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Stokes-Brinkman equations with diffusion and convection in thin tube structures

Antonio Gaudiello [a,∗,1], Grigory Panasenko [b]

[a] Dipartimento di Matematica e Fisica, Università degli Studi della Campania Luigi Vanvitelli, viale Abramo Lincoln n.5 - 81100 Caserta, Italy

[b] Institute of Applied Mathematics, Vilnius University, Naugarduko Str., 24, Vilnius, 03225, Lithuania

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Abstract

The steady state Stokes-Brinkman equations coupled with a system of diffusion-convection equations in a thin tube structure is considered. The Brinkman term differs from zero only in small balls near the ends of the tubes. The boundary conditions are: given pressure and concentrations at the inflow and outflow of the tube structure, the no slip boundary condition on the lateral boundary for the fluid, and Neumann type condition on the lateral boundary for the diffusion-convection equations. In this paper, the existence, uniqueness, and stability of the solution to such a problem are proved. Moreover, some *a priori* norm-estimates depending on the small thickness of the tubes are also provided. This model is well suited to describing thrombosis in blood vessels.

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Keywords: Navier-Stokes equations; Brinkman equations; Diffusion; Convection; Pressure boundary condition; Poiseuille flows

∗ Corresponding author.

E-mail addresses: antonio.gaudiello@unicampania.it (A. Gaudiello), grigory.panasenko@mif.vu.lt (G. Panasenko).

[1] He is also member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of Istituto Nazionale di Alta Matematica (INdAM).

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1. Introduction

In this paper, we consider a tube structure made of a finite union of thin cylinders that can simulate a network of blood vessels (see subsection 1.1 for the definition of such a structure and Fig. 2).

In such a structure, we consider the steady state Stokes-Brinkman equations coupled with a system of diffusion-convection equations, where the following boundary conditions are assumed: given pressure and concentrations at the inflow and outflow of the tube structure, the no slip boundary condition on the lateral boundary for the fluid, and the homogeneous Neumann boundary condition on the lateral boundary for the diffusion-convection equations (see subsection 1.2 for the formulation of the problem).

The purpose of this work is to study the existence, uniqueness, and stability of the solution to such a problem. Moreover, some *a priori* norm-estimates depending on the small thickness of the tubes are provided (for a related problem in an infinite domain which does not depend on ε see [6]).

The Newtonian rheology for the fluid motion in thin structures corresponding to the stationary and nonstationary Navier-Stokes or Stokes equations was considered by many authors (for instance, see [19] and references therein for an exhaustive view on the argument), while few papers studied non-Newtonian models (for instance, see [7], [8], [9], [13], and [17]). On the other hand, purely Newtonian rheology is not perfectly adequate for the description of the blood flow with clot formation zones (see [10] and [21] for the most important models for the blood). This prompted us to study Stokes-Brinkman equations coupled with a system of diffusion-convection equations. Really, the modeling of zones of thrombus formation could be better described by the Brinkman equations combining the Stokes description of the fluid motion with the Darcy filtration law. Indeed, the external part of the thrombus behaves as a porous medium, but approaching the surface of the thrombus it corresponds better to a Newtonian fluid. Moreover, the permeability of the clot tissue depends on the concentrations of the cells and substances, that is why the viscous flow is governed by nonlinear equations, when the Brinkman term depends on the concentrations [22]. Also the problem is non-linear because the velocity of the fluid motion equations appears as a coefficient in the convective terms of the diffusion-convection equations.

The paper is planned as follows. Subsection 1.1 is devoted to describe the thin tube structure, subsection 1.2 to introduce the problem, subsection 1.3 to state the main result (existence, uniqueness, and *a priori* norm-estimates of the solution to the problem) which will be proved in section 2, where also stability results are proved.

Full dimension numerical computations of flows in networks of thin tubes require huge computer resources. To reduce these resources and accelerate computations one uses asymptotic analysis where the small parameter is the ratio of thickness of thin tubes to their length (for instance, see [3], [11], [12], [15], [16], [19], and [20]). Using *a priori* norm-estimates obtained in the present paper, a forthcoming paper will be devoted to the construction of the asymptotic expansion justified by error estimate for Stokes-Brinkman equations with diffusion and convection in a thin tube structure.

1.1. Definition of a thin tube structure

Let us recall the definitions of the tube structure and its graph given in [14] and [16] (for structures made of elastic rods or plates, see [2], [4], and [16]).

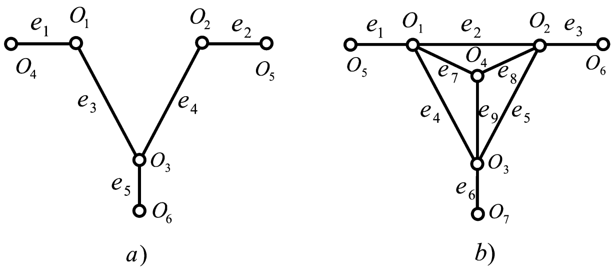

Fig. 1. Graphs of tube structures.

Definition 1.1. Let $O_1, O_2, ..., O_N$ be N different points in $\mathbb{R}^n, n = 2, 3,$ and $e_1, e_2, ..., e_M$ be M closed segments each connecting two of these points (i.e. each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, ..., N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. A point O_i is called a node, if it is the common end of at least two edges and O_i is called a vertex, if it is the end of the only one edge. Any two edges can intersect only at the common node. The set of vertices is supposed to be non-empty.

Denote

$$\mathcal{B} = \bigcup_{j=1}^{M} e_j$$

the union of edges and assume that $\mathcal{B}$ is a connected set (see Fig. 1). Each point O_i, a node or a vertex, with all edges containing O_i as an end point, is called the bundle $\mathcal{B}_i$. For instance, Fig. 1 a) presents the graph as a union of edges $e_1, ..., e_5$, points O_1, O_2, O_3 are the nodes, points O_4, O_5, O_6 are the vertices, O_1 with edges e_1 and e_3 form bundle $\mathcal{B}_1$. Fig. 1 b) presents the graph as a union of edges $e_1, ..., e_9$, points O_1, O_2, O_3, O_4 are the nodes, points O_5, O_6, O_7 are the vertices.

Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_i x_n^{(e)}$ has the direction of the ray $[O_i O_j)$; the second one has the origin in O_j and the opposite direction, i.e. $O_j \tilde{x}_n^{(e)}$ is directed over the ray $[O_j O_i)$.

Below in various situations, we choose one or another coordinates system denoting the local variable in both cases by $x^{(e)}$ and pointing out which end is taken as the origin of the coordinate system.

With every edge e_j we associate a bounded domain $\sigma^j \subset \mathbb{R}^{n-1}$ containing the origin and having C^2- smooth boundary $\partial\sigma^j$, $j = 1, ..., M$. For every edge $e_j = e$ and associated $\sigma^j = \sigma^{(e)}$ we denote by $\Pi_\varepsilon^{(e)}$ the cylinder

$$\Pi_\varepsilon^{(e)} = \left\{ x^{(e)} \in \mathbb{R}^n \ : \ x_n^{(e)} \in (0, |e|), \ \frac{x^{(e)\prime}}{\varepsilon} \in \sigma^{(e)} \right\},$$

where $x^{(e)\prime} = (x_1^{(e)}, ..., x_{n-1}^{(e)})$, $|e|$ is the length of the edge e and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_i, as well as the domains σ^j, do not depend on ε.

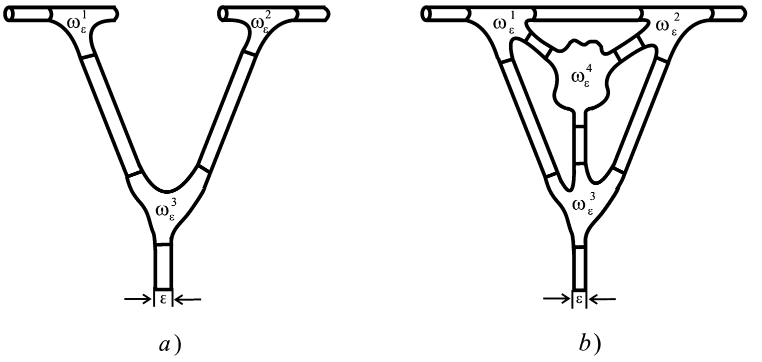

Fig. 2. Tube structures.

Let $O_1, ..., O_{N_1}$ be nodes and $O_{N_1+1}, ..., O_N$ be vertices. Let $\omega^1, ..., \omega^{N_1}$ be bounded independent of ε domains in $\mathbb{R}^n$ containing the origin of $\mathbb{R}^n$; introduce the nodal domains

$$\omega_\varepsilon^i = \left\{ x \in \mathbb{R}^n \ : \ \frac{x - O_i}{\varepsilon} \in \omega^i \right\}.$$

Every vertex O_j is the end of one and only one edge e_k which will also be denoted as e_{O_j}; we will re-denote as well the domain σ^k associated to this edge as σ^{O_j}. Notice that the subscript k may be different from j.

Definition 1.2. By a tube structure, we call the following domain

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} \Pi_\varepsilon^{(e_j)}\Big) \bigcup \Big(\bigcup_{i=1}^{N_1} \omega_\varepsilon^i\Big).$$

Suppose that it is a connected set and that the boundary ∂B_ε of B_ε is C^2-smooth except for the parts of the boundary which are the boundary of the bases

$$\gamma_\varepsilon^i = \{\varepsilon^{-1} x^{(e)\prime} \in \sigma^{O_i}, \ x_n^{(e)} = 0\}$$

of cylinders $\Pi_\varepsilon^{(e)}$, $i = N_1 + 1, ..., N$ (see Fig. 2).

Let r_1 be the maximal diameter of domains ω^i, $i = 1, ..., N_1$, denote $r = r_1 + 1$.

1.2. Formulation of the problem

For any vector-field $\mathbf{u}$ defined on some surface with normal vector $\mathbf{n}$ denote

$$\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n},$$

the tangential component of the vector $\mathbf{u}$. Here and below we use the bold script for the vectors.

Let

$$\Gamma_\varepsilon = \partial B_\varepsilon \setminus \bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j$$

be the lateral surface of the domain B_ε. In the tube structure B_ε we define the spaces

$$\widehat{W}^{1,2}(B_\varepsilon) = \Big\{ \boldsymbol{\eta} \in W^{1,2}(B_\varepsilon) : \boldsymbol{\eta}|_{\Gamma_\varepsilon} = 0,\ \boldsymbol{\eta}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j} = 0,\ j = N_1+1, \dots, N \Big\},$$
$$\widehat{K}^{1,2}(B_\varepsilon) = \Big\{ \boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon) \, : \, \mathrm{div}\boldsymbol{\eta} = 0 \Big\}.$$

We denote $B(O, R)$ the open ball in $\mathbb{R}^n$ with center O and radius R.

Let us introduce a vector-valued function $\mathbf{f}$ in $L^2(B_\varepsilon)$ and L scalar functions in $L^2(B_\varepsilon)$,

$$g_k = g_{k,0} - \sum_{q=1}^{n} \frac{\partial}{\partial x_q} g_{k,q},$$

with $g_{k,0} \in L^2(B_\varepsilon)$, and $g_{k,q} \in W^{1,2}(B_\varepsilon)$, vanishing on the part of the boundary Γ_ε, $k = 1, ..., L$, $q = 1, ..., n$. Denote $D = \nabla + \nabla^T$.

Let us consider the following boundary value problem for the steady-state Stokes equations in a tube structure B_ε

$$\begin{cases} -\mathrm{div}\Big(\nu_\varepsilon(x) D\mathbf{u}_\varepsilon\Big) + R_\varepsilon(x, \mathbf{c}_\varepsilon)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}(x) \ \text{ in } B_\varepsilon, \\ \mathrm{div}\mathbf{u}_\varepsilon = 0 \ \text{ in } B_\varepsilon, \\ -\mathrm{div}\Big(M_k \nabla c_{k,\varepsilon} - c_{k,\varepsilon}\mathbf{u}_\varepsilon\Big) = g_k(x) \ \text{ in } B_\varepsilon, \ \ k = 1, ..., L, \\ \mathbf{u}_\varepsilon = 0 \ \text{ on } \partial B_\varepsilon \setminus \bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{u}_{\varepsilon\boldsymbol{\tau}} = 0 \ \text{ on } \gamma_\varepsilon^j, \ j = N_1+1, ..., N, \\ p_\varepsilon = p^j \ \text{ on } \gamma_\varepsilon^j, \ j = N_1+1, ..., N, \\ \dfrac{\partial \mathbf{c}_\varepsilon}{\partial \mathbf{n}} = 0 \ \text{ on } \partial B_\varepsilon \setminus \bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{c}_\varepsilon = \mathbf{c}^{0,j} \ \text{ on } \gamma_\varepsilon^j, \ j = N_1+1, ..., N. \end{cases} \tag{1.1}$$

In problem (1.1), $\nu_\varepsilon \in C(\overline{B_\varepsilon})$ is a function (effective dynamical viscosity related to the porosity) greater than some positive constant independent of ε, ν_ε is equal to a positive constant $\nu^{(0)}$ everywhere except for the balls $B(O_l, r\varepsilon)$ where is equal to given functions $\nu^{(l)}\Big(\dfrac{x - O_l}{\varepsilon}\Big)$, $l = 1, ..., N_1$, with $\nu^{(l)} \in C^1(B(0, r))$; $R_\varepsilon \in C^1(\overline{B_\varepsilon} \times \mathbb{R}^L)$ is a $n \times n$ matrix-valued function (resistance, inverse to effective permeability of porous medium) equal to zero everywhere except for the balls $B(O_l, r\varepsilon)$ where $R_\varepsilon(x, \cdot)$ is equal to given functions $R^{(l)}\Big(\dfrac{x - O_l}{\varepsilon}, \cdot\Big)$, with $R^{(l)} \in C^1(B(0, r) \times \mathbb{R}^L)$ $n \times n$ non-negative symmetric matrix-valued bounded functions, Lipschitz with respect to the second variable; $\mathbf{c}_\varepsilon = (c_{1,\varepsilon}, ..., c_{L,\varepsilon})$; $\mathbf{n}$ is the unit (external to B_ε)

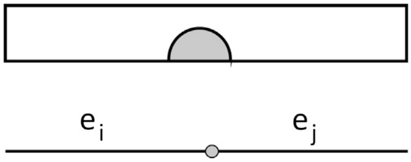

Fig. 3. A thrombus can be considered as a zone of connection of two co-linear cylinders.

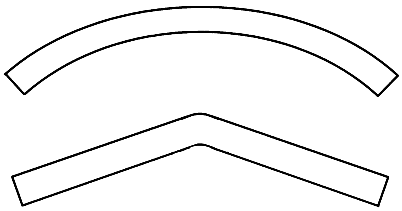

Fig. 4. Approximation of curved cylinders by a chain of straight ones.

normal vector to ∂B_ε, $\partial_{\mathbf{n}} g = \nabla g \cdot \mathbf{n}$ is the normal derivative of g; M_k, $k = 1, ..., L$, are some given positive constants independent of ε, $p^j \in \mathbb{R}$ and $\mathbf{c}^{0,j} \in \mathbb{R}^L$, $j = N_1 + 1, ..., N$, are some given constants independent of ε. In what follows, we assume that $\mathbf{c}^{0,N} = \mathbf{0}$ and it is not restrictive.

The Brinkman equation was rigorously derived from the Navier-Stokes equation in a porous medium in [1] and was extensively studied in fluid mechanics [5]. This model describes the Newtonian flow in the tubes combined with the fluid filtration process through the zones $B(O_l, r\varepsilon)$, simulating the eventual clots or thrombi [22]. In these zones ν_ε stands for the effective dynamical velocity taking into account the porosity of the clot, while R_ε stands for the inverse to the effective permeability of the clot. From physical sense $c_q^{0,j}$, $q = 1, \cdots, L$, belongs to the interval $[0, 1]$. Usually in applications, $\mathbf{f} = 0$, $g_{k,q} = 0$.

Note that in applications the thrombus formation may occur in the middle of a vessel. This situation is taken into consideration by our model: the center of the thrombus is considered as a node which is an end point of two co-linear edges (see Fig. 3).

Also in applications some slightly curved vessels can be replaced in the idealized geometry B_ε by a chain of thin straight cylinders connected by smooth junction domains (see Fig. 4).

Alternatively, we need to introduce curved vessels instead of the straight ones. Such generalization hasn't important influence for the proof of the existence, uniqueness, and estimates results but makes much more technical construction of the asymptotic expansion of the solution.

Note that from the boundary condition $\mathbf{u}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j} = 0$ and the divergence equation $\mathrm{div}\mathbf{u} = 0$, it follows that $-\nu_\varepsilon(x)\partial_{\mathbf{n}}\mathbf{u} \cdot \mathbf{n}|_{\gamma_\varepsilon^j} = 0$. Thus the boundary condition $p_\varepsilon = p^j$ is equivalent to $-\nu_\varepsilon(x)\partial_{\mathbf{n}}\mathbf{u}_\varepsilon \cdot \mathbf{n} + p_\varepsilon = p^j$.

Let us define a weak solution to problem (1.1) as a couple $(\mathbf{u}_\varepsilon, \mathbf{c}_\varepsilon)$, $\mathbf{u}_\varepsilon \in \widehat{K}^{1,2}(B_\varepsilon)$, $\mathbf{c}_\varepsilon \in W^{1,2}(B_\varepsilon)$, such that $\mathbf{c}_\varepsilon = \mathbf{c}^{0,j}$ on γ_ε^j, $j = N_1 + 1, ..., N$, satisfying the integral identity

$$\begin{aligned}
&\int\limits_{B_\varepsilon} \Big(\frac{1}{2}\nu_\varepsilon(x) D\mathbf{u}_\varepsilon \cdot D\boldsymbol{\eta} + R_\varepsilon(x,\mathbf{c}_\varepsilon)\mathbf{u}_\varepsilon\cdot\boldsymbol{\eta}\Big)\,dx \\
&\qquad + \sum_{j=N_1+1}^{N} p^j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}\cdot\mathbf{n}\,d\sigma - \int\limits_{B_\varepsilon} \mathbf{f}\cdot\boldsymbol{\eta}\,dx \\
&+ \int\limits_{B_\varepsilon} \sum_{k=1}^{L} (M_k \nabla c_{k,\varepsilon} - c_{k,\varepsilon}\mathbf{u}_\varepsilon)\cdot\nabla\zeta_k - g_{k,0}\zeta_k\,dx \\
&\qquad - \sum_{k=1}^{L}\sum_{q=1}^{n} \int\limits_{B_\varepsilon} g_{k,q}\frac{\partial}{\partial x_q}\zeta_k\,dx = 0,
\end{aligned} \tag{1.2}$$

for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$, and for every $\zeta_k \in W^{1,2}(B_\varepsilon)$, $k = 1, ..., L$, vanishing on γ_ε^j, $j = N_1 + 1, ..., N$. Here and below for any two $n \times n$ matrices A and B having entries a_{ij} and b_{ij} denote $A \cdot B$ the sum $\sum_{i,j=1}^{n} a_{ij} b_{ij}$.

Introduce $p^{j*} = p^j - p^N$, $j = N_1 + 1, ..., N$. Consider an equivalent weak formulation: find a couple $(\mathbf{u}_\varepsilon, \mathbf{c}_\varepsilon)$, $\mathbf{u}_\varepsilon \in \widehat{K}^{1,2}(B_\varepsilon)$, $\mathbf{c}_\varepsilon \in W^{1,2}(B_\varepsilon)$, such that $\mathbf{c}_\varepsilon = \mathbf{c}^{0,j}$ on γ_ε^j, $j = N_1 + 1, ..., N$, satisfying the integral identity

$$\begin{aligned}
&\int\limits_{B_\varepsilon} \Big(\frac{1}{2}\nu_\varepsilon(x) D\mathbf{u}_\varepsilon \cdot D\boldsymbol{\eta} + R_\varepsilon(x,\mathbf{c}_\varepsilon)\mathbf{u}_\varepsilon\cdot\boldsymbol{\eta}\Big)\,dx \\
&\qquad + \sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}\cdot\mathbf{n}\,d\sigma - \int\limits_{B_\varepsilon} \mathbf{f}\cdot\boldsymbol{\eta}\,dx \\
&+ \int\limits_{B_\varepsilon} \sum_{k=1}^{L} (M_k \nabla c_{k,\varepsilon} - c_{k,\varepsilon}\mathbf{u}_\varepsilon)\cdot\nabla\zeta_k - g_{k,0}\zeta_k\,dx \\
&\qquad - \sum_{k=1}^{L}\sum_{q=1}^{n} \int\limits_{B_\varepsilon} g_{k,q}\frac{\partial}{\partial x_q}\zeta_k\,dx = 0,
\end{aligned} \tag{1.3}$$

for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$, and for every $\zeta_k \in W^{1,2}(B_\varepsilon)$, $k = 1, ..., L$, vanishing on γ_ε^j, $j = N_1 + 1, ..., N$.

The equivalence of these formulations follows from the identity

$$\sum_{j=N_1+1}^{N-1} p^{j*} \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}\cdot\mathbf{n}\,d\sigma = \sum_{j=N_1+1}^{N} p^j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}\cdot\mathbf{n}\,d\sigma,$$

which is a corollary of the relation

$$\sum_{j=N_1+1}^{N} \int_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\, d\sigma = 0,$$

for the solenoidal vector-valued function $\boldsymbol{\eta}$.

1.3. Main result: existence and uniqueness of the solution

Let us introduce several constants independent of ε.

1. There exists a constant $C_0^* > 0$, independent of ε, such that for any function $v \in W^{1,2}(B_\varepsilon)$ vanishing on γ_ε^N and for any function $w \in W^{1,2}(B_\varepsilon)$ vanishing on Γ_ε, the lateral part of the boundary of the thin tube structure, the following inequalities hold:

$$\begin{cases} \|v\|_{L^2(B_\varepsilon)} \le C_0^* \|\nabla v\|_{L^2(B_\varepsilon)} \\ \|w\|_{L^2(B_\varepsilon)} \le \varepsilon C_0^* \|\nabla w\|_{L^2(B_\varepsilon)}, \end{cases} \tag{1.4}$$

where $C_0^* > 0$ is the first introduced constant independent of ε (called Poincaré-Friendrichs constant, see [16]).

Also, we will use the constants of the embedding theorems. There exists a constant $C_1^* > 0$, independent of ε, such that for any function $v \in W^{1,2}(B_\varepsilon)$ vanishing on γ_ε^N and for any function $w \in W^{1,2}(B_\varepsilon)$ vanishing on Γ_ε the following inequalities hold:

$$\begin{cases} \|v\|_{L^4(B_\varepsilon)} \le \varepsilon^{-n/4} C_1^* \|\nabla v\|_{L^2(B_\varepsilon)} \\ \|w\|_{L^4(B_\varepsilon)} \le \varepsilon^{1-n/4} C_1^* \|\nabla w\|_{L^2(B_\varepsilon)}, \end{cases} \tag{1.5}$$

where $C_1^* > 0$ is the second introduced constant independent of ε (called embedding constant, see [17], Lemma 3.2).

2. Considering an *a priori* estimate for a solution to the linear problem

$$\begin{cases} -\mathrm{div}\Big(\nu_\varepsilon(x) D\mathbf{u}_\varepsilon\Big) + \nabla p_\varepsilon = \mathbf{f}(x) \ \text{ in } B_\varepsilon, \\ \mathrm{div}\mathbf{u}_\varepsilon = 0 \ \text{ in } B_\varepsilon, \\ \mathbf{u}_\varepsilon = 0 \ \text{ on } \partial B_\varepsilon \setminus \bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{u}_{\varepsilon\boldsymbol{\tau}} = 0 \ \text{ on } \gamma_\varepsilon^j, \ j = N_1+1, ..., N, \\ p_\varepsilon = p^j \ \text{ on } \gamma_\varepsilon^j, \ j = N_1+1, ..., N, \end{cases} \tag{1.6}$$

we get

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)} \le C_2^* \Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)} \Big), \tag{1.7}$$

where $C_2^* > 0$ is the third introduced constant independent of ε. Here $\mathbf{f} \in L^2(B_\varepsilon)$, $p^j \in \mathbb{R}$, $j = N_1+1, ..., N$. The proof is similar to the proof of Theorem 2.1 in [18].

3. Considering an *a priori* estimate for a solution to the linear problem

$$\begin{cases} -\operatorname{div}\Big(M_k \nabla c_{k,\varepsilon}\Big) = g_{k0} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} g_{ki} \text{ in } B_\varepsilon, \quad k = 1, ..., L, \\ \dfrac{\partial \mathbf{c}_\varepsilon}{\partial \mathbf{n}} = 0 \text{ on } \partial B_\varepsilon \setminus \displaystyle\bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{c}_\varepsilon = \mathbf{c}^{0,j} \text{ on } \gamma_\varepsilon^j, \; j = N_1+1, ..., N, \end{cases} \tag{1.8}$$

we get ([16])

$$\|\nabla \mathbf{c}_\varepsilon\|_{L^2(B_\varepsilon)} \le C_3^*\Big(\varepsilon^{(n-1)/2} \sum_{j=N_1+1}^{N-1} |\mathbf{c}^{0,j}| + \sum_{k=1}^{L}\sum_{q=0}^{n} \|g_{k,q}\|_{L^2(B_\varepsilon)}\Big), \tag{1.9}$$

where $C_3^* > 0$ is the fourth introduced constant independent of ε. Here, $g_{k,0} \in L^2(B_\varepsilon)$, $g_{k,q} \in W^{1,2}(B_\varepsilon)$, vanishes on the part of the boundary Γ_ε, $k = 1, ..., L$, $q = 1, ..., n$, $\mathbf{c}^{0,j} \in \mathbb{R}^L$, $j = N_1 + 1, ..., N$.

4. The definition of R_ε ensures the existence of a constant C_4^*, independent of ε, such that

$$\max_{x\in B_\varepsilon, \mathbf{c}\in\mathbb{R}^L} |R_\varepsilon(x, \mathbf{c})|_2 \le C_4^*, \tag{1.10}$$

where $|\cdot|_2$ is the Euclidean norm of a matrix. Also we assume that λ is a uniform bound of the Lipschitz constant of the function R_ε with respect to the second argument.

5. Denote

$$\begin{aligned} r_\varepsilon = \max\Big\{ & 2C_2^*\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big), \\ & 2C_3^*\Big(\varepsilon^{(n-1)/2} \sum_{j=N_1+1}^{N} |\mathbf{c}^{0,j}| + \sum_{k=1}^{L}\sum_{q=0}^{n} \|g_{k,q}\|_{L^2(B_\varepsilon)}\Big)\Big\}, \end{aligned} \tag{1.11}$$

and assume that

$$r_\varepsilon < \min\Big\{(2C_3^* nL(C_1^*)^2)^{-1}, (C_2^*(C_1^*)^2\lambda)^{-1}\Big\}\, \varepsilon^{-1+n/2}, \tag{1.12}$$

and that

$$\varepsilon \le \min\Big\{(2C_2^* C_4^* C_0^*)^{-1}, 1\Big\}. \tag{1.13}$$

The following theorem states the main result of this paper.

Theorem 1.1. *Let ε be a positive parameter satisfying (1.13). Let $\mathbf{f} \in L^2(B_\varepsilon)$, $g_{k,0} \in L^2(B_\varepsilon)$, $k = 1, ..., L$, $g_{k,q} \in W^{1,2}(B_\varepsilon)$, with $g_{k,q}|_{\Gamma_\varepsilon} = 0$, $k = 1, ..., L$, $q = 1, ..., n$, $p_j^* \in \mathbb{R}$, $j = N_1 + 1, ..., N-1$, $\mathbf{c}^{0,j} \in \mathbb{R}^L$, $j = N_1 + 1, ..., N$, $\mathbf{c}^{0,N} = \mathbf{0}$, be satisfying (1.12) with r_ε defined by*

(1.11). *Then, problem* (1.1) *admits a unique weak solution* $(\mathbf{u}_\varepsilon, \mathbf{c}_\varepsilon) \in \widehat{K}^{1,2}(B_\varepsilon) \times W^{1,2}(B_\varepsilon)$ *satisfying*

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)} \leq r_\varepsilon, \quad \|\nabla \mathbf{c}_\varepsilon\|_{L^2(B_\varepsilon)} \leq r_\varepsilon. \tag{1.14}$$

Remark 1.1. In applications normally the right-hand sides vanish: $\mathbf{f} = \mathbf{0}$, $g_{k,q} = 0$, $k = 1, ..., L$, $q = 0, ..., n$, while the scaling of the unknown functions $\mathbf{u}_\varepsilon$ and $\mathbf{c}_\varepsilon$ is governed by the order of the boundary value functions $p_j^* \in \mathbb{R}$, $j = N_1 + 1, ..., N - 1$, and $\mathbf{c}^{0,j} \in \mathbb{R}^L$, $j = N_1 + 1, ..., N$. In this case the assumptions of Theorem 1.1 (1.12) and (1.13) are satisfied, for example, when constants p_j^* satisfy the following condition

$$p_j^* = O(\varepsilon^{-1+\beta}), \quad \beta > 0,$$

while the concentrations $\mathbf{c}^{0,j}$ are of order of one. These assumptions correspond to the velocity tending to zero as ε tends to zero, so that in the diffusion-convection equation the diffusion term dominates. However, the magnitude of the pressure is not small, it is of order $O(\varepsilon^{-1+\beta})$, $\beta > 0$, and limit junctions conditions of Kirchhoff type appear for pressures (see [19]), so that they are very important for the correct flow computations. In the forthcoming paper we construct and justify the complete asymptotic expansion of the solution, evaluate the error of the asymptotic approximations and evaluate the influence of the Brinkman term on the diffusion-convection equation.

Note that if functions $\mathbf{f}$, $g_{k,q}$, $k = 1, ..., L, q = 0, ..., n$, have norms of order of one in L^∞ instead of being equal to zero, the assumption (1.12) still holds true.

We will also prove the existence and uniqueness of the pressure function $p_\varepsilon \in L^2(B_\varepsilon)$ for (1.1) such that

$$\begin{gathered}\int\limits_{B_\varepsilon} \Big(\frac{1}{2}\nu_\varepsilon(x) D\mathbf{u}_\varepsilon \cdot D\boldsymbol{\eta} + R_\varepsilon(x, \mathbf{c}_\varepsilon)\mathbf{u}_\varepsilon \cdot \boldsymbol{\eta}\Big)\, dx + \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\, d\sigma \\ -\int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta}\, dx = \int\limits_{B_\varepsilon} p_\varepsilon \operatorname{div}\boldsymbol{\eta}\, dx,\end{gathered} \tag{1.15}$$

for every $\boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon)$. We prove an a priori estimate for the pressure. We also prove the continuity of the solution with respect to data.

2. Proof of the main results

2.1. *Proof of Theorem 1.1*

Proof. The proof follows the fixed point theorem argument. Consider the closed balls $B_w = \{\mathbf{v} \in \widehat{K}^{1,2}(B_\varepsilon) : \|\nabla \mathbf{v}\|_{L^2(B_\varepsilon)} \leq r_\varepsilon\}$ and $B_b = \{\mathbf{d} \in (W^{1,2}(B_\varepsilon))^L :\ d = 0 \text{ on } \gamma_\varepsilon^N, \quad \|\nabla \mathbf{d}\|_{L^2(B_\varepsilon)} \leq r_\varepsilon\}$.

Consider the fixed point operator $T : B_w \times B_b \to B_w \times B_b$ such that $T(\mathbf{v}, \mathbf{d})$ is a couple $(\mathbf{w}, \mathbf{b})$ solution to the following problem

$$
\begin{aligned}
&\int_{B_\varepsilon}\Big(\frac{1}{2}\nu_\varepsilon(x)D\mathbf{w}\cdot D\boldsymbol{\eta}+\sum_{k=1}^{L}M_k\nabla b_k\cdot\nabla\zeta_k+R_\varepsilon(x,\mathbf{d})\mathbf{v}\cdot\boldsymbol{\eta}\Big)\,dx\\
&\qquad=-\sum_{j=N_1+1}^{N-1}p_j^*\int_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,d\sigma+\int_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,dx\\
&+\int_{B_\varepsilon}\sum_{k=1}^{L}\big(d_k\mathbf{v}\cdot\nabla\zeta_k+g_{k,0}\zeta_k\big)\,dx+\sum_{k=1}^{L}\sum_{q=1}^{n}\int_{B_\varepsilon}g_{k,q}\frac{\partial}{\partial x_q}\zeta_k\,dx,
\end{aligned}
\tag{2.1}
$$

for every $\boldsymbol{\eta}\in\widehat{K}^{1,2}(B_\varepsilon)$, and for every $\zeta_k\in W^{1,2}(B_\varepsilon)$, $k=1,\dots,L$, vanishing on γ_ε^j, $j=N_1+1,\dots,N$; $\mathbf{b}=\mathbf{c}^{0,j}$ on γ_ε^j, $j=N_1+1,\dots,N$, $\mathbf{c}^{0,N}=\mathbf{0}$.

1. Let us prove that if $(\mathbf{v},\mathbf{d})\in B_w\times B_b$ then $T(\mathbf{v},\mathbf{d})\in B_w\times B_b$. Indeed, taking $\zeta_k=0$ in (2.1) we get the weak formulation for the problem of the type (1.6) and applying the estimate (1.7), we get

$$
\|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)}\le C_2^*\Big(\varepsilon^{n/2}\sum_{j=N_1+1}^{N-1}|p_j^*|+\varepsilon\|\tilde{\mathbf{f}}\|_{L^2(B_\varepsilon)}\Big),
$$

where $\tilde{\mathbf{f}}=\mathbf{f}-R_\varepsilon(x,\mathbf{d})\mathbf{v}$. So, thanks to (1.10) and (1.4), one has

$$
\|\tilde{\mathbf{f}}\|_{L^2(B_\varepsilon)}\le\|\mathbf{f}\|_{L^2(B_\varepsilon)}+C_4^*\|\mathbf{v}\|_{L^2(B_\varepsilon)}\le\|\mathbf{f}\|_{L^2(B_\varepsilon)}+C_4^*C_0^*\varepsilon r_\varepsilon.
$$

Consequently,

$$
\|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)}\le r_\varepsilon,
$$

thanks to definition (1.11) and assumption (1.13).

Then, taking $\boldsymbol{\eta}=0$ in (2.1) and applying (1.9), we get

$$
\begin{aligned}
\|\nabla\mathbf{b}\|_{L^2(B_\varepsilon)}\le C_3^*\Bigg(&\varepsilon^{(n-1)/2}\sum_{j=N_1+1}^{N-1}|\mathbf{c}^{0,j}|+\\
&\sum_{k=1}^{L}\sum_{q=0}^{n}\|g_{k,q}\|_{L^2(B_\varepsilon)}+\sum_{k=1}^{L}\sum_{i=1}^{n}\|d_kv_i\|_{L^2(B_\varepsilon)}\Bigg).
\end{aligned}
\tag{2.2}
$$

Using the Hölder inequality and (1.5) we get

$$
\begin{aligned}
&\|d_kv_j\|_{L^2(B_\varepsilon)}\le\|d_k\|_{L^4(B_\varepsilon)}\|v_j\|_{L^4(B_\varepsilon)}\\
&\le(C_1^*)^2\varepsilon^{1-n/2}\|\nabla d_k\|_{L^2(B_\varepsilon)}\|\nabla v_j\|_{L^2(B_\varepsilon)}\le(C_1^*)^2\varepsilon^{1-n/2}r_\varepsilon^2.
\end{aligned}
\tag{2.3}
$$

Eventually, combining (2.2) and (2.3), and using (1.11) and (1.12), we obtain

$$\|\nabla \mathbf{b}\|_{L^2(B_\varepsilon)} \le C_3^*\Big(\varepsilon^{(n-1)/2}\sum_{j=N_1+1}^{N-1}|\mathbf{c}^{0,j}| + \sum_{k=1}^{L}\sum_{q=0}^{n}\|g_{k,q}\|_{L^2(B_\varepsilon)} + nL(C_1^*)^2\varepsilon^{1-n/2}r_\varepsilon^2\Big) \le r_\varepsilon.$$

2. Let us prove that T is a contraction.

Introduce

$$\alpha_\varepsilon = \max\Big\{C_2^*(C_1^*)^2\varepsilon^{1-n/2}r_\varepsilon\lambda, \quad \varepsilon C_2^* C_4^* C_0^*, \quad C_3^* nL(C_1^*)^2\varepsilon^{1-n/2}r_\varepsilon\Big\}. \tag{2.4}$$

Due to (1.12) and (1.13), $\alpha_\varepsilon < 1$.

Let $(\mathbf{v}_1, \mathbf{d}_1)$ and $(\mathbf{v}_2, \mathbf{d}_2)$ be two couples from $B_w \times B_b$ and let $(\mathbf{w}_i, \mathbf{b}_i)$ be $T(\mathbf{v}_i, \mathbf{d}_i)$, $i = 1, 2$. Subtracting the integral identities (2.1) with $\zeta_k = 0$ and using (1.7), the Hölder inequality, the definition of λ, (1.10), (1.5), (1.4), and (2.4), we get the estimate

$$\begin{aligned}
\|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\|_{L^2(B_\varepsilon)} &\le C_2^*\Big(\|R_\varepsilon(x, \mathbf{d}_1)\mathbf{v}_1 - R_\varepsilon(x, \mathbf{d}_2)\mathbf{v}_2\|_{L^2(B_\varepsilon)}\Big)\\
&\le C_2^*\Big(\lambda\|\mathbf{d}_1 - \mathbf{d}_2\|_{L^4(B_\varepsilon)}\|\mathbf{v}_2\|_{L^4(B_\varepsilon)} + C_4^*\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(B_\varepsilon)}\Big)\\
&\le C_2^*\Big((C_1^*)^2\varepsilon^{1-n/2}r_\varepsilon\lambda\|\nabla(\mathbf{d}_1 - \mathbf{d}_2)\|_{L^2(B_\varepsilon)} + \varepsilon C_4^* C_0^*\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^2(B_\varepsilon)}\Big)\\
&\le \alpha_\varepsilon\Big(\|\nabla(\mathbf{d}_1 - \mathbf{d}_2)\|_{L^2(B_\varepsilon)} + \|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^2(B_\varepsilon)}\Big).
\end{aligned}$$

Subtracting the integral identities (2.1) with $\boldsymbol{\eta} = 0$ and using (1.9), the Hölder inequality, (1.5), and (2.4), we get the estimate

$$\begin{aligned}
&\|\nabla(\mathbf{b}_1 - \mathbf{b}_2)\|_{L^2(B_\varepsilon)}\\
&\le C_3^* nL\Big(\|\mathbf{d}_1 - \mathbf{d}_2\|_{L^4(B_\varepsilon)}\|\mathbf{v}_1\|_{L^4(B_\varepsilon)} + \|\mathbf{d}_2\|_{L^4(B_\varepsilon)}\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^4(B_\varepsilon)}\Big)\\
&\le C_3^* nL(C_1^*)^2\varepsilon^{1-n/2}r_\varepsilon\Big(\|\nabla(\mathbf{d}_1 - \mathbf{d}_2)\|_{L^2(B_\varepsilon)} + \|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^2(B_\varepsilon)}\Big)\\
&\le \alpha_\varepsilon\Big(\|\nabla(\mathbf{d}_1 - \mathbf{d}_2)\|_{L^2(B_\varepsilon)} + \|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^2(B_\varepsilon)}\Big).
\end{aligned}$$

So, T is a contraction and applying the fixed point theorem we get the assertion of Theorem 1.1. □

2.2. Continuous dependence of the velocity and concentration on the data

Theorem 2.1. *Let* $\mathbf{f}^{(1)}, \mathbf{f}^{(2)} \in L^2(B_\varepsilon)$, $g_{k,0}^{(1)}, g_{k,0}^{(2)} \in L^2(B_\varepsilon)$, $k = 1, ..., L$, $g_{k,q}^{(1)}, g_{k,q}^{(2)} \in W^{1,2}(B_\varepsilon)$, $g_{k,q}^{(1)}|_{\Gamma_\varepsilon} = 0$, $g_{k,q}^{(2)}|_{\Gamma_\varepsilon} = 0$, $k = 1, ..., L$, $q = 1, ..., n$, $p_j^{*(1)}, p_j^{*(2)} \in \mathbb{R}$, $j = N_1 + 1, ..., N - 1$, $\mathbf{c}^{0,j,(1)}, \mathbf{c}^{0,j,(2)} \in \mathbb{R}^L$, $j = N_1 + 1, ..., N$, $\mathbf{c}^{0,N,(1)} = \mathbf{c}^{0,N,(2)} = \mathbf{0}$, *be data of two problems of the form (1.1), where* $\mathbf{f}$, g_k, p_j^*, $\mathbf{c}^{0,j}$ *are replaced by* $\mathbf{f}^{(1)}$, $g_k^{(1)}$, $p_j^{*(1)}$, $\mathbf{c}^{0,j,(1)}$ *and* $\mathbf{f}^{(2)}$, $g_k^{(2)}$, $p_j^{*(2)}$, $\mathbf{c}^{0,j,(2)}$, *respectively. Assume that conditions (1.12), (1.13), are satisfied for both sets of* *data. Let* $(\mathbf{u}_\varepsilon^{(1)}, \mathbf{c}_\varepsilon^{(1)})$ *and* $(\mathbf{u}_\varepsilon^{(2)}, \mathbf{c}_\varepsilon^{(2)})$ *be solutions (from Theorem 1.1) to these problems corresponding to the data marked* (1) *and* (2) *respectively, satisfying inequalities* $\|\nabla \mathbf{u}_\varepsilon^{(i)}\|_{L^2(B_\varepsilon)} \le r_\varepsilon$, $\|\nabla \mathbf{c}_\varepsilon^{(i)}\|_{L^2(B_\varepsilon)} \le r_\varepsilon$, $i = 1, 2$. *Then, the following estimate holds true.*

$$
\begin{aligned}
&\|\nabla \mathbf{u}_\varepsilon^{(1)} - \nabla \mathbf{u}_\varepsilon^{(2)}\|_{L^2(B_\varepsilon)} + \|\nabla \mathbf{c}_\varepsilon^{(1)} - \nabla \mathbf{c}_\varepsilon^{(2)}\|_{L^2(B_\varepsilon)} \\
&\le \frac{1}{1-\alpha_\varepsilon}\left(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^{*(1)} - p_j^{*(2)}| + \varepsilon \|\mathbf{f}^{(1)} - \mathbf{f}^{(2)}\|_{L^2(B_\varepsilon)}\right. \\
&\left. + \varepsilon^{(n-1)/2} \sum_{j=N_1+1}^{N} |\mathbf{c}^{0,j,(1)} - \mathbf{c}^{0,j,(2)}| + \sum_{k=1}^{L} \sum_{q=0}^{n} \|g_{k,q}^{(1)} - g_{k,q}^{(2)}\|_{L^2(B_\varepsilon)}\right).
\end{aligned}
\tag{2.5}
$$

Proof. Subtracting the integral identity (2.1) for $\mathbf{w} = \mathbf{v} = \mathbf{u}_\varepsilon^{(1)}$ from the same identity for $\mathbf{w} = \mathbf{v} = \mathbf{u}_\varepsilon^{(2)}$ with test function $\zeta_k = 0$, and then subtracting (2.1) for $\mathbf{b} = \mathbf{d} = \mathbf{c}_\varepsilon^{(1)}$ from the same identity with $\mathbf{b} = \mathbf{d} = \mathbf{c}_\varepsilon^{(2)}$ taking $\eta = \mathbf{0}$, and applying the same arguments as in the second part of the proof of Theorem 1.1, we get estimate (2.5). □

2.3. *Reconstruction of the pressure*

Let us reconstruct the pressure p_ε. We will use the following theorem proved in [18], see also [19].

Theorem 2.2. *Let* Φ *be a linear bounded functional defined on the space* $\widehat{W}^{1,2}(B_\varepsilon)$, $\boldsymbol{\eta} \mapsto \Phi(\boldsymbol{\eta})$ *vanishing on the subspace* $\widehat{K}^{1,2}(B_\varepsilon)$. *Then there exists a unique function* $p \in L^2(B_\varepsilon)$ *such that* $\Phi(\boldsymbol{\eta})$ *can be presented in a form* $\int_{B_\varepsilon} p(x) \operatorname{div} \boldsymbol{\eta}(x) dx$.

Taking $\zeta_k = 0$ in (1.2) we get the weak formulation of the form $\Phi(\boldsymbol{\eta}) = 0$ for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$, where

$$
\begin{aligned}
\Phi(\boldsymbol{\eta}) = &\int_{B_\varepsilon} \left(\frac{1}{2}\nu_\varepsilon(x) D\mathbf{u}_\varepsilon \cdot D\boldsymbol{\eta} + R_\varepsilon(x, \mathbf{c}_\varepsilon)\mathbf{u}_\varepsilon \cdot \boldsymbol{\eta}\right) dx \\
&+ \sum_{j=N_1+1}^{N} p_j \int_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\, d\sigma - \int_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta}\, dx.
\end{aligned}
\tag{2.6}
$$

Applying Theorem 2.2, we get the existence of a function $p_\varepsilon \in L^2(B_\varepsilon)$ such that

$$
\Phi(\boldsymbol{\eta}) = \int_{B_\varepsilon} p_\varepsilon \operatorname{div} \boldsymbol{\eta}\, dx, \quad \forall \boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon).
\tag{2.7}
$$

To evaluate the pressure we will use the following result proved in Lemma 2.8 in [18] (see also [19]).

Lemma 2.1. *Let h be a function in $L^2(B_\varepsilon)$. Then the divergence equation*

$$\mathrm{div}\mathbf{w}(x) = h(x), \quad x \in B_\varepsilon, \tag{2.8}$$

admits at least one solution $\mathbf{w} \in \widehat{W}^{1,2}(B_\varepsilon)$, satisfying the estimate

$$\|\nabla \mathbf{w}\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2}\|h\|_{L^2(B_\varepsilon)}. \tag{2.9}$$

Here constant c is independent of ε.

Taking in (2.7) $\boldsymbol{\eta}$ solution to equation (2.8) with $h = p_\varepsilon$, we get

$$\Phi(\boldsymbol{\eta}) = \int_{B_\varepsilon} p_\varepsilon(x)\mathrm{div}\boldsymbol{\eta}(x)dx = \int_{B_\varepsilon} p_\varepsilon^2(x)dx.$$

On the other hand see Lemma 2.3.16 in [19],

$$\Phi(\boldsymbol{\eta}) \le C_5^*\|\nabla \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)}\|\nabla \boldsymbol{\eta}\|_{L^2(B_\varepsilon)}$$
$$+C_6^*\Big(\varepsilon^{n/2}\sum_{j=N_1+1}^{N}|p_j| + \varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big)\|\nabla \boldsymbol{\eta}\|_{L^2(B_\varepsilon)},$$

with the constants C_5^* and C_6^* independent of ε. Applying now estimate (2.9) for $\|\nabla \boldsymbol{\eta}\|_{L^2(B_\varepsilon)}$ and (1.14) for $\|\nabla \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)}$, we get the following assertion.

Theorem 2.3. *The following estimate holds*

$$\|p_\varepsilon\|_{L^2(B_\varepsilon)} \le C_7^*\varepsilon^{-3/2}\Bigg(\varepsilon^{n/2}\sum_{j=N_1+1}^{N}|p_j| + \varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}$$
$$+\varepsilon^{(n-1)/2}\sum_{j=N_1+1}^{N}|\mathbf{c}^{0,j}| + \sum_{k=1}^{L}\sum_{q=0}^{n}\|g_{k,q}\|_{L^2(B_\varepsilon)}\Bigg).$$

3. Conclusion

In this paper, we introduce a thin tube structure B_ε, where ε is a small positive parameter describing the thickness of the tubes. A boundary value problem for the Stokes-Brinkman equation coupled with the diffusion-convection equation is considered in B_ε. The boundary conditions are: given pressure and concentrations at the inflow and outflow of B_ε, the no slip boundary condition on the lateral boundary of B_ε for the fluid, and Neumann type condition on the lateral boundary of B_ε for the diffusion-convection equations. This problem is well suited to describing thrombosis in blood vessels. The existence, uniqueness, and stability of the solution to such a problem are proved. Moreover, some *a priori* norm-estimates depending on ε are also provided. In particular, these results hold true in real life applications, where the internal forces are null and the given pressures at the inflow and outflow of B_ε can also depend on ε with order $O(\varepsilon^{-1+\beta})$, $\beta > 0$. By starting from the results and in particular from *a priori* norm-estimates obtained in the present paper, in a forthcoming paper we will construct the asymptotic expansion in B_ε, justified by error estimate, for Stokes-Brinkman equations with diffusion and convection.

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Data availability

No data was used for the research described in the article.

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