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Vacuum Stability Conditions for New $SU(2)$ Multiplets

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We consider the addition to the Standard Model of a scalar $SU(2)$ multiplet Δ_n with dimension n going from 1 to 6. The multiplet Δ_n is assumed to have null vacuum expectation value and an arbitrary (free) hypercharge. We determine the shape of the phase space for the new terms that appear in the scalar potential; we observe in particular that, in the case of a 6-plet, the phase space is slightly concave along one of its boundaries. We determine the bounded-from-below and vacuum stability conditions on the scalar potential for each value of n.

Subject Index B40, B46, B53, B57

1. Introduction

The Higgs particle was discovered at the Large Hadron Collider in 2012 [1,2]. Since then, exploration of the interactions of that particle has shown that they are quite close to the predictions of the Standard Model (SM) [3]. This either confirms that the breaking of the gauge symmetry of the SM and the generation of the fermion masses are effected solely by a single scalar doublet of $SU(2)$, or else it suggests the presence of an "alignment" mechanism [4,5] that allows a more complex scalar sector—such as in the two-Higgs-doublet model (2HDM) [6]—to mimic the SM predictions—despite the absence of a symmetry enforcing such alignment. Moreover, the almost-exact prediction of the SM

$$m_W = c_w m_Z \tag{1}$$

—where m_W and m_Z are the masses of the gauge bosons $W^\pm$ and Z^0, respectively, and c_w is the cosine of the weak mixing angle—strongly suggests that only $SU(2)$ doublets, and possibly also singlets, have vacuum expectation values (VEVs) [7]. So, the scalar sector of any extension of the SM is currently already rather strongly constrained. On the other hand, there is no reason—but for Occam's razor—why the scalar sector of a spontaneously broken $SU(2) \times U(1)$ gauge theory should consist of only one $SU(2)$ doublet. One can entertain the speculation that larger $SU(2)$ multiplets exist—even if they have zero VEVs because of Eq. (1), and even if they are so large that they do not couple to the known fermions. In particular, large extra $SU(2)$ multiplets

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may be useful, even if they have no VEVs, to alter Eq. (1) through the radiative ("oblique") corrections that they produce, if that Eq. (1) is observed to be slightly off the mark [8].

In this paper we consider the possibility that the scalar sector of an $SU(2) \times U(1)$ gauge theory consists of one hypercharge-$1/2$ $SU(2)$ doublet Φ, which has a VEV, and another $SU(2)$ multiplet Δ_n, with weak isospin J and $n = 2J + 1$ components, that has null VEV and a free (arbitrary) hypercharge. The latter assumption means that the theory has a global $U(1)$ symmetry $\Delta_n \to \Delta_n \exp(i\vartheta)$; since Δ_n has zero VEV, that symmetry remains unbroken, and no Goldstone boson arises from it.[1]

A difficult problem that one faces when one considers any extension of the scalar sector of the SM is to find the bounded-from-below (BFB) conditions[2] on the scalar potential (SP) V. This is a mathematical problem that is very easy to state—what are the conditions on the coupling constants of the quartic part V_4 of the SP,[3] such that V_4 can be negative for *no* configuration of the scalar fields[4]—but surprisingly difficult to solve even for modest extensions of the SM. The problem has only been solved for the two-Higgs-doublet model [11,12], for some constrained forms of the three-Higgs-doublet model [13–17], for a few models with $SU(2)$ triplets [18–20], and for a few other rather simple models [21–23]. Additionally, there are in the literature BFB conditions for the $SU(3) \times U(1)$ electroweak model [24] and for models with colored scalars [25].

There are papers on general methods for deriving BFB conditions [26–28]. Among other methods, geometric and group-theoretic approaches may sometimes be utilized. These include analyzing the potential on the *orbit space* [29–34], applying *stratification theory* (the classification of extrema by symmetry patterns) [14,35–37], and using *boundary conditions and convexity (copositivity) criteria* [25–27,38]. The orbit space approach seeks to find a coordinate system tailored to the symmetries of the potential, often transforming the problem of the minimization of V_4 into one of analyzing geometric shapes (cones, polyhedra, etc.) within that space. Due to its versatility, many modern analyses of multiscalar potentials—including those using the P-matrix formalism [21,22]—either explicitly or implicitly utilize orbit-space reasoning.

In this paper, we analyze the gauge orbit space (which we call the *phase space*) to derive BFB constraints for two $SU(2)$ scalar multiplets Φ and Δ_n. When the potential is linear in the phase space variables, it is sufficient to consider the convex hull of the orbit space to determine the BFB conditions [25]—concave stretches of boundary do not matter. We note, however, that for two multiplets the potential has the form in Eq. (26) below, and then one needs, at least in principle,[5] to take into account the concave stretches of the boundary in detail.

One further aim of this paper is to fill a gap in our understanding of vacuum stability in multiscalar models, at least partially. (We term as "vacuum stability conditions" the conditions for

[1] Two other recent papers that consider the addition to the SM of $SU(2)$ scalar multiplets with dimension up to $n = 5$ are Refs. [9,10]. Note, however, that in those papers, those multiplets are supposed to acquire VEVs, contrary to what we assume here.

[2] Some people call "vacuum stability conditions" what we call "BFB conditions." We reserve the former term for the conditions stemming from another rationale (see below).

[3] We assume that the SP is renormalizable.

[4] If V_4 is negative for some values of the scalar fields, then by multiplying all the fields by an ever larger positive real number κ one makes $V_4 \to \kappa^4 V_4$, which is ever more negative, and this means that V is not BFB and therefore has no minimum, that is, the theory lacks vacuum state.

[5] We do not do that explicitly in the case $n = 6$, because the concavity existing in that case is extremely slight.

the desired minimum of the potential to be its absolute *global* minimum, and not just a local minimum. Note that, contrary to the BFB conditions, which only affect the quartic part V_4 of V, the vacuum stability conditions affect the whole V including its quadratic part.) Building on the results of Refs. [39,43], we derive analytical vacuum stability constraints for the scalar potentials of SM extensions through scalar multiplets up to $n = 6$. The detailed expressions and explanations provided in the paper may be useful for readers attempting to apply the methodology to other models.

The plan of this paper is as follows. After writing down in Section 2 the most general renormalizable SP for our model, we delimit in Section 3 the extent of the freedom of that SP—which is smaller than its many scalar fields suggest, because the SP is renormalizable, that is, it contains no higher powers of the fields than four. That allows us to derive, in Section 4, necessary and sufficient (n&s) conditions for V_4 to be BFB when $n \leq 5$—and almost n&s conditions when $n = 6$. In Section 5 we consider the vacuum stability conditions and in Section 6 we summarize our achievements. Appendix A collects useful results from a previous paper by two of us [39], Appendix B considers *Ansätze* for the fields in the cases $n = 5$ and $n = 6$, and Appendix C solves a technical mathematical problem that often arises in the main body of the paper.

2. The scalar potential

We write the Higgs doublet of the SM as

$$\Phi = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{2}$$

where a and b are complex scalar fields. Then,

$$\tilde{\Phi} = \begin{pmatrix} b^* \\ -a^* \end{pmatrix} \tag{3}$$

is also an $SU(2)$ doublet. Let I_z be the third component of isospin of the generic field z. One has $I_a = -I_b = 1/2$. The SP of the SM is

$$V_{\mathrm{SM}} = \mu_1^2 F_1 + \frac{\lambda_1}{2} F_1^2, \tag{4}$$

where

$$F_1 = |a|^2 + |b|^2 \equiv A + B \tag{5}$$

is $SU(2)$-invariant. In general, we denote the squared modulus of the generic field z by the corresponding capital letter Z, that is,$Z \equiv |z|^2$.

In the models that we consider in this paper there is just one scalar $SU(2)$ multiplet beyond Φ; we call that extra multiplet Δ_n, where the integer $n = 1, \ldots, 6$ denotes the dimension of the irreducible representation of $SU(2)$ embodied by Δ_n. Of course $n = 2J + 1$, where J is the weak isospin of Δ_n. If $n \geq 3$ and an electrically neutral component of Δ_n has a VEV, then the gauge-boson masses do not obey Eq. (1). Indeed [7],

$$m_W^2 = g^2 \sum_{\varphi_{JY}} |v_{JY}|^2 \left(J^2 - Y^2 + J\right), \tag{6a}$$

$$m_Z^2 = \frac{g^2}{\cos^2\theta_w} \sum_{\varphi_{JY}} |v_{JY}|^2 \left(2Y^2\right), \tag{6b}$$

where the sum is performed over all neutral fields φ_{JY} with isospin J, hypercharge Y, and VEV v_{JY}. In Eq. (6a,b), g is the $SU(2)$ gauge coupling constant.

In order to keep the relation (1) valid, we assume that all the neutral fields except b have null VEV. Partly in order to guarantee this, we also assume that the models enjoy $U(1)$ symmetries $\Delta_n \to \exp(i\vartheta)\,\Delta_n$; those symmetries prevent couplings either of the form $\Phi^2\Delta_n$ or of the form $\Phi^3\Delta_n$, that might induce a VEV for a component of Δ_n.[6] With this additional symmetry, the SPs become simpler and are given by a general form explained in Ref. [39] and given in detail in Appendix A. The most important takeaway is that

$$\text{when } n = 1,\ \ V = V_{\text{SM}} + \mu_2^2 F_2 + \frac{\lambda_2}{2} F_2^2 + \lambda_3 F_1 F_2; \tag{7a}$$

$$\text{when } n = 2,\ \ V = V_{\text{SM}} + \mu_2^2 F_2 + \frac{\lambda_2}{2} F_2^2 + \lambda_3 F_1 F_2 + \lambda_4 F_4; \tag{7b}$$

$$\text{when either } n = 3 \text{ or } n = 4,\ \ V = V_{\text{SM}} + \mu_2^2 F_2 + \frac{\lambda_2}{2} F_2^2 + \lambda_3 F_1 F_2 + \lambda_4 F_4 + \lambda_5 F_5; \tag{7c}$$

$$\text{when either } n = 5 \text{ or } n = 6,\ \ V = V_{\text{SM}} + \mu_2^2 F_2 + \frac{\lambda_2}{2} F_2^2 + \lambda_3 F_1 F_2 + \sum_{k=4}^{6} \lambda_k F_k. \tag{7d}$$

In Eq. (7a–d), the F_k are $SU(2)$-invariant polynomials of the fields of Φ and of Δ_n; their functional forms depend on the dimension of Δ_n.

3. Phase spaces

The vacuum structure of a scalar potential may be analyzed geometrically by studying the space spanned by the invariants of the theory, which we call the *phase space*[7]. In particular, the conditions for boundedness from below and vacuum stability may be derived from the shape of the phase space. However, for arbitrary field configurations, the invariants F_i may become unbounded, leading to a loss of information about the boundary of the phase space. In order to avoid this, we introduce the following set of dimensionless $SU(2)$-invariants[8]

$$r \equiv \frac{F_1}{F_2}, \qquad \gamma_5 \equiv \frac{F_5}{F_2^2}, \qquad \gamma_6 \equiv \frac{F_6}{F_2^2}, \qquad \delta \equiv \frac{F_4}{F_1 F_2}, \tag{9}$$

enabling us to rewrite the SP as

$$V = \mu_1^2 F_1 + \mu_2^2 F_2 + \left[\frac{\lambda_1}{2} r^2 + \Pi(\delta)\, r + \frac{\Xi(\gamma_5, \gamma_6)}{2}\right] F_2^2, \tag{10}$$

where

$$\Pi(\delta) \equiv \lambda_3 + \lambda_4\, \delta, \tag{11a}$$

$$\Xi(\gamma_5, \gamma_6) \equiv \lambda_2 + 2\lambda_5\gamma_5 + 2\lambda_6\gamma_6. \tag{11b}$$

In the following subsections, we derive the shape of the space spanned by the $SU(2)$-invariants γ_5, γ_6, and δ. We will refer to this space as *the* phase space, even though, in fact, it is just a subspace of the true phase space that also includes the unbounded invariants F_1 and

[6] Such couplings would anyway be forbidden by the $SU(2)$ gauge symmetry, together with renormalizability, for $n > 4$.

[7] Other authors call it the "orbit space" [29–34].

[8] We employ the same notation as Ref. [39], except that our δ is rescaled by a factor of $J/2$. Therefore, when comparing the results between the two papers one should take into account that

$$\delta_{\text{here}} = \frac{J}{2}\, \delta_{\text{Ref.[39]}}. \tag{8}$$

F_2. Since $\delta = \gamma_5 = \gamma_6 = 0$ for the case $n = 1$, its corresponding phase space is zero-dimensional, and we start at $n = 2$.

3.1. $n = 2$

In the case $n = 2$, that is, $J = 1/2$, both γ_5 and γ_6 are zero, hence the phase space is spanned by a single dimensionless parameter δ. From its definition in Eq. (9) we find

$$\delta = \frac{1}{4} - \frac{|ad - bc|^2}{2\,(A+B)\,(C+D)} \tag{12a}$$

$$= -\frac{1}{4} + \frac{|ac^* + bd^*|^2}{2\,(A+B)\,(C+D)}. \tag{12b}$$

Since the second terms in the right-hand sides of Eqs. (12a) and (12b) are non-negative,

$$|\delta| \le \frac{1}{4}. \tag{13}$$

3.2. $n = 3$

In the case $n = 3$, that is, $J = 1$, the phase space has two dimensionless parameters δ and γ_5. One may show that [39]

$$1 - 4\delta^2 - 3\gamma_5 \ge 0, \tag{14}$$

which simultaneously bounds δ and γ_5; the latter moreover is, by definition, non-negative [39]. Therefore,

$$0 \ \le \ |\delta| \le \frac{\sqrt{1 - 3\gamma_5}}{2}, \tag{15a}$$

$$0 \ \le \ \gamma_5 \le \frac{1}{3}. \tag{15b}$$

The parameter γ_5 is a function of the three fields c, d, and e through

$$\gamma_5 = \frac{\left|2ce - d^2\right|^2}{3\,(C+D+E)^2}. \tag{16}$$

The upper bound (15b) on γ_5 is saturated, for instance, if $c = e = 0$, or else if $d = 0$ and $c = e$. This case was first rigorously treated in Ref. [19], after an initial attempt in Ref. [18].

3.3. $n = 4$

In the case $n = 4$, that is,$J = 3/2$, the phase space once again has two parameters δ and γ_5. One may show that they are simultaneously bounded by the condition [39]

$$9 - 16\,\delta^2 - 20\gamma_5 \ge 0, \tag{17}$$

which substitutes Eq. (14) of case $n = 3$. This means that now

$$0 \ \le \ |\delta| \le \frac{\sqrt{9 - 20\gamma_5}}{4}, \tag{18a}$$

$$0 \ \le \ \gamma_5 \le \frac{9}{20}, \tag{18b}$$

instead of Eq. (15a,b). The parameter γ_5 is a function of the four fields $c, \ldots, f$ through

$$\gamma_5 = \frac{2|\sqrt{3}\,ce - d^2|^2 + 2|\sqrt{3}\,df - e^2|^2 + |3cf - de|^2}{5(C+D+E+F)^2}. \tag{19}$$

The upper bound (18b) on γ_5 is saturated, for instance, when $d = e = 0$ and $c = f$.

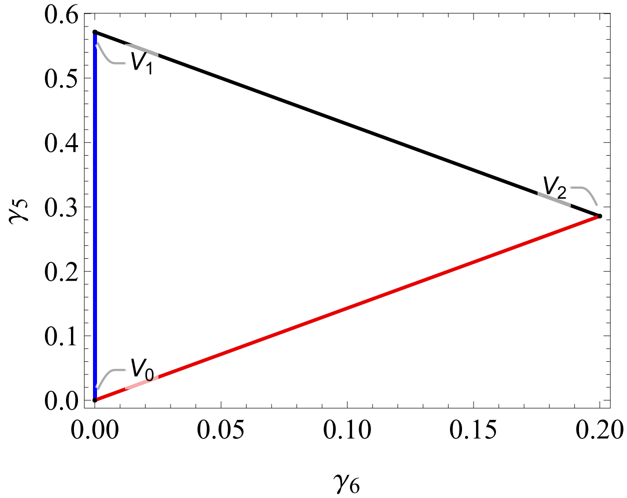

Fig. 1. Boundaries of the phase space for the case $n = 5$, plotted in the (γ_6, γ_5) plane. The blue straight line is given by Eq. (B1b), the red straight line is given by Eq. (B3), and the black straight line is given by Eq. (B6). The vertices are given by Eq. (22a–c).

3.4. $n = 5$

In the case $n = 5$, that is,$J = 2$, the phase space has three dimensionless parameters δ, γ_5, and γ_6. The condition [39]

$$4 - 4\delta^2 - 7\gamma_5 - 10\gamma_6 \geq 0 \tag{20}$$

simultaneously bounds δ, γ_5, and γ_6. The upper bound on the magnitude of δ now reads

$$|\delta| \leq \frac{\sqrt{4 - 7\gamma_5 - 10\gamma_6}}{2}. \tag{21}$$

The invariants γ_5 and γ_6 may be written in terms of F_2, F_5, and F_6, which in turn are given in terms of the five fields $c, \ldots g$ by Eqs. (A16a–f). Since F_2, F_5, and F_6 are non-negative, γ_5 and γ_6 are non-negative too. In order to gain a grasp on how large γ_5 and γ_6 may be, we have considered in section B.1 of Appendix B three *Ansätze* for $c, \ldots, g$. With these three *Ansätze* we have constructed a triangle in the (γ_6, γ_5) plane, which we depict in Fig. 1. The sides of that triangle are given by Eqs. (B1b), (B3), and (B6), and are plotted in blue, red, and black, respectively. The vertices of the triangle are

$$V_0 = (0,\ 0)\,, \tag{22a}$$

$$V_1 = \left(0,\ \frac{4}{7}\right), \tag{22b}$$

$$V_2 = \left(\frac{1}{5},\ \frac{2}{7}\right). \tag{22c}$$

Numerically generating random complex values for the five fields $c, \ldots, g$ and therefrom computing γ_5 and γ_6 by means of Eqs. (9) and (A16), one finds that all the (γ_6, γ_5) thus obtained are inside the above-mentioned triangle. So, the range of$\boldsymbol{(\gamma_6, \gamma_5)}$is the triangle with vertices$\boldsymbol{V_0}$**,** $\boldsymbol{V_1}$, and$\boldsymbol{V_2}$in Eq. (22a–c).

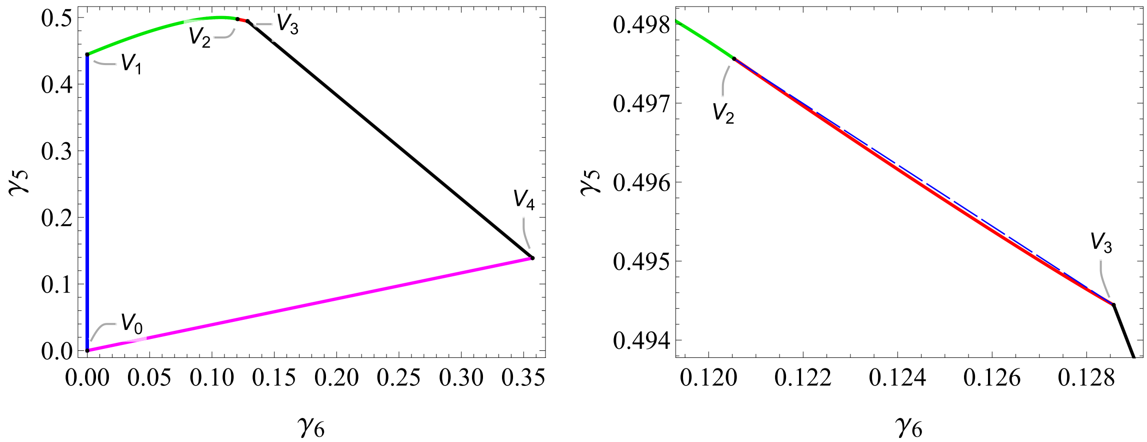

Fig. 2. Left: Boundaries of the phase space for the case $n = 6$, plotted in the (γ_6, γ_5) plane. The points $V_0, \ldots, V_4$ are given by Eq. (25a–e). The blue straight line is given by Eq. (B7b); the green curve is given by Eq. (B8b); the red curve is given in parametric form by Eqs. (B14); the black straight line is given by Eq. (B17b); the magenta straight line is given by Eq. (B19). Right: zoomed-in view of the region of points V_2 and V_3. The dashed blue straight line is given by Eq. (B16).

3.5. $n = 6$

In the case $n = 6$, that is, $J = 5/2$, the phase space once again has parameters δ, γ_5, and γ_6. One may show that these $SU(2)$-invariants are simultaneously bounded by the condition [39]

$$25 - 16\delta^2 - 36\gamma_5 - 56\gamma_6 \geq 0, \tag{23}$$

instead of by inequality (20). In turn, the upper bound on the magnitude of δ now reads

$$|\delta| \leq \frac{\sqrt{25 - 36\gamma_5 - 56\gamma_6}}{4} \tag{24}$$

instead of inequality (21).

The invariants γ_5 and γ_6 may be written in terms of the six fields $c, \ldots h$ through Eqs. (9) and (A18); they are non-negative because F_2, F_5, and F_6 are also non-negative. In order to find out the range spanned by γ_5 and γ_6, we have first considered five *Ansätze* for the fields, given in detail in section B.2 of Appendix B. With these five *Ansätze* we have formed the boundary of a domain in the (γ_6, γ_5) plane. That domain is depicted in the left panel of Fig. 2 and has five vertices[9]:

$$V_0 = (0,\ 0)\,; \tag{25a}$$

$$V_1 = \left(0,\ \frac{4}{9}\right); \tag{25b}$$

$$V_2 = (\varrho,\ \varsigma)\,; \tag{25c}$$

$$V_3 = \left(\frac{9}{70},\ \frac{89}{180}\right); \tag{25d}$$

$$V_4 = \left(\frac{5}{14},\ \frac{5}{36}\right). \tag{25e}$$

[9]Vertex V_2 is not really a vertex; it is just the point where the curves (B8b) and (B14) meet. At that point, the two curves have the same slopes but different second derivatives. Indeed, the curve (B8b) is a *convex* boundary of the (γ_6, γ_5) domain, while the curve (B14) is a *concave* boundary.

To better illustrate the slight concavity of the boundary connecting vertices V_2 and V_3, the right panel of Fig. 2 shows a zoomed-in view of this region. The blue dashed line represents the approximation of Eq. (B14a,b) by the straight line in Eq. (B16).

Numerically generating random complex values for the six fields $c, \ldots, h$ and therefrom computing γ_5 and γ_6 by means of Eqs. (A18) and (9), one finds that all the points (γ_6, γ_5) thus generated are inside the above-mentioned domain, and indeed fill it completely. So, the range of (γ_6, γ_5) is the area bounded by the lines (B7b), (B8b), (B14), (B17b), and (B19). This area is very slightly *concave* at one of its borders, namely, at the line (B14).[10]

4. Boundedness from below

The SP in Eq. (10) is BFB if its quartic part (V_4) is non-negative for every possible field configuration, that is for every point in phase space. Following the method outlined in Refs. [26,39], we rewrite the quartic part of the SP as

$$\frac{V_4}{F_2^2} = \frac{1}{2}\,(r, 1) \begin{pmatrix} \lambda_1 & \Pi(\delta) \\ \Pi(\delta) & \Xi(\gamma_5, \gamma_6) \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix}, \tag{26}$$

where $\Pi(\delta)$ and $\Xi(\gamma_5, \gamma_6)$ have been defined in Eq. (11a,b). Since r is strictly positive, the SP in Eq. (10) is BFB if and only if the 2×2 matrix in Eq. (26) is copositive [26]. Therefore, we must ensure that the following conditions hold for every δ, γ_5, and γ_6:

$$\lambda_1 \geq 0, \tag{27a}$$

$$\Xi(\gamma_5, \gamma_6) \geq 0, \tag{27b}$$

$$\widetilde{\Gamma}(\delta, \gamma_5, \gamma_6) \equiv \Pi(\delta) + \sqrt{\lambda_1\, \Xi(\gamma_5, \gamma_6)} \geq 0. \tag{27c}$$

As stated in Ref. [19], it is *necessary* that $\Xi(\gamma_5, \gamma_6)$ and $\widetilde{\Gamma}(\delta, \gamma_5, \gamma_6)$ are non-negative everywhere on the boundary of the phase space for conditions (27) to hold. In addition, if the absolute minima of those functions lie inside the phase space, then one must require them to be non-negative. Regarding $\Xi(\gamma_5, \gamma_6)$ given in Eq. (11b), we note that it is monotonic in both γ_5 and γ_6, so its minimum necessarily lies at the boundary of the phase space. On the other hand, $\widetilde{\Gamma}(\delta, \gamma_5, \gamma_6)$ is monotonic in δ, hence its minimum is attained at the boundary of the phase space in the δ direction. We affect the minimization with respect to δ firstly and fix

$$\delta = -\frac{\lambda_4}{|\lambda_4|}\, \Sigma(\gamma_5, \gamma_6), \tag{28}$$

where, by using Eqs. (13), (15a), (18a), (21), and (24), we have

$$\Sigma(\gamma_5, \gamma_6) = 0 \quad \text{when } n = 1, \tag{29a}$$

$$\Sigma(\gamma_5, \gamma_6) = \frac{1}{4} \quad \text{when } n = 2, \tag{29b}$$

$$\Sigma(\gamma_5, \gamma_6) = \frac{\sqrt{1 - 3\gamma_5}}{2} \quad \text{when } n = 3, \tag{29c}$$

$$\Sigma(\gamma_5, \gamma_6) = \frac{\sqrt{9 - 20\gamma_5}}{4} \quad \text{when } n = 4, \tag{29d}$$

[10] The possibility that the phase space is concave at some of its boundaries was already made clear in Fig. 2 of Ref. [25]. That was in a model with a four-dimensional phase space; here we find the same feature in a simpler model with a two-dimensional phase space.

$$\Sigma\left(\gamma_5, \gamma_6\right) = \frac{\sqrt{4 - 7\gamma_5 - 10\gamma_6}}{2} \quad \text{when } n = 5, \tag{29e}$$

$$\Sigma\left(\gamma_5, \gamma_6\right) = \frac{\sqrt{25 - 36\gamma_5 - 56\gamma_6}}{4} \quad \text{when } n = 6. \tag{29f}$$

Then, the functions that we ought to minimize are $\Xi\left(\gamma_5, \gamma_6\right)$ of Eq. (11b) and

$$\Gamma\left(\gamma_5, \gamma_6\right) = \lambda_3 - |\lambda_4|\ \Sigma\left(\gamma_5, \gamma_6\right) + \sqrt{\lambda_1\, \Xi\left(\gamma_5, \gamma_6\right)}. \tag{30}$$

In the following sections, we deduce the necessary and sufficient conditions for the SP to be BFB when $n = 1, \ldots, 6$. We note that the conditions presented in Ref. [39] for the cases $n = 1, \ldots, 4$ are necessary and sufficient, but those for the cases $n = 5$ and $n = 6$ are just *necessary*. We chose to write them for all n for the sake of completeness.

4.1. *The cases* $n = 1, \ldots, 4$

$n = 1$:

As previously noted, the phase space for the case $n = 1$ is zero-dimensional. Setting $\delta = \gamma_5 = \gamma_6 = 0$ in Eq. (27a–c), those BFB conditions become

$$\lambda_1 \geq 0, \tag{31a}$$

$$\lambda_2 \geq 0, \tag{31b}$$

$$\lambda_3 + \sqrt{\lambda_1 \lambda_2} \geq 0. \tag{31c}$$

These conditions are in agreement with those derived for the complex singlet extension of the SM, for instance in Ref. [40].

$n = 2$:

In the case $n = 2$ we have $\gamma_5 = \gamma_6 = 0$. Consequently,

$$\Xi\left(\gamma_5, \gamma_6\right) = \lambda_2, \quad \Gamma\left(\gamma_5, \gamma_6\right) = \lambda_3 - \frac{|\lambda_4|}{4} + \sqrt{\lambda_1 \lambda_2}. \tag{32}$$

Enforcing the conditions (27a–c), the BFB conditions read

$$\lambda_1 \geq 0, \tag{33a}$$

$$\lambda_2 \geq 0, \tag{33b}$$

$$\lambda_3 - \frac{|\lambda_4|}{4} + \sqrt{\lambda_1 \lambda_2} \geq 0. \tag{33c}$$

If one chooses to use the usual notation for the case of the $U(1)$-symmetric 2HDM, given in Eq. (A7), then conditions (33a–c) read

$$\bar{\lambda}_1 \geq 0, \tag{34a}$$

$$\bar{\lambda}_2 \geq 0, \tag{34b}$$

$$\bar{\lambda}_3 + \frac{\bar{\lambda}_4}{2} - \frac{\left|\bar{\lambda}_4\right|}{2} + \sqrt{\bar{\lambda}_1 \bar{\lambda}_2} \geq 0. \tag{34c}$$

Condition (34c) is equivalent to

$$\bar{\lambda}_3 + \sqrt{\bar{\lambda}_1 \bar{\lambda}_2} \geq 0, \tag{35a}$$

$$\bar{\lambda}_3 + \bar{\lambda}_4 + \sqrt{\bar{\lambda}_1 \bar{\lambda}_2} \geq 0. \tag{35b}$$

Therefore, our conditions (33a–c) are equivalent to

$$\bar{\lambda}_1 \geq 0, \tag{36a}$$

$$\bar{\lambda}_2 \geq 0, \tag{36b}$$

$$\bar{\lambda}_3 \geq -\sqrt{\bar{\lambda}_1 \bar{\lambda}_2}, \tag{36c}$$

$$\bar{\lambda}_3 + \bar{\lambda}_4 \geq -\sqrt{\bar{\lambda}_1 \bar{\lambda}_2}, \tag{36d}$$

which are the textbook BFB conditions for the $U(1)$-symmetric 2HDM [6,41].

$n = 3$: In the case $n = 3$ only γ_6 is 0, so we need to minimize the functions

$$\Xi\left(\gamma_5\right) = \lambda_2 + 2\lambda_5\gamma_5, \tag{37a}$$

$$\Gamma\left(\gamma_5\right) = \lambda_3 - \frac{|\lambda_4|}{2}\sqrt{1 - 3\gamma_5} + \sqrt{\lambda_1}\sqrt{\lambda_2 + 2\lambda_5\gamma_5}. \tag{37b}$$

As stated before, $\Xi\left(\gamma_5\right)$ is monotonic, so it is non-negative everywhere in phase space if it is non-negative at its end-points. Explicitly, we require

$$\begin{cases} \Xi(0) \geq 0, \\ \Xi\left(\dfrac{1}{3}\right) \geq 0, \end{cases} \Leftrightarrow \begin{cases} \lambda_2 \geq 0, \\ \lambda_2 + \dfrac{2}{3}\lambda_5 \geq 0. \end{cases} \tag{38}$$

Furthermore, $\Gamma\left(\gamma_5\right)$ is of the form of the function analyzed in Appendix C, with

$$k = \frac{1}{2}, \quad p = 1, \quad s = 3, \quad w = 2\lambda_5, \quad v = \lambda_2, \quad \alpha = 0, \quad \beta = \frac{1}{3}. \tag{39}$$

For it to be non-negative everywhere inside the γ_5 domain $[\alpha, \beta]$ it is necessary to require

$$\lambda_3 - \frac{|\lambda_4|}{2} + \sqrt{\lambda_1\lambda_2} \geq 0, \tag{40a}$$

$$\lambda_3 + \sqrt{\lambda_1\left(\lambda_2 + \frac{2}{3}\lambda_5\right)} \geq 0. \tag{40b}$$

Furthermore, assuming condition (38) to hold, it is sufficient to *exclude* the situation where

$$\lambda_5 < -\frac{3\lambda_4^2}{8\lambda_1}, \tag{41a}$$

$$\lambda_5 < -\frac{3}{4}\sqrt{\frac{\lambda_2}{\lambda_1}}\,|\lambda_4|\,, \tag{41b}$$

$$\lambda_3 < -\sqrt{\frac{\left(2\lambda_5 + 3\lambda_2\right)\left(8\lambda_5\lambda_1 + 3\lambda_4^2\right)}{24\lambda_5}}. \tag{41c}$$

Despite their differing appearance, the n&s conditions presented in Eqs. (38), (40), and (41) are Boolean-equivalent to those originally derived in Ref. [19] and later confirmed in Refs. [39,42]. We have confirmed this equivalence both numerically and algebraically.

$n = 4$: In the case $n = 4$, $\gamma_6 = 0$ and the functions that we need to minimize take the form

$$\Xi\left(\gamma_5\right) = \lambda_2 + 2\lambda_5\gamma_5, \tag{42a}$$

$$\Gamma(\gamma_5) = \lambda_3 - \frac{|\lambda_4|}{4}\sqrt{9 - 20\gamma_5} + \sqrt{\lambda_1}\sqrt{\lambda_2 + 2\lambda_5\gamma_5}. \tag{42b}$$

Again, $\Xi(\gamma_5)$ is monotonic in γ_5, implying that it is non-negative everywhere for $\gamma_5 \in [0, 9/20]$ if and only if it is non-negative both for $\gamma_5 = 0$ and for $\gamma_5 = 9/20$, that is

$$\lambda_2 \geq 0, \tag{43a}$$

$$\lambda_2 + \frac{9\lambda_5}{10} \geq 0. \tag{43b}$$

Furthermore, $\Gamma(\gamma_5)$ is of the form of the function analyzed in Appendix C, with

$$k = \frac{1}{4}, \quad p = 9, \quad s = 20, \quad w = 2\lambda_5, \quad v = \lambda_2, \quad \alpha = 0, \quad \beta = \frac{9}{20}. \tag{44}$$

Therefore, for it to be non-negative everywhere inside the γ_5 domain $[0, 9/20]$ it is necessary to require

$$\lambda_3 - \frac{3}{4}|\lambda_4| + \sqrt{\lambda_1\lambda_2} \geq 0, \tag{45a}$$

$$\lambda_3 + \sqrt{\lambda_1\left(\lambda_2 + \frac{9\lambda_5}{10}\right)} \geq 0. \tag{45b}$$

Furthermore, one must *exclude* the situation where

$$\lambda_5 < -\frac{5\lambda_4^2}{8\lambda_1}, \tag{46a}$$

$$\lambda_5 < -\frac{5}{6}\sqrt{\frac{\lambda_2}{\lambda_1}}|\lambda_4|, \tag{46b}$$

$$\lambda_3 < -\sqrt{\frac{(9\lambda_5 + 10\lambda_2)\left(8\lambda_1\lambda_5 + 5\lambda_4^2\right)}{80\lambda_5}}. \tag{46c}$$

4.2. *The case $n = 5$*

For $n = 5$ the functions that we need to minimize depend on both γ_5 and γ_6 and read

$$\Xi(\gamma_5, \gamma_6) = \lambda_2 + 2\lambda_5\gamma_5 + 2\lambda_6\gamma_6, \tag{47a}$$

$$\Gamma(\gamma_5, \gamma_6) = \lambda_3 - \frac{\sqrt{4 - 7\gamma_5 - 10\gamma_6}}{2}|\lambda_4| + \sqrt{\lambda_1}\sqrt{\lambda_2 + 2\lambda_5\gamma_5 + 2\lambda_6\gamma_6}. \tag{47b}$$

In this case the phase space is two-dimensional. Its boundary consists of three straight-line segments. From the monotonicity of $\Xi(\gamma_5, \gamma_6)$, it follows that it is sufficient to ensure that it is non-negative at the three vertices specified in Eq. (22). Thus,

$$\lambda_2 \geq 0, \tag{48a}$$

$$\lambda_2 + \frac{8}{7}\lambda_5 \geq 0, \tag{48b}$$

$$\lambda_2 + \frac{2}{35}\lambda_5' \geq 0, \tag{48c}$$

where we conveniently introduced

$$\lambda_5' \equiv 10\lambda_5 + 7\lambda_6. \tag{49}$$

Regarding $\Gamma(\gamma_5, \gamma_6)$ in Eq. (47b), we begin by observing that that function does not admit any extremum in the interior of the phase space, since the system of equations

$$\frac{\partial \Gamma}{\partial \gamma_5} = \frac{\partial \Gamma}{\partial \gamma_6} = 0 \tag{50}$$

has no solution. Therefore, if there is a minimum, it must reside at one of the boundaries of the phase space defined by Eqs. (B1b), (B3), and (B6).

At the boundary (B1b), we set $\gamma_6 = 0$ and $\Gamma(\gamma_5, \gamma_6)$ becomes

$$\Gamma(\gamma_5, 0) = \lambda_3 - \frac{\sqrt{4 - 7\gamma_5}}{2}|\lambda_4| + \sqrt{\lambda_1}\sqrt{\lambda_2 + 2\lambda_5\gamma_5}. \tag{51}$$

Equation (51) has the form of the function studied in Appendix C with

$$k = \frac{1}{2}, \quad p = 4, \quad s = 7, \quad w = 2\lambda_5, \quad v = \lambda_2, \quad \alpha = 0, \quad \beta = \frac{4}{7}. \tag{52}$$

Therefore, $\Gamma(\gamma_5, 0)$ is non-negative for all $\gamma_5 \in [0, 4/7]$ if we assume

$$\lambda_3 - |\lambda_4| + \sqrt{\lambda_1\lambda_2} \geq 0, \tag{53a}$$

$$\lambda_3 + \sqrt{\lambda_1\left(\lambda_2 + \frac{8}{7}\lambda_5\right)} \geq 0, \tag{53b}$$

and moreover if we exclude any situation where

$$\lambda_5 < -\frac{7\lambda_4^2}{8\lambda_1} \tag{54a}$$

$$\lambda_5 < -\frac{7}{8}\sqrt{\frac{\lambda_2}{\lambda_1}}|\lambda_4|, \tag{54b}$$

$$\lambda_3 < -\sqrt{\frac{(8\lambda_5 + 7\lambda_2)\left(8\lambda_5\lambda_1 + 7\lambda_4^2\right)}{56\lambda_5}}. \tag{54c}$$

At the boundary (B3) we take $\gamma_5 = (10/7)\gamma_6$ to obtain

$$\Gamma\left(\frac{10}{7}\gamma_6, \gamma_6\right) = \lambda_3 - |\lambda_4|\sqrt{1 - 5\gamma_6} + \sqrt{\lambda_1}\sqrt{\lambda_2 + \frac{2}{7}\lambda_5'\gamma_6}. \tag{55}$$

This function has the form of the function studied in Appendix C, with

$$k = 1, \quad p = 1, \quad s = 5, \quad w = \frac{2}{7}\lambda_5', \quad v = \lambda_2, \quad \alpha = 0, \quad \beta = \frac{1}{5}, \tag{56}$$

Therefore, it is non-negative $\forall \gamma_6 \in [0, 1/5]$ if we require

$$\lambda_3 - |\lambda_4| + \sqrt{\lambda_1\lambda_2} \geq 0, \tag{57a}$$

$$\lambda_3 + \sqrt{\lambda_1\left(\lambda_2 + \frac{2}{35}\lambda_5'\right)} \geq 0, \tag{57b}$$

and if we exclude the situation where

$$\lambda_5' < -\frac{35\lambda_4^2}{2\lambda_1}, \tag{58a}$$

$$\lambda_5' < -\frac{35}{2}\sqrt{\frac{\lambda_2}{\lambda_1}}|\lambda_4|, \tag{58b}$$

$$\lambda_3 < -\sqrt{\frac{(2\lambda_5' + 35\lambda_2)\left(2\lambda_5'\lambda_1 + 35\lambda_4^2\right)}{70\lambda_5'}}. \tag{58c}$$

At the boundary (B6) the function $\Gamma(\gamma_5, \gamma_6)$ takes the form

$$\Gamma\left(-\frac{10}{7}\gamma_6 + \frac{4}{7},\ \gamma_6\right) = \lambda_3 + \sqrt{\lambda_1}\sqrt{\lambda_2 + \frac{8}{7}\lambda_5 + \left(-\frac{20}{7}\lambda_5 + 2\lambda_6\right)\gamma_6}. \tag{59}$$

This function is monotonic in γ_6 and the n&s conditions for it to be non-negative $\forall \gamma_6 \in [0, 1/5]$ are conditions (53b) and (57b).

To summarize, the n&s BFB conditions for the case $n = 5$ are (48), (53), (57), and furthermore one must impede both conditions (54) and conditions (58).

4.3. *The case $n = 6$*

When $n = 6$, the functions that we need to minimize read

$$\Xi(\gamma_5, \gamma_6) = \lambda_2 + 2\lambda_5\gamma_5 + 2\lambda_6\gamma_6, \tag{60a}$$

$$\Gamma(\gamma_5, \gamma_6) = \lambda_3 - \frac{\sqrt{25 - 36\gamma_5 - 56\gamma_6}}{4}|\lambda_4| + \sqrt{\lambda_1}\sqrt{\lambda_2 + 2\lambda_5\gamma_5 + 2\lambda_6\gamma_6}. \tag{60b}$$

Neither of these functions admits extrema inside the phase space, so their minima must reside at its boundary. That boundary has two curved segments, namely, Eqs. (B8b) and (B14). We firstly require the two functions to be non-negative at the vertices of the phase space, given in Eq. (25a–). The following *necessary* BFB conditions are then found:

$$\lambda_2 \geq 0, \tag{61a}$$

$$\lambda_2^{(1)} \equiv \lambda_2 + \frac{8}{9}\lambda_5 \geq 0, \tag{61b}$$

$$\lambda_2^{(2)} \equiv \lambda_2 + 2\varsigma\lambda_5 + 2\varrho\lambda_6 \geq 0, \tag{61c}$$

$$\lambda_2^{(3)} \equiv \lambda_2 + \frac{89}{90}\lambda_5 + \frac{9}{35}\lambda_6 \geq 0, \tag{61d}$$

$$\lambda_2^{(4)} \equiv \lambda_2 + \frac{5}{18}\lambda_5 + \frac{5}{7}\lambda_6 \geq 0, \tag{61e}$$

and

$$\lambda_3 + \sqrt{\lambda_1\lambda_2} - \frac{5}{4}|\lambda_4| \geq 0, \tag{62a}$$

$$\lambda_3 + \sqrt{\lambda_1\lambda_2^{(1)}} - \frac{3}{4}|\lambda_4| \geq 0, \tag{62b}$$

$$\lambda_3 + \sqrt{\lambda_1\lambda_2^{(2)}} - \frac{5\left(24\sqrt{5} - 41\right)}{436}|\lambda_4| \geq 0, \tag{62c}$$

$$\lambda_3 + \sqrt{\lambda_1\lambda_2^{(3)}} \geq 0, \tag{62d}$$

$$\lambda_3 + \sqrt{\lambda_1\lambda_2^{(4)}} \geq 0. \tag{62e}$$

If we ignore the slight concavity of segment (B14) and instead describe the boundary connecting vertices V_2 and V_3 by the linear Eq. (B16),[11] then the boundary of phase space consists of four straight segments and just one curved one. The function $\Xi(\gamma_5, \gamma_6)$ is monotonic along

[11]This implies that we are exploring a phase space slightly larger than the true one, and as a result, the BFB conditions we impose are, strictly speaking, necessary instead of n&s conditions. Nevertheless, we

a straight segment, and therefore the four straight segments do not generate any further BFB condition besides conditions (61a–e). At the curved segment (B8b) one has

$$\widetilde{\Xi}\,(\lambda_6) \equiv \Xi\left(\frac{2\left(1+7\gamma_6+\sqrt{1-7\gamma_6}\right)}{9},\ \gamma_6\right) = \left(\lambda_2+\frac{4}{9}\lambda_5\right)+\left(\frac{28}{9}\lambda_5+2\lambda_6\right)\gamma_6 + \frac{4\lambda_5}{9}\sqrt{1-7\gamma_6}, \tag{63}$$

which must be non-negative $\forall\gamma_6 \in [0, \varrho]$. One must avoid the situation where $\widetilde{\Xi}\,(\lambda_6)$ has a minimum $\mu \in [0, \varrho]$ with $\widetilde{\Xi}\,(\mu) < 0$. This means that one must *impede* the situation where

$$7\lambda_5 + 9\lambda_6 < 0, \tag{64a}$$

$$7(5\sqrt{5}-8)\lambda_5 - 54\lambda_6 < 0, \tag{64b}$$

$$\lambda_2^{(1)} + \frac{2\,(7\lambda_5+9\lambda_6)^2}{63\Lambda} < 0, \tag{64c}$$

where

$$\Lambda \equiv 14\lambda_5 + 9\lambda_6. \tag{65}$$

Since γ_5 and γ_6 are linearly dependent along a straight segment, $\Gamma(\gamma_5, \gamma_6)$ can be cast into the form of the function analyzed in Appendix C for each one of the four straight segments of the boundary. The parameters are

$$\overline{V_0\,V_1}:\ k=\frac{1}{4},\quad p=25,\quad s=36,\quad w=2\lambda_5,\quad v=\lambda_2,\quad \alpha=0,\quad \beta=\frac{4}{9}, \tag{66a}$$

$$\overline{V_2\,V_3}:\ k=\frac{1}{4},\quad p=\frac{225(2\,999-912\sqrt{5})}{39\,961},\quad s=\frac{70}{9}\,p,\quad w=\frac{7(71\,677-38\,000\sqrt{5})}{119\,883}\lambda_5+2\lambda_6,\quad v=\lambda_2+\frac{25(6\,485+4\,104\sqrt{5})}{359\,649}\lambda_5,\quad \alpha=\varrho,\quad \beta=\frac{9}{70}, \tag{66b}$$

$$\overline{V_3\,V_4}:\ k=0,\quad w=-\frac{28}{9}\lambda_5+2\lambda_6,\quad v=\lambda_2+\frac{25}{18}\lambda_5,\quad \alpha=\frac{9}{70},\quad \beta=\frac{5}{14}, \tag{66c}$$

$$\overline{V_0\,V_4}:\ k=\frac{1}{4},\quad p=25,\quad s=70,\quad w=\frac{7}{9}\lambda_5+2\lambda_6,\quad v=\lambda_2,\quad \alpha=0,\quad \beta=\frac{5}{14}. \tag{66d}$$

The straight segment connecting V_3 to V_4 has $k = 0$, so conditions (62d) and (62e) are n&s to guarantee that $\Gamma(\gamma_5, \gamma_6) \geq 0$ everywhere on that segment. For the other three cases, one should exclude any values of $\lambda_1, \ldots, \lambda_6$ where all the conditions (C10), (C11), and (C12) are satisfied—for each set of parameters $\{k, p, s, v, w, \alpha, \beta\}$ in Eqs. (66a), (66b), and (66d). Notice that $p - s\beta = 0$ for the segments $\overline{V_2\,V_3}$ and $\overline{V_0\,V_4}$, and therefore for those segments condition (C11b) automatically holds.

have verified that this approximation has a negligible practical impact. We generated approximately 3.6×10^9 random sets of λ_i within the ranges $\lambda_{1,2} \in [0, 4\pi]$ and $\lambda_{3,4,5,6} \in [-4\pi, 4\pi]$. Among these, around 10^9 sets satisfied our necessary BFB conditions. Notably, we did not encounter a single case where using the true concave boundary, instead of its straight approximation, would have changed our conclusions regarding any of these 10^9 points.

On the convex segment connecting V_1 and V_2 we introduce

$$x \equiv \sqrt{1 - 7\gamma_6}, \tag{67}$$

with $x \in \left[\sqrt{1 - 7\varrho}, 1\right]$. Then, the functions that we ought to minimize on this segment are[12]

$$\Xi\left(\frac{2(1 + 7\gamma_6 + \sqrt{1 - 7\gamma_6})}{9}, \gamma_6\right) = g(x) \equiv \left(\lambda_2 + \frac{4\lambda_5}{9} + \frac{2\Lambda}{63}\right) + \frac{4\lambda_5}{9}x - \frac{2\Lambda}{63}x^2, \tag{68a}$$

$$\Gamma\left(\frac{2(1 + 7\gamma_6 + \sqrt{1 - 7\gamma_6})}{9}, \gamma_6\right) = f(x) \equiv \lambda_3 - \frac{4x - 1}{4}|\lambda_4| + \sqrt{\lambda_1\, g(x)}. \tag{68b}$$

Their first and second derivatives read

$$g'(x) = \frac{4\lambda_5}{9} - \frac{4\Lambda}{63}x, \tag{69a}$$

$$g''(x) = -\frac{4\Lambda}{63}, \tag{69b}$$

$$f'(x) = -|\lambda_4| + \frac{g'(x)}{2}\sqrt{\frac{\lambda_1}{g(x)}}, \tag{69c}$$

$$f''(x) = \frac{2\, g(x)\, g''(x) - [g'(x)]^2}{4}\sqrt{\frac{\lambda_1}{g(x)^3}}. \tag{69d}$$

We have already seen that for $g(x)$ to be non-negative for all x in the interval $\left[\sqrt{1 - 7\varrho},\ 1\right]$ one just needs to enforce conditions (61b) and (61c) while avoiding conditions (64). For the function $f(x)$ to be non-negative everywhere in the same interval one must enforce conditions (62b) and (62c) while avoiding the situation where

$$f'(\sqrt{1 - 7\varrho}) < 0, \quad f'(1) > 0, \quad \text{and} \quad \exists \mu : \quad f'(\mu) = 0, \quad f(\mu) < 0. \tag{70}$$

The third condition (70) yields the solution

$$\mu = \frac{7\lambda_5}{\Lambda} - \frac{9\,|\lambda_4|}{\Lambda}\sqrt{\frac{7K}{2S}}, \tag{71}$$

which only exists if $K/S > 0$, where

$$K \equiv 2\,(7\lambda_5 + 3\lambda_6)^2 + 7\lambda_2\Lambda, \tag{72a}$$

$$S \equiv 63\lambda_4^2 + 2\lambda_1\Lambda. \tag{72b}$$

When the first and second conditions (70) are satisfied, μ is guaranteed to be a minimum and to lie inside the domain $\left[\sqrt{1 - 7\varrho},\ 1\right]$. Under these assumptions, the positivity of $f''(\mu)$ further specifies that

$$K < 0, \tag{73a}$$

$$S < 0. \tag{73b}$$

The remaining three conditions (70) yield, respectively

[12] Minimizing $g(x)$ and $f(x)$ with respect to x is equivalent to minimizing the functions in the left-hand side of Eq. (68a,b) with respect to γ_6 because the transformation $\gamma_6 \to x$ in Eq. (67) is *injective* in the domain $\gamma_6 \in [0, \varrho] \Leftrightarrow x \in \left[\sqrt{1 - 7\varrho}, 1\right]$.

$$-|\lambda_4| + \frac{2(7\lambda_5 - \Lambda\sqrt{1-7\varrho})}{63}\sqrt{\frac{\lambda_1}{\lambda_2^{(2)}}} < 0, \tag{74a}$$

$$-|\lambda_4| + \frac{2\,(7\lambda_5 - \Lambda)}{63}\sqrt{\frac{\lambda_1}{\lambda_2^{(1)}}} > 0, \tag{74b}$$

$$\lambda_3 - \frac{4\mu - 1}{4}|\lambda_4| + \sqrt{\lambda_1\, g(\mu)} < 0. \tag{74c}$$

Therefore, besides enforcing conditions (62b) and (62c), one must avoid any situation where all the following conditions hold:

$$\Lambda < -\frac{2\,(7\lambda_5 + 3\lambda_6)^2}{7\lambda_2}, \tag{75a}$$

$$\Lambda < -\frac{63\lambda_4^2}{2\lambda_1}, \tag{75b}$$

$$|\lambda_4| > \frac{2}{63}\sqrt{\frac{\lambda_1}{\lambda_2^{(2)}}}\,(7\lambda_5 - \Lambda\sqrt{1-7\varrho}), \tag{75c}$$

$$|\lambda_4| < -\frac{2}{63}\sqrt{\frac{\lambda_1}{\lambda_2^{(1)}}}\,(7\lambda_5 + 9\lambda_6)\,, \tag{75d}$$

$$\lambda_3 < \frac{4\mu - 1}{4}|\lambda_4| - \sqrt{\lambda_1\, g(\mu)}. \tag{75e}$$

In condition (75c), note that $\sqrt{1-7\varrho} = 6(5\sqrt{5}-4)/109$.

To summarize, the necessary BFB conditions for the case $n = 6$ are (61) and (62). Besides that, it is sufficient to *exclude* any situation where Eq. (64a–c) or Eq. (75a–e) hold, or where conditions (C10), (C11), and (C12) are satisfied for each set of parameters $\{k, p, s, v, w, \alpha, \beta\}$ in Eqs. (66a), (66b), and (66d).

5. Vacuum stability

Following Ref. [43], we classify the possible extrema of the scalar potential (SP) of our model in the following exhaustive way:

- The type-0 extremum has

$$\langle\Phi\rangle = 0, \quad \langle\Delta_n\rangle = 0. \tag{76}$$

- A type-I extremum has

$$\langle\Phi\rangle \neq 0, \quad \langle\Delta_n\rangle = 0. \tag{77}$$

- A type-II extremum has

$$\langle\Phi\rangle = 0, \quad \langle\Delta_n\rangle \neq 0. \tag{78}$$

- A type-III extremum has

$$\langle\Phi\rangle \neq 0, \quad \langle\Delta_n\rangle \neq 0. \tag{79}$$

In Ref. [43] two of us have proved that, if the SP of a renormalizable $SU(2)$-symmetric model with a scalar doublet Φ contains neither linear terms, nor trilinear terms, nor quadratic terms with two different multiplets, and if moreover Φ only interacts with any other scalar $SU(2)$

multiplet Δ_n through quartic invariants constructed out of

$$\Phi \otimes \widetilde{\Phi} \otimes \Delta_n \otimes \widetilde{\Delta}_n \tag{80}$$

—where $\widetilde{\Delta}_n$ stands for the $SU(2)$ multiplet with the same dimension of Δ_n and formed by the complex conjugates of the scalar fields of Δ_n—then a type-I *local minimum* of the SP has a lower value of the SP than any type-0 or type-III extremum. As a consequence, either the *global minimum* of the SP is that type-I local minimum, or it is a type-II minimum (if there is any). Since the models studied in this work fulfil the assumptions of that theorem, in order to ensure the stability of the type-I vacuum one just has to find the conditions which ensure that:

(1) The vacuum is a local minimum of the SP. This is equivalent to all the physical scalars having positive masses-squared.
(2) No type-II extremum has an expectation value of the SP lower than the one of the vacuum.

We concentrate here on the second task.

5.1. *Type-I vacuum*

The expectation value of the SP at the type-I vacuum is independent of the dimension of Δ_n; it is given by

$$\overline{V}_{\mathrm{I}} = \mu_1^2 \langle F_1 \rangle + \frac{\lambda_1}{2} \langle F_1 \rangle^2 , \tag{81}$$

where $\langle F_1 \rangle$ is the expectation value of the $SU(2)$ invariant defined in Eq. (5). The stationarity condition for $\overline{V}_{\mathrm{I}}$ is

$$\mu_1^2 = -\lambda_1 \langle F_1 \rangle . \tag{82}$$

Boundedness from below of the SP requires λ_1 to be positive. Hence, Eq. (82) tells us that μ_1^2 must be negative for the type-I vacuum to exist. Equations (81) and (82) imply that

$$\overline{V}_{\mathrm{I}} = -\frac{\left(\mu_1^2\right)^2}{2\lambda_1}. \tag{83}$$

5.2. *Type-II extremum*

At a type-II extremum, only Δ_n acquires a VEV. According to Eq. (10) with $F_1 = 0$, the expectation value of the SP is

$$\overline{V}_{\mathrm{II}} = \mu_2^2 \langle F_2 \rangle + \frac{\langle \Xi\left(\gamma_5, \gamma_6\right) \rangle}{2} \langle F_2 \rangle^2 , \tag{84}$$

where $\langle F_2 \rangle$ and $\langle \Xi\left(\gamma_5, \gamma_6\right) \rangle$ are the expectation values of the $SU(2)$ invariants F_2 and $\Xi\left(\gamma_5, \gamma_6\right)$ at that type-II extremum, respectively. Further note that

$$\langle \Xi\left(\gamma_5, \gamma_6\right) \rangle = \begin{cases} \lambda_2 \quad \Leftarrow \quad \text{either } n = 1 \text{ or } n = 2, \\ \lambda_2 + 2\lambda_5 \langle \gamma_5 \rangle \quad \Leftarrow \quad \text{either } n = 3 \text{ or } n = 4, \\ \lambda_2 + 2\lambda_5 \langle \gamma_5 \rangle + 2\lambda_6 \langle \gamma_6 \rangle \quad \Leftarrow \quad \text{either } n = 5 \text{ or } n = 6. \end{cases} \tag{85}$$

Effecting the minimization of $\overline{V}_{\mathrm{II}}$ in Eq. (84) relative to $\langle F_2 \rangle$, one finds that at a type-II extremum

$$\mu_2^2 = - \langle \Xi\left(\gamma_5, \gamma_6\right) \rangle \langle F_2 \rangle . \tag{86}$$

The SP being bounded from below necessitates that $\Xi\left(\gamma_5, \gamma_6\right)$ is non-negative, as we have extensively discussed in the previous section. Therefore, a type-II extremum only exists if $\mu_2^2 < 0$.

From Eqs. (84) and (86),

$$\overline{V}_{\mathrm{II}} = -\frac{\left(\mu_2^2\right)^2}{2\left\langle \Xi\left(\gamma_5, \gamma_6\right)\right\rangle}. \tag{87}$$

To ensure that the type-I vacuum lies below the type-II extremum with the lowest $\overline{V}_{\mathrm{II}}$, we require the parameters of the SP to be such that all possible $\langle \Xi(\gamma_5, \gamma_6)\rangle$ satisfy

$$\langle \Xi(\gamma_5, \gamma_6)\rangle > \lambda_1 \left(\frac{\mu_2^2}{\mu_1^2}\right)^2. \tag{88}$$

5.3. *Conditions for vacuum stability*

When Δ_n is either a singlet ($n = 1$) or a doublet ($n = 2$), $\langle \Xi(\gamma_5, \gamma_6)\rangle = \lambda_2$ and condition (88) simply implies that

$$\lambda_2 > \left(\frac{\mu_2^2}{\mu_1^2}\right)^2 \lambda_1. \tag{89}$$

This condition for the type-I extremum to be the global minimum of the SP is consistent with the results derived for the complex singlet extension of the SM in Ref. [40], and with the analogous condition found in Refs. [6,41,44] for the $U(1)$-symmetric 2HDM.

For larger n, $\langle \Xi(\gamma_5, \gamma_6)\rangle$ may take several values and we must investigate the space spanned by γ_5 and γ_6. The task of finding the values of $\langle \Xi(\gamma_5, \gamma_6)\rangle$ that lead to the smallest possible value of $\overline{V}_{\mathrm{II}}$ can be simplified by noting that $\langle \Xi(\gamma_5, \gamma_6)\rangle$ is linear in $\langle\gamma_5\rangle$ and $\langle\gamma_6\rangle$. Therefore, any type-II extremum must lie at the *boundary* of the phase space [34]. Furthermore, the minimum of the potential is attained at the points of phase space that extend the farthest in some direction. Hence, when the boundary consists of straight-line segments meeting at vertices, it suffices to evaluate $\langle \Xi(\gamma_5, \gamma_6)\rangle$ at those vertices to find the possible type-II extrema. Consequently,

$$\langle \Xi(\gamma_5, \gamma_6)\rangle = \begin{cases} \text{either } \lambda_2 \text{ or } \lambda_2 + \dfrac{2\lambda_5}{3} \Leftarrow n = 3, \\ \text{either } \lambda_2 \text{ or } \lambda_2 + \dfrac{9\lambda_5}{10} \Leftarrow n = 4, \\ \text{either } \lambda_2, \text{ or } \lambda_2 + \dfrac{8\lambda_5}{7}, \text{ or } \lambda_2 + \dfrac{4\lambda_5}{7} + \dfrac{2\lambda_6}{5} \Leftarrow n = 5. \end{cases} \tag{90}$$

Therefore, condition (89) must hold for all $n \leq 5$ and besides one must require that

$$n = 3 \Rightarrow \lambda_2 + \frac{2\lambda_5}{3} > \left(\frac{\mu_2^2}{\mu_1^2}\right)^2 \lambda_1, \tag{91a}$$

$$n = 4 \Rightarrow \lambda_2 + \frac{9\lambda_5}{10} > \left(\frac{\mu_2^2}{\mu_1^2}\right)^2 \lambda_1, \tag{91b}$$

$$n = 5 \Rightarrow \lambda_2 + \min\left(\frac{8\lambda_5}{7}, \frac{4\lambda_5}{7} + \frac{2\lambda_6}{5}\right) > \left(\frac{\mu_2^2}{\mu_1^2}\right)^2 \lambda_1. \tag{91c}$$

An important consistency check may be performed in the case $n = 3$, where, using the notation of Eq. (A11), the conditions given in Eqs. (89) and (91a) exactly coincide with those presented in Eq. (5.9) of Ref. [45].

For $n = 6$, determining the possible type-II extrema is more challenging because the boundary of the phase space contains both straight-line segments and a convex segment.[13] Note that

[13] For the moment, we are approximating the slightly concave parametric curve between the vertices V_2 and V_3 by a straight line, just as we did in the previous section.

$\langle \Xi (\gamma_5, \gamma_6) \rangle$ is still linear in both $\langle \gamma_5 \rangle$ and $\langle \gamma_6 \rangle$, so we introduce the direction of steepest descent $\vec{n} = -(\langle \gamma_6 \rangle, \langle \gamma_5 \rangle)$. If $\vec{n}$ points to one of the straight boundaries, then the farthest protruding points in the direction of $\vec{n}$ coincide with the vertices of the phase space given in Eq. (25). The possible type-II extrema then have

$$\langle \Xi (\gamma_5, \gamma_6) \rangle = \text{either } \lambda_2, \text{ or } \lambda_2 + \frac{8\lambda_5}{9}, \text{ or } \lambda_2 + 2\varsigma\lambda_5 + 2\varrho\lambda_6, \text{ or } \lambda_2 + \frac{89\lambda_5}{90} + \frac{9\lambda_6}{35},$$
$$\text{or } \lambda_2 + \frac{5\lambda_5}{18} + \frac{5\lambda_6}{7}. \tag{92}$$

However, if $\vec{n}$ points to the boundary connecting the vertices V_1 and V_2, then the farthest protruding point in the direction of $\vec{n}$ will lie somewhere on the convex boundary (B8b). In other words, we need to find the minimum of the function $\widetilde{\Sigma} (\gamma_6)$ defined in Eq. (63) in the range $\gamma_6 \in [0, \varrho]$. That was done already when studying the boundedness-from-below conditions in Section 4.3; more specifically, if

$$7(5\sqrt{5} - 8)\lambda_5 < 54\lambda_6 < -42\lambda_5, \tag{93}$$

then, from Eq. (64c), one more possible value of $\langle \Xi (\gamma_5, \gamma_6) \rangle$ is

$$\lambda_2 + \frac{2\,(7\lambda_5 + 3\lambda_6)^2}{7\,(14\lambda_5 + 9\lambda_6)}. \tag{94}$$

If we want to be more rigorous, then we must not neglect the curvature of the parametric curve (B14). We must then consider, for each set of values of λ_2, λ_5, and λ_6 the function of t

$$\langle \Xi (\gamma_5, \gamma_6) \rangle (t) = \lambda_2 + \frac{40t}{63\left(75t^4 - 100t^3 + 90t^2 + 12t + 3\right)} \times \tag{95a}$$
$$\Big[144t^2 \left(-165t^4 + 212t^3 - 102t^2 + 36t + 3\right) \lambda_6 \tag{95b}$$
$$+7\left(1125t^7 - 195t^6 + 821t^5 - 1371t^4 + 823t^3 \right. \tag{95c}$$
$$\left. +399t^2 + 175t + 15\right) \lambda_5 \Big], \tag{95d}$$

and we must look for minima of this function in the interval

$$\frac{3}{5} \leq t \leq \frac{3 + 2\sqrt{5}}{11}. \tag{96}$$

Any minima must be treated as extra possibilities for $\langle \Xi (\gamma_5, \gamma_6) \rangle$.

6. Conclusions and discussion

In this paper, we have considered the extension of the electroweak SM through a single scalar multiplet Δ_n of $SU(2)$. We have assumed that multiplet to enjoy a $U(1)$ symmetry $\Delta_n \to \Delta_n \exp(i\vartheta)$ and to have no VEV, so as not to perturb, at tree level, the successful SM prediciton (1) (although it does perturb it at loop level). We have analyzed the scalar potential V of this model with only two scalar multiplets—Δ_n and the SM doublet Φ—in order to find out the ranges of its various $SU(2)$-invariants, and thus, the conditions for it to be BFB (and thus be able to produce a vacuum state), and the conditions for our preferred vacuum state (where only Φ has a VEV) to be the global minimum of V.

With just one scalar $SU(2)$ multiplet Δ_n with n components, there are m quartic $SU(2)$-invariants in V, where $m = 1$ for either $n = 1$ or $n = 2$, $m = 2$ for either $n = 3$ or $n = 4$, $m = 3$ for either $n = 5$ or $n = 6$, and so on. The increasing number of invariants renders their ranges increasingly complicated to calculate. As demonstrated in Section 3, for $n \leq 4$, we were able to identify the phase space boundaries through direct algebraic manipulation of the invariants. To tackle the same problem for the cases $n = 5$ and $n = 6$, we first assigned random values to the scalar fields of Δ_n to map out the phase space, and only afterwards did we attempt to find suitable combinations of nonzero fields to characterize the boundaries analytically. It is important to clarify that, although numerical sampling both provided the initial insight and confirmed the final analytical result, our determination of the phase space boundaries is entirely analytical.

We have found that, starting with $n = 6$,[14] the phase space has some curved boundaries, at least one of which is *concave*. As shown in Eq. (B14), this concave boundary can only be expressed in parametric form as a ratio of high-degree polynomials. For practical purposes, we therefore approximate it throughout our analysis by using the straight-line segment given in Eq. (B16).

By using the analytic equations for all the boundaries of the phase space for the cases $n \leq 6$, we were able to deduce analytic n&s BFB conditions on V, and also analytic n&s conditions for our desired vacuum state—where Φ, but not Δ_n, has nonzero VEV—to be the absolute minimum of the potential. We assessed the accuracy of all our analytical results by performing numerical scans over the phase space. Note, however, that due to the straight-line approximation adopted for the concave boundary in the case $n = 6$, the phase space that we have worked with was slightly larger than the exact one. As a result, the conditions found by us in this case are, strictly speaking, necessary instead of n&s conditions. To evaluate the validity of this simplification, we scanned 10^9 sets of couplings λ_i and did not find a single instance where applying the exact concave boundary—rather than its linear approximation—would have altered our conclusions. This demonstrates the negligible practical impact of the approximation.

One may discuss the situation where the theory has, instead of the $U(1)$ symmetry $\Delta_n \to \Delta_n \exp(i\vartheta)$, the smaller (discrete) $\mathbb{Z}_2$ symmetry $\Delta_n \to -\Delta_n$. In the case where n is odd, that is, where J is integer, that makes no difference relative to the case with $U(1)$ symmetry. But, in the case where n is even and if additionally Δ_n has a specific hypercharge, there is an additional $SU(2) \times U(1)$-invariant term in the quartic part of the scalar potential. Indeed, if n is even, then the product $\Delta_n \otimes \Delta_n$ includes an $SU(2)$ triplet $(\Delta_n \otimes \Delta_n)_{\mathbf{3}}$. (The boldface subindex indicates the dimension of the $SU(2)$ representation.) Then, besides the F_4 term, which is the $SU(2)$-invariant in $\left(\Phi \otimes \tilde{\Phi}\right)_{\mathbf{3}} \otimes \left(\Delta_n \otimes \tilde{\Delta}_n\right)_{\mathbf{3}}$, there are further terms in the potential, namely either

$$\left[\left(\Phi \otimes \tilde{\Phi}\right)_{\mathbf{3}} \otimes (\Delta_n \otimes \Delta_n)_{\mathbf{3}}\right]_{\mathbf{1}} \tag{97}$$

if Δ_n has null hypercharge, or

$$\left[\left(\tilde{\Phi} \otimes \tilde{\Phi}\right)_{\mathbf{3}} \otimes (\Delta_n \otimes \Delta_n)_{\mathbf{3}}\right]_{\mathbf{1}} \tag{98}$$

if the hypercharge of Δ_n is the same as the one of Φ. (Besides the terms (97) and (98) there are their Hermitian conjugates.) (In the case where $n = 2$, that is, in the 2HDM, the term (98) is, in the standard notation, the term $\left(\phi_1^\dagger \phi_2\right)^2$ with coefficient λ_5.) So, the case with $\mathbb{Z}_2$ symmetry is, when n is even and Δ_n has hypercharge either 0 or $1/2$, more complicated than the case with

[14]We have also briefly explored the case $n = 7$, but we do not report on it in this paper (and we are not planning to do it elsewhere).

$U(1)$ symmetry, because there is then an additional dimensionless parameter—with denominator F_1F_2 just as δ in the last Eq. (9)—and therefore the phase space has an extra dimension.

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Appendix A: The definitions of F_2, F_4, F_5, and F_6

In this appendix, we recover relevant definitions and results of Ref. [39].

A.1 $n = 1$

If $n = 1$, that is, if $J = 0$, then Δ_1 contains a single complex scalar fields c, with $I_c = 0$:

$$\Delta_1 = (c), \quad \widetilde{\Delta}_1 = (c^*). \tag{A1}$$

Besides the $SU(2)$-invariant F_1 defined in Eq. (5), one may write

$$F_2 = |c|^2 \equiv C. \tag{A2}$$

The scalar potential is given by Eq. (7a).

A.2 $n = 2$

If $n = 2$, that is, if $J = 1/2$, then

$$\Delta_2 = \begin{pmatrix} c \\ d \end{pmatrix}, \quad \widetilde{\Delta}_2 = \begin{pmatrix} d^* \\ -c^* \end{pmatrix}, \tag{A3}$$

where c and d are complex scalar fields with $I_c = -I_d = 1/2$. We can build two $SU(2)$-invariants apart from F_1. Namely

$$F_2 = C + D, \tag{A4a}$$

$$F_4 = \frac{(A - B)(C - D)}{4} + \frac{ab^*c^*d + a^*bcd^*}{2}. \tag{A4b}$$

The SP is given by Eq. (7b). This model is identical to the $U(1)$-symmetric 2HDM. The notation that most authors use for the quartic part of the scalar potential of the latter is [6]

$$V_4 = \frac{\bar{\lambda}_1}{2}\left(\Phi^\dagger\Phi\right)^2 + \frac{\bar{\lambda}_2}{2}\left(\Delta_2^\dagger\Delta_2\right)^2 + \bar{\lambda}_3\left(\Phi^\dagger\Phi\right)\left(\Delta_2^\dagger\Delta_2\right) + \bar{\lambda}_4\left(\Phi^\dagger\Delta_2\right)\left(\Delta_2^\dagger\Phi\right). \tag{A5}$$

This is the same as our notation

$$V_4 = \frac{\lambda_1}{2}F_1^2 + \frac{\lambda_2}{2}F_2^2 + \lambda_3 F_1 F_2 + \lambda_4 F_4, \tag{A6}$$

with

$$\bar{\lambda}_1 = \lambda_1, \quad \bar{\lambda}_2 = \lambda_2, \quad \bar{\lambda}_3 + \frac{\bar{\lambda}_4}{2} = \lambda_3, \quad 2\bar{\lambda}_4 = \lambda_4. \tag{A7}$$

A.3 $n = 3$

If $n = 3$, that is if $J = 1$, then

$$\Delta_3 = \begin{pmatrix} c \\ d \\ e \end{pmatrix}, \quad \widetilde{\Delta}_3 = \begin{pmatrix} e^* \\ -d^* \\ c^* \end{pmatrix}, \tag{A8}$$

where c, d, and e are complex scalar fields with $I_c = -I_e = 1$ and $I_d = 0$. There are now three $SU(2)$-invariants that we can build, namely

$$F_2 = C + D + E, \tag{A9a}$$

$$F_4 = \frac{(A-B)(C-E)}{2} + \frac{ab^*(c^*d + d^*e) + a^*b(cd^* + de^*)}{\sqrt{2}}, \tag{A9b}$$

$$F_5 = \frac{\left|2ce - d^2\right|^2}{3}. \tag{A9c}$$

The SP is given by Eq. (7c).

The triplet Δ_3 is more usually written in the form

$$\Delta = \begin{pmatrix} -d/\sqrt{2} & c \\ -e & d/\sqrt{2} \end{pmatrix}, \tag{A10}$$

wherewith the potential is written as

$$V = \mu_1^2 \Phi^\dagger \Phi + \mu_2^2 \,\mathrm{tr}\left(\Delta^\dagger \Delta\right) + \frac{\lambda_1}{2}\left(\Phi^\dagger \Phi\right)^2 + \frac{\overline{\lambda}_2}{2}\left[\mathrm{tr}\left(\Delta^\dagger \Delta\right)\right]^2 \tag{A11a}$$

$$+\overline{\lambda}_3 \,\mathrm{tr}\left(\Delta^\dagger \Delta \Delta^\dagger \Delta\right) + \overline{\lambda}_4 \left(\Phi^\dagger \Phi\right) \mathrm{tr}\left(\Delta^\dagger \Delta\right) + \overline{\lambda}_5 \,\Phi^\dagger \Delta \Delta^\dagger \Phi. \tag{A11b}$$

The SP of Eq. (A11) is equivalent to the one of Eq. (7c) with

$$\overline{\lambda}_2 + 2\overline{\lambda}_3 = \lambda_2, \quad \overline{\lambda}_4 + \frac{1}{2}\overline{\lambda}_5 = \lambda_3, \quad \overline{\lambda}_5 = \lambda_4, \quad -\frac{2}{3}\overline{\lambda}_3 = \lambda_5. \tag{A12}$$

A.4 $n = 4$

If $n = 4$, that is, if $J = 3/2$, then

$$\Delta_4 = \begin{pmatrix} c \\ d \\ e \\ f \end{pmatrix}, \quad \widetilde{\Delta}_4 = \begin{pmatrix} f^* \\ -e^* \\ d^* \\ -c^* \end{pmatrix}, \tag{A13}$$

where c, d, e, and f are complex scalar fields with $I_c = -I_f = 3/2$ and $I_d = -I_e = 1/2$. The SP is still the one in Eq. (7c), but now with

$$F_2 = C + D + E + F, \tag{A14a}$$

$$F_4 = \frac{(A-B)(3C + D - E - 3F)}{4} \tag{A14b}$$

$$+\frac{ab^*\left(\sqrt{3}\,c^*d + 2d^*e + \sqrt{3}\,e^*f\right) + \mathrm{H.c.}}{2}, \tag{A14c}$$

$$F_5 = \frac{2|\sqrt{3}\,ce - d^2|^2 + 2|\sqrt{3}\,df - e^2|^2 + |3cf - de|^2}{5}. \tag{A14d}$$

A.5 n = 5

If $n = 5$, that is, if $J = 2$, then

$$\Delta_5 = \begin{pmatrix} c \\ d \\ e \\ f \\ g \end{pmatrix}, \quad \widetilde{\Delta}_5 = \begin{pmatrix} g^* \\ -f^* \\ e^* \\ -d^* \\ c^* \end{pmatrix}, \tag{A15}$$

where c, d, e, f, and g are complex scalar fields with $I_c = -I_g = 2$, $I_d = -I_f = 1$, and $I_e = 0$. The set of $SU(2)$-invariants now reads

$$F_2 = C + D + E + F + G, \tag{A16a}$$

$$F_4 = \frac{(A - B)(2C + D - F - 2G)}{2} \tag{A16b}$$

$$+ \left\{ ab^* \left[c^* d + f^* g + \sqrt{\frac{3}{2}} (d^* e + e^* f) \right] ll + \text{H.c.} \right\}, \tag{A16c}$$

$$F_5 = \frac{|2\sqrt{2}\, ce - \sqrt{3}\, d^2|^2 + |2\sqrt{2}\, eg - \sqrt{3}\, f^2|^2}{7} \tag{A16d}$$

$$+ \frac{2\left(\left|\sqrt{6}\, cf - de\right|^2 + \left|\sqrt{6}\, dg - ef\right|^2 + \left|2cg + df - e^2\right|^2 \right)}{7}, \tag{A16e}$$

$$F_6 = \frac{\left|2cg - 2df + e^2\right|^2}{5}, \tag{A16f}$$

and the full SP takes the form in Eq. (7d).

A.6 n = 6

If $n = 6$, that is, if $J = 5/2$, then

$$\Delta_6 = \begin{pmatrix} c \\ d \\ e \\ f \\ g \\ h \end{pmatrix}, \quad \widetilde{\Delta}_6 = \begin{pmatrix} h^* \\ -g^* \\ f^* \\ -e^* \\ d^* \\ -c^* \end{pmatrix}, \tag{A17}$$

where c, d, e, f, g, and h are complex scalar fields with $I_c = -I_h = 5/2$, $I_d = -I_g = 3/2$, and $I_e = -I_f = 1/2$. The SP is the one of Eq. (7d), with

$$F_2 = C + D + E + F + G + H, \tag{A18a}$$

$$F_4 = \frac{(A - B)(5C + 3D + E - F - 3G - 5H)}{4} \tag{A18b}$$

$$+ \text{Re} \left\{ ab^* \left[\sqrt{5}\, (c^* d + g^* h) + 2\sqrt{2}\, (d^* e + f^* g) + 3e^* f \right] \right\}, \tag{A18c}$$

$$F_5 = \frac{2\left(\left|\sqrt{5}\, ce - \sqrt{2}\, d^2\right|^2 + \left|\sqrt{5}\, fh - \sqrt{2}\, g^2\right|^2\right)}{9} + \frac{\left|\sqrt{5}\, cf - de\right|^2 + \left|\sqrt{5}\, eh - fg\right|^2}{3} \tag{A18d}$$

$$+\frac{2\left(\left|\sqrt{10}\,cg+df-\sqrt{2}\,e^2\right|^2+\left|\sqrt{10}\,dh+eg-\sqrt{2}\,f^2\right|^2\right)}{15} \tag{A18e}$$

$$+\frac{\left|5ch+7dg-4ef\right|^2}{45}, \tag{A18f}$$

$$F_6=\frac{\left|2\sqrt{5}\,cg-4\sqrt{2}\,df+3e^2\right|^2+\left|2\sqrt{5}\,dh-4\sqrt{2}\,eg+3f^2\right|^2+2\left|5ch-3dg+ef\right|^2}{35}. \tag{A18g}$$

Appendix B: Ansätze for the fields

In this appendix, we consider *Ansätze* for the fields in the cases with $n = 5$ and $n = 6$.

B.1 $n = 5$

With reference to Eqs. (A15) and (A16), we construct the following three *Ansätze*:

(1) $d = e = g = 0$: If only c and f are nonzero, then

$$\gamma_5=\frac{3F\,(4C+F)}{7\,(C+F)^2}, \tag{B1a}$$

$$\gamma_6=0. \tag{B1b}$$

By letting F vary from 0 to $2C$ one obtains $\gamma_5 \in [0,\ 4/7]$.

(2) $d = e = f = 0$: If only c and g are nonzero, then

$$\gamma_5=\frac{8CG}{7\,(C+G)^2}, \tag{B2a}$$

$$\gamma_6=\frac{4CG}{5\,(C+G)^2}. \tag{B2b}$$

It is obvious that in this case

$$\gamma_5=\frac{10}{7}\,\gamma_6, \tag{B3}$$

with $\gamma_5 \in [0,\ 2/7]$ and $\gamma_6 \in [0,\ 1/5]$.

(3) $d = f = 0$ and $g = -c$: In this case

$$F_2=2C+E, \tag{B4a}$$

$$F_5=\frac{16CE}{7}+\frac{2\left|2c^2+e^2\right|^2}{7}, \tag{B4b}$$

$$F_6=\frac{\left|2c^2-e^2\right|^2}{5}. \tag{B4c}$$

Assuming furthermore that $c^2 = ke^2$, with a *real* non-negative k, one obtains

$$\gamma_5=\frac{2}{7}\left[1+\frac{8k}{(2k+1)^2}\right], \tag{B5a}$$

$$\gamma_6=\frac{1}{5}\left(\frac{2k-1}{2k+1}\right)^2. \tag{B5b}$$

This gives

$$\gamma_5=-\frac{10}{7}\,\gamma_6+\frac{4}{7}, \tag{B6}$$

with $\gamma_6 \in [0,\ 1/5]$.

B.2 $n = 6$

With reference to Eqs. (A17) and (A18), we construct the following five *Ansätze*:

(1) $e = f = g = h = 0$: In this case

$$\gamma_5 = \frac{4}{9}\left(\frac{D}{C+D}\right)^2, \tag{B7a}$$

$$\gamma_6 = 0. \tag{B7b}$$

The extreme situations $d = 0$ and $c = 0$ show that $0 \le \gamma_5 \le 4/9$.

(2) $d = e = f = h = 0$: In this case

$$\gamma_6 = \frac{4CG}{7\,(C+G)^2}, \tag{B8a}$$

$$\gamma_5 = \frac{2\left(1 + 7\gamma_6 + \sqrt{1 - 7\gamma_6}\right)}{9}. \tag{B8b}$$

Equation (B8a) suggests that $0 \le \gamma_6 \le 1/7$, but in reality the curve (B8b) forms the boundary of the allowed region only for $0 \le \gamma_6 \le \varrho$, where

$$\varrho = \frac{5\left(1361 + 288\sqrt{5}\right)}{7 \times 109^2} \approx 0.121 \tag{B9}$$

is slightly less than $1/7 \approx 0.143$. The value of γ_5 corresponding through Eq. (B8b) to $\gamma_6 = \varrho$ is

$$\varsigma = \frac{20\left(1607 + 471\sqrt{5}\right)}{9 \times 109^2} \approx 0.498. \tag{B10}$$

(3) $d = f = h = 0$: In this case

$$F_2 = C + E + G, \tag{B11a}$$

$$F_5 = \frac{10CE + 4G^2}{9} + \frac{4\left|\sqrt{5}\,cg - e^2\right|^2 + 2EG}{15}, \tag{B11b}$$

$$F_6 = \frac{\left|2\sqrt{5}\,cg + 3e^2\right|^2 + 32EG}{35}. \tag{B11c}$$

Assuming furthermore that $e = \ell c$ and $g = -\sqrt{5}\,tc$, with *real* ℓ and t, one obtains

$$\gamma_5 = \frac{50\ell^2 + 500t^4 + 300t^2 + 12\ell^4 + 120\ell^2 t + 30\ell^2 t^2}{45\left(1 + \ell^2 + 5t^2\right)^2}, \tag{B12a}$$

$$\gamma_6 = \frac{100t^2 + 9\ell^4 - 60\ell^2 t + 160\ell^2 t^2}{35\left(1 + \ell^2 + 5t^2\right)^2}. \tag{B12b}$$

We are interested in the envelope of this family of parametric curves. The relevant solution to the envelope equation reads

$$\frac{\partial\gamma_6}{\partial t}\frac{\partial\gamma_5}{\partial\ell^2} - \frac{\partial\gamma_6}{\partial\ell^2}\frac{\partial\gamma_5}{\partial t} = 0 \ \Leftrightarrow\ \ell^2 = \frac{10t\left(-11t^2 + 6t + 1\right)}{15t^2 + 2t + 3}. \tag{B13}$$

Plugging back this value of ℓ^2 into Eq. (B12a,b), we obtain the parametric curve

$$\gamma_5 = \frac{20t\left(1125t^7 - 195t^6 + 821t^5 - 1371t^4 + 823t^3 + 399t^2 + 175t + 15\right)}{9\left(75t^4 - 100t^3 + 90t^2 + 12t + 3\right)^2}, \tag{B14a}$$

$$\gamma_6 = \frac{320t^3\left(-165t^4 + 212t^3 - 102t^2 + 36t + 3\right)}{7\left(75t^4 - 100t^3 + 90t^2 + 12t + 3\right)^2}. \tag{B14b}$$

The parametric curve (B14) intersects the curves (B8b) and (B17b), respectively, when

$$\gamma_6 = \varrho \Leftrightarrow t = \frac{3 + 2\sqrt{5}}{11}, \tag{B15a}$$

$$\gamma_6 = \frac{9}{70} \Leftrightarrow t = \frac{3}{5}. \tag{B15b}$$

In the interval $t \in [3/5, (3 + 2\sqrt{5})/11]$, the parametric curve (B14) may be approximated, to a high degree of accuracy, by the straight line[15]

$$\gamma_5 = \frac{21\left(71\,677 - 38\,000\sqrt{5}\right)\gamma_6 + 25\left(6\,485 + 4\,104\sqrt{5}\right)}{719\,298}. \tag{B16}$$

(4) $e = f = 0$, $c = h$, and $d = -g$: In this case

$$\gamma_6 = \frac{25C^2 + 9D^2 + 50CD}{70\,(C + D)^2}, \tag{B17a}$$

$$\gamma_5 = -\frac{14}{9}\gamma_6 + \frac{25}{36}. \tag{B17b}$$

The straight line (B17b) extends, according to Eq. (B17a), from $\gamma_6 = 9/70$ to $\gamma_6 = 5/14$. Note that when γ_5 and γ_6 are related through Eq. (B17b), δ is zero by virtue of Eq. (23).

(5) $d = e = f = g = 0$: In this case

$$\gamma_5 = \frac{5\,CH}{9\,(C + H)^2}, \tag{B18a}$$

$$\gamma_6 = \frac{10\,CH}{7\,(C + H)^2}. \tag{B18b}$$

It is clear that

$$\gamma_5 = \frac{7}{18}\gamma_6, \tag{B19}$$

with $5/14 \geq \gamma_6 \geq 0$ and $5/36 \geq \gamma_5 \geq 0$.

Appendix C: Minimization of a function

Consider the $\mathbb{R} \to \mathbb{R}$ function

$$f(\theta) = \lambda_3 - k\,|\lambda_4|\,\sqrt{p - s\theta} + \sqrt{\lambda_1}\,\sqrt{v + w\theta}, \tag{C1}$$

with real parameters λ_1, λ_3, λ_4, k, p, s, v, and w satisfying

$$\lambda_1 \geq 0, \quad k \geq 0, \quad p > 0, \quad s > 0. \tag{C2}$$

We are interested in the necessary and sufficient conditions that guarantee $f(\theta)$ to be non-negative everywhere inside the range $\theta \in [\alpha, \beta]$, with $\alpha \in [0, \beta]$ and $\beta \in [0, p/s]$. It is *necessary* that $f(\theta)$ be non-negative at the end-points of the domain, so the following two conditions must hold:

$$\lambda_3 - k\,|\lambda_4|\,\sqrt{p - s\alpha} + \sqrt{\lambda_1}\,\sqrt{v + w\alpha} \geq 0, \tag{C3a}$$

[15] This is the straight line connecting the points (ϱ, ς) and $(9/70, 89/180)$ in the (γ_6, γ_5) plane.

$$\lambda_3 - k\,|\lambda_4|\,\sqrt{p - s\beta} + \sqrt{\lambda_1}\,\sqrt{v + w\beta} \geq 0. \tag{C3b}$$

If $k = 0$, then $f(\theta)$ is monotonic, so conditions (C3) are both necessary and sufficient. If $k > 0$, then $f(\theta)$ has at most one extremum:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{\theta=\mu} = \frac{ks|\lambda_4|}{2\sqrt{p - s\mu}} + \frac{w\sqrt{\lambda_1}}{2\sqrt{v + w\mu}} = 0 \;\Rightarrow\; \mu = \frac{pw^2\lambda_1 - k^2s^2v\lambda_4^2}{sw\left(w\lambda_1 + k^2s\lambda_4^2\right)}. \tag{C4}$$

If that extremum is a minimum, then we must avoid the situation where $\mu \in [\alpha, \beta]$ and $f(\mu)$ is negative. Therefore, besides enforcing conditions (C3), it suffices to avoid any situation where there is a μ such that

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{\theta=\mu} = 0, \tag{C5a}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{\theta=\alpha} < 0, \tag{C5b}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{\theta=\beta} > 0, \tag{C5c}$$

$$f(\mu) < 0. \tag{C5d}$$

We aim to express conditions (C5) in terms of λ_1, λ_3, λ_4, k, p, s, v, and w. To analyze condition (C5a), we observe that, since $ks > 0$, $\mathrm{d}f/\mathrm{d}\theta$ can only vanish if

$$w < 0. \tag{C6}$$

In this case, the solution μ given in Eq. (C4) satisfies

$$p - s\mu = \frac{k^2s\lambda_4^2\,(pw + sv)}{w\left(w\lambda_1 + k^2s\lambda_4^2\right)}, \tag{C7a}$$

$$v + w\mu = \frac{w\lambda_1\,(pw + sv)}{s\left(w\lambda_1 + k^2s\lambda_4^2\right)}. \tag{C7b}$$

Since $k > 0$, $s > 0$, and $w < 0$, both $p - s\mu > 0$ and $v + w\mu > 0$ require

$$\frac{pw + sv}{w\lambda_1 + k^2s\lambda_4^2} < 0. \tag{C8}$$

One then obtains

$$\sqrt{p - s\mu} = -\frac{k\sqrt{s}\,|\lambda_4|}{w}\sqrt{\frac{w\,(pw + sv)}{w\lambda_1 + k^2s\lambda_4^2}}, \tag{C9a}$$

$$\sqrt{v + w\mu} = \sqrt{\frac{\lambda_1}{s}}\sqrt{\frac{w\,(pw + sv)}{w\lambda_1 + k^2s\lambda_4^2}}, \tag{C9b}$$

$$\left.\frac{\mathrm{d}^2f}{\mathrm{d}\theta^2}\right|_{\theta=\mu} = -\frac{w^2\sqrt{s}}{4k^2\lambda_1\lambda_4^2}\left[\sqrt{\frac{w\lambda_1 + k^2s\lambda_4^2}{w\,(pw + sv)}}\right]^3\left(w\lambda_1 + k^2s\lambda_4^2\right). \tag{C9c}$$

Therefore, in order that μ be a *minimum* of $f(\theta)$ one must have

$$w\lambda_1 + k^2s\lambda_4^2 < 0, \tag{C10a}$$

$$pw + sv > 0, \tag{C10b}$$

which is stronger than condition (C8). Notice that condition (C10a) actually implies condition (C6), so the latter is not needed any more.

Conditions (C5b) and (C5c) require μ to lie within the domain $[\alpha, \beta]$. Those conditions translate into

$$\frac{ks|\lambda_4|}{\sqrt{p - s\alpha}} + \frac{w\sqrt{\lambda_1}}{\sqrt{v + w\alpha}} < 0, \tag{C11a}$$

$$\frac{ks|\lambda_4|}{\sqrt{p - s\beta}} + \frac{w\sqrt{\lambda_1}}{\sqrt{v + w\beta}} > 0. \tag{C11b}$$

It is readily seen that condition (C11a) is equivalent to $\mu > \alpha$, and that condition (C11b) is equivalent to $\mu < \beta$, when μ is given by Eq. (C4).

Finally, $f(\mu)$ is negative when

$$\lambda_3 + \sqrt{\frac{(pw + sv)\left(w\lambda_1 + k^2 s\lambda_4^2\right)}{sw}} < 0. \tag{C12}$$

To summarize, the *necessary and sufficient* conditions for $f(\theta)$ to be non-negative in the whole interval $[\alpha, \beta]$ are conditions (C3) if $k = 0$. If $k > 0$, then one must furthermore *exclude* the situation where all five inequalities (C10), (C11), and (C12) simultaneously hold.

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