm{d}r = -(\theta^2/2)\sigma_+^2 x^{2H_+}$ similarly to the proof of Theorem 7.3(i).

(vi) We follow the proof of Theorem 7.9(iv). By the same change of variables, we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47). We split $W_{T,\gamma,\beta}(\theta) = W^-_{T,\gamma,\beta}(\theta) + W^0_{T,\gamma,\beta}(\theta) + W^+_{T,\gamma,\beta}(\theta)$ with the same $W^\pm_{T,\gamma,\beta}(\theta)$ being the remainder terms. Note that now $\delta_2 < \delta_1/2$, since $\gamma > \gamma_+$. Next, we split $W^0_{T,\gamma,\beta}(\theta) = W'_{T,\gamma,\beta}(\theta) + W''_{T,\gamma,\beta}(\theta)$, where

$$W'_{T,\gamma,\beta}(\theta) := \int_1^{T^{\delta_2/p}} g_T(r)f_T(r)\mathrm{d}r, \qquad W''_{T,\gamma,\beta}(\theta) := \int_{T^{\delta_2/p}}^{T^{\delta_1/2p}} g_T(r)f_T(r)\mathrm{d}r.$$

Analogously to the proof of Theorem 7.9(iv), we show the convergence $W'_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)\widehat{\sigma}_2^2 x$, where

$$\widehat{\sigma}_2^2 := c_f \frac{\delta_2}{p} = c_f\Big(\frac{1}{p} - \frac{\beta}{1-p}\Big). \tag{7.54}$$

Using (7.30) and $\int_{\mathbb{R}} h_T(r,u)\mathrm{d}u = x$ with $h_T(r,u) \le x(T^{\delta_2}/r^p)$, we get

$$\begin{aligned} |W''_{T,\gamma,\beta}(\theta)| &\le \frac{C}{\log T}\int_{T^{\delta_2/p}}^{T^{\delta_1/2p}} \frac{\mathrm{d}r}{r}\int_{\mathbb{R}} h_T^2(r,u)\mathrm{d}u \\ &\le \frac{CT^{\delta_2}}{\log T}\int_{T^{\delta_2/p}}^{T^{\delta_1/2p}} \frac{\mathrm{d}r}{r^{1+p}} = O((\log T)^{-1}) = o(1), \end{aligned}$$

which completes the proof of $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)\widehat{\sigma}_2^2 x = \log \mathrm{E}\exp\{\mathrm{i}\theta\widehat{\sigma}_2 B(x)\}$ as $T \to \infty$ for any $\theta \in \mathbb{R}$.

(vii) By the same change of variables as in part (i), we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where

$$g_T(r) := \int_{\mathbb{R}} T^{\delta_1}\Psi\big(\theta T^{-\delta_1/2}(r^{1-p} \wedge 1)h(u,r)\big)\mathrm{d}u$$

and where δ_1, $h(u,r)$, $f_T(r)$ are the same as in (i). Then $g_T(r) \to -(\theta^2/2)\int_{\mathbb{R}} h^2(u,r)\mathrm{d}u$ along with $\int_R h^2(u,r)\mathrm{d}u \le C\min(r^p, r^{2p})$ and (7.30) imply $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)c_f \int_0^\infty \int_{\mathbb{R}} (r^{1-p}\wedge 1)^2 h^2(u,r) r^{-(1+\alpha)}\mathrm{d}r\mathrm{d}u =: \log \mathrm{E}\exp\{\mathrm{i}\theta \widehat{Z}(x)\}$ as $T \to \infty$ for any $\theta \in \mathbb{R}$, by Lemma 7.12.

(viii) By the same change of variables as in part (iii) we obtain $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where $g_T(r) := \int_{\mathbb{R}} \Psi(\theta(r^{1-p}\wedge 1)h_T(u,r))\mathrm{d}u$ with $f_T(r)$, $\delta_2 = \gamma/\gamma_+ - 1 > 0$ and $h_T(u,r) := \int_0^x \mathbf{1}(u < T^{-\delta_2}t < u + r^p)\mathrm{d}t \to x\mathbf{1}(-r^p < u < 0)$ the same as in (iii). Using $\int_{\mathbb{R}} h_T(u,r)\mathrm{d}u = xr^p$ and $h_T(u,r) \le x$ yields $|g_T(r)| \le C\min(r^p, r^{2-p})$ from (7.30). Hence, by Lemma 7.12, it follows that

$$W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty \Psi(\theta x(r^{1-p}\wedge 1)) r^{p-(1+\alpha)}\mathrm{d}r =: \log \mathrm{E}\exp\{\mathrm{i}\theta x\widehat{Z}\}. \tag{7.55}$$

(ix) By the same change of variables as in the proof of part (iv), we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where

$$g_T(r) := \int_{\mathbb{R}} \Psi\big(\theta(\log T)^{-1/2}(r^{1-p}\wedge T^{\delta_1})h_T(u,r)\big)\mathrm{d}u$$

with δ_1, $\delta_2 := \gamma/\gamma_+ - 1 > 0$ and $h_T(u,r) := \int_0^x \mathbf{1}(u < T^{-\delta_2}t < u + r^p)\mathrm{d}t \to x\mathbf{1}(-r^p < u < 0) =: h(u,r)$ and $f_T(r)$ being the same as in (iv). We split $W_{T,\gamma,\beta}(\theta) = W^-_{T,\gamma,\beta}(\theta) + W^0_{T,\gamma,\beta}(\theta) + W^+_{T,\gamma,\beta}(\theta)$ and next prove that $W^-_{T,\gamma,\beta}(\theta) := \int_0^{T^{-\delta_2/p}} g_T(r)f_T(r)$ and $W^+_{T,\gamma,\beta}(\theta) := \int_1^\infty g_T(r)f_T(r)\mathrm{d}r$ are the remainder terms, whereas

$$\begin{aligned} W^0_{T,\gamma,\beta}(\theta) := \int_{T^{-\delta_2/p}}^1 g_T(r)f_T(r)\mathrm{d}r \ &\sim\ -\frac{\theta^2}{2}\frac{c_f}{\log T}\int_{T^{-\delta_2/p}}^1 \frac{\mathrm{d}r}{r^{1+p}}\int_{\mathbb{R}} h^2(u,r)\mathrm{d}u \\ &=\ -\frac{\theta^2}{2}\widetilde{\sigma}_+^2 x^2 =: W_{\gamma,\beta}(\theta), \end{aligned}$$

where the constant $\widetilde{\sigma}_+^2$ is given in (7.42). Using $\int_{\mathbb{R}} h_T(u,r)\mathrm{d}u = xr^p$ and $h_T(u,r) \le x\wedge(T^{\delta_2}r^p)$ along with (7.30), we show that

$$\begin{aligned} |W^+_{T,\gamma,\beta}(\theta)| &\le \frac{C}{(\log T)^{1/2}}\int_1^\infty r f_T(r)\mathrm{d}r = O((\log T)^{-1/2}) = o(1), \\ |W^-_{T,\gamma,\beta}(\theta)| &\le \frac{CT^{\delta_2}}{\log T}\int_0^{T^{-\delta_2/p}} r^2 f_T(r) = \frac{C}{T\log T}\int_0^{T^{1/p}} r^2 f(r)\mathrm{d}r \\ &= O((\log T)^{-1}) = o(1). \end{aligned}$$

To deal with the main term $W^0_{T,\gamma,\beta}(\theta)$, set $\widetilde{W}_{T,\gamma,\beta}(\theta) := -\frac{\theta^2}{2\log T}\int_{T^{-\delta_2/p}}^1 r^{2(1-p)} f_T(r)\mathrm{d}r$

$\int_{\mathbb{R}} h_T^2(u,r)\mathrm{d}u$. From $|\Psi(z) + z^2/2| \le |z|^3/6$, we obtain

$$\begin{aligned}
|W_{T,\gamma,\beta}(\theta) - \widetilde{W}_{T,\gamma,\beta}(\theta)| &\le \frac{C}{(\log T)^{3/2}} \int_{T^{-\delta_2/p}}^1 r^{3(1-p)} f_T(r)\mathrm{d}r \int_{\mathbb{R}} h_T^3(u,r)\mathrm{d}u \\
&\le \frac{C}{(\log T)^{3/2}} \int_{T^{-\delta_2/p}}^1 r^{3-2p} f_T(r)\mathrm{d}r \\
&= O((\log T)^{-3/2}) = o(1).
\end{aligned}$$

Finally, we consider $|\widetilde{W}_{T,\gamma,\beta}(\theta) - W_{\gamma,\beta}(\theta)| \le C(J'_T + J''_T)$, where

$$\begin{aligned}
J'_T &:= \frac{1}{\log T} \int_{T^{-\delta_2/p}}^1 r^{2(1-p)} f_T(r)\mathrm{d}r \int_{\mathbb{R}} |h_T^2(u,r) - h^2(u,r)|\mathrm{d}u, \\
J''_T &:= \frac{1}{\log T} \int_{T^{-\delta_2/p}}^1 r^{2-p}|f_T(r) - c_f r^{p-3}|\mathrm{d}r.
\end{aligned}$$

Using

$$\begin{aligned}
&\int_{\mathbb{R}} |h_T^2(u,r) - h^2(u,r)|\mathrm{d}u \\
&\qquad \le C \int_0^x \mathrm{d}t \int_{\mathbb{R}} |\mathbf{1}(u < T^{-\delta_2}t < u + r^p) - \mathbf{1}(-r^p < u < 0))|\mathrm{d}u \le CT^{-\delta_2}
\end{aligned}$$

we obtain $J'_T \le C(\log T)^{-1} T^{-\delta_2} \int_{T^{-\delta_2/p}}^1 r^{-(1+p)}\mathrm{d}r = O((\log T)^{-1}) = o(1)$. Then $J''_T = o(1)$ follows from (7.3), since $|f_T(r) - c_f r^{p-3}| \le \epsilon c_f r^{p-3}$ for all $r > \rho/T^{\gamma/2(1-p)}$ and some $\rho > 0$ if given any $\epsilon > 0$. This finishes the proof of $W_{T,\gamma,\beta}(\theta) \to -(\theta^2/2)\widetilde{\sigma}_+^2 x^2 = \log \mathrm{E}\exp\{\mathrm{i}\theta\widetilde{\sigma}_+^2 B_{1,1/2}(x,1)\}$ as $T \to \infty$ for any $\theta \in \mathbb{R}$.
The proof of Theorem 7.10 is complete. □

Proof of Theorem 7.11. (i) By the same change of variables as in Theorem 7.10(i), we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where

$$g_T(r) := \int_{\mathbb{R}} \Psi\big(\theta((T^{\delta_2} r^{1-p}) \wedge 1)h(u,r)\big)\mathrm{d}u \to \int_{\mathbb{R}} \Psi(\theta h(u,r))\mathrm{d}u =: g(r),$$

since $\delta_2 := (1-p)/p - \beta = \gamma_+/\alpha_+ - \beta > 0$ with $h(u,r)$, $f_T(r)$ being the same as in Theorem 7.10(i). Using (7.30) along with $\int_{\mathbb{R}} h(u,r)\mathrm{d}u = xr^p$ and $h(u,r) \le r^p$, we get $|g_T(r)| \le C\min(r^p, r^{2p})$. Hence $W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty \int_{\mathbb{R}} \Psi(\theta h(u,r)) r^{-(1+\alpha)}\mathrm{d}r\mathrm{d}u$ $=: \log \mathrm{E}\exp\{\mathrm{i}\theta I(x)\}$ by Lemma 7.12.

(ii), (iii) The proof is the same as that of Theorem 7.9(iii), (iv) respectively.

(iv) By the same change of variables as in Theorem 7.10(i), we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where $g(r) := \int_{\mathbb{R}} \Psi(\theta(r^{1-p} \wedge 1)h(u,r))\mathrm{d}u$ with $h(u,r)$, $f_T(r)$ being the same as in Theorem 7.10(i). Then $|g(r)| \le C\min(r^p, r^2)$ follows from (7.30). By Lemma 7.12, we get $W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\mathrm{d}r =: \log \mathrm{E}\exp\{\mathrm{i}\theta\widehat{I}(x)\}$.

(v) By the same change of variables as in Theorem 7.10(v), we rewrite $W_{T,\gamma,\beta}(\theta)$ as in (7.47), where $f_T(r)$, $g_T(r)$ are the same as in Theorem 7.10(v) except for

$\delta_1 = 0$. Then $g_T(r) \to g(r) := \int_{\mathbb{R}} \Psi(\theta r^{1-p} h(u,r))\mathrm{d}u$ and $|g_T(r)| \le C \min(r, r^2)$ from (7.30) lead to $W_{T,\gamma,\beta}(\theta) \to c_f \int_0^\infty g(r) r^{-(1+\alpha)}\mathrm{d}r = \log \mathrm{E} \exp\{\mathrm{i}\theta I_+(x,1)\}$ by Lemma 7.12, similarly to the proof of Theorem 7.6.

The proof of Theorem 7.11 is complete. $\square$

Chapter 8

Conclusions

In this last chapter, we review the main research contributions of this thesis.

- We identified three distinct limit regimes in the scheme of joint temporal-contemporaneous aggregation for independent copies of random-coefficient AR(1) process. We obtained three limit processes respectively. We showed that the process, arising in the 'intermediate' regime, admits a Poisson integral representation and can be regarded as a 'bridge' between the other two limit processes. The 'intermediate' limit of cumulative network traffic studied in [27, 34, 35, 55], though different, but has similar properties.

- We identified three different limit regimes in the scheme of joint temporal-contemporaneous aggregation for copies of random-coefficient AR(1) process, all driven by common innovations. We showed that a new process arising under 'intermediate' scaling can be regarded as a 'bridge' between the other two limit processes.

- We proved that the empirical process based on lag 1 sample autocorrelations of individual random-coefficient AR(1) series weakly converges to a generalized Brownian bridge under certain conditions. Applications of the obtained result arise in statistical inference from multiple random-coefficient AR(1) series, which are long enough so that lag 1 sample autocorrelations accurately estimate the unobservable AR coefficients. In particular, we justified testing with Kolmogorov–Smirnov statistic both simple and composite hypotheses, that AR coefficient is beta distributed.

- We proved that a nonlinear RF, defined as the Appell polynomial of some stationary linear LRD RF on $\mathbb{Z}^2$, may exhibit scaling transition. Such being the case, scaling transition occurs at the point $\gamma_0 > 0$, independent of the degree of the Appell polynomial even if the underlying linear RF is anisotropic.

- For the random grain model on $\mathbb{R}^2$ with LRD, we obtain two change-points $0 < \gamma_- < \gamma_+$ (distinct even in the $p = 1/2$ case) in its scaling limits, which shows that the concept of scaling transition requires further study. We showed that for $\gamma > \gamma_+$, the random grain model can have two different scaling limits, depending on α, p. We relate this dichotomy to the change from the vertical LRD to the vertical SRD property in the random grain model. A similar result holds for $0 < \gamma < \gamma_-$.

The following are some directions for future research.

- An interesting open problem concerns joint temporal-contemporaneous aggregation of independent copies of regime-switching AR(1) process, which combines the dependence structures of both random-coefficient AR(1) and network traffic models, see [62, 65].

- Another possible generalization concerns joint temporal-contemporaneous aggregation of random-coefficient AR(1) processes driven by innovations of infinite variance.

- If random AR coefficient a has a regularly varying density near the unit root, then $1/(1-a)$ is heavy-tailed distributed with the same index. We will adapt some Hill-type estimator of a tail index to the context of panel random-coefficient AR(1) data and study asymptotic properties of this estimator.

- One may ask if some random field model can have more than two change-points in the family of its scaling limits.

- It is of interest to obtain scaling transition for RFs on $\mathbb{R}^2$ as $\lambda \to 0$.

- For all models considered, it would be useful to strengthen the weak convergence of finite-dimensional distributions to the weak convergence in the space of functions.

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