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Randomly stopped sums with exponential-type distributions

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Abstract. Assume that $\{\xi_1, \xi_2, \ldots\}$ are independent and possibly nonidentically distributed random variables. Suppose that η is a nonnegative, nondegenerate at zero and integer-valued random variable, which is independent of $\{\xi_1, \xi_2, \ldots\}$. In this paper, we consider conditions for η and $\{\xi_1, \xi_2, \ldots\}$ under which the distribution of the random sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ belongs to the class of exponential distributions.

Keywords: class of exponential distributions, random sum, closure property.

1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v., i.e. an integer-valued, nonnegative and nondegenerate at zero r.v. In addition, suppose that r.v. η and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

Let $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of r.v.s $\xi_1, \xi_2, \ldots$. We denote the d.f. of S_η by F_{S_η} together with its tail $\overline{F}_{S_\eta}$. It is obvious that

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}(S_n > x)$$

for any positive x.

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In this paper, we consider possibly nonidentically distributed r.v.s $\xi_1, \xi_2, \dots$. We find conditions under which d.f. F_{S_η} belongs to the class of exponential distributions. If $F_{\xi_1}, F_{\xi_2}, \dots$ are different, then the various collections of conditions on d.f.s $\{F_{\xi_1}, F_{\xi_2}, \dots\}$, and the counting r.v. η imply exponentiality of d.f. F_{S_η}. Before discussing the properties of F_{S_η}, we recall some d.f. classes related to exponentiality.

- *For $\gamma > 0$, by $\mathcal{L}(\gamma)$ we denote the class of exponential d.f.s. It is said that $F \in \mathcal{L}(\gamma)$ if for any $y > 0$,*
$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}.$$
- *For $\gamma = 0$, the class $\mathcal{L}(0)$ is called the long-tailed distribution class and is denoted by $\mathcal{L}$.*
- *A d.f. F is $\mathcal{O}$-exponential ($F \in \mathcal{OL}$) if for any $y > 0$,*
$$\liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} > 0$$
or, equivalently,
$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

According to Proposition 2.6 by Albin and Sundén in [2], an absolutely continuous d.f. F belongs to the class $\mathcal{L}(\gamma)$ if and only if

$$F(x) = 1 - \exp\left\{-\int_{-\infty}^{x} \bigl(\alpha(u) + \beta(u)\bigr)\,\mathrm{d}u\right\} \quad \text{for } x \in \mathbb{R},$$

for some measurable functions α and β with $\alpha(u) + \beta(u) \geqslant 0$, for all $u \in \mathbb{R}$ such that

$$\lim_{u\to\infty} \alpha(u) = \gamma, \quad \lim_{x\to\infty} \int_{-\infty}^{x} \alpha(u)\,\mathrm{d}u = \infty \quad \text{and} \quad \lim_{x\to\infty} \int_{-\infty}^{x} \beta(u)\,\mathrm{d}u$$

exists. We note that each exponential distribution, each Erlang's distribution and each gamma distribution belong to the class $\mathcal{L}(\gamma)$ with some $\gamma > 0$.

It is easy to see that the following two inclusions hold:

$$\mathcal{L} \subset \mathcal{OL}, \qquad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

In [4, 5], Cline claimed that d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if r.v.s $\{\xi_1, \xi_2, \dots\}$ are identically distributed with d.f. $F_\xi \in \mathcal{L}(\gamma)$ and η is any counting r.v. Albin [1] constructed a counterexample and showed that Cline's result is false in general. In his paper [1], Albin stated that d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if r.v.s $\{\xi_1, \xi_2, \dots\}$ are identically distributed with common d.f. F_ξ belonging to the class $\mathcal{L}(\gamma)$

and $\mathbf{E}\mathrm{e}^{\delta\eta} < \infty$ for each $\delta > 0$. In order to prove his statement, Albin used the following implication for $c \in \mathbb{R}$:

$$\sup_{x\geqslant c} \frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant (1+\varepsilon)\,\mathrm{e}^{\gamma t}$$
$$\Longrightarrow \quad \sup_{x\geqslant n(c-t)+t} \frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant (1+\varepsilon)\,\mathrm{e}^{\gamma t}, \quad n \in \mathbb{N},$$

provided that $\varepsilon > 0$, $t \in \mathbb{R}$ and F is a d.f. from the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Here and later F^{*n} denotes n-fold convolution of d.f. F with itself. Unfortunately, if $\gamma > 0$, then the obtained relation holds for positive t only. Watanabe and Yamamuro (see [13, Remark 6.1]) showed that the above implication is incorrect in the case of positive γ and negative t. When $\gamma = 0$, the above implication for positive t is sufficient to prove the Albin's statement under the weaker restrictions on the counting r.v. η (see [11, Thm. 6]). The Albin's statement on conditions for which $F_{S_\eta} \in \mathcal{L}(\gamma)$ has remained only as a hypothesis in the case $\gamma > 0$. Watanabe and Yamamuro [13] do not prove the Albin's hypothesis in this case, they presented the following statement (see [13, Prop. 6.1]).

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent identically distributed r.v.s with a common d.f. F_ξ. If $F_\xi \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, then F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for each counting r.v. η distributed according to the Poisson law.*

The above result was generalized in [15], where the following statement was proved (see [15, Thm. 2.3]).

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent nonnegative r.v.s with a common d.f. F_ξ such that $F_\xi^{*\kappa} \in \mathcal{L}(\gamma)$ for some integer $\kappa \geqslant 1$ and some $\gamma \geqslant 0$. In addition, let counting r.v. η be independent of $\{\xi_1, \xi_2, \ldots\}$ and $\mathbf{P}(\eta \geqslant \kappa) > 0$. Then $F_{S_\eta} \in \mathcal{L}(\gamma)$ if any pair* (i), (ii) *or* (i), (iii) *of conditions holds, where*

(i) *for any $\varepsilon \in (0,1)$, there is an integer $M = M(\varepsilon)$ such that for $x \geqslant 0$,*

$$\sum_{k=M}^{\infty} \mathbf{P}(\eta = k+1)\overline{F_\xi^{*k}}(x) \leqslant \varepsilon \overline{F}_{S_\eta}(x),$$

(ii) $\overline{F}_\xi(x) = o(\overline{F_\xi^{*2}}(x))$,

(iii) *for all $t > 0$ and $1 \leqslant i \leqslant \kappa - 1$,*

$$\liminf_{x\to\infty} \frac{\overline{F_\xi^{*i}}(x-t)}{\overline{F_\xi^{*i}}(x)} \geqslant \mathrm{e}^{\gamma t}.$$

Motivated by the presented results, we also consider conditions for which d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Here the randomly stopped sum S_η contains independent but not necessarily identically distributed r.v.s. We suppose that some d.f.s from $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ belongs to the exponential class, and we find conditions for

$F_{\xi_1}, F_{\xi_2}, \dots$ and η such that the distribution of the randomly stopped sum S_η remains in the same class. In this work, we present three collections of such conditions. The proofs of the main results are based on ideas from the papers [6–8, 10, 13] and [15]. The similar results for class $\mathcal{L} = \mathcal{L}(0)$ were obtained in the papers [12] and [14].

The rest of the paper is organized as follows. In Section 2, we present our main results together with two examples of randomly stopped sums having some exponential distributions. Section 3 is a collection of auxiliary lemmas. The proofs of the three main results are presented in Section 4.

2 Main results and examples

At first, in this section, we present three theorems, which deal with situations when the d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. In Theorem 3, the case of a finitely supported counting r.v. η is considered, while Theorems 4 and 5 deal with the case of unbounded right tail of η. In Theorems 3 and 5, we consider nonnegative r.v.s, while in Theorem 4, r.v.s $\xi_1, \xi_2, \dots$ can be real valued.

Theorem 3. *Let $n \geqslant 1$ and $\{\xi_1, \xi_2, \dots, \xi_n\}$ be a collection of nonnegative independent r.v.s with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \dots, F_{\xi_n}\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots, \xi_n\}$ and having a finite support $\operatorname{supp}\eta \subseteq \{0, 1, \dots, n\}$. Then d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ for some nonrandom $1 \leqslant \nu \leqslant \min\{\operatorname{supp}\eta \setminus \{0\}\}$, and*

$$F_{\xi_k} \in \mathcal{L}(\gamma) \quad \textit{or} \quad \overline{F}_{\xi_k}(x) = o\bigl(\overline{F}_{\xi_\nu}(x)\bigr)$$

for each $k \in \{1, 2, \dots, \max\{\operatorname{supp}\eta\}\}$.

Theorem 4. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of real valued independent r.v.s such that for some $\gamma \geqslant 0$,*

$$\sup_{k \geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\longrightarrow} 0 \tag{1}$$

for each fixed $y \geqslant 0$. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$ such that

$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} \underset{k\to\infty}{\longrightarrow} 0. \tag{2}$$

Then $F_{S_\eta} \in \mathcal{L}(\gamma)$.

Here we observe that condition (1) is equivalent to the two-sided estimate

$$\mathrm{e}^{-\gamma y} \leqslant \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma y},$$

which holds for some $\gamma \geqslant 0$ and for each fixed $y \geqslant 0$.

In addition, we observe that condition (2) implies that $\mathbf{P}(\eta = k) > 0$ for all sufficiently large k.

Theorem 5. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of nonnegative independent r.v.s with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. D.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if there exist $\varkappa \geqslant 1$ and $1 \leqslant \nu \leqslant \varkappa$ such that*

(i) $\nu \leqslant \min\{\operatorname{supp}\eta \setminus \{0\}\}$,
(ii) $F_{\xi_\nu} \in \mathcal{L}(\gamma)$,
(iii) *for each* $1 \leqslant k \leqslant \varkappa$, $F_{\xi_k} \in \mathcal{L}(\gamma)$ *or* $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_\nu}(x))$,
(iv) *for each* $y \geqslant 0$,
$$\sup_{k \geqslant \varkappa+1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\longrightarrow} 0,$$
(v) $\mathbf{P}(\eta = k+1)/\mathbf{P}(\eta = k) \to 0$ *as* $k \to \infty$.

Further in this section, we present two examples, which illustrate several applications of our theorems. In both examples, we construct randomly stopped sums that belong to the class of exponential distributions.

Example 1. Suppose that we have a three-seasonal sequence of independent Erlang r.v.s with d.f.s from class $\mathcal{L}(2)$, i.e.

$$F_{\xi_k}(x) = \begin{cases} (1 - \mathrm{e}^{-2x}(1+2x))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 1 \bmod 3, \\ (1 - \mathrm{e}^{-2x}(1+2x+2x^2))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 2 \bmod 3, \\ (1 - \mathrm{e}^{-2x}(1+2x+2x^2+4x^3/3))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 0 \bmod 3. \end{cases}$$

In addition, suppose that the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and has Poisson distribution with parameter $\lambda > 0$.

It is clear that

$$\begin{aligned} &\sup_{k\geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-2y} \right| \\ &\quad = \max\left\{ \left| \frac{\overline{F}_{\xi_1}(x+y)}{\overline{F}_{\xi_1}(x)} - \mathrm{e}^{-2y} \right|, \left| \frac{\overline{F}_{\xi_2}(x+y)}{\overline{F}_{\xi_2}(x)} - \mathrm{e}^{-2y} \right|, \left| \frac{\overline{F}_{\xi_3}(x+y)}{\overline{F}_{\xi_3}(x)} - \mathrm{e}^{-2y} \right| \right\} \underset{x\to\infty}{\longrightarrow} 0 \end{aligned}$$

and

$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} = \frac{\lambda}{k+1} \underset{k\to\infty}{\longrightarrow} 0.$$

We see that all conditions of Theorem 3 are satisfied. Consequently, d.f. $F_{S_\eta} \in \mathcal{L}(2)$.

Example 2. Suppose that $\{\xi_1, \xi_2, \ldots\}$ is a sequence of nonnegative r.v.s such that

$$\begin{aligned} \overline{F}_{\xi_1}(x) &= \mathrm{e}^{-x}, \quad x \geqslant 0, \\ \overline{F}_{\xi_k}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y, \quad x \geqslant 0,\ k \in \{2, 3, \ldots, 10\}, \\ \overline{F}_{\xi_k}(x) &= \mathrm{e}^{-x}\left(1 + \frac{x}{k-10}\right), \quad x \geqslant 0,\ k \in \{11, 12, \ldots\}. \end{aligned}$$

In addition, let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ distributed according to law

$$\mathbf{P}(\eta = k) = \frac{1}{\hat{c}} \mathrm{e}^{-k^2}, \quad k \in \{0, 1, 2, \ldots\},$$

where

$$\hat{c} = \sum_{k=0}^{\infty} \mathrm{e}^{-k^2} \approx 1.3863.$$

The described sequence $\{\xi_1, \xi_2, \ldots\}$ and the counting r.v. η satisfy conditions of Theorem 5 with $\gamma = 1$, $\nu = 1$ and $\varkappa = 10$ because:

$$F_{\xi_1} \in \mathcal{L}(1), \qquad \operatorname{supp} \eta \setminus \{0\} = \mathbb{N},$$
$$\overline{F}_{\xi_k}(x) = o\bigl(\overline{F}_{\xi_1}(x)\bigr) \quad \text{if } k \in \{2, 3, \ldots, 10\},$$
$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} = \mathrm{e}^{-2k-1}, \quad k \in \mathbb{N},$$

and

$$\left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-y} \right| = \frac{y\mathrm{e}^{-y}}{x + k - 10}$$

for all $k \geqslant 11$, $x > 0$ and $y \geqslant 0$.

Consequently, d.f. F_{S_η} belongs to the class $\mathcal{L}(1)$ due to assertion of Theorem 5.

3 Auxiliary lemmas

In this section, we give all auxiliary assertions, which we use in the proofs of our main results. The first lemma was proved by Embrechts and Goldie (see [9, Thm. 3]).

Lemma 1. *Let F and G be two d.f.s, and let F belong to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Then convolution $F * G$ belongs to the class $\mathcal{L}(\gamma)$ if one of the following conditions holds*:

(i) *d.f. G belongs to the class $\mathcal{L}(\gamma)$,*
(ii) $\overline{G}(x) = o(\overline{F}(x))$.

The second lemma is the inhomogeneous case of the upper estimate, which was presented in the proof of Proposition 6.1 from [13].

Lemma 2. *Let $\xi_1, \xi_2, \ldots$ be real valued independent r.v.s such that*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+a)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma a} \tag{3}$$

for some $\gamma \geqslant 0$ and $a > 0$. Then, for any $\varepsilon \in (0, 1)$, there exists $b = b(a, \varepsilon) > 0$ such that

$$\overline{F}_{S_{n+1}}(x+a) \leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a} \overline{F}_{S_{n+1}}(x) + \overline{F}_{S_n}(x-b)$$

for all x and all $n \geqslant 1$.

Proof. For any x and any $b > 0$, we have

$$\begin{aligned}\overline{F}_{S_{n+1}}(x) &= \mathbf{P}(S_n + \xi_{n+1} > x)\\ &= \int\limits_{(-\infty,\, x-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) + \int\limits_{(x-b,\,\infty)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\ &=: \mathcal{J}_1(x,b) + \mathcal{J}_2(x,b). \end{aligned} \tag{4}$$

Condition (3) implies that

$$\sup_{n\geqslant 1} \frac{\overline{F}_{\xi_{n+1}}(x-y+a)}{\overline{F}_{\xi_{n+1}}(x-y)} \leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma a}$$

for any fixed $\varepsilon \in (0,1)$ if $y \leqslant x + a - b$ (then $x - y \geqslant b - a$) and b is sufficiently large. For such b, we get

$$\begin{aligned}\mathcal{J}_1(x+a,\, b) &= \int\limits_{(-\infty,\, x+a-b]} \frac{\overline{F}_{\xi_{n+1}}(x+a-y)}{\overline{F}_{\xi_{n+1}}(x-y)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\ &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(-\infty, x-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\ &\quad + (1+\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(x-b,\, x+a-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\ &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a}\big(\mathcal{J}_1(x,b) + \mathcal{J}_2(x,b)\big)\\ &= (1+\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x).\end{aligned}$$

On the other hand, it is obvious that

$$\mathcal{J}_2(x+a,\, b) \leqslant \int\limits_{(x+a-b,\infty)} \mathrm{d}F_{S_n}(y) \leqslant \overline{F}_{S_n}(x-b).$$

Therefore, for any $\varepsilon \in (0,1)$ and sufficiently large $b = b(a,\varepsilon)$, we obtain

$$\begin{aligned}\overline{F}_{S_{n+1}}(x+a) &= \mathcal{J}_1(x+a,b) + \mathcal{J}_2(x+a,b)\\ &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x) + \overline{F}_{S_n}(x-b).\end{aligned}$$

Lemma 2 is proved. □

The next lemma deals with the lower estimate of $\overline{F}_{S_{n+1}}(x+a)$ in the case of non-identical d.f.s $F_{\xi_1}, F_{\xi_2}, \dots$.

Lemma 3. *Let $\xi_1, \xi_2, \dots$ be real valued independent r.v.s such that*

$$\liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+a)}{\overline{F}_{\xi_k}(x)} \geqslant \mathrm{e}^{-\gamma a} \tag{5}$$

for some $\gamma \geqslant 0$ and $a > 0$. Then, for any $\varepsilon \in (0, 1/2)$, there exists $\hat{b} = \hat{b}(a, \varepsilon) > 0$ such that

$$\overline{F}_{S_{n+1}}(x+a) \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x) - \overline{F}_{S_n}(x-\hat{b})$$

for all x and $n \geqslant 1$.

Proof. Due to representation (4), we have

$$\overline{F}_{S_{n+1}}(x) = \mathcal{J}_1(x, \hat{b}) + \mathcal{J}_2(x, \hat{b})$$

for arbitrary real x and positive $\hat{b}$.

According to (5), for fixed $\varepsilon \in (0, 1/2)$, we have

$$\inf_{n\geqslant 1} \frac{\overline{F}_{\xi_{n+1}}(x-y+a)}{\overline{F}_{\xi_{n+1}}(x-y)} \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}$$

for all $y \leqslant x + a - \hat{b}$ and sufficiently large $\hat{b} = \hat{b}(a, \varepsilon)$.

Similarly as in the proof of Lemma 2, for such $\hat{b}$, we get

$$\begin{aligned}
\mathcal{J}_1(x+a,\, \hat{b}) &= \int\limits_{(-\infty,\, x+a-\hat{b}]} \frac{\overline{F}_{\xi_{n+1}}(x+a-y)}{\overline{F}_{\xi_{n+1}}(x-y)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(-\infty, x+a-\hat{b}]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(-\infty, x-\hat{b}]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&= (1-\varepsilon)\mathrm{e}^{-\gamma a}\mathcal{J}_1(x, \hat{b}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{F}_{S_{n+1}}(x+a) &\geqslant \mathcal{J}_1(x+a, \hat{b}) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}\mathcal{J}_1(x, \hat{b}) \\
&= (1-\varepsilon)\mathrm{e}^{-\gamma a}\big(\mathcal{J}_1(x, \hat{b}) + \mathcal{J}_2(x, \hat{b})\big) - (1-\varepsilon)\mathrm{e}^{-\gamma a}\mathcal{J}_2(x, \hat{b}) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x) - \overline{F}_{S_n}(x-\hat{b}),
\end{aligned}$$

and the assertion of Lemma 3 follows. □

The last auxiliary assertion is a mild generalization of Braverman's lemma (see [3, Lemma 1]).

Lemma 4. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s, and let $F_{\xi_1} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Then, for each $a > 0$, there exists a constant c_a such that*

$$\mathbf{P}(S_n > x - a) \leqslant c_a \mathbf{P}(S_n > x)$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof. The definition of the class $\mathcal{L}(\gamma)$ implies that

$$\overline{F}_{S_1}(x-a) = \overline{F}_{\xi_1}(x-a) \leqslant 2\mathrm{e}^{\gamma a}\overline{F}_{\xi_1}(x)$$

if $x > x_a$ and x_a is sufficiently large.

If $x \leqslant x_a$, then, obviously,

$$\frac{\overline{F}_{S_1}(x-a)}{\overline{F}_{S_1}(x)} = \frac{\overline{F}_{\xi_1}(x-a)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{1}{\overline{F}_{\xi_1}(x_a)}.$$

Consequently,

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_1}(x-a)}{\overline{F}_{S_1}(x)} \leqslant \max\left\{2\mathrm{e}^{\gamma a}, \frac{1}{\overline{F}_{\xi_1}(x_a)}\right\} =: c_a.$$

If $n \geqslant 2$, then, for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\overline{F}_{S_n}(x-a) &:= \mathbf{P}(\xi_1 + S_{2,n} > x - a) \\
&= \int_{-\infty}^{\infty} \frac{\overline{F}_{\xi_1}(x-y-a)}{\overline{F}_{\xi_1}(x-y)} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbf{P}(S_{2,n} \leqslant y) \\
&\leqslant \sup_{z\in\mathbb{R}} \frac{\overline{F}_{\xi_1}(z-a)}{\overline{F}_{\xi_1}(z)} \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbf{P}(S_{2,n} \leqslant y) \\
&\leqslant c_a \mathbf{P}(\xi_1 + S_{2,n} > x) \\
&= c_a \mathbf{P}(S_n > x).
\end{aligned}$$

So, the assertion of Lemma 4 follows. □

4 Proofs of main results

In this section, we present detailed proofs of all our main results. For these proofs, we use essentially approaches from [6, 10] and [13].

Proof of Theorem 3. For each positive x, we have

$$\overline{F}_{S_\eta}(x) = \sum_{k \in \operatorname{supp}\eta} \mathbf{P}(\eta = k)\overline{F}_{S_k}(x).$$

Since support $\operatorname{supp}\eta$ is finite, for an arbitrary positive y, we have

$$\min_{k \in \operatorname{supp}\eta} \left\{ \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} \leqslant \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \max_{k \in \operatorname{supp}\eta} \left\{ \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} \tag{6}$$

due to the two-sided inequality

$$\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m}\right\} \leqslant \frac{a_1 + a_2 + \ldots + a_m}{b_1 + b_2 + \cdots + b_m} \leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m}\right\},$$

provided that $a_i \geqslant 0$, $b_i > 0$, $i \in \{1, 2, \dots, m\}$ and $m \in \mathbb{N}$.

If $k \in \operatorname{supp}\eta$, then

$$S_k = \sum_{i \in \mathcal{K}} \xi_i + \sum_{i \notin \mathcal{K}} \xi_i,$$

where $\mathcal{K} = \{1 \leqslant i \leqslant k\colon\ F_{\xi_i} \in \mathcal{L}(\gamma)\}$.

Since $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ for some $1 \leqslant \nu \leqslant \min\{\operatorname{supp}\eta \setminus \{0\}\}$, the set of indices $\mathcal{K}$ is not empty. Lemma 1 implies that d.f. $F_{\mathcal{K}}$ of sum $\sum_{i \in \mathcal{K}} \xi_i$ belongs to the class $\mathcal{L}(\gamma)$.

Further, if $i^* \notin \mathcal{K}$, then $\overline{F}_{\xi_{i^*}}(x) = o(\overline{F}_{\xi_\nu}(x))$ because of the theorem's conditions. Therefore,

$$\frac{\overline{F}_{\xi_{i^*}}(x)}{\overline{F}_{\mathcal{K}}(x)} = \frac{\mathbf{P}(\xi_{i^*} > x)}{\mathbf{P}(\sum_{i \in \mathcal{K}} \xi_i > x)} \leqslant \frac{\overline{F}_{\xi_{i^*}}(x)}{\overline{F}_{\xi_\nu}(x)} \underset{x \to \infty}{\longrightarrow} 0,$$

and consequently, $F_{\mathcal{K}} * F_{\xi_{i^*}}$ belongs to the class $\mathcal{L}(\gamma)$ due to the second part of Lemma 1. Continuing our considerations, we get that d.f.

$$F_{S_k} = F_{\mathcal{K}} * \left\{ \prod_{i \notin \mathcal{K}}^{*} F_{\xi_i} \right\}$$

belongs to the class $\mathcal{L}(\gamma)$ as well for an arbitrary index $k \in \operatorname{supp}\eta$. Here $\prod^{*}_{i \notin \mathcal{K}} F_{\xi_i}$ denotes d.f. of sum $\sum_{i \notin \mathcal{K}} \xi_i$.

Consequently, the double estimate (6) implies the following two inequalities:

$$\limsup_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \max_{k \in \operatorname{supp}\eta} \left\{ \limsup_{x \to \infty} \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} = \mathrm{e}^{-\gamma y},$$
$$\liminf_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant \min_{k \in \operatorname{supp}\eta} \left\{ \liminf_{x \to \infty} \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} = \mathrm{e}^{-\gamma y}$$

for each positive y. The last two estimates finish the proof of Theorem 3. $\square$

Proof of Theorem 4. In order to show that $F_{S_\eta} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, it is sufficient to derive the following two estimates:

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \mathrm{e}^{-\gamma y}, \tag{7}$$

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant \mathrm{e}^{-\gamma y}, \tag{8}$$

which both should valid for each positive y.

(I) At first, we show inequality (7). For this, we suppose that y is an arbitrary positive number, and we choose $\varepsilon \in (0,1)$. According to condition (2), we have

$$\mathbf{P}(\eta = n+1) \leqslant \varepsilon \mathbf{P}(\eta = n) \tag{9}$$

for all $n \geqslant N = N(\varepsilon) \geqslant 2$. For such N, we get that

$$\overline{F}_{S_\eta}(x+y) = \sum_{n=1}^{N} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) + \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y). \tag{10}$$

Using Lemma 2, we obtain

$$\begin{aligned}\sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) \quad &\leqslant \sum_{n=N+1}^{\infty} (1+\varepsilon)\mathrm{e}^{-\gamma y}\,\mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\ &\quad + \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_{n-1}}(x-b)\end{aligned}$$

for some $b = b(y,\varepsilon) > 0$. This relation together with inequality (9) shows that

$$\begin{aligned}\overline{F}_{S_\eta}(x+y) \leqslant{}& \sum_{n=1}^{N} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) \\ &+ (1+\varepsilon)\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\ &+ \varepsilon \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n-1)\,\overline{F}_{S_{n-1}}(x-b).\end{aligned} \tag{11}$$

Condition (1) implies that

$$\mathrm{e}^{-\gamma u} \leqslant \liminf_{x\to\infty} \frac{\overline{F}_{\xi_k}(x+u)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(x+u)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma u}$$

for all fixed k and u. It follows that $\overline{F}_{\xi_k} \in \mathcal{L}(\gamma)$ for each k. Hence, according to Lemma 1, we obtain that $\overline{F}_{S_n} \in \mathcal{L}(\gamma)$ for each fixed $n \in \mathbb{N}$. Therefore,

$$\max_{1 \leqslant n \leqslant N} \frac{\overline{F}_{S_n}(x+y)}{\overline{F}_{S_n}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} \tag{12}$$

if $x \geqslant \hat{x} = \hat{x}(N, y, \varepsilon)$.

So, for the chosen N, b and for all $x \geqslant \hat{x}$, we have

$$\begin{aligned}
\overline{F}_{S_\eta}(x+y) \leqslant{}& (1+\varepsilon)\mathrm{e}^{-\gamma y} \sum_{n=1}^{N} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x) \\
&+ (1+\varepsilon)\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x) \\
&+ \varepsilon \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x-b) \\
={}& (1+\varepsilon)\mathrm{e}^{-\gamma y} \overline{F}_{S_\eta}(x) + \varepsilon \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x-b).
\end{aligned}$$

According to Lemma 4, $\overline{F}_{S_n}(x-b) \leqslant c_1 \overline{F}_{S_n}(x)$ for some positive constant $c_1 = c_1(b(y, \varepsilon))$.

Therefore,

$$\begin{aligned}
\overline{F}_{S_\eta}(x+y) &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} \overline{F}_{S_\eta}(x) + \varepsilon c_1 \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x) \\
&\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} \overline{F}_{S_\eta}(x) + \varepsilon c_1 \overline{F}_{S_\eta}(x)
\end{aligned}$$

for all sufficiently large x.

The last inequality implies that

$$\limsup_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} + \varepsilon c_1.$$

Since $\varepsilon \in (0,1)$ is arbitrarily chosen, the desired inequality (7) holds for each positive y.

(II) In this part, we show inequality (8). We fix positive y and choose $\varepsilon \in (0, 1/2)$. Let N be a natural number such that, for $n \geqslant N$, inequality (9) holds. Due to Lemma 3, we have

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x+y) \geqslant{}& \sum_{n=N+1}^{\infty} (1-\varepsilon)\mathrm{e}^{-\gamma y} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x) \\
&- \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_{n-1}}(x - \hat{b})
\end{aligned} \tag{13}$$

for some $\hat{b} = \hat{b}(y,\varepsilon) > 0$. Substituting estimates (9) and (13) into equality (10), we get

$$\begin{aligned}\overline{F}_{S_\eta}(x+y) \geqslant \sum_{n=1}^{N} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) \\ + (1-\varepsilon)\,\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\ - \varepsilon \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n-1)\,\overline{F}_{S_{n-1}}(x-\hat{b}).\end{aligned} \tag{14}$$

Since $F_{\xi_k} \in \mathcal{L}(\gamma)$ for each fixed k, using Lemma 1, we obtain $F_{S_n} \in \mathcal{L}(\gamma)$ for $n \in \mathbb{N}$. Similarly as in derivation of (12), this implies that

$$\min_{1 \leqslant n \leqslant N} \frac{\overline{F}_{S_n}(x+y)}{\overline{F}_{S_n}(x)} \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y} \tag{15}$$

if $x \geqslant \tilde{x} = \tilde{x}(N,y,\varepsilon)$.

For such $x \geqslant \tilde{x}$, due to (14) and (15), we have

$$\overline{F}_{S_\eta}(x+y) \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) - \varepsilon \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x-\hat{b}).$$

According to Lemma 4, we have that $\overline{F}_{S_n}(x-\hat{b}) \leqslant c_2\overline{F}_{S_n}(x)$ for some positive constant $c_2 = c_2(\hat{b}(y,\varepsilon))$. Therefore,

$$\overline{F}_{S_\eta}(x+y) \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) - \varepsilon c_2\overline{F}_{S_\eta}(x)$$

when $x \geqslant \tilde{x}$. This last estimate implies that

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y} - \varepsilon c_2$$

for an arbitrary $\varepsilon \in (0, 1/2)$.

Letting ε tend to zero, from the last inequality we get the desired estimate (8). The theorem is proved. □

Proof of Theorem 5. If $\varkappa = 1$, then the assertion of Theorem follows from Theorem 4. So, we suppose that $\varkappa \geqslant 2$, and we split our proof into two parts.

(I) If $\mathbf{P}(\eta \leqslant \varkappa) = 0$, then r.v. η has an infinite support

$$\operatorname{supp}\eta \subset \{\varkappa+1,\ \varkappa+2,\ \dots\}.$$

Conditions (i)–(iii) imply that $F_{S_\varkappa} \in \mathcal{L}(\gamma)$ due to Theorem 3. Since $F_{\xi_{\varkappa+1}} \in \mathcal{L}(\gamma)$ according to condition (iv), the convolution $F_{S_{\varkappa+1}} = F_{S_\varkappa} * F_{\xi_{\varkappa+1}}$ belongs to the class $\mathcal{L}(\gamma)$

as well due to Lemma 1. This and condition (iv) imply that

$$\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\hat{\xi}_k}(x+y)}{\overline{F}_{\hat{\xi}_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\longrightarrow} 0$$

for each fixed $y \geqslant 0$, where $\hat{\xi}_1 = S_{\varkappa+1}$, $\hat{\xi}_2 = \xi_{\varkappa+2}$, $\hat{\xi}_3 = \xi_{\varkappa+3}, \ldots$.

Let $\hat{\eta}$ be the counting r.v. defined by equality $\mathbf{P}(\hat{\eta} = k) = \mathbf{P}(\eta = \varkappa + k)$, where $k = 1, 2, \ldots$, and let $\hat{S}_n = \hat{\xi}_1 + \hat{\xi}_2 + \cdots + \hat{\xi}_n$ for each $n \geqslant 1$.

R.v.s $\{\hat{\xi}_1, \hat{\xi}_2, \ldots\}$ and $\hat{\eta}$ satisfy conditions of Theorem 4. Hence, d.f. $F_{\hat{S}_{\hat{\eta}}}$ belongs to the class $\mathcal{L}(\gamma)$. We observe that

$$\begin{aligned}
\overline{F}_{\hat{S}_{\hat{\eta}}}(x) &= \mathbf{P}(\hat{\eta} = 1)\mathbf{P}(\hat{S}_1 > x) + \sum_{k=2}^{\infty}\mathbf{P}(\hat{\eta} = k)\mathbf{P}(\hat{S}_k > x)\\
&= \mathbf{P}(\eta = \varkappa + 1)\mathbf{P}(S_{\varkappa+1} > x) + \sum_{k=2}^{\infty}\mathbf{P}(\eta = \varkappa + k)\mathbf{P}(S_{\varkappa+k} > x)\\
&= \overline{F}_{S_\eta}(x)
\end{aligned}$$

for an arbitrary nonnegative x. Consequently, F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ as well in the case under consideration.

(II) Let now $\mathbf{P}(\eta \leqslant \varkappa) > 0$. Since $\mathbf{P}(\eta \geqslant \varkappa + 1) > 0$ due to condition (v), we have that

$$\overline{F}_{S_\eta}(x) = \mathbf{P}(\eta \leqslant \varkappa)\overline{F}_{S_{\tilde{\eta}}}(x) + \mathbf{P}(\eta \geqslant \varkappa + 1)\,\overline{F}_{S_{\hat{\eta}}}(x) \tag{16}$$

for an arbitrary nonnegative x, where $\tilde{\eta}$ and $\hat{\eta}$ are two counting r.v.s independent of $\{\xi_1, \xi_2, \ldots\}$ with distributions

$$\begin{aligned}
\mathbf{P}(\tilde{\eta} = k) &= \frac{\mathbf{P}(\eta = k)}{\mathbf{P}(\eta \leqslant \varkappa)}, \quad k \in \{0, 1, \ldots, \varkappa\},\\
\mathbf{P}(\hat{\eta} = k) &= \frac{\mathbf{P}(\eta = k)}{\mathbf{P}(\eta \geqslant \varkappa + 1)}, \quad k \in \{\varkappa + 1,\ \varkappa + 2,\ \ldots\}.
\end{aligned}$$

Theorem 3 implies that $F_{S_{\tilde{\eta}}} \in \mathcal{L}(\gamma)$ because of the finiteness of support $\operatorname{supp}\tilde{\eta}$. The investigation analogous to that in part (II) implies that d.f. $F_{S_{\hat{\eta}}}$ belongs to the class $\mathcal{L}(\gamma)$ as well. Now the statement of theorem follows immediately from relation (16). Theorem 5 is proved. □

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