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Numbers Whose Powers Are Arbitrarily Close to Integers

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Abstract: In this paper, it is proved that, for any sequence of positive numbers ξ_n, $n = 1, 2, \ldots$, which does not converge to zero faster than the exponential function, and any sequence of positive numbers δ_n, $n = 1, 2, 3, \ldots$, there is an uncountable set of positive numbers S such that, for each $\alpha > 1$ in S, there are infinitely many $n \in \mathbb{N}$ for which the fractional parts $\{\xi_n \alpha^n\}$ are smaller than δ_n, regardless of how fast the sequence δ_n tends to zero. In particular, for any sequence bounded away from zero, namely, $\xi_n \geq \xi > 0$ for $n \geq 1$, it is shown that infinitely many integers n for which the inequality $\{\xi_n \alpha^n\} < \delta_n$ is true can be extracted from an arbitrary subsequence $\mathcal{N}$ of positive integers.

Keywords: fractional parts; powers of transcendental numbers; distribution modulo 1

MSC: 11B07; 11J20; 11J71

1. Introduction

Starting from Weyl's 1916 paper [1], various problems related to the distribution of the sequence of fractional parts

$$\{\xi \alpha^n\}, \quad n = 1, 2, 3, \ldots, \tag{1}$$

where $\xi > 0$ and $\alpha > 1$ are two real numbers, were studied. Weyl's result implies that, for each $\alpha > 1$, the sequence (1) is uniformly distributed for almost all $\xi > 0$. See also [2] for a more precise version of this result. In the opposite direction, Koksma [3] proved that, if $\xi > 0$ is fixed, then the sequence (1) is uniformly distributed for almost all $\alpha > 1$. In this respect, the exceptional α are Pisot and Salem numbers. Recall that an algebraic integer $\alpha > 1$ is called a Pisot number if its conjugates over $\mathbb{Q}$ other than α itself (if any) all lie in the open unit disc $|z| < 1$. An algebraic integer $\alpha > 1$ is called a Salem number if its degree over $\mathbb{Q}$ is an even number $d \geq 4$ and $d - 2$ of its conjugates lie on the unit circle $|z| = 1$. (Since such α is reciprocal its other conjugates are α and α^{-1}.) See, for instance, the paper of Pisot and Salem themselves [4], where they proved that, if $\xi = 1$ and $\alpha > 1$ is a Salem number, then the sequence (1) is everywhere dense in $[0, 1]$, but not uniformly distributed in $[0, 1]$. The monographs [5,6] contain some basic information about Pisot and Salem numbers, while in Smyth's review paper [7] there are more recent references. In some literature, Pisot numbers are also called Pisot–Vijayaraghavan numbers or PV numbers; see, for instance, some early papers of Vijayaraghavan on this subject [8–11].

For algebraic numbers α, at least something can be said about the distribution of (1). Extending an earlier result of Flatto, Lagarias, and Pollington for rational $\alpha > 1$ [12], in [13], it was proved that, for each $\xi > 0$ and each algebraic number $\alpha > 1$, there should be a gap between the largest and the smallest limit points of the sequence (1) which depends only on α except if α is a Pisot number or a Salem number when we need an extra

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condition $\xi \notin \mathbb{Q}(\alpha)$. This result is not only of interest itself but also has several applications. In particular, it seems to be useful in a so-called Erdős similarity conjecture [14]. The cases when $\xi \in \mathbb{Q}(\alpha)$ and α is a Pisot number or a Salem number were treated in [15,16], respectively; see also [17]. Nevertheless, for example, Mahler's $3/2$-problem [18], where he asks whether, for $\alpha = 3/2$, there is a so-called Z-number, namely, $\xi > 0$, such that all elements of (1) lie in $(0, 1/2)$, is unsolved (see [19–21]). However, the situation with any specific transcendental number α is more complicated and less known. For example, it is not known if (1) with $(\xi, \alpha) = (1, e)$ has one or more than one limit point. Determining whether there is a transcendental number $\alpha > 1$ for which the sequence $\{\alpha^n\}$, $n = 1, 2, 3, \ldots$, has only finitely many limit points is still a completely open problem as well.

We remark that the behavior of the sequence (1) for $\xi = 1$ and a transcendental number $\alpha > 1$ can be very different depending on α. In [22], it was shown that, for any sequence of real numbers r_n, $n = 1, 2, 3, \ldots$, and any $\varepsilon > 0$, there is a transcendental number $\alpha > 1$ such that

$$\|\alpha^n - r_n\| < \varepsilon \tag{2}$$

for all $n \in \mathbb{N}$. (Here, $\|y\|$ is the distance from $y \in \mathbb{R}$ to the nearest integer.) See also two subsequent papers [23,24]. In fact, if we want (2) to hold not for all $n \in \mathbb{N}$, but only for infinitely many n, then this follows from another paper of Koksma [25] with ε replaced by a sequence of positive numbers ε_n such that the series $\sum_{n=1}^{\infty} \varepsilon_n$ are divergent. In [22], it was also shown that, for any sequence of positive numbers δ_n, $n = 1, 2, 3, \ldots$, there is a transcendental number $\alpha > 1$ for which the inequality

$$\{\alpha^n\} < \delta_n$$

holds for infinitely many $n \in \mathbb{N}$. This time, there are no conditions or restrictions whatsoever on the rate of convergence of δ_n to zero.

In this paper, it will be shown that, even if we replace in (1) a fixed number $\xi > 0$ by any sequence of positive numbers ξ_n, $n = 1, 2, 3, \ldots$, which is not converging to zero faster than the exponential function, then there are "many" numbers $\alpha > 1$ such that $\{\xi_n \alpha^n\}$ is smaller than an arbitrary positive number δ_n for infinitely many $n \in \mathbb{N}$, regardless of how fast the sequence δ_n, $n = 1, 2, 3, \ldots$, converges to zero. (This type of sequence, specifically, with $\xi_n = 1/n$ and an integer $\alpha \geq 2$, was considered before; see [26], where their density in $[0, 1]$ was established, and [27–29].) Of course, the theorem stated below holds in the special case when $\xi_n = \xi > 0$ for each $n \in \mathbb{N}$ and, more generally, when ξ_n is bounded away from zero, namely, $\xi_n \geq \xi > 0$ for $n \in \mathbb{N}$.

Theorem 1. *Let $\boldsymbol{\delta} = \{\delta_1, \delta_2, \delta_3, \ldots\}$ and $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \ldots\}$ be two sequences of positive numbers such that*

$$\limsup_{n \to \infty} \frac{\log \xi_n}{n} \geq 0. \tag{3}$$

Then, for any interval $I = [a, b]$, where $1 \leq a < b$, there is an uncountable set $S(\boldsymbol{\delta}, \boldsymbol{\xi}, I) \subset I$ such that, for each $\alpha \in S(\boldsymbol{\delta}, \boldsymbol{\xi}, I)$, the inequalities

$$0 < \{\xi_n \alpha^n\} < \delta_n \tag{4}$$

hold for infinitely many $n \in \mathbb{N}$.

Note that the condition (3) of Theorem 1 cannot be omitted. For example, if $\tau > 0$, $\xi_n = e^{-\tau n}$ and $\delta_n = 1/n!$, then, for each $\alpha \in [1, e^\tau)$ and each sufficiently large $n \in \mathbb{N}$, we have

$$\{\xi_n \alpha^n\} = \{e^{(\log \alpha - \tau)n}\} = e^{(\log \alpha - \tau)n} > \frac{1}{n!} = \delta_n,$$

so there is no α in the interval $I = [1, e^\tau)$ for which the inequality (4) is true for infinitely many $n \in \mathbb{N}$. Similarly, if $\xi_n = \delta_n = 1/n!$, then there is no $\alpha > 1$ at all for which (4) is true for infinitely many $n \in \mathbb{N}$.

Of course, since the set $S(\boldsymbol{\delta}, \boldsymbol{\xi}, I)$ is uncountable and $\mathbb{Q}$ is countable, $S(\boldsymbol{\delta}, \boldsymbol{\xi}, I)$ contains an uncountable subset of transcendental numbers α with the property (4). Therefore, Theorem 1 is already more general than Theorem 3 of [22] for $\xi_n = 1$.

Replace each δ_j by

$$\delta_j' := \min(1/j, \delta_1, \ldots, \delta_j)$$

and set

$$\Phi_j = \lceil 1/\delta_j' \rceil$$

for every $j \in \mathbb{N}$. (Here and below, $\lceil y \rceil$ is the ceiling function, namely, the smallest integer greater than or equal to $y \in \mathbb{R}$.) It is clear then that $\Phi_1, \Phi_2, \Phi_3, \ldots$ is an unbounded nondecreasing sequence of positive integers such that each element of the sequence

$$U = \{1/\Phi_1, 1/\Phi_2, 1/\Phi_3, \ldots\}$$

does not exceed the corresponding element of the sequence $\boldsymbol{\delta}$. Therefore, in order to prove Theorem 1, it suffices to show that for each interval $I \subset \mathbb{R}_{>1}$, there is an uncountable set of real numbers $S \subset I$ such that, for every $\alpha \in S$, the inequalities

$$0 < \{\xi_n \alpha^n\} < \frac{1}{\Phi_n} \tag{5}$$

hold for infinitely many $n \in \mathbb{N}$.

We will prove the following more general statement:

Theorem 2. *Let $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \ldots\}$ be a sequence of positive numbers satisfying (3), and let $\Phi_1 \le \Phi_2 \le \Phi_3 \le \ldots$ be an unbounded sequence of positive integers. Then, for any interval $I = [a, b]$, $1 \le a < b$, and any real number $\eta > 1$, there is an uncountable set $S(\boldsymbol{\xi}, \Phi, I, \eta) \subset I$ such that, for each $\alpha \in S(\boldsymbol{\xi}, \Phi, I, \eta)$, the inequalities*

$$\eta^m < \xi_n \alpha^n < \eta^m + \frac{1}{\Phi_n} \tag{6}$$

hold for infinitely many pairs $(n, m) \in \mathbb{N}^2$.

It is clear that (6) implies (5), with, for example, $\eta = 2$, so Theorem 2 immediately implies Theorem 1.

We will derive Theorem 2 from the following proposition of independent interest.

Proposition 1. *Let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be a sequence of real numbers, and let $1 = \varepsilon_1 \ge \varepsilon_2 \ge \varepsilon_3 \ge \ldots$ be a sequence of positive numbers. Assume that $I = [a, b]$, where $0 \le a < b$, and let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$. Then there is an uncountable set $B \subset I$ such that, for each $\beta \in B$, the inequalities*

$$0 < \{n\beta - \gamma_n\} < \varepsilon_n \tag{7}$$

hold for infinitely many $n \in \mathcal{N}$.

Note that, if $\gamma_1 = \gamma_2 = \gamma_3 = \ldots = 0$ and if the sequence ε_n tends to zero faster than any constant power of $1/n$, then the numbers β satisfying (7) are Liouville numbers. Recall that a Liouville number is a real number whose irrationality exponent is infinite,

see p. 248 in [30]. This means that, for any $C > 1$, there is a pair of integers k, n, where $n > 1$, such that
$$0 < \left|\beta - \frac{k}{n}\right| < \frac{1}{n^C}.$$

Therefore, Proposition 1 is the construction of uncountably many Liouville type numbers with good approximation not just by rational fractions k/n, $k, n \in \mathbb{N}$, but by fractions with "moving numerator" $(k + \gamma_n)/n$. The author thanks Prof. Nikolay Moshchevitin for a useful advice towards this construction. Note that the approximation $(k + \gamma_n)/n$ to those special Liouville type numbers is with $k \in \mathbb{Z}$ and with n being not just in $\mathbb{N}$ but in any infinite sequence of positive integers $\mathcal{N}$. For instance, $\mathcal{N}$ can be the set of squares or the set of primes.

Next, we will prove Proposition 1 (Section 2) and then derive Theorem 2 from this proposition (Section 3). In Section 4, we will give a stronger version of Theorem 1 under a condition slightly stronger than that in (3). Then, in Section 5, we provide another application of Proposition 1. Section 6 contains some final remarks.

2. Proof of Proposition 1

We begin with the following simple observation:

Lemma 1. *Let $I = [a, b]$ be a closed real interval with $a < b$, and let $u < v$ be two real numbers. Then, for each sufficiently large positive integer n, there is an integer $k = k(n)$ such that $(k + u)/n, (k + v)/n \in I$.*

Proof. Take any integer n satisfying
$$n \geq \frac{v - u + 1}{b - a}. \tag{8}$$
Note that $n > 0$. Select
$$k = k(n) = \lceil na - u \rceil. \tag{9}$$

Then $k \geq na - u$, and hence $a \leq (k + u)/n$. Next, from (8) and (9), it follows that
$$\frac{k + v}{n} < \frac{na - u + 1 + v}{n} \leq \frac{na + n(b - a)}{n} = \frac{nb}{n} = b.$$
Therefore, the numbers $(k + u)/n$ and $(k + v)/n$ both belong to the interval I, which completes the proof of the lemma. □

Next, for any sequence of real numbers γ_n, $n = 1, 2, 3, \ldots$, and any infinite sequence of positive integers $\mathcal{N}$, we will prove the existence of a real number that is very close to the fraction $(k + \gamma_n)/n$ for infinitely many pairs (n, k), where $n \in \mathcal{N}$ and $k \in \mathbb{Z}$.

Lemma 2. *Let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be a sequence of real numbers, and let $1 = \varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \geq \ldots$ be a sequence of positive numbers. Let $I = [a, b]$, where $0 < a < b$, and let $\mathcal{N} \subseteq \mathbb{N}$ be infinite. Then, for any sequence $\mathbf{u} = \{u_1, u_2, u_3, \ldots\}$, where $u_j \in \{2, 4\}$ for each $j \in \mathbb{N}$, there is a positive real number $\beta(\mathbf{u}) \in I$ and a sequence $n_1 < n_2 < n_3 < \ldots$ in $\mathcal{N}$ such that, for every $j \in \mathbb{N}$, we have*
$$\frac{\varepsilon_{n_j}}{u_j + 1} \leq \{n_j \beta(\mathbf{u}) - \gamma_{n_j}\} \leq \frac{\varepsilon_{n_j}}{u_j}. \tag{10}$$

Proof. We will construct the number $\beta(\mathbf{u})$ using the method of nested intervals. Set $I_0 = I = [a, b]$. Take the least integer n_1 in $\mathcal{N}$ satisfying

$$n_1 \geq \frac{13}{10(b-a)}, \tag{11}$$

and set

$$u = \gamma_{n_1} + \frac{\varepsilon_{n_1}}{5} \quad \text{and} \quad v = \gamma_{n_1} + \frac{\varepsilon_{n_1}}{2}.$$

Note that

$$0 < v - u = \frac{3\varepsilon_{n_1}}{10} \leq \frac{3}{10},$$

so n_1 chosen in (11) satisfies the inequality (8). Choosing k_1 as in (9), namely,

$$k_1 = \lceil n_1 a - \gamma_{n_1} - \varepsilon_{n_1}/5 \rceil$$

and applying Lemma 1, we find that both endpoints of the interval

$$J_1 = [n_1^{-1}(k_1 + \gamma_{n_1} + \varepsilon_{n_1}/5), n_1^{-1}(k_1 + \gamma_{n_1} + \varepsilon_{n_1}/2)]$$

belong to the interval I_0. Consequently, as $u_1 \in \{2, 4\}$,

$$I_1 = [n_1^{-1}(k_1 + \gamma_{n_1} + \varepsilon_{n_1}/(u_1+1)), n_1^{-1}(k_1 + \gamma_{n_1} + \varepsilon_{n_1}/u_1)]$$

is its subinterval, so it satisfies $I_1 \subset I_0$. Furthermore, for any number $\zeta \in I_1$, we have

$$\frac{\varepsilon_{n_1}}{u_1+1} \leq n_1\zeta - \gamma_{n_1} - k_1 \leq \frac{\varepsilon_{n_1}}{u_1}.$$

From $\varepsilon_{n_1} \leq 1$ and $u_1 \in \{2, 4\}$, it follows that $k_1 = \lfloor n_1\zeta - \gamma_{n_1} \rfloor$ (where $\lfloor y \rfloor$ is the integral part of $y \in \mathbb{R}$), so (10) is true for $j = 1$ and any number ζ from the interval I_1.

We now argue by induction on j. Assume that $l \geq 1$ is an integer such that, for $j = 1, 2, \ldots, l$, there is a nested collection of intervals

$$I_j = [n_j^{-1}(k_j + \gamma_{n_j} + \varepsilon_{n_j}/(u_j+1)), n_j^{-1}(k_j + \gamma_{n_j} + \varepsilon_{n_j}/u_j)] = [a_j, b_j] \tag{12}$$

with uniquely chosen $n_1 < n_2 < n_3 < \ldots < n_l$ in $\mathcal{N}$ and $k_1, \ldots, k_l \in \mathbb{Z}$ such that

$$I_l \subseteq I_{l-1} \subseteq \cdots \subseteq I_1 \subseteq I_0 = [a, b].$$

For any $\zeta \in I_j$, we clearly have

$$\frac{\varepsilon_{n_j}}{u_j+1} \leq n_j\zeta - \gamma_{n_j} - k_j = \{n_j\zeta - \gamma_{n_j}\} \leq \frac{\varepsilon_{n_j}}{u_j},$$

so (10) is true for $j = 1, 2, \ldots, l$ and any number ζ from the interval I_l.

Next, we will show how to choose the interval I_{l+1} of the form

$$I_{l+1} = [n_{l+1}^{-1}(k_{l+1} + \gamma_{n_{l+1}} + \varepsilon_{n_{l+1}}/(u_{l+1}+1)), n_{l+1}^{-1}(k_{l+1} + \gamma_{n_{l+1}} + \varepsilon_{n_{l+1}}/u_{l+1})] \tag{13}$$

contained in I_l, with $k_{l+1} \in \mathbb{Z}$ and $n_{l+1} > n_l$ in $\mathcal{N}$. To this end, we will apply Lemma 1, with $a = a_l$ being the left endpoint of I_l, $b = b_l$ being the right endpoint of I_l, and the smallest integer $n = n_{l+1} > n_l$ in $\mathcal{N}$ satisfying

$$n_{l+1} \geq \frac{13}{10(b_l - a_l)}. \tag{14}$$

As above, applying Lemma 1 to

$$u = \gamma_{n_{l+1}} + \frac{\varepsilon_{n_{l+1}}}{5} \quad \text{and} \quad v = \gamma_{n_{l+1}} + \frac{\varepsilon_{n_{l+1}}}{2},$$

due to $0 < v - u \le 3/10$, we can choose an appropriate integer k_{l+1} by (9), namely,

$$k_{l+1} = \lceil n_{l+1} a_l - \gamma_{n_{l+1}} - \varepsilon_{n_{l+1}}/5 \rceil. \tag{15}$$

Then, by Lemma 1, both endpoints of the interval

$$J_{l+1} = [n_{l+1}^{-1}(k_{l+1} + \gamma_{n_{l+1}} + \varepsilon_{n_{l+1}}/5), n_{l+1}^{-1}(k_{l+1} + \gamma_{n_{l+1}} + \varepsilon_{n_{l+1}}/2)]$$

belong to I_l. Consequently, the subinterval I_{l+1} of J_{l+1}, which we defined in (13), satisfies $I_{l+1} \subset J_{l+1} \subseteq I_l$.

By this construction, since the length of I_j, namely, $\varepsilon_{n_j}/(u_j(u_j+1)n_j)$, tends to zero as $j \to \infty$, the unique point of the intersection $\cap_{j=1}^{\infty} I_j$ is the required positive real number $\beta(\mathbf{u})$. (It is clear that $\beta(\mathbf{u}) \in I_1 \subseteq I_0 = [a, b]$.) $\square$

We now show that the numbers $\beta(\mathbf{u})$ and $\beta(\mathbf{u}')$ are distinct for distinct vectors

$$(u_1, u_2, u_3, \dots) \quad \text{and} \quad (u_1', u_2', u_3', \dots).$$

Indeed, let ℓ be the smallest positive integer for which $u_\ell \neq u_\ell'$. Without restriction of generality, we may assume that $u_\ell = 2$ and $u_\ell' = 4$. Since $(u_1, \dots, u_{\ell-1}) = (u_1', \dots, u_{\ell-1}')$, the intervals I_j and I_j' constructed in (12) are the same for $j = 1, 2, \dots, \ell - 1$. Furthermore, the integers n_ℓ and k_ℓ are also the same. (In view of (14) and (15), they do not depend on u_ℓ.) Therefore, by (12) and $(u_\ell, u_\ell') = (2, 4)$, we find that

$$I_\ell = [n_\ell^{-1}(k_\ell + \gamma_{n_\ell} + \varepsilon_{n_\ell}/3), n_\ell^{-1}(k_\ell + \gamma_{n_\ell} + \varepsilon_{n_\ell}/2)]$$

and

$$I_\ell' = [n_\ell^{-1}(k_\ell + \gamma_{n_\ell} + \varepsilon_{n_\ell}/5), n_\ell^{-1}(k_\ell + \gamma_{n_\ell} + \varepsilon_{n_\ell}/4)].$$

Note that the intervals I_ℓ and I_ℓ' are disjoint. Since $\beta(\mathbf{u}) \in I_\ell$ and $\beta(\mathbf{u}') \in I_\ell'$, the numbers $\beta(\mathbf{u})$ and $\beta(\mathbf{u}')$ are distinct. In fact, we always have the inequality $\beta(\mathbf{u}') < \beta(\mathbf{u})$ if the vector $\mathbf{u}$ is lexicographically smaller than the vector $\mathbf{u}'$.

Clearly, there is a continuum of such distinct sequences $\mathbf{u}$ when $\mathbf{u}$ runs over all possible infinite sequences consisting of 2 and 4. As we have shown above, the numbers $\beta(\mathbf{u})$ are all distinct, so there are continuum of numbers $\beta(\mathbf{u})$. This completes the proof of Proposition 1, because in (10) we have $0 < \varepsilon_{n_j}/(u_j + 1)$ and $\varepsilon_{n_j}/u_j < \varepsilon_{n_j}$, so (7) is true for each $n = n_j$.

3. Proof of Theorem 2

Fix any $\eta > 1$ and an interval $I = [a, b]$, where $1 \le a < b$. Note that, without restriction of generality, we may assume that $a > 1$, because in case $a = 1$, one can consider the subinterval $[(a+b)/2, b]$ of I instead of I itself.

In order to apply Proposition 1, we will consider the sequence of positive numbers $\varepsilon_1 = 1$,

$$\varepsilon_n = \min\left(1, \frac{1}{\Phi_2 2! \xi_2}, \frac{1}{\Phi_3 3! \xi_3}, \dots, \frac{1}{\Phi_n n! \xi_n}\right)$$

for each $n \ge 2$. Then, $1 = \varepsilon_1 \ge \varepsilon_2 \ge \varepsilon_3 \ge \dots$ is a sequence of positive numbers. Let also

$$\gamma_n = -\frac{\log \xi_n}{\log \eta} \tag{16}$$

for each $n \in \mathbb{N}$.

Fix any ϵ in the interval $(0, \log a)$. Then, by (3), there is an infinite sequence $\mathcal{N} \subset \mathbb{N}$ such that, for each $n \in \mathcal{N}$, we have

$$\xi_n \geq e^{-\epsilon n}. \tag{17}$$

Furthermore, by (17), for $\alpha \geq a$, we have

$$\xi_n \alpha^n \to \infty \quad \text{as} \quad n \in \mathcal{N} \quad \text{tends to infinity.} \tag{18}$$

Now, by Proposition 1 applied to the interval

$$J = [\log a / \log \eta, \log b / \log \eta] \tag{19}$$

and (16), there is an uncountable in J set of positive numbers B such that, for each $\beta \in B$, the inequalities

$$0 < n\beta + \frac{\log \xi_n}{\log \eta} - m < \varepsilon_n \leq \frac{1}{\Phi_n n! \xi_n}$$

hold for infinitely many pairs (n, m), where $n \in \mathcal{N}$ and $m \in \mathbb{Z}$. Multiplying all this by $\log \eta > 0$, we derive that the inequalities

$$0 < n\beta \log \eta + \log \xi_n - m \log \eta < \frac{1}{\Phi_n (n-1)! \xi_n} \tag{20}$$

hold for infinitely many pairs (n, m), where $n \in \mathcal{N}$ and $m \in \mathbb{Z}$. Note that, by (17), we have

$$\Phi_n (n-1)! \xi_n \to \infty \quad \text{as} \quad n \in \mathcal{N} \quad \text{tends to infinity.} \tag{21}$$

Let S be the set of numbers of the form $\alpha = \eta^\beta$, where β runs over every element of B. Note that the map $x \mapsto \eta^x$ maps the interval J defined in (19) into the interval $[a, b]$. Therefore, the set S is a subset of $[a, b]$. Moreover, the set S is uncountable because so is the set B.

Consider the difference

$$\xi_n \alpha^n - \eta^m = e^{n\beta \log \eta + \log \xi_n} - e^{m \log \eta} = \eta^m (e^{n\beta \log \eta + \log \xi_n - m \log \eta} - 1).$$

By (20), the exponent here is in the interval $(0, 1/(\Phi_n (n-1)! \xi_n))$. Additionally, $\eta^m < \xi_n \alpha^n$ by (20) as well. Consequently, from (21), it follows that

$$0 < \xi_n \alpha^n - \eta^m = \eta^m (e^{n\beta \log \eta + \log \xi_n - m \log \eta} - 1) < \xi_n \alpha^n \cdot \frac{2}{\Phi_n (n-1)! \xi_n} = \frac{2\alpha^n}{\Phi_n (n-1)!} < \frac{1}{\Phi_n}$$

for a sufficient large $n \in \mathcal{N}$. Here, in view of (18), for each sufficiently large $n \in \mathcal{N}$, the corresponding integer m must be positive. This completes the proof of (6).

4. A Different Version of the Main Result

Note that, in the proof of Theorem 2, we did not use (3) but rather the condition (18), with a subsequence $\mathcal{N}$. Therefore, we can change the initial condition (3) for the sequence ξ_n, $n = 1, 2, 3, \ldots$, by the condition

$$\xi_n a^n \to \infty \quad \text{as} \quad n \in \mathcal{N} \quad \text{tends to infinity,} \tag{22}$$

where $\mathcal{N}$ is an arbitrary infinite sequence of positive integers and $a > 1$ is a fixed number. Observe that (22) is true for any infinite sequence $\mathcal{N} \subseteq \mathbb{N}$ and any $a > 1$ if, say, $\xi_n \geq \xi > 0$ for every $n \in \mathbb{N}$.

Then, by the argument given in Section 3, we obtain the following version of Theorem 1:

Theorem 3. *Let $I = [a, b]$ be an interval with $1 < a < b$. Assume that $\mathcal{N}$ is an infinite sequence of positive integers, and $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \dots\}$ is a sequence of positive numbers satisfying (22) with this $\mathcal{N}$. Then, for any sequence of positive numbers $\boldsymbol{\delta} = \{\delta_1, \delta_2, \delta_3, \dots\}$, there is an uncountable set $S(I, \mathcal{N}, \boldsymbol{\xi}, \boldsymbol{\delta}) \subset I$ such that, for each $\alpha \in S(I, \mathcal{N}, \boldsymbol{\xi}, \boldsymbol{\delta})$, the inequalities*

$$0 < \{\xi_n \alpha^n\} < \delta_n \tag{23}$$

hold for infinitely many $n \in \mathcal{N}$.

We omit the proof, since it is exactly the same as that above.

5. An Application of Proposition 1

Recently, in [31], we studied the following problem. Given $\theta \in \mathbb{R}_{>0} \setminus \mathbb{N}$, let $R_\theta(N)$ be the least nonzero value of $\|a^\theta\|$ as $a = 1, 2, \dots, N$. Define

$$E_\theta = \limsup_{N \to \infty} \frac{\log(1/R_\theta(N))}{\log N}.$$

In Theorem 5 of [31], we provided several estimates for the quantity E_θ for some θ. For example, it was shown that $E_{2/3} \geq 1$, with the equality holding under assumption of the *abc*-conjecture.

Then Iyer [32] showed that $E(\theta)$ can be infinite for some $\theta \in \mathbb{R}_{>0} \setminus \mathbb{N}$. This follows from Theorem 1.9 of [32], where it was shown that, for any sequence of positive numbers δ_n, $n = 1, 2, 3, \dots$, there are many $\tau \in \mathbb{R}_{>0} \setminus \mathbb{N}$ for which the inequalities

$$0 < \|n^\tau\| < \delta_n$$

hold for infinitely many $n \in \mathbb{N}$. Indeed, selecting $\delta_n = 1/n!$ and all the corresponding numbers τ, we see that $E_\tau = \infty$ for each of those τ, because $\log(n!)/\log(n) \to \infty$ as $n \to \infty$.

We will derive the following more general result:

Theorem 4. *Let $I = [a, b]$ be an interval with $0 < a < b$. Assume that $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \dots\}$ is a sequence of positive numbers satisfying*

$$\xi_n n^a \to \infty \quad \text{as} \quad n \to \infty. \tag{24}$$

Then, for any sequence of positive numbers $\boldsymbol{\delta} = \{\delta_1, \delta_2, \delta_3, \dots\}$, there is an uncountable set $W(I, \boldsymbol{\xi}, \boldsymbol{\delta}) \subset I$ such that, for each $\tau \in W(I, \boldsymbol{\xi}, \boldsymbol{\delta})$, the inequalities

$$0 < \{\xi_n n^\tau\} < \delta_n \tag{25}$$

hold for infinitely many $n \in \mathbb{N}$.

Proof. Without restriction of generality, we may assume that $\delta_n < 1$ for each $n \in \mathbb{N}$. In all that follows, it will be shown that (25) holds for infinitely many powers of 2, namely, the inequalities

$$0 < \{\xi_{2^n} 2^{n\tau}\} < \delta_{2^n} \tag{26}$$

are true for infinitely many $n \in \mathbb{N}$.

To this end, we will apply Proposition 1 to $I = [a, b]$,

$$\gamma_n = -\frac{\log \zeta_{2^n}}{\log 2},$$

and the sequence of positive numbers ε_n, $n = 1, 2, 3, \ldots$, where $\varepsilon_1 = 1$ and

$$\varepsilon_n = \min\left(\varepsilon_1, \ldots, \varepsilon_{n-1}, \frac{\delta_{2^n}}{\zeta_{2^n} 2^{nb}}\right) \tag{27}$$

for $n = 2, 3, 4, \ldots$. It is clear that $1 = \varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \geq \ldots$.

By Proposition 1, it follows that there is an uncountable in I set of positive numbers W such that, for each $\tau \in W$, the inequalities

$$0 < n\tau + \frac{\log \xi_{2^n}}{\log 2} - m < \varepsilon_n$$

hold for infinitely many pairs (n, m), where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Multiplying by $\log 2$, we obtain that the inequalities

$$0 < n\tau \log 2 + \log \xi_{2^n} - m \log 2 < \varepsilon_n \log 2 \tag{28}$$

hold for infinitely many pairs (n, m), where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Now, we consider the difference

$$\xi_{2^n} 2^{n\tau} - 2^m = e^{n\tau \log 2 + \log \xi_{2^n}} - e^{m \log 2} = 2^m (e^{n\tau \log 2 + \log \xi_{2^n} - m \log 2} - 1).$$

By (28), the exponent here is in the interval $(0, \varepsilon_n \log 2)$. Additionally, $2^m < \xi_{2^n} 2^{n\tau}$ by (28), and $\xi^{2^n} 2^{nb} \varepsilon_n \leq \delta_{2^n}$ by (27). Therefore,

$$0 < \xi_{2^n} 2^{n\tau} - 2^m = 2^m (e^{n\tau \log 2 + \log \xi_{2^n} - m \log 2} - 1) < \xi_{2^n} 2^{n\tau} \cdot (2\varepsilon_n \log 2) < \xi_{2^n} 2^{nb} \varepsilon_n \leq \delta_{2^n}$$

for each of those $n \in \mathbb{N}$.

Here, we have

$$\zeta_{2^n} 2^{n\tau} \to \infty \quad \text{as} \quad n \to \infty,$$

because

$$\zeta_{2^n} 2^{na} \to \infty \quad \text{as} \quad n \to \infty.$$

Consequently, as

$$2^m > \xi_{2^n} 2^{n\tau} - \delta_{2^n} > \xi_{2^n} 2^{n\tau} - 1,$$

the corresponding integer m must be positive for each sufficiently large $n \in \mathbb{N}$, and hence $2^m \in \mathbb{Z}$.

Combined with $0 < \delta_{2^n} < 1$, this implies that 2^m is the integer part of the number $\xi_{2^n} 2^{n\tau}$. Thus, for infinitely many $n \in \mathbb{N}$, we have

$$0 < \{\xi_{2^n} 2^{n\tau}\} = \xi_{2^n} 2^{n\tau} - 2^m < \delta_{2^n}$$

which is (26). This completes the proof of the theorem. $\square$

Note that we cannot omit the condition (24). Indeed, select, for instance, $\xi_n = \delta_n = 1/n!$. Then, for each $\tau > 0$, we have $0 < \xi_n n^\tau < 1$ for each sufficiently large $n \in \mathbb{N}$. Therefore, for each $\tau > 0$ and all sufficiently large $n \in \mathbb{N}$, we have

$$\{\xi_n n^\tau\} = \xi_n n^\tau > \xi_n = \frac{1}{n!} = \delta_n,$$

so (25) does not hold for $\tau > 0$.

6. Concluding Remarks

In particular, Theorem 3 implies that, for any $\xi > 0$ and any sequence of positive numbers δ_n, $n = 1, 2, 3, \ldots$, there are uncountably many $\alpha > 1$ for which the inequalities

$$0 < \{\xi\alpha^p\} < \delta_p$$

hold for infinitely many primes p, and uncountably many $\gamma > 1$ for which the inequalities

$$0 < \{\xi\gamma^{n^2}\} < \delta_{n^2}$$

hold for infinitely many $n \in \mathbb{N}$.

On the other hand, we do not know whether our method can be extended to conclude the same as stated in Theorem 3 with inequality (23) replaced by

$$0 < \{\xi_n\alpha^n - \eta_n\} < \delta_n,$$

where η_n, $n = 1, 2, 3, \ldots$, is an arbitrary sequence of real numbers. This problem is open even if $\xi_n = 1$ for $n \in \mathbb{N}$. More precisely, we do not know whether for any sequence of real numbers η_n, $n = 1, 2, 3, \ldots$, and any sequence of positive numbers δ_n, $n = 1, 2, 3, \ldots$, there is a real number $\alpha > 1$ for which we have

$$0 < \{\alpha^n - \eta_n\} < \delta_n$$

for infinitely many $n \in \mathbb{N}$.

Similarly, with respect to Theorem 4, we may ask whether for any sequence of real numbers η_n, $n = 1, 2, 3, \ldots$, and any sequence of positive numbers δ_n, $n = 1, 2, 3, \ldots$, there is a real number $\tau > 0$ for which the inequalities

$$0 < \{n^\tau - \eta_n\} < \delta_n$$

hold for infinitely many $n \in \mathbb{N}$.

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