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Elementary theory of cubics and quartics

Juozas Juvencijus Mačys

Institute of Mathematics and Informatics, Vilnius University
Akademijos str. 4, LT-08663, Vilnius, Lithuania
E-mail: juozas.macys@mii.vu.lt

Abstract. It is proved that cubic and quartic equations in real numbers can be solved elementarily, avoiding complex numbers and derivatives. Corresponding algorithms are presented.

Keywords: cubic equation, quartic equation, rational solutions, factorization.

The school with the third and fourth degree equations encounters rarely (actually only when they have rational solutions). They are not considered in mathematical classes or even in facultative courses – the widespread belief is that they are needed to deal with complex numbers (see [1]). The purpose of this paper – to show that to solve elementary these equations is perfectly possible, and school knowledge is sufficient for this.

Cubic equations. General form of cubic (third degree) equation is

$$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0). \tag{1}$$

The equation can be divided by $a \neq 0$, and this form of the equation is called general one too:

$$x^3 + bx^2 + cx + d = 0. \tag{2}$$

The equation can be further simplified: entering a new variable y by substitution $x = y + k$ and selecting parameter k one can achieve that the equation has no longer a member of the second degree. Really, after inserting $x = y + k$ in equation (2), only the two first addends x^3 and bx^2 give the terms with y^2. Because

$$x^3 + bx^2 = x^2(x + b) = (y^2 + 2yk + k^2)(y + k + b),$$

then only terms $ky^2 + by^2 + 2ky^2$ have y^2. Thus y^2 disappears if $3k + b = 0$, i.e., $k = -b/3$. Thus, for equation (2) had not a quadratic member, the substitution can be made $x = y - b/3$. The resulting equation is called reduced one (the variable here again is denoted by x):

$$x^3 + px + q = 0. \tag{3}$$

The number of solutions. Cubic equation (3) always has at least one solution: on the left standing cubic function $f(x)$ is positive for sufficiently large x and negative for large negative x, so $f(x)$ graph crosses the x-axis. Cubic equation may have one, two or three solutions – for example, equations

$$(x - 1)^3 = 0, \quad (x - 1)^2(x - 2) = 0, \quad (x - 1)(x - 2)(x - 3) = 0$$

have sets of solutions, respectively, $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$.

More than three solutions cubic equation can have not. Indeed, suppose that equation (3) has a solution α. This means that $\alpha^3 + p\alpha + q = 0$. Now (3) it is easy to factorize:

$$\begin{aligned} x^3 + px + q &= x^3 + px + q - \alpha^3 - p\alpha - q \\ &= x^3 - \alpha^3 + px - p\alpha = (x-\alpha)(x^2 + x\alpha + \alpha^2 + p). \end{aligned}$$

Thus, other solutions of equation (3) we find from the quadratic equation

$$x^2 + x\alpha + \alpha^2 + p = 0. \tag{4}$$

So equation (4) can add to α 0, 1 or maximum 2 new (i.e., not coinciding with α) solutions of equation (3).

We see that the cubic equation solving and determination of number of solutions knowing at least one solution becomes trivial. That is why it is very easy to solve the cubic equation, which has a rational solution. By the way, it is easy to find rational solutions of every equation of any degree with rational coefficients. For example, if the coefficients of an equation are integers and the first coefficient is 1 (such a form we can give to every equation with rational coefficients), then rational solutions can be only integers – positive and negative divisors of the free member (see [2]). So it is worth the solving of every equation with rational coefficients always begin from rational solutions.

Case $\boldsymbol{p}$> 0. The cubic equation (3) can be further simplified to make the modulus of coefficient $p \neq 0$ equal to 3 (when $p = 0$, then equation (3) becomes $x^3 = -q$ and comprises a unique solution $x = -\sqrt[3]{q}$).

Again, we use a simple linear substitution $x = ky$ and properly select k. Our equation (3) is converted into

$$k^3y^3 + pky + q = 0, \quad y^3 + py/k^2 + q/k^3 = 0.$$

In the case $p > 0$ we choose k as follows: $p/k^2 = 3$, $k^2 = p/3$, $k = \sqrt{p/3}$. Thus substitution $x = y\sqrt{p/3}$ makes the coefficient of y equal to 3. The free member becomes $q/k^3 = -3\sqrt{3}q/(p\sqrt{p})$. Noting it for brevity sake $-2m$ (in other words, denoting $-3\sqrt{3}q/(2p\sqrt{p}) = m$), we have equation

$$y^3 + 3y - 2m = 0. \tag{5}$$

Now we try find some solution. Let us do substitution $y = z - 1/z$:

$$\begin{gathered} (z - 1/z)^3 + 3(z - 1/z) - 2m = 0, \quad z^3 - 1/z^3 - 2m = 0, \\ z^6 - 2mz^3 = 1, \quad z^6 - 2mz^3 + m^2 = 1 + m^2, \quad \left(z^3 - m\right)^2 = 1 + m^2, \\ z^3 - m = \pm\sqrt{m^2+1}, \quad z^3 = m \pm \sqrt{m^2+1}, \quad z = \sqrt[3]{m \pm \sqrt{m^2+1}}. \end{gathered}$$

Both of these z values give the same solution of equation (5)

$$y = \sqrt[3]{\sqrt{m^2+1} + m} + \sqrt[3]{\sqrt{m^2+1} - m}.$$

Let's make sure that equation (5) has no more solutions. Recall equation (4) – now it looks like this:

$$y^2 + y\alpha + \alpha^2 + 3 = 0, \quad \text{where } \alpha = \sqrt[3]{\sqrt{m^2+1}+m} + \sqrt[3]{\sqrt{m^2+1}-m}.$$

The discriminant of the quadratic equation is equal to

$$\alpha^2 - 4(\alpha^2 + 3) = -3\alpha^2 - 12.$$

Since it is negative, equation (5) has no more solutions. Thus, equation (3) in the case $p > 0$ has the only solution. We find it going back from y to $x = y\sqrt{p/3}$.

Case $\boldsymbol{p}$< 0. It is more difficult to solve equation (3), when p is negative.We apply substitution $x = ky$ again. Now in the equation

$$y^3 + py/k^2 + q/k^3 = 0$$

we choose k as $p/k^2 = -3$, $k^2 = -p/3$, $k = \sqrt{-p/3}$. Denoting the resulting free member $-2m$, we have the equation

$$y^3 - 3y - 2m = 0. \tag{6}$$

We apply substitution $y = z + 1/z$:

$$(z + 1/z)^3 - 3(z + 1/z) - 2m = 0, \quad z^3 + 1/z^3 - 2m = 0, \quad z^6 - 2mz^3 = -1,$$
$$(z^3 - m)^2 = m^2 - 1. \tag{7}$$

Here awaits us a surprise – equation (7) has solutions not always, and we have to consider three subcases: $m^2 > 1$, $m^2 = 1$ and $m^2 < 1$.

Subcase $p < 0$, $m^2 > 1$ (the unique solution). If $m^2 > 1$, then

$$z^3 - m = \pm\sqrt{m^2 - 1}, \quad z = \sqrt[3]{m \pm \sqrt{m^2 - 1}},$$

and both values of z give the same solution:

$$y = z + 1/z = \sqrt[3]{m + \sqrt{m^2-1}} + \sqrt[3]{m - \sqrt{m^2-1}}.$$

Let us convince that this solution is unique. The modulus of this solution is greater than 2:

$$y^2 = (z + 1/z)^2 = z^2 + 2 + 1/z^2 = z^2 - 2 + 1/z^2 + 4 = (z - 1/z)^2 + 4 > 4.$$

Really, $y^2 = 4$ would mean $y = \pm 2$, and then from equation (6)

$$2m = y^3 - 3y = y(y^2 - 3) = \pm 2, \quad m = \pm 1$$

(but in our case must be $m^2 > 1$). Other solutions could be given by (4) equation, which now looks like this:

$$y^2 + y\alpha + \alpha^2 - 3 = 0, \quad \text{where } |\alpha| > 2.$$

The discriminant of it

$$\alpha^2 - 4(\alpha^2 - 3) = 12 - 3\alpha^2 = 3(4 - \alpha^2)$$

is negative, and equation (4) has no solutions. Thus, equation (6) in case of $p < 0$, $m^2 > 1$ has the unique solution.

Subcase $p < 0$, $m^2 = 1$ (two solutions). If $m^2 = 1$, then from (7) equation $z^3 = m$. When $m = 1$, then $z = 1$, $y = z + 1/z = 2$. Equation (6) is transformed into $y^3 - 3y - 2 = 0$, which can be factorized even without equation (4) treatment:

$$y^3 - 3y - 2 = 0, \quad (y - 2)(y^2 + 2y + 1) = 0, \quad (y - 2)(y + 1)^2 = 0.$$

Thus, (6) (and (3)) in this case has two solutions. Now we return to x.

When $m = -1$, then $z = -1$, $y = -2$. Equation (6) becomes

$$y^3 - 3y + 2 = 0, \quad (y + 2)(y^2 - 2y + 1) = 0, \quad (y + 2)(y - 1)^2 = 0,$$

and as well (hence equation (3)) has two solutions. We return to x.

Subcase $p < 0$, $m^2 < 1$ (three solutions). If $m^2 < 1$, then equation (7) has no solutions, and substitution $y = z + 1/z$ did not find a solution. But here helps trigonometry: the substitution $y = 2z$ puts equation (6) in the form $8z^3 - 6z - 2m = 0$, i.e. $4z^3 - 3z - m = 0$. Do substitution $z = \cos\varphi$:

$$4\cos^3\varphi - 3\cos\varphi - m = 0, \quad \cos 3\varphi = m.$$

This equation has solutions specifically when $|m| < 1$ (that is $m^2 < 1$). Further,

$$3\varphi = 2k\pi \pm \arccos m, \quad \varphi = \frac{2}{3}k\pi \pm \frac{1}{3}\arccos m,$$

$$z = \cos\varphi = \cos\left(\frac{2}{3}k\pi \pm \frac{1}{3}\arccos m\right), \quad y = 2z = 2\cos\left(\frac{2}{3}k\pi \pm \frac{1}{3}\arccos m\right), \quad k \in \mathbb{Z}.$$

Based on the reduction formulas we make sure we get only three different y values (more than three solutions cubic equation can have not!):

$$2\cos\left(\frac{1}{3}\arccos m\right), \quad 2\cos\left(\frac{2}{3}\pi + \frac{1}{3}\arccos m\right), \quad 2\cos\left(\frac{4}{3}\pi + \frac{1}{3}\arccos m\right).$$

We return to x. General cubic equation (3) is solved.

An example. Consider a concrete equation in the form (3):

$$x^3 - 3x - 4 = 0. \tag{8}$$

Solve it anew, "without theory". (Of course, the theory is always born of concrete examples.) Think of the school formulas having something like $x^3 - 3x$. There comes to mind the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b)$$

and its particular case $(a \pm 1/a)^3 = a^3 \pm 1/a^3 \pm 3(a \pm 1/a)$.

The last formula offers that in equation (8) it is worth to use substitution $x = y + 1/y$. Then the equation turns into

$$(y+1/y)^3 - 3(y+1/y) - 4 = 0,$$
$$y^3 + 1/y^3 + 3(y+1/y) - 3(y+1/y) - 4 = 0, \quad y^3 + 1/y^3 - 4 = 0.$$

We have the equation, quadratic with respect to y^3, so we can solve it (you can even not use formula of the quadratic equation solutions):

$$y^6 - 4y^3 + 1 = 0, \quad (y^3-2)^2 = 3, \quad y^3 - 2 = \pm\sqrt{3},$$
$$y^3 = 2 \pm \sqrt{3}, \quad y = \sqrt[3]{2 \pm \sqrt{3}}.$$

The first value of y returning to $x = y + 1/y$ gives

$$x = \sqrt[3]{2+\sqrt{3}} + 1/\sqrt[3]{2+\sqrt{3}} = \sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}}$$

(the same solution gives the other value of y too). Still need to make sure that there are no other solutions (by the use of the substitution $x = y + 1/y$ you can not find the solutions $x \in (-2, 2)$, because $|y + 1/y| = |y| + 1/|y| = (\sqrt{|y|} - 1/\sqrt{|y|})^2 + 2 \geq 2$). Equation (8) can be factorized dividing by $x - \sqrt[3]{2+\sqrt{3}} - \sqrt[3]{2-\sqrt{3}}$, but it would be quite an unpleasant job. It performs for us the formula (4), which becomes as follows: $x^2 + x\alpha + \alpha^2 - 3 = 0$, where $\alpha = \sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}}$. The discriminant of this equation $\alpha^2 - 4(\alpha^2 - 3) = 12 - 3\alpha^2$ is negative because $\alpha^2 = (y+1/y)^2 = (y-1/y)^2 + 4 > 4$. Thus, equation (8) has a unique solution.

The Cardano formula. It is now clear that examined ways you can immediately apply to equation (3).

Case 1: $q^2/4 + p^3/27 > 0$. Perform substitution $x = \sqrt[3]{y} - p/(3\sqrt[3]{y})$. Find y, then return to x:

$$x = \sqrt[3]{-q/2 + \sqrt{q^2/4 + p^3/27}} + \sqrt[3]{-q/2 - \sqrt{q^2/4 + p^3/27}}$$

(this is so called the Cardano formula). The solution is unique.

Case 2: $q^2/4 + p^3/27 = 0$. Cardano's formula gives the solution $-\sqrt[3]{4q}$, then equation (4) gives the solution $\sqrt[3]{q/2}$. When $q = 0$ (and hence $p = 0$), we obtain a unique solution $x = 0$. When $q \neq 0$, we obtain two solutions.

Case 3: $q^2/4 + p^3/27 < 0$. Obviously, $p < 0$. Performing substitution $x = \sqrt{4|p|/3}\cos\varphi$, we obtain the equation $4\cos^3\varphi - 3\cos\varphi = q\sqrt{27}/(2p\sqrt{|p|})$, $\cos 3\varphi = q\sqrt{27}/(2p\sqrt{|p|})$. Since $27q^2 < -4p^3$, then module of right side is less than 1. We finde φ, and then all three solutions of equation (3):

$$x = \sqrt{4|p|/3}\cos\left(\frac{1}{3}\arccos\left(3q\sqrt{3}/2p\sqrt{|p|}\right)\right),$$
$$\alpha = \sqrt{4|p|/3}\cos\left(\frac{2}{3}\pi + \frac{1}{3}\arccos\left(3q\sqrt{3}/2p\sqrt{|p|}\right)\right),$$
$$\alpha = \sqrt{4|p|/3}\cos\left(\frac{4}{3}\pi + \frac{1}{3}\arccos\left(3q\sqrt{3}/2p\sqrt{|p|}\right)\right).$$

As a rule namely formulas of this section you find in manuals (cf. [2]), so one can get solutions without solving equation (3).

Quartic equation. Common fourth degree equation is

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \quad (a \neq 0).$$

It can be divided by a, so it is enough to consider the equation

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

After substitution $x = y - a/4$, we eliminate of it a member of the third degree, so it is enough to know how to solve the equation

$$x^4 + bx^2 + cx + d = 0. \tag{9}$$

If the fourth degree equation has rational solutions (at least one rational solution α), it is simple to solve it: the left-hand side can be factorized again, separating the $x - \alpha$, and it remains to solve the cubic equation. Consider (9) equation (when it does not have rational solutions; naturally, we can also deal with when it has).

If $c = 0$, equation (9) is biquadrate, and adopting the new variable $x^2 = y$ it becomes quadratic one. By the way, it is easy to decompose biquadrate trinomial $x^4 + bx^2 + d$. If $b^2/4 - d \geq 0$, then

$$\begin{aligned} x^4 + bx^2 + d &= (x^2 + b/2)^2 - (b^2/4 - d) \\ &= \big(x^2 + b/2 + \sqrt{b^2/4 - d}\big)\big(x^2 + b/2 - \sqrt{b^2/4 - d}\big). \end{aligned}$$

If $b^2/4 - d < 0$, then $d > b^2/4$ (i.e. d positive), $\sqrt{d} > b/2$, $2\sqrt{d} - b > 0$, and

$$\begin{aligned} x^4 + bx^2 + d &= (x^2 + \sqrt{d})^2 - x^2(2\sqrt{d} - b) \\ &= \Big(x^2 + \sqrt{d} + x\sqrt{2\sqrt{d} - b}\Big)\Big(x^2 + \sqrt{d} - x\sqrt{2\sqrt{d} - b}\Big). \end{aligned}$$

If $c \neq 0$, then the left side of equation (9) is always possible to write as the difference of two squares:

$$(x^2 + m)^2 - p(x + n)^2 = 0 \quad (p > 0).$$

Thus the left-hand side can be factorized into two quadratic trinomials,

$$(x^2 + x\sqrt{p} + m + n\sqrt{p})(x^2 - x\sqrt{p} + m - n\sqrt{p}),$$

and you solve two quadratic equations.

Left to figure out how to find a suitable m, p and n values. In equation

$$(x^2 + m)^2 - p(x + n)^2 = x^4 + bx^2 + cx + d$$

removing brackets and comparing the coefficients of x^2, x^1 and x^0, we have the system of equations for suitable m, p, n to find:

$$2m - p = b, \quad -2np = c, \quad m^2 - pn^2 = d. \tag{10}$$

Since $c \neq 0$, from second equation $p \neq 0$, then $n = -c/(2p)$. From first equation $m = (b+p)/2$. After inserting the third equation becomes $(b+p)^2/4 - c^2/(4p) = d$. After multiplying by $4p$, we get the cubic equation (cf. [1, 3])

$$p^3 + 2bp^2 - 4dp + b^2p - c^2 = 0.$$

This equation has positive solution (namely, this new observation is the fourth degree equation solving key). Really, the function $f(p)$ on the left at $p = 0$ is equal to $f(0) = -c^2$, namely negative (recall $c \neq 0$) and for the sufficiently large positive p it is positive, so the graph of $f(p)$ intersects p axis at any point $p_0 \in (0, \infty)$. Since we can solve the cubic equation, we find positive solution p_0, receiving the solution of the system (10) $(m, p, n) = (b/2 + p_0/2, p_0, -\frac{1}{2}c/p_0)$. Thus, we are able to decompose the left side of equation (9) and, consequently, to solve any fourth order equation. It may have $1, 2, 3, 4$ or no solutions (e.g. $x^4 + 1 = 0$ has no solutions).

It turns out that to solve the general equation of the fifth degree is not possible. However, the most important statement (the fundamental theorem of algebra) remains correct [1]: *each n-th degree polynomial is a product of linear and quadratic factors.*

References

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REZIUMĖ

Elementari trečiojo ir ketvirtojo laipsnių lygčių teorija

J.J. Mačys

Įrodyta, kad trečiojo ir ketvirtojo laipsnio lygtis realiųjų skaičių aibėje galima išspręsti elementariai – nesiremiant nei kompleksiniais skaičiais, nei išvestinėmis. Pateikti atitinkami algoritmai.

Raktiniai žodžiai: trečiojo laipsnio lygtys, ketvirtojo laipsnio lygtys, racionalieji sprendiniai, daugianarių skaidymas.