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# Discrete Limit Bohr–Jessen Type Theorem for the Epstein Zeta-Function in Short Intervals

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**Abstract**

We prove a probabilistic limit theorem for the Epstein zeta-function $\zeta(s;Q)$ in the interval $[N, N+M]$ as $N \to \infty$, using discrete shifts $\zeta(\sigma + ikh; Q)$, where $h > 0$ and $\sigma > \frac{n-1}{2}$ are fixed. Here, $Q$ is a positive-definite $n \times n$ matrix, and the interval length $M$ satisfies $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$. The limit measure is given explicitly. This theorem is the first result in short intervals for $\zeta(s;Q)$. The obtained theorem improves the known results established for the interval of length $N$. Since the considered probability measures are defined in terms of frequency, theorems in short intervals have a certain advantage in the detection of $\zeta(\sigma + ikh; Q)$ with a given property, as well as in the characterization of the asymptotic behaviour of $\zeta(s;Q)$ in general.

**Keywords:** Bohr–Jessen theorem; Haar measure; Epstein zeta-function; convergence in distribution; weak convergence of probability measures; short intervals

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## 1. Introduction

Let $s = \sigma + it$ be a complex variable. The term "zeta-function" steams from the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

Thus, zeta-functions in the classical sense are analytic functions of a complex variable in some half-plane $\sigma > \sigma_0$ defined by Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

with coefficients $a_m$ of certain arithmetical nature, and have meromorphic continuation to the region $\sigma < \sigma_0$. In other words, zeta-functions are various generalizations of the function $\zeta(s)$.

The function $\zeta(s)$ appears to be quite simple; however, its value distribution is especially complicated. This is clearly illustrated by the Riemann hypothesis on the location of the nontrivial zeros of $\zeta(s)$. Roughly speaking, the value distribution of zeta-functions is chaotic, and information on their majority concrete values is insufficient. On the other hand, in the context of certain problems, precise information on values of $\zeta(s)$ is not necessary: certain average data are sufficient. This situation led H. Bohr to the idea that statistical methods could be applied for investigation of $\zeta(s)$ (see ref. [1]). By examining values of $\zeta(s)$ on vertical lines $\sigma + it$ with a fixed $\sigma$, one can take measurable sets $A \subset \mathbb{C}$ and consider the frequency of $t$ such that $\zeta(\sigma + it) \in A$. The first result of such a kind, for fixed $\sigma > 1$, was obtained in [2]. Denote by J the Jordan measure on the real line, and let $R$ be a rectangle with edges parallel to the coordinate axis. Then, the main result of [2] states that the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\{t \in [0, T] : \log \zeta(\sigma + it) \in R\}$$

exists and depends only on $R$ and $\sigma$. The case for $\sigma > \frac{1}{2}$ is more complicated than that of $\sigma > 1$, as it depends on the possible zeros of $\zeta(s)$. Therefore, in [3], the existence of the limit for the frequency

$$\frac{1}{T} \mathrm{J}\{t \in [0, T] : \sigma + it \in G, \log \zeta(\sigma + it) \in R\}$$

as $T \to \infty$ was considered. Here,

$$G = \left\{ s \in \mathbb{C} : \sigma > \frac{1}{2} \right\} \setminus \bigcup_{\varrho=\beta+i\gamma} \left\{ s = \sigma + i\gamma : \frac{1}{2} < \sigma \leq \beta \right\},$$

where the union is taken over all zeros $\varrho$ of $\zeta(s)$ located in the strip $\left\{ s \in \mathbb{C} : \sigma \in \left(\frac{1}{2}, 1\right) \right\}$. The proofs of these results are based on a theory of convex curves developed by the authors themselves [4].

Bohr–Jessen theorems were further developed in [5,6], leading to probabilistic limit theorems on weakly convergent probability measures. Let $\mathbb{X}$ be a topological space with Borel $\sigma$-field $\mathcal{B}(\mathbb{X})$, and let $P$ and $P_n$, $n \in \mathbb{N}$, be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Recall that $P_n$ converges weakly to $P$ as $n \to \infty$, denoted $P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$, if for every bounded continuous real-valued function $g$ on $\mathbb{X}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \, \mathrm{d}P_n = \int_{\mathbb{X}} g \, \mathrm{d}P.$$

In this terminology, Bohr–Jessen theorems can be rephrased as follows: on the space $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_\sigma$ such that, for fixed $\sigma > \frac{1}{2}$, and

$$P_{T,\sigma}(A) \stackrel{\mathrm{def}}{=} \frac{1}{T} \mathrm{meas}\{t \in [0, T] : \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

the relation $P_{T,\sigma} \xrightarrow[T\to\infty]{\mathrm{w}} P_\sigma$ holds (see, for example, refs. [7,8]). Here and throughout, $\mathrm{meas}\{*\}$ denotes the Lebesgue measure on the real line.

A new stage in the development of the probabilistic theory of zeta-functions was initiated by B. Bagchi. In his thesis [9], he created a theory of probabilistic limit theorems in the space of analytic functions and applied it to prove universality property of zeta-functions on approximation of classes of analytic functions. In his theorems, Bagchi proposed a new way for identification of the limit measure. In [10], we applied Bagchi's method to prove a limit theorem for the Epstein zeta-function.

Suppose that $Q$ is a positive-definite quadratic matrix of order $n \in \mathbb{N}$. The Epstein zeta-function $\zeta(s;Q)$ is defined, for $\sigma > \frac{n}{2}$, by the series

$$\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} \left(\underline{x}^{\mathrm{T}} Q \underline{x}\right)^{-s},$$

where $\underline{x}^{\mathrm{T}}$ is the transpose of $\underline{x}$. Moreover, $\zeta(s;Q)$ has analytic continuation to the whole complex plane, except for the point $s = \frac{n}{2}$, which is a simple pole with residue $\pi^{n/2}(\Gamma(\frac{n}{2})\sqrt{\det Q})^{-1}$, where $\Gamma(s)$ is the Euler gamma-function. The function $\zeta(s;Q)$ was introduced and studied in [11]. Its author aimed to define the most general zeta-function with the functional equation of the Riemann type, i.e.,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \quad s \in \mathbb{C}.$$

The attempt was successful: Epstein proved the following functional equation

$$\pi^{-s}\Gamma(s)\zeta(s;Q) = \sqrt{\det Q}\,\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right)\zeta\left(\frac{n}{2}-s;Q^{-1}\right), \quad s \in \mathbb{C},$$

for $\zeta(s;Q)$, where $Q^{-1}$ denotes the inverse of $Q$.

We observe that in a particular case, the function $\zeta(s;Q)$ can be expressed as an ordinary Dirichlet series. Actually, suppose that $\underline{x}^{\mathrm{T}} Q \underline{x} \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$, and, for $m \in \mathbb{N}$, $r_Q(m)$ is the number of $\underline{x} \in \mathbb{Z}^n$ such that $\underline{x}^{\mathrm{T}} Q \underline{x} = m$. Then, we have

$$\zeta(s;Q) = \sum_{m=1}^{\infty} \frac{r_Q(m)}{m^s}, \quad \sigma > \frac{n}{2}. \tag{1}$$

For example, for the $n$-dimensional unit matrix $I_n$ and $\sigma > 1$, $\zeta(s;I_1) = 2\zeta(2s)$, $\zeta(s;I_2) = 4\zeta(s)L(s,\chi_4)$, where $L(s,\chi)$ is the Dirichlet $L$-function, and $\chi_4$ is the non-principal Dirichlet character modulo 4.

Note that the Epstein zeta-function is not only an object of interest in pure mathematics, particularly in algebra and algebraic number theory, but it also has practical applications. In fact, the function $\zeta(s;Q)$ appears in crystallography [12], as well as in physics, including quantum-field theory [13], and in studies related to energy and temperature [14,15].

The function $\zeta(s;Q)$ is, in general, significantly more complicated than the Riemann zeta-function. For example, the analytic continuation and functional equation of $\zeta(s;Q)$ depend essentially on the signature and rank of the real quadratic form $Q$ on $\mathbb{R}^n$: when $Q$ is positive-definite, the function $\zeta(s;Q)$ admits a meromorphic continuation to the whole complex plane and satisfies a classical functional equation. In contrast, when $Q$ is indefinite, the analytic behavior of $\zeta(s;Q)$ becomes more intricate due to the presence of a continuous spectrum and its connections to non-holomorphic automorphic forms, such as Maass forms. Therefore, it is difficult to expect any general results about the value distribution characteristics of $\zeta(s;Q)$ for all matrices $Q$.

As previously mentioned, a Bohr–Jessen type theorem for $\zeta(s;Q)$ was obtained under the following restrictions in [10]. Based on (1), it was shown in [16] that

$$\zeta(s;Q) = \zeta(s;E_Q) + \zeta(s;F_Q),$$

where $\zeta(s;E_Q)$ and $\zeta(s;F_Q)$ are zeta-functions of a certain Eisenstein series and a cusp form, respectively, related to the coefficients $r_Q(m)$. Moreover, even for $n \geqslant 4$, it is known that the mentioned Eisenstein series is a modular form of weight $\frac{n}{2}$ and level $q \in \mathbb{N}$ such that $q(2Q)^{-1}$ is an integral matrix [17], and then the zeta-function $\zeta(s;E_Q)$ is a certain combination of Dirichlet $L$-functions [18]. This leads to the representation

$$\zeta(s;Q)=\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{a_{kl}}{k^s l^s}L(s,\chi_k)L\Big(s-\frac{n}{2}+1,\psi_l\Big)+\sum_{m=1}^{\infty}\frac{b_Q(m)}{m^s}, \tag{2}$$

where $k$ and $l$ run over positive divisors of the level $q$, and $K$ and $L$ are finite integers; the characters $\chi_k$, $1\leqslant k\leqslant K$, and $\psi_l$, $1\leqslant l\leqslant L$ are the pairwise nonequivalent Dirichlet characters modulo $q/k$ and $q/l$, respectively, and $L(s,\chi_k)$ and $L(s,\psi_l)$ are the corresponding Dirichlet $L$-functions. The coefficients $a_{kl}\in\mathbb{C}$ are certain complex numbers. Furthermore, the series with coefficients $b_Q(m)$ is absolutely convergent for $\sigma>\frac{n-1}{2}$.

The formulation of the main limit theorem relies on the following object

$$\Omega=\prod_{p\in\mathbb{P}}\{s\in\mathbb{C}:|s|=1\},$$

where $\mathbb{P}$ is the set of all prime numbers. With the product topology and operation of pairwise multiplication, $\Omega$ is a compact topological group, and this leads to the probability space $(\Omega,\mathcal{B}(\Omega),\mu)$, where $\mu$ is the probability Haar measure on $(\Omega,\mathcal{B}(\Omega))$. Denote by $\omega=(\omega(p):p\in\mathbb{P})$ the elements of $\Omega$, and extend $\omega(p)$ to the set $\mathbb{N}$ by using the formula

$$\omega(m)=\prod_{\substack{p^l|m\\ p^{l+1}\nmid m}}\omega^l(p),\quad m\in\mathbb{N}.$$

Now, on the probability space $(\Omega,\mathcal{B}(\Omega),\mu)$, define the complex-valued random element

$$\begin{aligned}\zeta(\sigma,\omega;Q)\ &=\ \sum_{k=1}^{K}\sum_{l=1}^{L}\frac{a_{kl}\omega(k)\omega(l)}{k^\sigma l^\sigma}L(\sigma,\omega,\chi_k)L\Big(\sigma-\frac{n}{2}+1,\omega,\psi_l\Big)\\ &\quad+\sum_{m=1}^{\infty}\frac{b_Q(m)\omega(m)}{m^\sigma},\quad \sigma>\frac{n-1}{2},\end{aligned} \tag{3}$$

where

$$L(\sigma,\omega,\chi_k)=\prod_p\Big(1-\frac{\chi_k(p)\omega(p)}{p^\sigma}\Big)^{-1}\text{ and }L\Big(\sigma-\frac{n}{2}+1,\omega,\varphi_l\Big)=\prod_p\Big(1-\frac{\varphi_l(p)\omega(p)}{p^{\sigma-n/2+1}}\Big)^{-1}.$$

Denote by $P_{\zeta,\sigma}$ the distribution

$$P_{\zeta,\sigma}(A)=\mu\{\omega\in\Omega:\zeta(\sigma,\omega;Q)\in A\},\quad A\in\mathcal{B}(\mathbb{C}),$$

of the random element $\zeta(\sigma,\omega;Q)$. Then, the main result of [10] is the following.

**Proposition 1** (Theorem 2 of [10]). *Suppose that $\sigma>\frac{n-1}{2}$ is fixed. Then, the probability measure*

$$\frac{1}{T}\text{meas}\{t\in[0,T]:\zeta(\sigma+it;Q)\in A\},\quad A\in\mathcal{B}(\mathbb{C}),$$

*converges weakly to the measure $P_{\zeta,\sigma}$ as $T\to\infty$.*

In [19], a discrete version of Proposition 1 is presented. To state it, some definitions are needed. A number $h$ is called type 1 if the number $\exp\{\frac{2\pi k}{h}\}$ is irrational for all $k \in \mathbb{N}$. In the opposite case, $h$ is type 2. Then, in this case, there exists the smallest $r_0$ such that

$$\exp\left\{\frac{2\pi r_0}{h}\right\} = \frac{a}{b} \text{ with cooprimes } a, b \in \mathbb{N}.$$

Let $\mathbb{P}_0$ be a subset of $\mathbb{P}$:

$$\mathbb{P}_0 = \left\{p \in \mathbb{P} : \prod_{p\in\mathbb{P}} p^{l_p} = \frac{a}{b}, \quad l_p \neq 0\right\}.$$

Then, $\#\mathbb{P}_0 < \infty$, where $\#A$ denotes the cardinality of the set $A$.

We now return to the group $\Omega$. For $h > 0$, let $\Omega_h$ be the closed subgroup of $\Omega$ generated by $\left\{p^{-ih} : p \in \mathbb{P}\right\}$. Then, $\Omega_h$, as $\Omega$, is a compact group. Therefore, on $(\Omega_h, \mathcal{B}(\Omega_h))$, the probability Haar measure $\mu_h$ can be defined. It is known (see Lemma 4.2.2 of [9] and Lemma 1 of [20]) that

$$\Omega_h = \begin{cases} \Omega & \text{if } h \text{ is of type 1,} \\ \{\omega \in \Omega : \omega(a) = \omega(b)\} & \text{otherwise.} \end{cases}$$

Let $\zeta_h(\sigma, \omega; Q)$, for $\omega \in \Omega_h$, be the complex-valued random element on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), \mu_h)$ given by the Representation (3), and $P^h_{\zeta,\sigma}$ denote its distribution. Let $N$ run over the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, in [19], the following statement has been obtained.

**Proposition 2** (Theorem 1 of [19])**.** *Suppose that $\sigma > \frac{n-1}{2}$ and $h > 0$ are fixed. Then, the probability measure*

$$\frac{1}{N}\#\{0 \leqslant k \leqslant N : \zeta(\sigma + ikh; Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

*converges weakly to the measure $P^h_{\zeta,\sigma}$ as $N \to \infty$.*

In Propositions 1 and 2, the weak convergence of probability measures defined by frequencies in the intervals $[0, T]$ and $[0, N]$ is discussed. It is well known that frequencies in short intervals contain more information on the discussed objects. This motivates the refinement of the limit theorems in Propositions 1 and 2, leading to limit theorems in short intervals, i.e., intervals of length shorter than $T$ and $N$. A result of this type, corresponding to Proposition 1, was obtained in [21].

**Proposition 3** (Theorem 2 of [21])**.** *Suppose that $\sigma > \frac{n-1}{2}$ is fixed, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then, the probability measure*

$$\frac{1}{H}\text{meas}\{t \in [T, T+H] : \zeta(\sigma + it; Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

*converges weakly to the measure $P_{\zeta,\sigma}$ as $T \to \infty$.*

The purpose of this paper is to develop a more complex version of Proposition 2 in short intervals. For brevity, we use the notation

$$D_{N,M}(\dots) = \frac{1}{M+1}\#\{N \leqslant k \leqslant N + M : \dots\}, \quad M \in \mathbb{N},$$

where the dots indicate a condition to be satisfied by $k$. The main result is the following theorem.

**Theorem 1.** *Suppose that $\sigma > \frac{n-1}{2}$ and $h > 0$ are fixed, and $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$. Then, the probability measure*

$$P^h_{\zeta,\sigma,N,M}(A) = D_{N,M}(\zeta(\sigma + ikh; Q) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

*converges weakly to the measure $P^h_{\zeta,\sigma}$ as $N \to \infty$.*

Theorem 1 is theoretical, it extends and develops Bohr–Jessen's ideas. Recall that even the eminent mathematician Atle Selberg, who was awarded the Fields Medal, devoted much attention to probabilistic limit theorems for zeta-functions; see [22,23]. Note that results concerning short intervals are highly valued in analytic number theory, especially those related to the distribution of zeros of zeta-functions and prime numbers. Moreover, problems in short intervals are more complicated. Theorem 1 continues investigations in this direction, providing new results that confirm the chaotic behaviour of the function $\zeta(s; Q)$. The results presented here could be useful for researchers studying this function in applied mathematics.

Theorem 1 will be proved in Section 3; the cases where $h$ is type 1 and type 2 will be considered separately. Before that, we prove limit lemmas (Lemmas 7 and 8) for probability measures defined on the one-dimensional torus $\Omega_h$, the structure of which depends on the arithmetic nature of the number $h$. Using Lemmas 7 and 8, we obtain limit lemmas (Lemmas 11 and 13) for probability measures defined in terms of $\zeta_r(s; Q)$ involving absolutely convergent Dirichlet series. Section 2 is devoted to certain discrete mean estimates for $\zeta(s; Q)$ in short intervals. Lemma 6 occupies a central place with respect to short intervals. It shows that the functions $\zeta(s; Q)$ and $\zeta_r(s; Q)$ are close in the mean in such intervals.

## 2. Estimates in Short Intervals

We start with recalling the mean square estimate for the Hurwitz zeta-function in short intervals. Let $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha]$, for $\sigma > 1$, is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and has meromorphic continuation to the whole complex plane with the unique simple pole at the point $s = 1$ with residue 1.

**Lemma 1** (Theorem 2 of [24])**.** *Suppose that $\alpha \in (0, 1)$ and $\frac{1}{2} < \sigma \leqslant \frac{7}{12}$ are fixed, and $T^{27/82} \leqslant H \leqslant T^{\sigma}$. Then, uniformly in $H$, the estimate*

$$\int_{T-H}^{T+H} |\zeta(\sigma + it, \alpha)|^2 \mathrm{d}t \ll_{\sigma,\alpha} H$$

*holds.*

We recall that the notation $z \ll_\theta x$, $z \in \mathbb{C}$, $x > 0$ shows that there is a constant $c = c(\theta) > 0$ such that $|z| \leqslant cx$.

Let $\chi$ be an arbitrary Dirichlet character modulo $q \in \mathbb{N}$, and let $L(s,\chi)$ be the corresponding Dirichlet $L$ function. For $\sigma > 1$, the function $L(s,\chi)$ is given by the series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

The function has analytic continuation to the whole complex plane if $\chi$ is a non-principal character, and has a simple pole at the point $s = 1$ with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right), \quad p \in \mathbb{P},$$

if $\chi$ is the principal character modulo $q$ ($\chi$ is the principal character modulo $q$ if $\chi(m) = 1$ for all $m \in \mathbb{N}$ coprime to $q$).

**Lemma 2.** *Suppose that $\frac{1}{2} < \sigma \leqslant \frac{7}{12}$ is fixed, and $T^{27/82} \leqslant H \leqslant T^{\sigma}$. Then, uniformly in $H$ the estimate*

$$\int_{T-H}^{T+H} |L(\sigma + it, \chi)|^2 \mathrm{d}t \ll_{\sigma,q} H$$

*holds.*

**Proof.** We use the formula

$$L(s,\chi) = \frac{1}{q^s} \sum_{k=1}^{q} \chi(k) \zeta\left(s, \frac{k}{q}\right), \tag{4}$$

see, for example, [25]. Since $|\chi(k)| \leqslant 1$, we have

$$L(s,\chi) \ll_q \sum_{k=1}^{q} \left|\zeta\left(s, \frac{k}{q}\right)\right| \ll_q \max_{1 \leqslant k \leqslant q} \sum_{k=1}^{q} \left|\zeta\left(s, \frac{k}{q}\right)\right|.$$

Therefore,

$$\int_{T-H}^{T+H} |L(\sigma + it, \chi)|^2 \mathrm{d}t \ll_q \sum_{k=1}^{q} \int_{T-H}^{T+H} \left|\zeta\left(\sigma + it, \frac{m}{q}\right)\right|^2 \mathrm{d}t \ll_{\sigma,q} H$$

in virtue of Lemma 1. □

For our purposes, we need a discrete version of Lemma 2. For this, we will apply the Gallagher lemma on the connection of continuous and discrete mean squares of certain functions (see refs. [26,27]).

**Lemma 3** (Lemma 1.4 of [27])**.** *For $\delta > 0$, suppose that $T_1, T_2 \geqslant \delta$, $\mathcal{A} \neq \emptyset$ is a finite set, $\mathcal{A} \subset [T_1 + \delta/2, T_1 + T_2 - \delta/2]$, and, for $t \in \mathcal{A}$,*

$$M_\delta(t) = \sum_{\substack{\tau \in \mathcal{A} \\ |t-\tau| < \delta}} 1.$$

*Let $g(t)$ be a continuous function in the interval $[T_1, T_1 + T_2]$, which has a continuous derivative $g'(t)$ in $(T_1, T_1 + T_2)$. Then, the inequality*

$$\sum_{t \in \mathcal{A}} M_\delta^{-1}(t) |g(t)|^2 \leqslant \frac{1}{\delta} \int_{T_1}^{T_1+T_2} |g(t)|^2 \mathrm{d}t + \left( \int_{T_1}^{T_1+T_2} |g(t)|^2 \mathrm{d}t \int_{T_1}^{T_1+T_2} |g'(t)|^2 \mathrm{d}t \right)^{1/2}$$

*is valid.*

**Lemma 4.** *Suppose that* $\frac{1}{2} < \sigma \leqslant \frac{7}{12}$ *and* $h > 0$ *are fixed:* $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$ *and* $|t| \leqslant \log^2(Nh)$. *Then, uniformly in M, the estimate*

$$\sum_{k=N}^{N+M} |L(\sigma + ikh + it, \chi)|^2 \mathrm{d}t \ll_{\sigma,q,h} M(1+|t|)$$

*holds.*

**Proof.** We will apply Lemmas 2 and 3. In Lemma 3, take $\delta = 1$, $T_1 = N - \frac{1}{2}$, $T_2 = M + 1$, $\mathcal{A} = \{k \in \mathbb{N} : k \in [N, N+M]\}$, and $g(\tau) = L(\sigma + it + i\tau, \chi)$. Then, obviously,

$$M_\delta(k) = \sum_{\substack{m \in \mathcal{A} \\ |m-k|<1}} 1 = 1.$$

Therefore, in view of Lemma 3,

$$\sum_{k=N}^{N+M} |L(\sigma + ikh + it, \chi)|^2 \ll \int_{N-1/2}^{N+M+1/2} |L(\sigma + it + ih\tau, \chi)|^2 \mathrm{d}\tau$$
$$+ \left( \int_{N-1/2}^{N+M+1/2} |L(\sigma + it + ih\tau, \chi)|^2 \mathrm{d}t \int_{N-1/2}^{N+M+1/2} \left|L'(\sigma + it + ih\tau, \chi)\right|^2 \mathrm{d}t \right)^{1/2}. \quad (5)$$

It is easily seen that

$$\int_{N-1/2}^{N+M+1/2} |L(\sigma + it + ih\tau, \chi)|^2 \mathrm{d}\tau \ll_h \int_{(N-1/2)h-|t|}^{(N+M+1/2)h+|t|} |L(\sigma + i\tau, \chi)|^2 \mathrm{d}\tau. \quad (6)$$

Moreover, since $(M + \frac{1}{2})h + |t| \geqslant Mh + |t| \geqslant (Nh)^{27/82}$ and $(M + \frac{1}{2})h + |t| \leqslant (Nh)^{1/2} + \frac{h}{2} + \log^2(Nh) \leqslant (Nh)^\sigma$ because $\sigma > \frac{1}{2}$. These remarks allow us to apply Lemma 2. Thus, in view of Estimate (6),

$$\int_{N-1/2}^{N+M+1/2} |L(\sigma + it + ih\tau, \chi)|^2 \mathrm{d}\tau \ll_{\sigma,q,h} Mh + \frac{h}{2} + |t| \ll_{\sigma,q,h} M(1+|t|). \quad (7)$$

Application of the Cauchy integral formula

$$L'(s + it + ih\tau, \chi) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{L(z + it + ih\tau, \chi)}{(z-s)^2} \mathrm{d}z,$$

where $\mathcal{L}$ is a suitable simple closed contour enclosing $s$, and (7) gives

$$\int_{N-1/2}^{N+M+1/2} \left|L'(\sigma + it + ih\tau, \chi)\right|^2 \mathrm{d}\tau \ll_{\sigma,q,h} M(1+|t|).$$

From this and estimates (5) and (7), the lemma follows. □

This result ensures that we can bound discrete means analogously to continuous ones, which is essential for our approximation step.

Now, we will utilise Lemma 4 for approximation of the function $\zeta(s;Q)$ by a certain simple function involving absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, $r \in \mathbb{N}$, and

$$w_r(m) = \exp\left\{ -\left(\frac{m}{r}\right)^{\theta} \right\}, \quad m \in \mathbb{N}.$$

Return to the representation (2), and define

$$L_r(s_n, \psi_l) = \sum_{m=1}^{\infty} \frac{\psi_l(m) w_r(m)}{m^{s_n}}, \quad s_n = s - \frac{n}{2} + 1.$$

In virtue of the exponential decreasing with respect to $m$ for $w_r(m)$, the series for $L_r(s_n, \psi_l)$ is absolutely convergent in any half-plane $\sigma > \sigma_0$ with finite $\sigma_0$. Using $L_r(s_n, \psi_l)$, we introduce

$$\zeta_r(s;Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s, \chi_k) L_r(s_n, \psi_l) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^s},$$

and we will approximate $\zeta(s;Q)$ by $\zeta_r(s;Q)$. For this, we will use a certain integral representation for $L_r(s_n, \psi_l)$. Set

$$\kappa_r(s) = \frac{1}{\theta} \Gamma\left(\frac{s}{\theta}\right) r^s,$$

where $\theta > \frac{1}{2}$ is the number from the definition of $w_r(m)$.

**Lemma 5.** *The representation*

$$L_r(s_n, \psi_l) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} L(s_n + z, \psi_l) \kappa_r(z) \mathrm{d}z$$

*is valid.*

**Proof.** By the Mellin formula

$$\frac{1}{2\pi i} \int_{b - i\infty}^{b + i\infty} \Gamma(z) a^{-z} \mathrm{d}z = \mathrm{e}^{-a}, \quad a, b > 0,$$

and definitions of $w_r(m)$ and $\kappa_r(z)$, we have

$$w_r(m) = \frac{1}{2\pi i} \int_{1 - i\infty}^{1 + i\infty} \Gamma(z) \left(\frac{m}{r}\right)^{-\theta z} \mathrm{d}z = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \frac{1}{\theta} \Gamma\left(\frac{r}{\theta}\right) \frac{r^z}{m^z} \mathrm{d}z = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \kappa_r(z) \frac{\mathrm{d}z}{m^z}.$$

Therefore,

$$\frac{\psi_l(m) w_r(m)}{m^{s_n}} = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \frac{\psi_l(m)}{m^{s_n + z}} \kappa_r(z) \mathrm{d}z. \quad (8)$$

Since, for the Gamma function, the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \quad (9)$$

uniformly in $\sigma \in [\sigma_1, \sigma_2]$ with arbitrary $\sigma_1 < \sigma_2$ is valid, and $\Re s_n + \theta > 1$ for $\sigma > \frac{n-1}{2}$, from (8) we find that

$$L_r(s_n, \psi_l) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \left( \sum_{m=1}^{\infty} \frac{\psi_l(m)}{m^{s_n + z}} \right) \kappa_r(z) \mathrm{d}z = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} L(s_n + z, \psi_l) \kappa_r(z) \mathrm{d}z.$$

□

**Lemma 6.** *Suppose that $\sigma > \frac{n-1}{2}$ is fixed, and $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$. Then,*

$$\lim_{r \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=N}^{N+M} |\zeta(\sigma + ikh; Q) - \zeta_r(\sigma + ikh; Q)| = 0.$$

**Proof.** We begin by proving that

$$\lim_{r \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=N}^{N+M} |L(\sigma_n + ikh, \psi_l) - L_r(\sigma_n + ikh, \psi_l)| = 0, \quad (10)$$

where $\sigma_n = \sigma - \frac{n}{2} + 1$. First, we consider the case $\frac{n-1}{2} < \sigma \leqslant \frac{n}{2} + \frac{1}{4}$. To proceed, we will apply Lemma 5 and the residue theorem. Let $\varepsilon > 0$ be a small fixed number, take $\theta = \frac{n-1}{2} + \varepsilon$, and $\frac{n-1}{2} + 2\varepsilon \leqslant \sigma \leqslant \frac{n}{2} + \frac{1}{4}$. Moreover, let $\theta_1 = \frac{n-1}{2} + \varepsilon - \sigma$. Clearly, $\theta_1 < 0$ and $\theta_1 \geqslant \varepsilon - \frac{3}{4}$. Therefore, the function

$$L(s_n + z, \psi_l) \kappa_r(z)$$

in the strip $\theta_1 \leqslant \Re z \leqslant \theta$ has a simple pole at the point $z = 0$, which is the pole of $\Gamma\left(\frac{z}{\theta}\right)$, and a simple pole at the point $z = 1 - s_n$ of $L(s_n + z, \psi_l)$ if $\psi_l$ is the principal character modulo $q/l$. These observations, Lemma 5, and the residue theorem show that

$$L_r(s_n, \psi_l) - L(s_n, \psi_l) = \frac{1}{2\pi i} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} L(s_n + z, \psi_l) \kappa_r(z) \mathrm{d}z + R_r(s_n),$$

where

$$R_r(s_n) = \operatorname*{Res}_{z = 1 - s_n} L(s_n + z, \psi_l) \kappa_r(z) = \begin{cases} \kappa_r(1 - s_n) & \text{if } \psi_l \text{ is the principal character,} \\ 0 & \text{otherwise.} \end{cases}$$

From the latter equality, we derive

$$\begin{aligned} L_r(s_n, \psi_l) - L(s_n, \psi_l) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} L\left(s_n + \frac{n-1}{2} + \varepsilon - \sigma + iv, \psi_l\right) \\ &\quad \times \kappa_r\left(\frac{n-1}{2} + \varepsilon - \sigma + iv\right) \mathrm{d}v \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} L\left(\frac{1}{2} + \varepsilon + it + iv, \psi_l\right) \kappa_r\left(\frac{n-1}{2} + \varepsilon - \sigma + iv\right) \mathrm{d}v \\ &\quad + \kappa_r(1 - s_n). \end{aligned}$$

Hence,

$$\frac{1}{M+1} \sum_{k=N}^{N+M} \left| L_r\left(\sigma + ikh - \frac{n}{2} + 1, \psi_l\right) - L\left(\sigma + ikh - \frac{n}{2} + 1, \psi_l\right) \right|$$

$$
\begin{aligned}
&\ll \int_{-\infty}^{\infty}\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\right|\right)\left|\kappa_r\left(\frac{n-1}{2}+\varepsilon-\sigma+iv\right)\right|\mathrm{d}v \\
&+\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\kappa_r\left(1-\sigma-ikh+\frac{n-1}{2}-1\right)\right| \\
=&\left(\int_{-\infty}^{-\log^2 Nh}+\int_{\log^2 Nh}^{\infty}+\int_{-\log^2 Nh}^{\log^2 Nh}\right)\frac{1}{M+1}\sum_{k=N}^{N+M}\left|L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\right| \\
&\times\left|\kappa_r\left(\frac{n-1}{2}+\varepsilon-\sigma+iv\right)\right|\mathrm{d}v+\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\kappa_r\left(1-\sigma-ikh+\frac{n}{2}-1\right)\right|. \qquad (11)
\end{aligned}
$$

By the definition of $\kappa_r(s)$ and Estimate (9), we obtain

$$
\kappa_r\left(\frac{n-1}{2}+\varepsilon-\sigma+iv\right)\ll r^{(n-1)/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|v|\right\}\ll_\varepsilon r^{-\varepsilon}\exp\{-c_1|v|\},\quad c_1>0. \qquad (12)
$$

Moreover, it is known (see, for example, ref. [28]) that, for $\sigma>\frac{1}{2}$,

$$
\zeta(\sigma+it,\alpha)\ll_{\sigma,\alpha}|t|^{\frac{1}{2}},\quad \sigma\geqslant\frac{1}{2}.
$$

Hence, in view of Representation (4),

$$
L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\ll_{\varepsilon,q}(kh)^{\frac{1}{2}}+|v|^{\frac{1}{2}}. \qquad (13)
$$

Therefore, by Equation (12),

$$
\begin{aligned}
&\left(\int_{-\infty}^{\log^2 Nh}+\int_{\log^2 Nh}^{\infty}\right)\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\right|\right) \\
&\times\left|\kappa_r\left(\frac{n-1}{2}+\varepsilon-\sigma+iv\right)\right|\mathrm{d}v \\
\ll_{\varepsilon,q}\ & r^{-\varepsilon}\left(\int_{-\infty}^{\log^2 Nh}+\int_{\log^2 Nh}^{\infty}\right)\left((2N)^{\frac{1}{2}}+|v|^{\frac{1}{2}}\right)\exp\{-c_1|v|\}\mathrm{d}v \\
\ll_{\varepsilon,q,h}\ & r^{-\varepsilon}N^{\frac{1}{2}}\exp\{-c_2\log^2 Nh\},\quad c_2>0.
\end{aligned}
$$

For $v\in[-\log^2 Nh,\log^2 Nh]$, we apply Lemma 4. Thus,

$$
\begin{aligned}
\frac{1}{M+1}\sum_{k=N}^{N+M}\left|L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\right| &\ll \left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\right|^2\right)^{1/2} \\
&\ll_{\varepsilon,q,h} (1+|v|)^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{-\log^2 Nh}^{\log^2 Nh}\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|L\left(\frac{1}{2}+\varepsilon+ikh+iv,\psi_l\right)\right|\right)\left|\kappa_r\left(\frac{n-1}{2}+\varepsilon-\sigma+iv\right)\right|\mathrm{d}v \\
\ll_{\varepsilon,q,h}\ & r^{-\varepsilon}\int_{-\log^2 Nh}^{\log^2 Nh}(1+|v|)^{\frac{1}{2}}\exp\{-c_1|v|\}\mathrm{d}v
\end{aligned}
$$

$$\ll_{\varepsilon,q,h} \quad r^{-\varepsilon} \int_{-\infty}^{\infty} (1+|v|)^{\frac{1}{2}} \exp\{-c_1|v|\} \mathrm{d}v \ll_{\varepsilon,q} r^{-\varepsilon}. \tag{14}$$

Using Estimate (9) again yields

$$\kappa_r\Big(1-\sigma-ikh+\frac{n}{2}-1\Big) \ll r^{-\sigma+n/2-1} \exp\{-c_3kh\} \ll r^{1/2-2\varepsilon} \exp\{-c_3kh\}, \quad c_3 > 0.$$

Hence,

$$\frac{1}{M+1}\sum_{k=N}^{N+M}\Big|\kappa_r\Big(-\sigma-ikh+\frac{n}{2}\Big)\Big| \ll r^{1/2-2\varepsilon}\frac{1}{M+1}\sum_{k=N}^{N+M}\exp\{-c_3kh\} \ll r^{1/2-2\varepsilon}\exp\{-c_3Nh\}.$$

This, together with (11), (13), and (14), shows that the left-hand side of (10) is estimated as

$$\ll_{\varepsilon,q,h} r^{-\varepsilon} N^{\frac{1}{2}} \exp\{-c_2 \log^2 Nh\} + r^{-\varepsilon} + r^{1/2-2\varepsilon} \exp\{-c_3Nh\}.$$

Thus, letting $N \to \infty$ and subsequently $r \to \infty$, we obtain the Equality (10).

The case of $\sigma > \frac{n}{2} + \frac{1}{4}$ is simpler due to the absolute convergence of the series for $L(\sigma_n, \psi_l)$. Thus, with small $\varepsilon > 0$, we have

$$L(\sigma-\varepsilon+ikh+iv, \psi_l) \ll_{\sigma,\varepsilon,q} 1,$$

for $\sigma - \varepsilon \geqslant \frac{n}{2} + \frac{1}{4}$. Hence,

$$\frac{1}{M+1}\sum_{k=N}^{N+M}\Big|L_r\Big(\sigma+ikh-\frac{n}{2}+1,\psi_l\Big)-L\Big(\sigma+ikh-\frac{n}{2}+1,\psi_l\Big)\Big|$$
$$\ll_{\sigma,\varepsilon,q} \int_{-\infty}^{\infty}|\kappa_r(-\varepsilon+iv)|+\frac{1}{M+1}\sum_{k=N}^{N+M}\Big|\kappa_r\Big(1-\sigma-ikh+\frac{n}{2}-1\Big)\Big|,$$

and this together with (9) proves the lemma in this case. □

## 3. Proof of the Theorem 1

For the proof of Theorem 1, we will apply the method of Fourier transforms, the preservation of weak convergence of probability measures under certain mappings, as well as a connection of weak convergence and convergence in distribution. For the identification of the limit measure, we will apply the results of [10,19].

We start with a limit lemma for probability measures on $(\Omega, \mathcal{B}(\Omega))$. For $A \in \mathcal{B}(\Omega)$, set

$$Q^h_{N,M}(A) = D_{N,M}\Big(\Big(p^{-ikh} : p \in \mathbb{P}\Big) \in A\Big).$$

**Lemma 7.** *Suppose that $h > 0$ is type 1, and $M \to \infty$ as $N \to \infty$. Then, $Q^h_{N,M}$ converges weakly to the Haar measure $\mu$ as $N \to \infty$.*

**Proof.** The group $\Omega$ is compact, as the product of compact sets. Therefore, it provides a reason to study the Fourier transform of the measure $Q^h_{N,M}$, and, from its convergence, derive the weak convergence for $Q^h_{N,M}$. Denote by $\hat{\Omega}$ the dual group, or character group, of $\Omega$. It is a well-known and widely used fact that $\hat{\Omega}$ is isomorphic to the group

$$G \stackrel{\text{def}}{=} \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p,$$

where $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathbb{P}$, and an element $\underline{l} = (l_p : l_p \in \mathbb{N}) \in G$, where only a finite number of integers $l_p$ are non-zero, acts on $\hat{\Omega}$ by the formula

$$\omega \longmapsto \omega^{\underline{l}} = \prod_{p \in \mathbb{P}} \omega^{l_p}, \quad \omega = (\omega(p) : p \in \mathbb{P}) \in \Omega.$$

From this, it follows that the characters of $\Omega$ are given by the product

$$\prod_{p \in \mathbb{P}}^{*} \omega^{l_p}(p),$$

where the sign "*" shows that only a finite number of $l_p \neq 0$. This implies that the Fourier transform $f_{N,M}(\underline{l})$ of the measure $Q^h_{N,M}$ is

$$f_{N,M}(\underline{l}) = \int_{\Omega} \left( \prod_{p \in \mathbb{P}}^{*} \omega^{l_p}(p) \right) \mathrm{d}Q^h_{N,M}.$$

Hence, the definition of the measure $Q^h_{N,M}$ yields

$$f_{N,M}(\underline{l}) = \frac{1}{M+1} \sum_{k=N}^{N+M} \left( \prod_{p \in \mathbb{P}}^{*} p^{-ikhl_p} \right) = \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\left\{ -ikh \sum_{p \in \mathbb{P}}^{*} l_p \log p \right\}. \tag{15}$$

For brevity, let

$$L_h(\underline{l}) \stackrel{\mathrm{def}}{=} h \sum_{p \in \mathbb{P}}^{*} l_p \log p.$$

We observe that the requirement that $h$ is type 1 is equivalent to the linear independence over the field of rational numbers $\mathbb{Q}$ for the multi-set $\left\{ (\log p : p \in \mathbb{P}), \frac{2\pi}{h} \right\} \stackrel{\mathrm{def}}{=} A_h$. Actually, if the set $A_h$ is linearly dependent over $\mathbb{Q}$, then there exist the numbers $k_1, \ldots, k_r \in \mathbb{Z}$, and $m \in \mathbb{Z} \backslash \{0\}$ that

$$k_1 \log p_1 + \cdots + k_r \log p_r + \frac{2\pi m}{h} = 0,$$

and

$$\exp\left\{ \frac{2\pi m}{h} \right\} = p_1^{k_1} \cdots p_r^{k_r},$$

and this contradicts the definition of type 1. Thus, $h$ of type 1 implies the linear independence of $A_h$. Similarly, if $h$ is not type 1, then there exists $m_0 \in \mathbb{Z} \backslash \{0\}$ such that

$$\exp\left\{ \frac{2\pi m_0}{h} \right\} = p_1^{k_1} \cdots p_r^{k_r}$$

is a rational number. Hence, the set $A_h$ is linearly dependent over $\mathbb{Q}$. This shows that linear independence of $A_h$ implies the type 1 for $h$.

The above remarks imply that

$$hL_h(\underline{l}) = 2\pi m$$

with $m \in \mathbb{Z}$ if and only if $\underline{l} = \underline{0}$ and $m = 0$. Hence, in view of Expression (15),

$$f_{N,M}(\underline{l}) = 1 \tag{16}$$

if and only if $\underline{l} = \underline{0}$. If $\underline{l} \neq \underline{0}$, then $L_h(\underline{l}) \neq 0$, and (15) gives

$$f_{N,M}(\underline{l}) = \frac{\exp\{-iNL_h(\underline{l})\} - \exp\{-i(N+M+1)L_h(\underline{l})\}}{(M+1)(1 - \exp\{-iL_h(\underline{l})\})}.$$

Hence, for $\underline{l} \neq \underline{0}$,

$$\lim_{N\to\infty} f_{N,M}(\underline{l}) = 0. \tag{17}$$

Let

$$f(\underline{l}) = \begin{cases} 1 & \text{if } \underline{l} = \underline{0}, \\ 0 & \text{if } \underline{l} \neq \underline{0}. \end{cases}$$

Then, Equalities (16) and (17) show that

$$\lim_{N\to\infty} f_{N,M}(\underline{l}) = f(\underline{l}). \tag{18}$$

Since $f(\underline{l})$ is the Fourier transform of the Haar measure $\mu$ on $(\Omega, \mathcal{B}(\Omega))$, and $\Omega$, as a compact group, is the Lévy group, in view of Theorem 1.4.2 of [29] and Equation (18), we determine that the measure $Q^h_{N,M}$ converges weakly to $\mu$ as $N \to \infty$. $\square$

Now consider the case with $h$ of type 2.

**Lemma 8.** *Suppose that $h > 0$ is type 2, and $M \to \infty$ as $N \to \infty$. Then, $Q^h_{N,M}$ converges weakly to the Haar measure $\mu^h$ as $N \to \infty$.*

**Proof.** It is known, see the proof of Lemma 4 of [19], that in this case, characters of $\Omega_h$ are of the form

$$\prod_{p\in\mathbb{P}\setminus\mathbb{P}_0} \omega^{l_p}(p) \prod_{p\in\mathbb{P}_0} \omega^{l_p+l\beta_p}(p), \quad l \in \mathbb{Z},$$

and only a finite number of integers $l_p$ are not zero. Here,

$$\mathbb{P}_0 = \left\{ p \in \mathbb{P} : \prod_{p\in\mathbb{P}} p^{\alpha_p} = \frac{a}{b}, \quad \alpha_p \neq 0 \right\}$$

is a finite subset of $\mathbb{P}$. Hence, the Fourier transform $f_{N,M}(\underline{l})$ of $Q^h_{N,M}$ is

$$\begin{aligned} f_{N,M}(\underline{l}) &= \frac{1}{M+1} \sum_{k=N}^{N+M} \prod_{p\in\mathbb{P}\setminus\mathbb{P}_0} p^{-ikhl_p} \prod_{p\in\mathbb{P}_0} p^{-ikh(l_p+l\beta_p)} \\ &= \frac{1}{M+1} \sum_{k=N}^{N+M} \prod_{p\in\mathbb{P}\setminus\mathbb{P}_0} p^{-ikhl_p} \prod_{p\in\mathbb{P}_0} p^{-ikhl_p} \end{aligned} \tag{19}$$

due to the equality

$$\prod_{p\in\mathbb{P}_0} p^{-ikhl\beta_p} = \frac{a^{-ikhl}}{b^{-ikhl}} = 1, \quad l \in \mathbb{Z}, \tag{20}$$

since $\omega(a) = \omega(b)$ for $\omega \in \Omega_h$. Taking into account (19) and (20), we find that

$$f_{N,M}(\underline{l}) = 1 \tag{21}$$

for $l_p = 0$, $p \in \mathbb{P}\setminus\mathbb{P}_0$, and $l_p = l\beta_p$, $p \in \mathbb{P}_0$.

Now, suppose that $l_p \neq 0$ for some $p \in \mathbb{P}\setminus\mathbb{P}_0$, or there is $l \in \mathbb{Z}$ such that $l_p \neq l\beta_p$ for all $p \in \mathbb{P}_0$. Then, we have

$$\exp\left\{ -ih\left( \sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} l_p \log p + \sum_{p\in\mathbb{P}_0} (l_p + l\beta_p) \log p \right) \right\} \neq 1. \tag{22}$$

Actually, if (22) is not true, then

$$\exp\left\{\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} l_p \log p + \sum_{p\in\mathbb{P}_0} (l_p + l\beta_p)\log p\right\} = \exp\left\{\frac{2\pi l_0}{h}\right\} \tag{23}$$

with some integer $l_0$. If $l_0$ is a multiple of $r_0$ ($r_0$ is from the definition of type 2), then

$$\exp\left\{\frac{2\pi l_0}{h}\right\} = \prod_{p\in\mathbb{P}_0} p^{l_1\beta_p}$$

with $l_1 \in \mathbb{Z}$. Hence, by (23),

$$\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} l_p \log p + \sum_{p\in\mathbb{P}_0} (l_p + l_2\beta_p)\log p = 0$$

with some integer $l_2$, and this contradicts the linear independence over $Q$ of logarithms of prime numbers. If $l_0$ is not a multiple of $r_0$, then the number

$$\exp\left\{\frac{2\pi l_0}{h}\right\}$$

is irrational, and this contradicts (23) since

$$\exp\left\{\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} l_p \log p + \sum_{p\in\mathbb{P}_0} (l_p + l\beta_p)\log p\right\} = \prod_{p\in\mathbb{P}\setminus\mathbb{P}_0} p^{l_p} \prod_{p\in\mathbb{P}_0} p^{l_p + l\beta_p},$$

is a rational number. These contradictions show that inequality (22) is true. Let, for brevity,

$$\hat{L}_h(\underline{l}, l) = \sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} l_p \log p + \sum_{p\in\mathbb{P}_0} (l_p + l\beta_p)\log p.$$

Then, (19) and (22) yield that

$$\begin{aligned} f_{N,M}(\underline{l}) &= \frac{1}{M+1}\sum_{k=N}^{N+M} \exp\left\{-ikh\left(\sum_{p\in\mathbb{P}\setminus\mathbb{P}_0} l_p \log p + \sum_{p\in\mathbb{P}_0} (l_p + l\beta_p)\log p\right)\right\} \\ &= \frac{\exp\{-ihN\hat{L}_h(l_p, l)\} - \exp\{-ih(N+M+1)\hat{L}_h(l_p, l)\}}{(M+1)(1-\exp\{-ih\hat{L}_h(l_p, l)\})}, \quad l \in \mathbb{Z}. \end{aligned}$$

This and Equality (21) show that

$$\lim_{N\to\infty} f(\underline{l}) = \begin{cases} 1 & \text{if } l_p = 0 \text{ for } p \in \mathbb{P}\setminus\mathbb{P}_0, \text{ and } l_p = l\beta_p \text{ for all } p \in \mathbb{P}_0, \\ 0 & \text{otherwise,} \end{cases}$$

and the proof of the lemma is complete because the right-hand side of the latter equality is the Fourier transform of $\mu^h$. □

Lemma 7 serves as an important component for the proof of a limit lemma for the function $\zeta_r(s; Q)$. For this, we recall one simple but useful property of weak convergence of probability measures.

Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two topological spaces, and $g : \mathbb{X}_1 \longmapsto \mathbb{X}_2$ a continuous mapping. Then, $g$ is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable, i.e.,

$$\mathcal{B}^{-1}(\mathbb{X}_2) \subset \mathcal{B}(\mathbb{X}_1).$$

Then, every probability measure $P$ on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$ induces the unique probability measure $Pg^{-1}$ on $(\mathbb{X}_2, \mathcal{B}(\mathbb{X}_2))$ defined by

$$Pg^{-1}(A) = P(g^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_2),$$

where $g^{-1}(A)$ is the preimage of $A$.

**Lemma 9** (Section 5 of [30])**.** *Let $P_n$, $n \in \mathbb{N}$, and $P$ be probability measures on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$, and $g : \mathbb{X}_1 \longmapsto \mathbb{X}_2$ be a continuous mapping. Suppose that $P_n \xrightarrow[n\to\infty]{\text{w}} P$. Then, $P_n g^{-1} \xrightarrow[n\to\infty]{\text{w}} Pg^{-1}$.*

For $\omega \in \Omega$, define

$$\begin{aligned}\zeta_r(s, \omega; Q) &= \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}\omega(k)\omega(l)}{k^s l^s} L(s, \omega, \chi_k) L\Big(s - \frac{n}{2} + 1, \omega, \psi_l\Big) \\ &+ \sum_{m=1}^{\infty} \frac{b_Q(m)\omega(m)}{m^s},\end{aligned}$$

and, for $A \in \mathcal{B}(\mathbb{C})$, set

$$P^h_{N,M,r,\sigma}(A) = D_{N,M}(\zeta_r(\sigma + ikh; Q) \in A).$$

**Lemma 10.** *Suppose that $h > 0$ is type 1, $\sigma > \frac{n-1}{2}$, and $M \to \infty$ as $N \to \infty$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_{r,\sigma}$ such that $P^h_{N,M,r,\sigma} \xrightarrow[N\to\infty]{\text{w}} P_{r,\sigma}$.*

**Proof.** Consider the mapping $g_{r,\sigma} : \Omega \longmapsto \mathbb{C}$ given by the formula

$$g_{r,\sigma}(\omega) = \zeta_r(\sigma, \omega; Q), \quad \omega \in \Omega.$$

All functions involved in the definition of $\zeta_r(\sigma, \omega; Q)$ are absolutely convergent, hence, uniform in $\omega$. Therefore, $g_{r,\sigma}$ is a continuous mapping. Moreover, we have

$$g_{r,\sigma}\Big(p^{-ikh} : p \in \mathbb{P}\Big) = \zeta_r(\sigma + ikh; Q).$$

Therefore,

$$P^h_{N,M,r,\sigma}(A) = D_{N,M}\Big(\Big(p^{-ikh} : p \in \mathbb{P}\Big) \in g^{-1}_{r,\sigma}A\Big) = Q^h_{N,M}(g^{-1}A)$$

for all $A \in \mathcal{B}(\mathbb{C})$. Thus, we have $P^h_{N,M,r,\sigma} = Q^h_{N,M} g^{-1}_{r,\sigma}$. This, continuity of $g_{r,\sigma}$ and Lemmas 7 and 9 proves that $P^h_{N,M,r,\sigma} \xrightarrow[N\to\infty]{\text{w}} \mu g^{-1}_{r,\sigma} \stackrel{\text{def}}{=} P_{r,\sigma}$. □

A separate analysis of weak convergence for the measure $P_{r,\sigma}$ as $n \to \infty$ is unnecessary, as it has already been investigated in [19].

**Lemma 11.** *Suppose that $\sigma > \frac{n-1}{2}$ is fixed and $h > 0$ is type 1. Then, $P_{r,\sigma} \xrightarrow[r\to\infty]{\text{w}} P_{\zeta,\sigma}$.*

We notice that the statement of the lemma has been obtained in the proof of Theorem 2 of [10]. For identification of the limit measure of $P_{r,\sigma}$ as $r \to \infty$, elements of the ergodic theory have been involved; more precisely, the classical Birkhoff–Khintchine ergodic theorem (see, ref. [31]) has been applied.

Now, we resume the analysis of the case where $h$ is type 2. Repeating the proof of Lemma 10 and using Lemma 8 leads to the following statement.

**Lemma 12.** *Suppose that $h > 0$ is type 2. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P^h_{r,\sigma}$ such that $P^h_{N,M,r,\sigma} \xrightarrow[N\to\infty]{\text{w}} P^h_{r,\sigma}$.*

**Proof.** We notice that, in the case of type 2, the mapping $g_{r,\sigma}$ depends on $h$. Thus, $P^h_{r,\sigma} = \mu^h g^{-1}_{r,\sigma}$. □

The measure $P^h_{r,\sigma}$ does not depend on $M$. Thus, we may use the results of [19].

**Lemma 13** ([19], pp. 12–14)**.** *Suppose that $\sigma > \frac{n-1}{2}$ is fixed and $h > 0$ is type 2. Then $P^h_{r,\sigma} \xrightarrow[r\to\infty]{\text{w}} P^h_{\zeta,\sigma}$.*

To complete the proof of Theorem 1, one statement on convergence in distribution is needed. Let $X_n$, $n \in \mathbb{N}$, and $X$ be $\mathbb{X}$-valued random elements on a certain probability space $(\hat{\Omega}, \mathcal{B}, \mathcal{Q})$, and $P_n(A) = \mathcal{Q}\{\hat{\omega} \in \hat{\Omega} : X_n \in A\}$ and $P(A) = \mathcal{Q}\{\hat{\omega} \in \hat{\Omega} : X \in A\}$, $A \in \mathcal{B}(\mathbb{C})$ be the corresponding distributions. We say that $X_n$ converges to $X$ in distribution $\big(X_n \xrightarrow[n\to\infty]{\mathcal{D}} X\big)$ if $P_n \xrightarrow[n\to\infty]{\text{w}} P$.

**Lemma 14** (Theorem 4.2 of [30])**.** *Let $Y_n$ and $X_{ln}$, $n, l \in \mathbb{N}$, be $(\mathbb{X}, \varrho)$-valued random elements on the probability space $(\hat{\Omega}, \mathcal{B}, \mathcal{Q})$, and the space $\mathbb{X}$ is separable. Suppose that, for every $l \in \mathbb{N}$,*

$$X_{ln} \xrightarrow[n\to\infty]{\mathcal{D}} X_l \quad \text{and} \quad X_l \xrightarrow[l\to\infty]{\mathcal{D}} X.$$

*Moreover, let, for every $\varepsilon > 0$,*

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathcal{Q}\{\varrho(X_{ln}, Y_n) \geqslant \varepsilon\} = 0.$$

*Then, $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.*

**Proof of Theorem 1.** *Case for $h$ of type 1.* On a certain probability space $(\hat{\Omega}, \mathcal{B}, \mathcal{Q})$, define a random variable $\xi^h_{N,M}$ having the distribution

$$\mathcal{Q}\Big\{\xi^h_{N,M} = kh\Big\} = \frac{1}{M+1}, \quad k = N, N+1, \dots, M.$$

For $\sigma > \frac{n-1}{2}$, set

$$X^h_{N,M,r} = X^h_{N,M,r}(\sigma) = \zeta_r(\sigma + i\xi^h_{N,M}; Q),$$

and

$$Y^h_{N,M} = Y^h_{N,M}(\sigma) = \zeta(\sigma + i\xi^h_{N,M}; Q).$$

In virtue of Lemma 10, it follows that

$$X^h_{N,M,r} \xrightarrow[N\to\infty]{\mathcal{D}} X_r, \tag{24}$$

where $X_r = X_r(\sigma)$ is the $\mathbb{C}$-valued random element with the distribution $P_{r,\sigma}$. Moreover, in view of Lemma 11,

$$X_r \xrightarrow[r\to\infty]{\mathcal{D}} P_{\zeta,\sigma}. \tag{25}$$

Now, we apply Lemma 6 and find that, for fixed $\sigma > \frac{n-1}{2}$ and $\varepsilon > 0$,

$$\lim_{r\to\infty} \limsup_{N\to\infty} \mathcal{Q}\Big\{\Big|Y^h_{N,M}(\sigma) - X^h_{N,M,r}(\sigma)\Big| \geqslant \varepsilon\Big\}$$

$$= \lim_{r\to\infty}\limsup_{N\to\infty}\frac{1}{(M+1)\varepsilon}\sum_{k=N}^{N+M}|\zeta(\sigma+ikh;Q)-\zeta_r(\sigma+ikh;Q)|=0.$$

The later equality, relations (24) and (25) together with Lemma 14 show that

$$Y_{N,M}\xrightarrow[N\to\infty]{\mathcal{D}}P_{\zeta,\sigma},$$

i.e., $P_{N,M,\sigma}\xrightarrow[N\to\infty]{\mathrm{w}}P_{\zeta,\sigma}$.

*Case for h of type 2* . We repeat the proof of Theorem 1 in the case of $h$ of type preserving the notations with minor changes.

By Lemma 12, we have

$$X^h_{N,M,r}(\sigma)\xrightarrow[N\to\infty]{\mathcal{D}}X^h_r(\sigma), \tag{26}$$

where $X^h_r(\sigma)$ is the $\mathbb{C}$-valued random element with the distribution $P^h_{r,\sigma}$, and from Lemma 13 it follows that

$$X^h_r(\sigma)\xrightarrow[r\to\infty]{\mathcal{D}}P^h_{\zeta,\sigma}. \tag{27}$$

Using Lemma 6 yields, for $\sigma>\frac{n-1}{2}$ and $\varepsilon>0$,

$$\lim_{r\to\infty}\limsup_{N\to\infty}\mathcal{Q}\left\{\left|Y^h_{N,M}(\sigma)-X^h_{N,M,r}(\sigma)\right|\geqslant\varepsilon\right\}=0. \tag{28}$$

Now, (26)–(28) and Lemma 14 prove the theorem in the case of $h$ of type 2, i.e.,

$$P^h_{N,M,\sigma}\xrightarrow[N\to\infty]{\mathrm{w}}P^h_{\zeta,\sigma}.$$

The proof of Theorem 1 is complete. □

## 4. Conclusions

Assuming that $Q$ is a positive-definite quadratic matrix of order $n\in 2\mathbb{N}$, $n\geqslant 4$, and that $\underline{x}^TQ\underline{x}\in\mathbb{Z}$ for all $\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}$, we considered asymptotic properties of the frequency of the values $\zeta(\sigma+ikh;Q)$ of the Epstein zeta-function in the interval $[N,N+M]$, where $h^{-1}(Nh)^{27/82}\leqslant M\leqslant h^{-1}(Nh)^{1/2}$, as $N\to\infty$, and $h>0$ and $\sigma>\frac{n-1}{2}$ are fixed. Applying a probabilistic technique, we obtained that the above frequency converges weakly to the distribution of a certain explicitly given complex-valued random element. This random element depends on the arithmetic of the number $h$. The result obtained improves our earlier result, proved for the interval $[0,N]$, and is closely connected to the mean square estimate

$$\int_{T-H}^{T+H}|L(\sigma+it,\chi)|^2\mathrm{d}t\ll_\sigma H,\quad \sigma>\frac{1}{2}, \tag{29}$$

which holds in short intervals ($H=o(T)$ as $T\to\infty$) for Dirichlet $L$-functions $L(s,\chi)$. We believe that the lower bound for $M$ can be decreased by using a more precise estimate for $H$ in (29).

In the future, we plan to extend the results of this paper to the space $H(D)$, i.e., to obtain weak convergence of probability measures

$$\frac{1}{H}\mathrm{meas}\{t\in[T,T+H]:\zeta(s+i\tau;Q)\in A\}\quad\text{and}\quad D_{N,M}(\zeta(s+i\tau;Q)\in A)$$

with $A\in\mathcal{B}(H(D))$ in short intervals ($H=o(T)$ as $T\to\infty$, and $H=o(N)$ as $N\to\infty$), where $D=\{s\in\mathbb{C}:\frac{n-1}{2}<\sigma<\frac{n}{2}\}$, and $H(D)$ denotes the space of analytic functions on $D$ endowed with the topology of uniform convergence on compact sets. Such limit theorems and equivalents of weak convergence lead to the universality of $\zeta(s;Q)$ in short intervals. Note that the universality theorem in short intervals is an important step towards the effectivization of universality, i.e., the detection of approximating shifts $\zeta(s+i\tau;Q)$ for a given analytic function.

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