https://doi.org/10.15388/vu.thesis.834
https://orcid.org/0009-0008-2717-0380

VILNIUS UNIVERSITY

Paulius Virbalas

Problems on the Degrees of Algebraic Numbers and Relations Between Algebraic Conjugates

DOCTORAL DISSERTATION

Natural Sciences,
Mathematics (N 001)

VILNIUS 2025

The dissertation was prepared between 2021 and 2025 at Vilnius University.

Academic supervisor – Prof. Habil. Dr. Artūras Dubickas (Vilnius University, Natural Sciences, Mathematics, N 001).

Academic consultant – Prof. Dr. Paulius Drungilas (Vilnius University, Natural Sciences, Mathematics, N 001).

This doctoral dissertation will be defended in a public meeting of the Dissertation Defence Panel:

Chairman – Prof. Habil. Dr. Antanas Laurinčikas (Vilnius University, Natural Sciences, Mathematics, N 001).

Members:

Prof. Dr. Ramūnas Garunkštis (Vilnius University, Natural Sciences, Mathematics, N 001),

Assoc. Prof. Dr. Andrius Grigutis (Vilnius University, Natural Sciences, Mathematics, N 001),

Assoc. Prof. Dr. Alar Leibak (Tallinn University of Technology, Natural Sciences, Mathematics, N 001),

Prof. Dr. Darius Šiaučiūnas (Vilnius University, Natural Sciences, Mathematics, N 001).

The dissertation shall be defended at a public meeting of the Dissertation Defence Panel at 16:00 on 23rd October 2025 in Room 102 of the Faculty of Mathematics and Informatics, Vilnius University.

Address: Naugarduko st. 24, Vilnius, Lithuania.

Tel. +370 5219 3050; mif@mif.vu.lt.

The text of this dissertation can be accessed at the library of Vilnius University, as well as on the website of Vilnius University:

www.vu.lt/lt/naujienos/ivykiu-kalendorius

https://doi.org/10.15388/vu.thesis.834
https://orcid.org/0009-0008-2717-0380

VILNIAUS UNIVERSITETAS

Paulius Virbalas

Uždaviniai apie algebrinių skaičių laipsnius ir sąryšius tarp algebrinių jungtinių

DAKTARO DISERTACIJA

Gamtos mokslai,
Matematika (N 001)

VILNIUS 2025

Disertacija rengta 2021–2025 metais Vilniaus universitete.

Mokslinis vadovas – prof. habil. dr. Artūras Dubickas (Vilniaus universitetas, gamtos mokslai, matematika, N 001).
Mokslinis konsultantas – prof. dr. Paulius Drungilas (Vilniaus universitetas, gamtos mokslai, matematika, N 001).

Gynimo taryba:
Pirmininkas – prof. habil. dr. Antanas Laurinčikas (Vilniaus universitetas, gamtos mokslai, matematika, N 001).
Nariai:
prof. dr. Ramūnas Garunkštis (Vilniaus universitetas, gamtos mokslai, matematika, N 001),
doc. dr. Andrius Grigutis (Vilniaus universitetas, gamtos mokslai, matematika, N 001),
doc. dr. Alaras Leibakas (Talino technologijos universitetas, gamtos mokslai, matematika, N 001),
prof. dr. Darius Šiaučiūnas (Vilniaus universitetas, gamtos mokslai, matematika, N 001).

Disertacija ginama viešame Gynimo tarybos posėdyje 2025 m. spalio mėn. 23 d. 16:00 val. Vilniaus universiteto Matematikos ir informatikos fakulteto 102 auditorijoje.
Adresas: Naugarduko g. 24, Vilnius, Lietuva.
Tel. +3705219 3050; el. paštas mif@mif.vu.lt.

Disertaciją galima peržiūrėti Vilniaus universiteto bibliotekoje ir Vilniaus universiteto interneto svetainėje adresu:
https://www.vu.lt/naujienos/ivykiu-kalendorius

Contents

Introduction

A complex number α is called an *algebraic number* if it is a root of some nonzero polynomial $p(x)$ with rational coefficients, that is, if $p(\alpha) = 0$ for some $p(x) \in \mathbb{Q}[x]$. Algebraic numbers extend the rational numbers, as every rational number q is a root of the linear polynomial $x - q$, and hence algebraic. Irrational numbers such as $\sqrt{2}$ or the golden ratio $\phi = (1 + \sqrt{5})/2$ are also algebraic, since they are the roots of the polynomials $x^2 - 2$ and $x^2 + x - 1$, respectively. On the other hand, in 1882 Lindemann [34] proved that there is no polynomial $p(x) \in \mathbb{Q}[x]$ such that $p(\pi) = 0$. This established that the constant π is not algebraic; numbers of this kind are called *transcendental*.

The origins of algebraic numbers can be traced to the problem of finding algebraic solutions of polynomial equations, which became a central theme of algebra during the Renaissance. Explicit formulas, such as the well-known solution formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the quadratic equation $ax^2 + bx + c = 0$, were discovered for equations of degree up to four. The case of quintics and higher degrees remained unresolved until the 19th century, when the pioneering works of Ruffini, Abel, and Galois showed that no general algebraic solution formula exists for equations of degree five or higher. While this closed the centuries-old problem, it opened the way to new approaches in mathematics. For example, the three classical geometry problems of Ancient Greece – angle trisection, duplication of the cube, and squaring the circle – were finally resolved in the 19th century through algebraic methods. The systematic study of algebraic numbers as independent objects began in the 1870s with the work of Dedekind. Building on ideas introduced by Kummer in his attempts to prove Fermat's Last Theorem, Dedekind formalized

the notions of algebraic numbers and algebraic integers, introduced the concept of algebraic number fields, and laid the foundation for modern algebraic number theory. Algebraic numbers continue to play an important role in modern mathematics, with applications in areas such as algebraic geometry, number theory, and cryptography.

One of the fundamental characteristics of an algebraic number is its degree. For every algebraic number α, there exists a unique monic polynomial $p(x) \in \mathbb{Q}[x]$ of minimal degree such that $p(\alpha) = 0$. The polynomial $p(x)$ is called the *minimal polynomial* of α, and the *degree* of α, denoted by $\deg(\alpha)$, is defined as the degree of $p(x)$. It has been known since the time of Dedekind that the set of algebraic numbers is closed under addition and multiplication; that is, for any algebraic numbers α and β, both $\alpha + \beta$ and $\alpha \cdot \beta$ are also algebraic. Subsequently, given the degrees of α and β, it is natural to ask what possible values the degrees of $\alpha + \beta$ and $\alpha \cdot \beta$ can attain. Nonetheless, until the 21st century, this question attracted comparatively little scholarly attention. A systematic approach to the so-called degree problem was introduced by Drungilas, Dubickas, and Smyth [6] in 2012 and later applied in a series of other studies. The first part of the thesis is devoted to deriving new results on this topic.

The degree of an algebraic number α is also equal to the number of distinct roots its minimal polynomial $p(x)$ has (here and throughout, it is implicitly assumed that we work over the field of rational numbers $\mathbb{Q}$). The roots of $p(x)$ are called *algebraic conjugates* of α. For example, take $\sqrt{2}$ and its minimal polynomial $x^2 - 2$. Then $\sqrt{2}$ and $-\sqrt{2}$ are algebraic conjugates of $\sqrt{2}$. In the general case, consider an algebraic number α of degree $d \geqslant 2$, and denote its distinct algebraic conjugates by $\alpha_1, \ldots, \alpha_d$. In the thesis we mainly focus on linear relations between algebraic conjugates of the form

$$k_1\alpha_1 + \ldots + k_d\alpha_d = 0, \tag{1}$$

where $k_1, \ldots, k_d$ are integer coefficients. It is well known from classical algebra that if all the coefficients are equal, then for any positive integer d, there exist distinct algebraic conjugates of degree d that satisfy (1). Accordingly, such relations are referred to as *trivial*. Under exceptional circumstances, however, a relation of the form (1) may hold even when

some of the coefficients are distinct. In such cases, it is said that there exists a *non-trivial* additive relation between algebraic conjugates of α. For example, in 1982 Girstmair proved the existence of an algebraic number α of degree 9 whose distinct algebraic conjugates satisfy

$$4\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - 2\alpha_6 - 2\alpha_7 - 2\alpha_8 - 2\alpha_9 = 0.$$

Since the introduction of this topic in the 1980s through the works of Girstmair [18] and Smyth [39], significant results have been obtained, both theoretical and numerical. Nevertheless, the phenomenon of non-trivial relations remains relatively little understood even for algebraic numbers of low degree. The investigations in this area form the second part of the thesis.

Aims and problems

Throughout this thesis, the terms *degree*, *minimal polynomial*, and *algebraic conjugates* are understood to mean the degree over $\mathbb{Q}$, the minimal polynomial over $\mathbb{Q}$, and the algebraic conjugates over $\mathbb{Q}$, respectively.

The first part of the thesis addresses three related questions concerning the degrees of algebraic numbers. Before addressing these questions, let us recall that any subfield K of the complex numbers $\mathbb{C}$ can be viewed as a vector space over the field of rational numbers $\mathbb{Q}$. The dimension of this vector space, denoted by $[K:\mathbb{Q}]$, is called the degree K. If $[K:\mathbb{Q}] < \infty$, then K is called a number field. For every such K, there exists an algebraic number α such that $K = \mathbb{Q}(\alpha)$. In this case, the degree of K coincides with the degree of α, that is, $[K:\mathbb{Q}] = \deg(\alpha)$. Finally, for number fields K and L, the compositum KL is the smallest number field containing both K and L.

- Let K and L be number fields of degrees a and b, respectively. What are the possible values for the degree of compositum KL?
- Let α and β be algebraic numbers of degrees a and b, respectively. What are the possible values for the degree of $\alpha+\beta$?
- Let α and β be algebraic numbers of degrees a and b, respectively. What are the possible values for the degree of $\alpha\beta$?

The principal aim is to provide a complete solution to all three questions in the case $a = b = p$, where p is a prime number. In addition, we seek to obtain partial results for the three questions in situations where at least one of a or b is not prime.

The second part of the thesis investigates non-trivial relations between algebraic conjugates. In particular, let α be an algebraic number of degree d, and let $\alpha_1, \ldots, \alpha_d$ denote distinct algebraic conjugates of α over $\mathbb{Q}$. The two central questions are the following.

- For which integers $k_1, \ldots, k_d$, not all equal, does there exist an additive relation
$$k_1\alpha_1 + \ldots + k_d\alpha_d = 0?$$

- For which integers $k_1, \ldots, k_d$, not all equal, does there exist a multiplicative relation
$$\alpha_1^{k_1} \cdot \ldots \cdot \alpha_d^{k_d} = 1?$$

The principal aim is to describe non-trivial additive relations between algebraic conjugates of degree $d \leqslant 8$. Moreover, we seek to show how every additive relation can be transformed into a multiplicative one.

Layout of the dissertation

This doctoral dissertation is based on six articles, published or accepted for publication in journals indexed in the Clarivate Analytics Web of Science (CA WoS) database (see the next page).

Chapter 1 presents an overview of the degree problems considered in Article A1, Article A2, and Article A3. Chapter 2 presents a summary of non-trivial relations between algebraic conjugates investigated in Article A4, Article A5, and Article A6. Copies of the articles are attached at the end of the thesis.

For copyright reasons, the published articles are omitted from this publicly accessible version. They are included only in the official printed dissertation.

Articles (listed in the order of presentation)

[A1] VIRBALAS, P. Compositum of two number fields of prime degree. *New York J. Math. 29* (2023), 171–192.

[A2] VIRBALAS, P. Compositum of two number fields of prime power degree. *Lith. Math. J. 65* (2025), 445–454.

[A3] VIRBALAS, P. Degree of the product of two algebraic numbers one of which is of prime degree. *Mathematics 11* (2023), Paper No. 1485, 16 pp.

[A4] VIRBALAS, P. Linear relations between three conjugate algebraic numbers of low degree. *J. Korean Math. Soc. 62* (2025), 253–284.

[A5] VIRBALAS, P. Linear relations between three algebraic conjugates of degree twice a prime. *Glas. Mat. Ser. III* (to appear).

[A6] DUBICKAS, A., AND VIRBALAS, P. Additive and multiplicative relations with algebraic conjugates. *Rev. Un. Mat. Argentina* (to appear).

Chapter 1

Three degree problems about algebraic numbers

1.1 Literature review

Research on the three related degree problems about algebraic numbers originated from a question posed by Dubickas in 2007: *"Given two algebraic numbers α and β with degrees a and b, respectively, what are the possible values for the degree of $\alpha+\beta$?"* Independently, the same question was raised at the online mathematical forum MathOverflow[1] in 2010. With the aim of investigating this problem systematically, the following definitions were introduced by Drungilas, Dubickas, and Smyth in 2012 [6]. A triplet $(a,b,c) \in \mathbb{N}^3$ is called:

- *compositum-feasible* if there exist number fields K and L of degrees a and b (over $\mathbb{Q}$), respectively, such that the degree of their compositum KL is c;
- *sum-feasible* if there exist algebraic numbers α and β of degrees a and b (over $\mathbb{Q}$), respectively, such that the degree of $\alpha+\beta$ is c;
- *product-feasible* if there exist algebraic numbers α and β of degrees a and b (over $\mathbb{Q}$), respectively, such that the degree of $\alpha\cdot\beta$ is c.

For example, since

$$[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2, \quad [\mathbb{Q}(\sqrt{3}):\mathbb{Q}]=2, \quad \text{and} \quad [\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}]=4,$$

the triplet $(2,2,4)$ is compositum-feasible. This triplet is also sum-feasible and product-feasible. Indeed, by taking $\alpha=\sqrt{2}$ and $\beta=1+\sqrt{3}$,

[1] `http://mathoverflow.net/questions/30151/`

one can verify that α and β are both quadratic, while

$$\alpha+\beta=1+\sqrt{2}+\sqrt{3} \quad \text{and} \quad \alpha\cdot\beta=\sqrt{2}+\sqrt{6}$$

are both quartic. However, the three problems described above are not equivalent. For instance, from

$$\alpha=e^{2\pi i/3}, \quad \beta=\sqrt[3]{2}, \quad \text{and} \quad \alpha\cdot\beta=e^{2\pi i/3}\cdot\sqrt[3]{2}, \tag{1.1}$$

one can deduce that the triplet $(2,3,3)$ is product-feasible [35]. In contrast, the findings of Isaacs [26] imply that the triplet of the form $(2,3,c)$ is sum-feasible if and only if $c=6$. Thus, the triplet $(2,3,3)$ is not sum-feasible (it can be shown that it is not compositum-feasible either). The main result describing the relationship between triplets of different feasibility types is due to Drungilas and Dubickas [4]. Let $\mathcal{C}$ denote the set of all compositum-feasible triplets, $\mathcal{S}$ – the set of all sum-feasible triplets, and $\mathcal{P}$ – the set of all product-feasible triplets. Then the following chain of strict inclusions holds

$$\mathcal{C}\subsetneq\mathcal{S}\subsetneq\mathcal{P}. \tag{1.2}$$

Hence, from (1.2) it follows, for example, that every sum-feasible triplet is also a product-feasible triplet, but not vice versa.

At the time of writing this thesis, the classification of all triplets

$$(a,b,c)\in\mathcal{C} \quad \text{or} \quad (a,b,c)\in\mathcal{S} \quad \text{or} \quad (a,b,c)\in\mathcal{P} \tag{1.3}$$

remains out of reach. In fact, the classification of compositum-feasible triplets and sum-feasible triplets has been completed only for triplets (a,b,c) satisfying $a\leqslant b\leqslant c$ and $b\leqslant 9$ (see [5, 6, 7]). The multiplicative problem, arguably more difficult than its additive analogue, was extensively investigated by Maciulevičius in [35], where the groundwork was laid for the classification of all product-feasible triplets (a,b,c) with $a\leqslant b\leqslant c$ and $b\leqslant 7$ (the classification was finalized in his joint work with Dubickas [13]). Throughout the process, several important constructive results have been derived through which large families of triplets satisfying (1.3) can be generated. For instance, if $(a,b,c)\in\mathcal{C}$, then for any prime number p and non-negative integers u,v,w such that $\max(u,v)\leqslant w\leqslant u+v$, one always has $(ap^u,bp^v,cp^w)\in\mathcal{C}$ (the analogous property also holds within the sets $\mathcal{S}$ and $\mathcal{P}$) [6]. On the other hand,

sometimes it can be highly non-trivial to decide whether a particular triplet (a,b,c) is of some feasibility type or not. Notable examples are the triplets $(6,6,8)$ and $(4,6,8)$, which were at the center of consideration in [5] and [13], respectively.

We conclude the literature review with the most recent developments in this area. In [6], the conjecture was raised that for two compositum-feasible (respectively, sum-feasible or product-feasible) triplets (a,b,c) and (a',b',c'), the triplet (aa',bb',cc') is also compositum-feasible (respectively, sum-feasible or product-feasible). While no progress has been made on the additive and multiplicative cases, there is some conceptual evidence to support the claim that the conjecture is true in the compositum case [4]. If confirmed, this would imply that $\mathcal{C}$ forms a semigroup under pointwise multiplication. In connection with this, Maciulevičius initiated the study in [35] of the so-called irreducible compositum-feasible triplets that act as building blocks within the set $\mathcal{C}$. In particular, he identified the first non-trivial infinite family of irreducible compositum-feasible triplets, namely $(n,n,n(n-1))$ for all positive integers $n \geqslant 2$.

1.2 Research objectives

In their 2013 study, Drungilas, Dubickas, and Luca observed that *"even a natural question of describing which values $[KL:\mathbb{Q}]$ can take if K and L are two extensions of prime degree p over $\mathbb{Q}$ remains open"*. This remark motivates the following research objective.

Problem 1.1. *To determine all compositum-feasible, sum-feasible, and product-feasible triplets of the form (p,p,c), where p is a prime number.*

Among the three degree problems, the multiplicative one appears to be the least understood. Subsequently, the second research objective of this chapter focuses on identifying properties that could be applied in the study of product-feasible triplets.

Problem 1.2. *To discover new conditions under which the degree of the product $\alpha \cdot \beta$ is maximal possible, i.e.,*

$$\deg(\alpha \cdot \beta) = \deg(\alpha) \cdot \deg(\beta).$$

1.3 Main results and their actuality

In Article A1, the complete solution to Problem 1.1 is established.

Theorem 1.1. *Let p be a prime number. If the triplet (p,p,c) is compositum-feasible, then $c = ps$ and one of the following occurs:*

(i) $s = p$;

(ii) s *is a divisor of* $(p-1)$;

(iii) $p = 11$, $s = 6$;

(iv) $p = \frac{q^d-1}{q-1}$, $s = q^{d-1}$ *for some prime power* q *and* $d \geqslant 3$.

To illustrate Theorem 1.1, we provide a numerical example for the special case $p = 13$.

Corollary 1.2. *Let K and L be two number fields, both of degree* 13*. If c denotes the degree of compositum KL, then*

$$c = 13s \quad \textit{with} \quad s \in \{1,2,3,4,6,9,12,13\}.$$

A notable case of Corollary 1.2 is $s = 9$, which corresponds to part (iv) of Theorem 1.1, since

$$13 = \frac{3^3-1}{3-1} \quad \text{and} \quad 9 = 3^2.$$

The triplet $(13,13,13\cdot 9)$ is the second smallest example of a compositum-feasible triplets arising from part (iv) of Theorem 1.1. The smallest such example, namely $(7,7,7\cdot 4)$, appears in the work of Drungilas, Dubickas, and Luca, accompanied by the remark *"while there is no obvious reason for this at all, the triplet* $(7,7,28)$ *is compositum-feasible"* [5]. Thus, one of the main contributions of Article A1 is that it provides a precise explanation of why these exceptional triplets are compositum-feasible.

Part (iv) of Theorem 1.1 is closely linked with properties of the so-called projective groups formed by transformations of projective spaces. In that sense, Theorem 1.1 offers another example where primes of the form $p = (q^d-1)/(q-1)$ play a significant role (for other applications, see [16, 23, 27, 37]). It is conjectured that there are infinitely many such primes [28].

It turns out that for triplets of the form (p,p,c), the solutions to the additive and multiplicative problems can, with minor adjustments, be derived from the solution to the compositum problem given in Theorem 1.1.

Theorem 1.3. *Let p be a prime number. If the triplet (p,p,c) is sum-feasible, then it is also compositum-feasible or $c=1$.*

Theorem 1.4. *Let p be a prime number. If the triplet (p,p,c) is product-feasible, then it is also compositum-feasible or $c \in \{1, p-1\}$.*

We illustrate how Theorem 1.3 and Theorem 1.4 can be applied in the case $p=13$.

Corollary 1.5. *Let α and β be two algebraic numbers, both of degree 13. If c denotes the degree of $\alpha+\beta$, then*

$$c = 13s \quad \textit{with} \quad s \in \{1,2,3,4,6,9,12,13\} \quad \textit{or} \quad c=1.$$

Corollary 1.6. *Let α and β be two algebraic numbers, both of degree 13. If c denotes the degree of $\alpha \cdot \beta$, then*

$$c = 13s \quad \textit{with} \quad s \in \{1,2,3,4,6,9,12,13\} \quad \textit{or} \quad c \in \{1,12\}.$$

With the aim of exploring the extent to which the methods applied to solve Problem 1.1 can be generalized further, some attempts have been made to extend the classification of compositum-feasible triplets (p^k,p^k,c) from $k=1$ to $k \geqslant 2$. The findings presented in Article A2 constitute only a partial solution in this direction. To state the result precisely, it is necessary to introduce some notation. For a number field K, let $K_{\mathcal{G}}$ denote the Galois closure of K over $\mathbb{Q}$, and let $\mathrm{Gal}(K_{\mathcal{G}}/\mathbb{Q})$ be the corresponding Galois group. We also say that a compositum-feasible triplet (a,b,c) is induced by number fields K and L if

$$[K:\mathbb{Q}] = a, \quad [L:\mathbb{Q}] = b, \quad \text{and} \quad [KL:\mathbb{Q}] = c.$$

Note that the same compositum-feasible triplet (a,b,c) may be induced by distinct pairs (K,L).

Theorem 1.7. *Let p be a prime and $k \geqslant 2$. Suppose that (p^k,p^k,c) is a compositum-feasible triplet induced by number fields K and L. Also,*

assume that both $\mathrm{Gal}(K_{\mathcal{G}}/\mathbb{Q})$ *and* $\mathrm{Gal}(L_{\mathcal{G}}/\mathbb{Q})$ *are almost simple groups. Then* $c = p^k s$ *and one of the following occurs:*

(i) $s \in \{1,\ p^k-1,\ p^k\}$;

(ii) $p = 2,\ k \geqslant 4,\ s \in \{2^{k-1}-1,\ 2^{k-1}\}$;

(iii) $p = k = 3,\ s \in \{10, 16\}$;

(iv) $p^k = \frac{q^d-1}{q-1},\ s \in \left\{\frac{q^{d-1}-1}{q-1},\ q^{d-1}\right\}$, *where* q *is a prime power and* $d \geqslant 3$.

It is important to note that unless $p = k = 2$, Theorem 1.7 does not suffice to determine all compositum-feasible triplets of the form (p^k, p^k, c). By other methods, Drungilas and Maciulevičius have listed all compositum feasible triplets of the form $(2^3, 2^3, c)$ and $(3^2, 3^2, c)$ in [7]. By connecting their approach with Theorem 1.7, we derive the following.

Theorem 1.8. *The triplet* $(2^4, 2^4, c)$ *is compositum-feasible if and only if*

$$c = 2^4 \cdot s \quad \textit{and} \quad s \in \{1, \ldots, 16\} \setminus \{11, 13\}.$$

In summary, the obtained results in Article A1 and Article A2 provide new tools that can be applied in the classification of triplets belonging to the set $\mathcal{C}$, $\mathcal{S}$, or $\mathcal{P}$.

The next findings are related to Problem 1.2, which focuses on finding new conditions under which the degree of $\alpha \cdot \beta$ attains its maximum value. It is well known that

$$\deg(\alpha+\beta) \leqslant \deg(\alpha)\cdot\deg(\beta) \quad \text{and} \quad \deg(\alpha\cdot\beta) \leqslant \deg(\alpha)\cdot\deg(\beta).$$

In 1970, Isaacs [26] showed that if a and b are coprime, then the degree of $\alpha+\beta$ becomes maximal possible, i.e., we have an equality

$$\deg(\alpha+\beta) = \deg(\alpha)\cdot\deg(\beta).$$

Isaacs's result is particularly useful in the study of sum-feasible triplets. Indeed, if a and b are coprime, then (a, b, c) is sum-feasible if and only if $c = ab$. In contrast, the example (1.1) provided in the previous section demonstrates that the triplet $(2, 3, c)$ can be product-feasible even if $c \neq 2 \cdot 3$. Thus, the coprimality condition is no longer sufficient to ensure the equality

$$\deg(\alpha\cdot\beta) = \deg(\alpha)\cdot\deg(\beta).$$

In Article A3, several conditions are found under which the degree of $\alpha \cdot \beta$ is maximal possible. These conditions are rather technical and apply only in very special cases. Nevertheless, they are sufficient for the purposes of our research, which investigates product-feasible triplets of the form (a,p,c), where p is a prime number.

Theorem 1.9. *Let $p > 2$ be a prime number and let m be a positive integer such that $p \nmid m$. Suppose that α, β are algebraic numbers such that $\deg(\alpha) = m$, $\deg(\beta) = p$, and the minimal polynomial of β over $\mathbb{Q}$ is $f(x)$. If $f(x) \neq x^p + c$ for some $c \in \mathbb{Q}$, then*

$$\deg(\alpha \cdot \beta) = \deg(\alpha) \cdot \deg(\beta) = mp.$$

Theorem 1.9 also supplements some observations made by Weintraub on primitive elements of field extensions [42]. In particular, if α and β are algebraic numbers satisfying the assumptions of Theorem 1.9, then as an immediate corollary follows that $\alpha \cdot \beta$ is a primitive element of $\mathbb{Q}(\alpha, \beta)$, i.e., $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha \cdot \beta)$. For example, take

$$\alpha = \sqrt{\sqrt{2} + 2} \quad \text{and} \quad \beta = 1 + \sqrt[3]{2} + \sqrt[3]{4}.$$

Calculations with SageMath [41] show that the minimal polynomials of α and β are

$$x^4 - 4x^2 + 2 \quad \text{and} \quad x^3 - 3x^2 - 3x - 1,$$

respectively. We see that $\deg(\alpha) = 4$, $\deg(\beta) = 3$, and the minimal polynomial of β is not of the form $x^3 + c$. Therefore, Theorem 1.9 implies that the minimal polynomial of $\alpha \cdot \beta$ has degree equal to $4 \cdot 3 = 12$, and that the generating element of the composite field extension $\mathbb{Q}(\alpha, \beta)$ can be chosen to be $\alpha \cdot \beta$.

The second condition is purely numerical, and does not require any information about the minimal polynomials of α or β.

Theorem 1.10. *Let $p > 2$ be a prime number and let m be a positive integer such that $p \nmid m$ and $p - 1 \nmid m$. Suppose that α and β are algebraic numbers such that $\deg(\alpha) = m$ and $\deg(\beta) = p$. Then*

$$\deg(\alpha \cdot \beta) = \deg(\alpha) \cdot \deg(\beta) = mp.$$

Theorem 1.10 leads to the following corollary, which is directly applicable to the study of product-feasible triplets.

Corollary 1.11. *Let $p > 2$ be a prime number and let a be a positive integer such that $p \nmid a$ and $p-1 \nmid a$. Then the triplet (a,p,c) is product-feasible if and only if $c = ap$.*

1.4 Methods

In this thesis, the three degree problems are investigated primarily through group-theoretic methods. Indeed, any algebraic number α of degree d over $\mathbb{Q}$ can be associated with the unique monic irreducible polynomial $p(x) \in \mathbb{Q}[x]$ of degree d such that $p(\alpha) = 0$. The d distinct roots of $p(x)$ can be permuted in ways that preserve the field structure over $\mathbb{Q}$; the set of such permutations forms a group under composition. Thus, within the framework of Galois theory, it becomes possible to investigate questions about algebraic numbers through the so-called Galois groups. At the core of this approach lies the theorem of Drungilas, Dubickas, and Luca [5], which states that if the triplet (a,b,c) is compositum-feasible, then there exists a transitive permutation group G of degree c that has two subgroups H_1 and H_2 satisfying

$$[G : H_1] = a, \quad [G : H_2] = b, \quad \text{and} \quad [G : H_1 \cap H_2] = c. \tag{1.4}$$

Note, however, that the reverse implication does not necessarily hold, i.e., even if the group G with the properties described in (1.4) exists, one cannot automatically conclude that the triplet (a,b,c) is compositum-feasible. This is because (1.4) does not necessarily imply that there exist number fields K and L such that

$$[K : \mathbb{Q}] = a, \quad [L : \mathbb{Q}] = b, \quad \text{and} \quad [KL : \mathbb{Q}] = c. \tag{1.5}$$

For the reverse implication to hold, the group G has to be realizable over $\mathbb{Q}$, that is, it has to occur as the Galois group of some normal extension over $\mathbb{Q}$ (see the inverse Galois problem [36]).

The described framework, together with group classification theorems of Feit [17], Guralnick [24], and the recent work of Jones and Sezer [27], forms the basis of techniques applied in Article A1 and Article A2. Galois theory plays a central role in Article A3 as well, although the final

steps in the proofs there rely on constructing various types of multiplicative relations between algebraic numbers. The details of this approach are described in Chapter 2, where the topic of the so-called non-trivial relations between algebraic conjugates is investigated.

1.5 Future research directions

The successful resolution of Problem 1.1 was closely tied to progress in the classification of transitive permutation groups. While the methods used to solve it may have reached their current limits, we believe that, with suitable modifications, the group-theoretic approach can still yield significant advances on this topic. In particular, the classification of all finite simple groups offers promising opportunities for further developments in the compositum problem.

With respect to Problem 1.2, the conditions under which the product of two algebraic numbers attains the maximal possible degree have been identified only under restrictive assumptions. Therefore, the problem remains open in the general case.

Chapter 2

Non-trivial relations between algebraic conjugates

2.1 Literature review

Let α be an algebraic number of degree d. If $p(x)$ is the minimal polynomial of α, then $p(x)$ has d distinct roots (recall that we always work over $\mathbb{Q}$) denoted $\alpha_1, \ldots, \alpha_d$, that are called algebraic conjugates of α. An additive relation

$$k_1\alpha_1 + \ldots + k_d\alpha_d \in \mathbb{Q} \tag{2.1}$$

is called *non-trivial* if it holds for some $k_1, \ldots, k_d \in \mathbb{Q}$, not all equal. Similarly, a multiplicative relation

$$\alpha_1^{k_1} \cdot \ldots \cdot \alpha_d^{k_d} \in \mathbb{Q} \tag{2.2}$$

is called *non-trivial* if it holds for some $k_1, \ldots, k_d \in \mathbb{Z}$, not all equal [2]. It is well known from the theory of symmetric polynomials that both (2.1) and (2.2) hold trivially if all k_i are equal. In contrast, it is much less understood what causes non-trivial relations such as

$$33\alpha_i - 7\alpha_j + 37\alpha_k \in \mathbb{Q} \quad \text{or} \quad \alpha_i^3 \cdot \alpha_j^4 \cdot \alpha_k^5 \in \mathbb{Q}$$

to appear (here $\alpha_i, \alpha_j, \alpha_k$ denote distinct algebraic conjugates of some algebraic number α of degree $d \geqslant 3$). After suitable substitutions, non-trivial relations in (2.1) and (2.2) can usually be assumed to be of the form

$$k_1\alpha_1 + \ldots + k_d\alpha_d = 0, \qquad k_1, \ldots, k_d \in \mathbb{Z} \tag{2.3}$$

and

$$\alpha_1^{k_1} \cdot \ldots \cdot \alpha_d^{k_d} = 1, \qquad k_1, \ldots, k_d \in \mathbb{Z} \tag{2.4}$$

respectively [8]. Accordingly, only non-trivial relations of the form given in (2.3) or (2.4) are investigated in this thesis.

The first major result on non-trivial additive relations is due to Kurbatov [30], who found that non-trivial additive relations between algebraic conjugates of prime degree do not exist. In contrast, consider an algebraic number $\alpha = \sqrt[p]{2}$. Since the minimal polynomial of α is $x^p - 2$, the degree of α is equal to p. If $p > 2$, observe that for any other root of $x^p - 2$, say α', there always exists a non-trivial multiplicative relation between algebraic conjugates of prime degree p, namely

$$\alpha^p \cdot \alpha'^{-p} = 1.$$

From the above considerations it follows that the additive and multiplicative problems are not equivalent. There has been some effort, especially by Girstmair [19], to develop a unified approach for both problems. However, the exact links between non-trivial additive and multiplicative relations have been described only in the case of prime degree p, primarily by Drmota and Skałba [3].

The simplest non-trivial additive relation is

$$\alpha_i + \alpha_j = 0, \tag{2.5}$$

where α_i, α_j denote distinct algebraic conjugates of some algebraic number α of degree $d > 2$. If $p(x)$ is the minimal polynomial of α, from (2.5) it follows that

$$(x - \alpha_i)(x - \alpha_j) = (x - \alpha_i)(x + \alpha_i) = x^2 - \alpha_i^2$$

is a factor of $p(x)$. Hence, it is not difficult to deduce that (2.5) occurs if and only if the minimal polynomial of α is of the form $p(x) = g(x^2)$. Clearly, in this case the degree of α must be even. So far, this remains the only non-trivial additive relation that has been completely solved. In contrast, it is still not known what the sufficient and necessary conditions are for the non-trivial relation

$$\alpha_i + \alpha_j - \alpha_k = 0 \tag{2.6}$$

to hold (here $\alpha_i, \alpha_j, \alpha_k$ denote distinct algebraic conjugates of some algebraic number α of degree $d \geqslant 3$). Suppose that G is the Galois group

corresponding to the normal closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, and that d is the degree of α. Usually, relations like in (2.6) are investigated by imposing restrictions on G or d. For example, Dixon showed in [2] that if G is abelian, then (2.6) holds only if d is a multiple of 6. In the case $d = 6$, without any assumptions on the structure of G, Dubickas and Jankauskas [11] found that (2.6) occurs if and only if the minimal polynomial of α is of the form

$$p(x) = x^6 + 2ax^4 + a^2x^2 + b$$

for some $a, b \in \mathbb{Q}$. Various aspects of relation (2.6) are investigated in [20, 21, 22, 25, 31, 33], while the works in [9, 12, 18, 29, 32] address different types of non-trivial additive relations.

The current state of knowledge in the area of non-trivial multiplicative relations is similar. Only the multiplicative version of (2.5), namely

$$\alpha_i \cdot \alpha_j = 1, \tag{2.7}$$

has been solved. In particular, if $p(x)$ is the minimal polynomial of α whose two distinct conjugates satisfy (2.7), then the degree d of $p(x)$ is even and

$$p(x) = x^d \cdot p(1/x). \tag{2.8}$$

Thus, $p(x)$ is the so-called self-reciprocal polynomial – its coefficients are the same when read forwards or backwards. The irreducible polynomial

$$p(x) = x^4 + 3x^3 + 2x^2 + 3x + 1 = x^4 \cdot p(1/x)$$

is one such example. Due to the role played by roots of unity, the problem of non-trivial relations is more complex in the multiplicative case than in the additive one. Some differences between the two are explored by Baron, Drmota, and Skałba in [1], where the multiplicative version of (2.6), namely

$$\alpha_i \cdot \alpha_j \cdot \alpha_k^{-1} = 1$$

was investigated. Other studies on multiplicative relations appear in [2, 8, 10, 38, 43].

We conclude the literature review with one of the most general re-

sults in this research area. Based on empirical evidence, Smyth in 1986 conjectured that if integers $n_1,\dots,n_k$ satisfy certain explicit conditions [39], then there exist algebraic numbers α and β such that

$$\alpha_1^{n_1}+\ldots+\alpha_k^{n_k}=0 \quad \text{and} \quad \beta_1^{n_1}\cdot\ldots\cdot\beta_k^{n_k}=1,$$

where $\alpha_1,\dots,\alpha_k$ denote some conjugates of α (not necessarily distinct), and $\beta_1,\dots,\beta_k$ denote some conjugates of β (not necessarily distinct). Note that in this setting, no information is provided about the degrees of α or β. The conjecture was finally proved in 2025 by Ellenberg and Hardt [15].

2.2 Research objectives

In this thesis, the topic of non-trivial relations is mostly investigated for algebraic numbers of low degree. The following two problems are concerned only with additive relations.

Problem 2.1. *To find all non-trivial additive relations between three algebraic conjugates of degree $d\leqslant 8$.*

Problem 2.2. *To describe all non-trivial additive relations between algebraic conjugates of degree $d=4$.*

In contrast, the next problem is of a more general nature. Suppose that for some algebraic number α of degree d and integers $k_1,\dots,k_d$, there exists a non-trivial additive relation

$$k_1\alpha_1+\ldots+k_d\alpha_d=0$$

between distinct algebraic conjugates $\alpha_1,\dots,\alpha_d$ of α. If there exists an algebraic numbers β of degree d, whose algebraic conjugates $\beta_1,\dots,\beta_d$ satisfy

$$\beta_1^{k_1}\cdot\ldots\cdot\beta_d^{k_d}=1$$

with the same $k_1,\dots,k_d$, then it is said that an additive relation has a corresponding multiplicative relation.

Problem 2.3. *To show that for every non-trivial additive relation between algebraic conjugates of degree d, there exists a corresponding non-trivial multiplicative relation.*

2.3 Main results and their actuality

In Article A4, the complete solution to Problem 2.1 is presented. The notation for groups in the following theorem is taken from the transitive groups database [40].

Theorem 2.1. *Assume that a,b,c are non-zero integers and d is an integer satisfying $d \leqslant 8$. Then there exists an algebraic number α of degree d with Galois group $G \subseteq S_d$, and three distinct conjugates $\alpha_i, \alpha_j, \alpha_k$ satisfying*

$$a\alpha_i + b\alpha_j + c\alpha_k = 0$$

if and only if d, G and a,b,c (or their permutation) satisfy one of the following:

(i) $d = 3,\ G \in \{C_3,\ S_3\}$ *and* $a = b = c$;

(ii) $d = 6,\ G \in \{C_6,\ S_3,\ D_6,\ C_3 \times S_3,\ S_3^2,\ C_3^2 \rtimes C_4,\ S_3 \wr C_2\}$ *and* $a = b = c$;

(iii) $d = 6,\ G \in \{C_6,\ S_3,\ D_6\}$ *and* $a = b = -c$;

(iv) $d = 6,\ G = S_3$ *and* $a^2 + b^2 - ab = c^2$;

(v) $d = 8,\ G = D_4$ *and* $a^2 + b^2 = c^2$.

From Theorem 2.1 one can deduce, for example, that if $p(x)$ is an irreducible polynomial of degree 8 whose three distinct roots $\alpha_i, \alpha_j, \alpha_k$ satisfy

$$a\alpha_i + b\alpha_j + c\alpha_k = 0$$

for some non-zero integers a,b,c, then necessarily $a^2 + b^2 = c^2$ (up to permutation of a,b,c). Moreover, the Galois group of $p(x)$ must be isomorphic to the dihedral group D_4 of order 8. Conversely, for any non-zero integers a,b,c satisfying $a^2 + b^2 = c^2$, it is possible to construct an algebraic number α satisfying part (v) of Theorem 2.1. For example, for the solution $(3,4,5)$ of $a^2 + b^2 = c^2$, it can be checked with the computer algebra system SageMath [41] that the polynomial

$$p(x) = x^8 + 580608x^4 + 1074954240000$$

is irreducible over $\mathbb{Q}$ and three distinct roots of $p(x)$ satisfy

$$3\alpha_i + 4\alpha_j + 5\alpha_k = 0.$$

Parts (iv) and (v) of Theorem 2.1 describe previously unknown exceptional additive relations, whereas those in (i), (ii), and (iii) had been discovered earlier by other researchers. In particular, Lalande found in [31] that the cyclic group C_6, the symmetric group S_3, and the dihedral group D_6 admit the relation

$$\alpha_i + \alpha_j - \alpha_k = 0.$$

Part (iii) of Theorem 2.1 implies that if $d = 6$, then these three groups are, in fact, the only ones. Also, from the work of Dubickas and Jankauskas [11], the exact form of the sextic polynomial $p(x)$ inducing the relations

$$\alpha_i + \alpha_j - \alpha_k = 0 \quad \text{or} \quad \alpha_i + \alpha_j + \alpha_k = 0$$

is known. Parts (ii) and (iii) of Theorem 2.1 supplement their findings by identifying which transitive groups can occur as Galois groups of $p(x)$.

In the process of solving Problem 2.1, one often encounters the nontrivial relation

$$\alpha_i + \alpha_j + \alpha_k = 0, \tag{2.9}$$

which corresponds to part (ii) of Theorem 2.1. In [11], Dubickas and Jankauskas demonstrated that for any degree d, divisible by 3, it is possible to construct an algebraic number α of degree d whose three conjugates satisfy (2.9). However, some algebraic numbers of degree d have three distinct conjugates summing to zero, even when d is not divisible by 3. Determining the minimal value of such d remains an open problem. Based on the findings in [11] and [14], it is known that $10 \leqslant d \leqslant 20$. In relation to this question, the following result is obtained in Article A5.

Theorem 2.2. *Let α be an algebraic number of degree $2p$, where $p \geqslant 5$ is a prime number. Then*

$$\alpha_i + \alpha_j + \alpha_k \neq 0$$

for any three algebraic conjugates $\alpha_i, \alpha_j, \alpha_k$ of α.

Theorem 2.2 together with some auxiliary results narrows down the possibilities for the minimal value of degree d.

Corollary 2.3. *Let d be the smallest positive integer, not divisible by 3, for which there exists an algebraic number of degree d whose three algebraic conjugates add up to zero. Then either $d = 16$ or $d = 20$.*

Until the presentation of this thesis, the problem of finding all non-trivial additive relations of degree d has been solved only if d is a prime number. Problem 2.2 marks the first attempt to solve it for some composite degree d. In particular, building on some earlier findings by Kitaoka [29], Article A6 provides a complete description of non-trivial additive relations in the case $d = 4$.

Theorem 2.4. *Suppose that α is an algebraic number of degree 4 whose distinct algebraic conjugates $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy some non-trivial additive relation. Set $r = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. If $r \neq 0$, then (up to re-indexation) all non-trivial additive relations between the conjugates of α are described by $\mathbb{Z}$-multiples of*

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0,$$

while if $r = 0$, then (up to re-indexation) all non-trivial additive relations between the conjugates of α are described by $\mathbb{Z}$-linear combinations of

$$\alpha_1 + \alpha_2 = 0 \quad \textit{and} \quad \alpha_3 + \alpha_4 = 0.$$

Thus, from Theorem 2.4 it is not difficult to deduce that, for example, there is no algebraic number α of degree 4 whose conjugates satisfy the relation

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = 0.$$

Suppose that α is an algebraic number of degree 4 whose distinct algebraic conjugates $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy some non-trivial additive relations. Hence

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 = 0$$

for some integers k_1, k_2, k_3, k_4, not all equal. From Theorem 2.4 it follows

that for $d = 4$, there are three linear forms, namely

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= x_1 + x_2 - x_3 - x_4, \\ f_2(x_1, x_2, x_3, x_4) &= x_1 + x_2, \\ f_3(x_1, x_2, x_3, x_4) &= x_3 + x_4 \end{aligned}$$

so that

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in \sum_{j \in S} \mathbb{Z} f_j(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

where $S \subseteq \{1,2,3\}$ is such that $f_j(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$ for each $j \in S$. In particular, we always have either

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in \mathbb{Z} f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

with

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0$$

or

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in \mathbb{Z} f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) + \mathbb{Z} f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

with

$$f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 + \alpha_2 = 0 \quad \text{and} \quad f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_3 + \alpha_4 = 0.$$

The next theorem reveals that for $d = 6$ it is impossible to describe all non-trivial relations with a finite list of $\mathbb{Z}$-linear forms.

Theorem 2.5. *Let m be any positive integer, and let*

$$f_i(x_1, \ldots, x_6) = \sum_{j=1}^{6} u_{ij} x_j,$$

$i = 1, \ldots, m$, be a set of m linear forms such that for each i the coefficients $u_{i1}, \ldots, u_{i6} \in \mathbb{Z}$ are not all equal. Then, there are infinitely many algebraic numbers α of degree 6 whose conjugates $\alpha_1, \ldots, \alpha_6$ satisfy a nontrivial additive relation

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0 \quad \textit{with} \quad k_1, k_2, k_3 \in \mathbb{Z},$$

so that

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \notin \sum_{j\in S} \mathbb{Z} f_j(\alpha_1, \ldots, \alpha_6)$$

for every choice of $S \subseteq \{1, \ldots, m\}$ *subject to* $f_j(\alpha_1, \ldots, \alpha_6) = 0$ *for each* $j \in S$.

We conclude this section with the following result from Article A6, which establishes the link between additive and multiplicative non-trivial relations.

Theorem 2.6. *Assume that there exists an algebraic number* α *of degree* d *with conjugates* $\alpha_1, \ldots, \alpha_d$ *and some* $k_1, \ldots, k_d \in \mathbb{Z}$*, not all equal, such that*

$$k_1\alpha_1 + \cdots + k_d\alpha_d = 0.$$

Then, there exists an algebraic number β *of degree* d *with conjugates* $\beta_1, \ldots, \beta_d$ *such that*

$$\beta_1^{k_1} \ldots \beta_d^{k_d} = 1.$$

For illustration purposes, recall from part (v) of Theorem 2.1 that there exists an algebraic number α of degree 8 such that three of its conjugates, say $\alpha_i, \alpha_j, \alpha_k$, satisfy the non-trivial additive relation

$$3\alpha_i + 4\alpha_j + 5\alpha_k = 0.$$

Theorem 2.6 guarantees that there exists an algebraic number β of degree 8 such that three of its conjugates, say $\beta_i, \beta_j, \beta_k$, satisfy the multiplicative relation

$$\beta_i^3 \cdot \beta_j^4 \cdot \beta_k^5 = 1.$$

Moreover, the proof of Theorem 2.6 is constructive; that is, it provides an explicit method for constructing the conjugates of β from the conjugates of α. This complements the techniques used by Girstmair in [19], where the link between additive and multiplicative relations was investigated by non-constructive arguments.

2.4 Methods

The methods applied in this thesis to study non-trivial relations are based on various results from group theory. The key observation is

that the Galois group of a polynomial $p(x)$ imposes restrictions on the possible non-trivial relations between the roots of $p(x)$. Indeed, suppose that α is an algebraic number of degree d with the minimal polynomial $p(x)$ and the corresponding Galois group G. Denote by $\alpha_1, \ldots, \alpha_d$ all distinct roots of $p(x)$, and assume that for some integers $k_1, \ldots, k_d$ the following additive relation holds

$$k_1\alpha_1 + \ldots + k_d\alpha_d = 0. \tag{2.10}$$

Every element σ of G acts as a permutation on the roots of $p(x)$. Thus, application of σ on (2.10) yields another additive relation

$$k_1\alpha_{\sigma(1)} + \ldots + k_d\alpha_{\sigma(d)} = 0. \tag{2.11}$$

Moreover, any linear combination of (2.10) and (2.11) produces yet another additive relation. By the described process, it is usually possible to obtain some relation that is known to be impossible unless the coefficients $k_1, \ldots, k_d$ satisfy certain strict conditions. Reasoning of this type is extensively applied in Article A4 and Article A5. Of course, the fact that for some integers $k_1, \ldots, k_d$, the equality (2.10) does not lead to any contradiction from the viewpoint of group theory, is not sufficient to conclude that there exist algebraic conjugates $\alpha_1, \ldots, \alpha_d$ satisfying it. The actual construction of $\alpha_1, \ldots, \alpha_d$ is obtained through the use of the normal basis theorem. The techniques based on various applications of this theorem play an essential role in Article A6.

2.5 Future research directions

With respect to Problem 2.1, the choice to study non-trivial relations only between three algebraic conjugates

$$a\alpha_i + b\alpha_j + c\alpha_k = 0 \tag{2.12}$$

is motivated by the observation that (2.12) implies

$$\pm a \pm b \pm c \neq 0.$$

The analogous implication is no longer true for non-trivial relations between four algebraic conjugates. Thus, new techniques must be devel-

oped if one seeks to generalize Theorem 2.1 and determine all non-trivial additive relations between algebraic conjugates of degree $d \leqslant 8$.

With respect to Problem 2.2, the results obtained in Theorem 2.4 and Theorem 2.5 show that the structures governing all non-trivial additive relations between algebraic conjugates of degrees $d = 4$ and $d = 6$ are essentially different. There is some evidence that for $d = 8$, the situation is analogous to the case $d = 6$, as described by Theorem 2.5. To our knowledge, no reasonable guess can be made as to which case occurs for $d = 9$.

Finally, with respect to Problem 2.3, it has already been observed that the converse of Theorem 2.6 is not true. Thus, it remains an open problem to find conditions under which a multiplicative non-trivial relation has a corresponding additive relation.

Conclusions

The following is a summary of the key results established in this thesis.

- A complete classification is obtained of compositum-feasible, sum-feasible, and product-feasible triplets of the form (p, p, c), where p is a prime number.
- New conditions are established under which the product of two algebraic numbers attains the maximal possible degree.
- A complete classification is provided of non-trivial additive relations between three algebraic conjugates of degree $d \leqslant 8$.
- It is shown that there exists a finite list of $\mathbb{Z}$-linear forms that cover all non-trivial additive relations between algebraic conjugates of degree $d = 4$, whereas the analogous statement is proved to be false for $d = 6$.
- Using a constructive proof, it is demonstrated that every non-trivial additive relation between algebraic conjugates of degree d has a corresponding non-trivial multiplicative relation.

Bibliography

[1] Baron, G., Drmota, M., and Skałba, M. Polynomial relations between polynomial roots. *J. Algebra 177*, 3 (1995), 827–846.

[2] Dixon, J. D. Polynomials with nontrivial relations between their roots. *Acta Arith. 82*, 3 (1997), 293–302.

[3] Drmota, M., and Skałba, M. *On multiplicative and linear independence of polynomial roots*. Hölder-Pichler-Tempsky, Vienna, 1991.

[4] Drungilas, P., and Dubickas, A. On degrees of three algebraic numbers with zero sum or unit product. *Colloq. Math. 143*, 2 (2016), 159–167.

[5] Drungilas, P., Dubickas, A., and Luca, F. On the degree of compositum of two number fields. *Math. Nachr. 286*, 2-3 (2013), 171–180.

[6] Drungilas, P., Dubickas, A., and Smyth, C. A degree problem for two algebraic numbers and their sum. *Publ. Mat. 56*, 2 (2012), 413–448.

[7] Drungilas, P., and Maciulevičius, L. A degree problem for the compositum of two number fields. *Lith. Math. J. 59*, 1 (2019), 39–47.

[8] Dubickas, A. On the degree of a linear form in conjugates of an algebraic number. *Illinois J. Math. 46*, 2 (2002), 571–585.

[9] Dubickas, A. Additive relations with conjugate algebraic numbers. *Acta Arith. 107*, 1 (2003), 35–43.

[10] Dubickas, A. Multiplicative relations with conjugate algebraic numbers. *Ukraïn. Mat. Zh. 59*, 7 (2007), 890–900.

[11] Dubickas, A., and Jankauskas, J. Simple linear relations between conjugate algebraic numbers of low degree. *J. Ramanujan Math. Soc. 30*, 2 (2015), 219–235.

[12] Dubickas, A., and Jankauskas, J. Linear relations with conjugates of a Salem number. *J. Théor. Nombres Bordeaux 32*, 1 (2020), 179–191.

[13] Dubickas, A., and Maciulevičius, L. The product of a quartic and a sextic number cannot be octic. *Open Math. 22*, 1 (2024), Paper No. 20230184, 10 pp.

[14] Dubickas, A., and Smyth, C. Polynomials with three distinct zeros summing to zero. *Amer. Math. Monthly 113* (2006), 941–942.

[15] Ellenberg, J. S., and Hardt, W. Smyth's conjecture and a non-deterministic hasse principle. Preprint at arXiv:2503.15833, (2025), 31 pp.

[16] Estes, D., Guralnick, R., Schacher, M., and Straus, E. Equations in prime powers. *Pacific J. Math. 118*, 2 (1985), 359–367.

[17] Feit, W. *Some consequences of the classification of finite simple groups*. Amer. Math. Soc., Providence, RI, 1980.

[18] Girstmair, K. Linear dependence of zeros of polynomials and construction of primitive elements. *Manuscripta Math. 39*, 1 (1982), 81–97.

[19] Girstmair, K. Linear relations between roots of polynomials. *Acta Arith. 89*, 1 (1999), 53–96.

[20] Girstmair, K. The Galois relation $x_1 = x_2 + x_3$ and Fermat over finite fields. *Acta Arith. 124*, 4 (2006), 357–370.

[21] Girstmair, K. The Galois relation $x_1 = x_2 + x_3$ for finite simple groups. *Acta Arith. 127*, 3 (2007), 301–303.

[22] Girstmair, K. The Galois relations $x_1 = x_2 + x_3$ and $x_1 = x_2x_3$ for certain solvable groups. *Ann. Sci. Math. Québec 32*, 2 (2008), 171–174.

[23] Guralnick, R. M. Subgroups inducing the same permutation representation. *J. Algebra 81*, 2 (1983), 312–319.

[24] Guralnick, R. M. Subgroups of prime power index in a simple group. *J. Algebra 81*, 2 (1983), 304–311.

[25] Hardt, W., and Yin, J. Linear relations among Galois conjugates over $\mathbb{F}_q(t)$. *Res. Number Theory 8*, 2 (2022), Paper No. 34, 17 pp.

[26] Isaacs, I. M. Degrees of sums in a separable field extension. *Proc. Amer. Math. Soc. 25* (1970), 638–641.

[27] Jones, G. A., and Sezer, S. Permutation groups of prime power degree and p-complements. Preprint at arXiv:2402.13672, (2024), 19 pp.

[28] Jones, G. A., and Zvonkin, A. K. Groups of prime degree and the Bateman-Horn Conjecture. *Expo. Math. 41*, 1 (2023), 1–19.

[29] Kitaoka, Y. Notes on the distribution of roots modulo a prime of a polynomial. *Unif. Distrib. Theory 12*, 2 (2017), 91–117.

[30] Kurbatov, V. A. On equations of prime degree. *Mat. Sb. N.S. 43* (1957), 349–366.

[31] Lalande, F. Relations linéaires entre les racines d'un polynôme et anneaux de Schur. *Ann. Sci. Math. Québec 27*, 2 (2003), 169–175.

[32] Lalande, F. La relation linéaire $a = b + c + \cdots + t$ entre les racines d'un polynôme. *J. Théor. Nombres Bordeaux 19*, 2 (2007), 473–484.

[33] Lalande, F. À propos de la relation galoisienne $x_1 = x_2 + x_3$. *J. Théor. Nombres Bordeaux 22*, 3 (2010), 661–673.

[34] Lindemann, F. Über die zahl π. *Math. Ann. 20* (1882), 213–225.

[35] Maciulevičius, L. On the degree of product of two algebraic numbers. *Mathematics 11* (2023), Paper No. 2131, 11 pp.

[36] Malle, G., and Matzat, B. H. *Inverse Galois theory.* Springer, Berlin, 2018.

[37] Perlis, R. On the class numbers of arithmetically equivalent fields. *J. Number Theory 10*, 4 (1978), 489–509.

[38] Smyth, C. J. Conjugate algebraic numbers on conics. *Acta Arith. 40*, 4 (1982), 333–346.

[39] Smyth, C. J. Additive and multiplicative relations connecting conjugate algebraic numbers. *J. Number Theory 23*, 2 (1986), 243–254.

[40] The LMFDB Collaboration. The L-functions and modular forms database, 2025. `http://www.lmfdb.org`.

[41] The Sage Developers. SageMath, the Sage Mathematics Software System (Version 10.7), 2025. `https://www.sagemath.org`.

[42] Weintraub, S. H. Observations on primitive, normal, and subnormal elements of field extensions. *Monatsh. Math. 162*, 2 (2011), 239–244.

[43] Zheng, T. A fast algorithm for computing multiplicative relations between the roots of a generic polynomial. *J. Symbolic Comput. 104* (2021), 381–401.

Santrauka (Summary in Lithuanian)

Įvadas

Kompleksinis skaičius α vadinamas *algebriniu*, jei jis yra nenulinio polinomo $p(x)$ su racionaliaisiais koeficientais šaknis. Taigi, $p(\alpha) = 0$ su tam tikru $p(x) \in \mathbb{Q}[x]$. Algebriniai skaičiai praplečia racionaliųjų skaičių sąvoką, nes kiekvienas racionalusis skaičius q yra tiesinio polinomo $x - q$ šaknis. Iracionalieji skaičiai, tokie kaip $\sqrt{2}$ ar aukso pjūvis $\varphi = \frac{1+\sqrt{5}}{2}$, taip pat yra algebriniai, nes jie yra atitinkamai polinomų $x^2 - 2$ ir $x^2 + x - 1$ šaknys. Kita vertus, XIX a. buvo įrodyta, kad nei Oilerio konstanta e, nei π nėra algebriniai. Tokie skaičiai vadinami *transcendentiniais*.

Algebrinių skaičių istorija neatsiejama nuo žymių matematikos uždavinių. Renesanso laikotarpiu didelį susidomėjimą įgavo klausimas, ar įmanoma polinominių lygčių sprendinius išreikšti tos lygties koeficientais naudojant vien tik aritmetinius (sudėties, atimties, daugybos, dalybos) bei šaknų traukimo veiksmus. Kitaip tariant, ar egzistuoja sprendinių formulės, kaip kad

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.13}$$

kvadratinės lygties $ax^2 + bx + c = 0$ atveju, aukštesniojo laipsnio lygtims? XVI a. italų matematikų Tartalijos (Tartaglia), Kardano (Cardano) ir Ferario (Ferrari) įžvalgos leido išvesti sprendinių formules trečio ir ketvirto laipsnio lygtims. Naujo proveržio teko laukti daugiau nei du šimtmečius, kol Rufinio (Ruffini) ir Abelio (Abel) įrodymai patvirtino tai, ką kiek anksčiau jau numanė ir Lagranžas (Lagrange) – bendru atveju atsakymas į nagrinėjamą klausimą yra neigiamas. Pavyzdžiui, lygties $x^5 - x - 1 = 0$ sprendinių neįmanoma išreikšti (2.13) tipo formule. Išsamų šio klausimo sprendimą 1830 m. aprašė dvidešimtmetis prancūzų genijus Galua (Galois). Jo idėjų pagrindu susiformavo matematinė sritis – Galua teorija, kurios kalba formuluojama modernioji algebra. Šioje

disertacijoje nagrinėjami uždaviniai apie algebrinius skaičius taip pat yra Galua teorijos taikymo pavyzdžiai.

Viena iš pagrindinių charakteristikų, nusakančių algebrinį skaičių α, yra jo laipsnis. Tarp visų polinomų su racionaliaisiais koeficientais egzistuoja vienintelis mažiausio laipsnio normuotas polinomas $p(x)$, su kuriuo $p(\alpha) = 0$. Toks $p(x)$ vadinamas algebrinio skaičiaus α *minimaliuoju polinomu*, o skaičiaus α *laipsnis*, žymimas $\deg(\alpha)$, apibrėžiamas kaip polinomo $p(x)$ laipsnis. Pavyzdžiui, $\sqrt{7}$ minimalusis polinomas yra $x^2 - 7$, tad $\deg(\sqrt{7}) = 2$. Jau XIX a. buvo žinoma, kad algebrinių skaičių aibė yra uždara sudėties ir daugybos atžvilgiu. Kitaip tariant, bet kuriems algebriniams skaičiams α ir β, suma $\alpha + \beta$ bei sandauga $\alpha \cdot \beta$ taip pat yra algebriniai skaičiai. Natūraliai kyla klausimas, ar įmanoma nustatyti galimas $\alpha + \beta$ bei $\alpha \cdot \beta$ laipsnių reikšmes žinant α ir β laipsnius (bet ne pačius skaičius) iš anksto? Metodiškai laipsnių uždavinys buvo pradėtas tyrinėti tik 2012 metais pasirodžius Drungilo, Dubicko ir Smito (Smyth) [6] darbams. Nauji rezultatai šia tema sudaro pirmąją disertacijos dalį.

Algebrinio skaičiaus α laipsnis taip pat nurodo, kiek α minimalusis polinomas $p(x)$ turi skirtingų šaknų (atkreipiame dėmesį, kad tiek laipsnio, tiek minimaliojo polinomo sąvokos disertacijoje apibrėžiamos virš racionaliųjų skaičių kūno $\mathbb{Q}$). Tokio $p(x)$ šaknys vadinamos skaičiaus α *algebriniais jungtiniais*. Prisiminkime pavyzdį

$$\alpha = \sqrt{7}, \quad p(x) = x^2 - 7, \quad \deg(\alpha) = 2.$$

Šiuo atveju matome, kad $\sqrt{7}$ ir $-\sqrt{7}$ yra skaičiaus $\sqrt{7}$ algebriniai jungtiniai. Bendru atveju algebrinio skaičiaus α, kurio laipsnis lygus d, algebrinius jungtinius žymėsime $\alpha_1, \ldots, \alpha_d$. Antrojoje disertacijos dalyje nagrinėjami tiesiniai (adityvieji) sąryšiai tarp α algebrinių jungtinių

$$k_1\alpha_1 + \ldots + k_d\alpha_d = 0, \tag{2.14}$$

kai $k_1, \ldots, k_d$ yra sveikieji skaičiai. Jei visi koeficientai $k_1, \ldots, k_d$ yra tarpusavyje lygūs, tai remiantis klasikine algebros teorija nesunku parodyti, kad kiekvienam d galime rasti be galo daug d-tojo laipsnio algebrinių skaičių α, kurių algebriniai jungtiniai tenkina (2.14) sąryšį. Dėl šios priežasties, tokie sąryšiai vadinami *trivialiais*. Priešingai, jei lygybėje (2.14) bent du iš koeficientų $k_1, \ldots, k_d$ yra skirtingi, tai sakome,

jog egzistuoja *netrivialus* adityvusis sąryšis tarp α algebrinių jungtinių. Pažymėtina, kad netrivialūs adityvieji sąryšiai tarp algebrinių jungtinių susiformuoja tik išskirtiniais atvejais, o juos nulemiančios priežastys išlieka mažai suprastos. Pirmieji tyrimai šia tema pasirodė XX a. devintajame dešimtmetyje Girstmajerio (Girstmair) [18] ir Smito [38] darbuose. Pavyzdžiui, 1982m. Girstmajeris įrodė, kad egzistuoja devinto laipsnio algebrinis skaičius α, kurio algebriniai jungtiniai tenkina netrivialų adityvųjį sąryšį

$$4\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - 2\alpha_6 - 2\alpha_7 - 2\alpha_8 - 2\alpha_9 = 0.$$

Nors per paskutinius dešimtmečius buvo paskelbtas ne vienas reikšmingas rezultatas apie sąryšius tarp algebrinių jungtinių, netrivialių sąryšių fenomenas nėra pilnai išnagrinėtas net tarp mažo laipsnio algebrinių skaičių. Nauji rezultatai šioje srityje sudaro antrąją disertacijos dalį.

Tikslai ir uždaviniai

Akcentuojame, kad disertacijose vartodami terminus *laipsnis*, *minimalusis polinomas*, *algebriniai jungtiniai* visada turime omenyje: laipsnis virš $\mathbb{Q}$, minimalusis polinomas virš $\mathbb{Q}$, algebriniai jungtiniai virš $\mathbb{Q}$, atitinkamai.

Disertacijos tyrimų objektas – algebrinių skaičių laipsniai ir netrivialūs sąryšiai tarp algebrinių jungtinių. Pirmos disertacijos dalies tikslas – tyrinėti tris susijusius klausimus apie algebrinių skaičių laipsnius. Prieš pereidami prie šių klausimų prisiminkime, kad bet koks kompleksinių skaičių $\mathbb{C}$ pokūnis K gali būti suvokiamas kaip tiesinė erdvė virš racionaliųjų skaičių kūno $\mathbb{Q}$. Šiame kontekste K dimensiją virš $\mathbb{Q}$ (žymime $[K : \mathbb{Q}]$), vadiname kūno K laipsniu. Jei $[K : \mathbb{Q}] < \infty$, tai K vadiname *skaičių kūnu*. Kiekvienam tokiam K egzistuoja algebrinis skaičius α, su kuriuo $K = \mathbb{Q}(\alpha)$. Šiuo atveju K laipsniu galime laikyti algebrinio skaičiaus α laipsnį, t.y. galioja lygybė $[K : \mathbb{Q}] = \deg(\alpha)$. Galiausiai, dviejų skaičių kūnų K ir L *kompozitu* KL vadiname mažiausią tokį skaičių kūną, kuriam priklauso tiek K, tiek L.

- Tegu K ir L – skaičių kūnai, kurių laipsniai atitinkamai lygūs a ir b. Kokias reikšmes gali įgyti kompozito KL laipsnis?

  Jei KL laipsnis gali įgyti reikšmę c, tai natūraliųjų skaičių trejetas

(a,b,c) vadinamas *C-trejetu* (*compositum-feasible triplet*).

- Tegu α ir β – algebriniai skaičiai, kurių laipsniai atitinkamai lygūs a ir b. Kokias reikšmes gali įgyti sumos $\alpha+\beta$ laipsnis?

  Jei $\alpha+\beta$ laipsnis gali įgyti reikšmę c, tai natūraliųjų skaičių trejetas (a,b,c) vadinamas *S-trejetu* (*sum-feasible triplet*).

- Tegu α ir β – algebriniai skaičiai, kurių laipsniai atitinkamai lygūs a ir b. Kokias reikšmes gali įgyti sandaugos $\alpha\cdot\beta$ laipsnis?

  Jei $\alpha\cdot\beta$ laipsnis gali įgyti reikšmę c, tai natūraliųjų skaičių trejetas (a,b,c) vadinamas *P-trejetu* (*product-feasible triplet*).

Siekiant tiksliau apibrėžti aprašytų klausimų tyrinėjimo kryptį, suformuluojami pagrindiniai uždaviniai.

1 uždavinys. *Suklasifikuoti visus C-trejetus, S-trejetus ir P-trejetus, kurių pavidalas yra (p,p,c), čia p – fiksuotas pirminis skaičius.*

2 uždavinys. *Atrasti naujų rezultatų, kurie galėtų būti tiesiogiai pritaikyti pratęsiant dabartines C-trejetų, S-trejetų ir P-trejetų klasifikacijas.*

Antros disertacijos dalies tikslas – tyrinėti netrivialius sąryšius tarp skaičių, vadinamų algebriniais jungtiniais. Prisiminkime, kad algebrinio skaičiaus α, kurio laipsnis lygus d, minimalusis polinomas turi lygiai d (skirtingų) šaknų $\alpha_1,\ldots,\alpha_d$, kurios ir sudaro α algebrinius jungtinius. Disertacijoje nagrinėjami dviejų tipų sąryšiai.

- Jei su tam tikrais sveikaisiais koeficientais $k_1,\ldots,k_d$ (tarp kurių bent du yra skirtingi) galioja lygybė

$$k_1\alpha_1+\ldots+k_d\alpha_d=0, \tag{2.15}$$

  tai (2.15) vadinamas *netrivialiu adityviuoju sąryšiu* tarp skaičiaus α algebrinių jungtinių.

- Jei su tam tikrais sveikaisiais koeficientais $k_1,\ldots,k_d$ (tarp kurių bent du yra skirtingi) galioja lygybė

$$\alpha_1^{k_1}\cdot\ldots\cdot\alpha_d^{k_d}=1, \tag{2.16}$$

  tai (2.16) vadinamas *netrivialiu multiplikatyviuoju sąryšiu* tarp skaičiaus α algebrinių jungtinių.

Konkreti tyrimų apie netrivialius sąryšius kryptis nusakoma dviem uždaviniais.

3 uždavinys. *Nustatyti netrivialius adityviuosius sąryšius tarp skaičiaus α algebrinių jungtinių, kai α laipsnis d neviršija* 8.

Sakome, kad adityvusis sąryšis turi *atitinkamą* multiplikatyvųjį sąryšį, jei d-tojo laipsnio skaičiaus α algebriniams jungtiniams, tenkinantiems (2.15) lygybę su tam tikrais $k_1, \ldots, k_d$, egzistuoja d-tojo laipsnio algebrinis skaičius β, kurio algebriniai jungtiniai tenkina multiplikatyvųjį sąryšį su tais pačiais $k_1, \ldots, k_d$, t.y.

$$\beta_1^{k_1} \cdot \ldots \cdot \beta_d^{k_d} = 1.$$

4 uždavinys. *Konstruktyviu įrodymu pagrįsti, kad kiekvienas netrivialus adityvusis sąryšis turi atitinkamą multiplikatyvųjį sąryšį.*

Rezultatų apžvalga ir jų aktualumas

Disertacijos rezultatai paskelbti (arba priimti publikavimui) šešiuose tarptautiniuose mokslo leidiniuose, turinčiuose cituojamumo rodiklį Clarivate Analytics Web of Science (CA WoS). Tyrimai, aprašyti publikacijose A1, A2 ir A3, sudaro pirmąją disertacijos dalį, kurioje nagrinėjami trys glaudžiai susiję klausimai apie algebrinių skaičių laipsnius. Tyrimai, aprašyti publikacijose A4, A5 ir A6, sudaro antrąją disertacijos dalį, kurioje nagrinėjami netrivialūs sąryšiai tarp algebrinio skaičiaus α algebrinių jungtinių.

Publikacijos A1 rezultatai

2013 m. publikacijoje Drungilas, Dubickas ir Luka (Luca) pastebi, kad *„net natūralus klausimas, kokias reikšmes gali įgyti $[KL:\mathbb{Q}]$, kai K ir L abu yra pirminio laipsnio p plėtiniai virš $\mathbb{Q}$, lieka atviras“* [5]. Straipsnyje A1 šis klausimas yra pilnai atsakomas naudojant C-trejetų terminologiją.

1 teorema. *Tegu p – pirminis skaičius. Jei trejetas (p,p,c) yra C-trejetas, tai $c = ps$ ir galioja vienas iš žemiau pateiktų atvejų:*

(i) $s = p$;

(ii) s *yra* $(p-1)$ *daliklis;*

(iii) $p=11,\ s=6$*;*

(iv) $p=\frac{q^d-1}{q-1},\ s=q^{d-1}$, q *– pirminio skaičiaus laipsnis,* $d\geqslant 3$.

1 teoremą iliustruosime konkrečiu pavyzdžiu, kai $p=13$.

2 išvada. *Tegu* K *ir* L *– skaičių kūnai, kurių abiejų laipsniai lygūs* 13. *Tarkime, jog* c *žymi* K *ir* L *kompozito* KL *laipsnį. Tuomet*

$$c=13s \quad ir \quad s\in\{1,2,3,4,6,9,12,13\}.$$

Pastebėkime, kad kai $p=13$ ir $s=9$, tai turime 1 teoremos (iv) atvejo atitiktį, nes

$$13=\frac{3^3-1}{3-1} \quad \text{ir} \quad 9=3^2.$$

Pirminiai skaičiai, kurių pavidalas $p=(q^d-1)/(q-1)$, atlieka svarbų vaidmenį įvairiuose skaičių teorijos uždaviniuose [16, 23, 27, 37]. Šiuo atžvilgiu 1 teorema atskleidžia naują tokios formos pirminių skaičių pritaikymo sritį.

Nors ankstesniame skyriuje apibrėžtų C-trejetų, S-trejetų ir P-trejetų uždaviniai nėra ekvivalentūs, tam tikrais aspektais jie yra glaudžiai susiję. Pažymėkime visų C-trejetų, S-trejetų ir P-trejetų aibes atitinkamai $\mathcal{C}$, $\mathcal{S}$ ir $\mathcal{P}$. Drungilo, Dubicko ir Smito darbuose [4, 6] buvo įrodyta, kad

$$\mathcal{C}\subsetneq\mathcal{S}\subsetneq\mathcal{P}.$$

Vadinasi, kiekvienas C-trejetas yra tuo pačiu ir S-trejetas, o kiekvienas S-trejetas savo ruožtu yra ir P-trejetas (kita vertus, atvirkštiniai teiginiai nėra teisingi). Kai trejetų forma yra (p,p,c), publikacijoje A1 parodome, kad šias įžvalgas galima suformuluoti dar tiksliau.

3 teorema. *Tegu* p *– pirminis skaičius. Jei trejetas* (p,p,c) *yra S-trejetas, tai jis yra ir C-trejetas arba* $c=1$.

4 teorema. *Tegu* p *– pirminis skaičius. Jei trejetas* (p,p,c) *yra P-trejetas, tai jis yra ir C-trejetas arba* $c\in\{1,p-1\}$.

3 ir 4 teoremas iliustruosime pavyzdžiu, kai $p=13$.

5 išvada. *Tegu α ir β – algebriniai skaičiai, kurių abiejų laipsniai lygūs* 13. *Jei c žymi $\alpha+\beta$ laipsnį, tai*

$$c = 13s \quad ir \quad s \in \{1,2,3,4,6,9,12,13\}$$

arba

$$c = 1.$$

6 išvada. *Tegu α ir β – algebriniai skaičiai, kurių abiejų laipsniai lygūs* 13. *Jei c žymi $\alpha\cdot\beta$ laipsnį, tai*

$$c = 13s \quad ir \quad s \in \{1,2,3,4,6,9,12,13\}$$

arba

$$c \in \{1,12\}.$$

Publikacijos A2 rezultatai

Siekiant įvertinti publikacijoje A1 taikytų metodų apibendrinimo galimybes, publikacijoje A2 bandoma pratęsti C-trejetų, kurių forma yra (p^k,p^k,c), klasifikaciją nuo $k=1$ iki $k\geqslant 2$. Pilnai išspręsti suformuluoto uždavinio nepavyksta – randamas tik dalinis sprendimas tam tikrų sąlygų kontekste. Norint aprašyti šias sąlygas, tenka įvesti keletą žymėjimų. Nagrinėjant kūnų plėtinį $K/\mathbb{Q}$, simboliu $K_{\mathcal{G}}$ žymėsime šio plėtinio Galua uždarinį, o simboliu $\mathrm{Gal}(K_{\mathcal{G}}/\mathbb{Q})$ – atitinkamą Galua grupę. Taip pat sakysime, kad C-trejetas (a,b,c) yra *indukuotas* skaičių kūnų K ir L, jeigu

$$[K:\mathbb{Q}]=a, \quad [L:\mathbb{Q}]=b \quad \text{ir} \quad [KL:\mathbb{Q}]=c.$$

Atkreipiame dėmesį, kad tas pats C-trejetas (a,b,c) gali būti indukuotas skirtingų porų (K,L). Galiausiai, *beveik pirmine* grupe vadinsime tokią grupę G, kuri yra įsiterpusi tarp pirminės grupės S ir visų S automorfizmų grupės $\mathrm{Aut}(S)$, t.y.

$$S \subseteq G \subseteq \mathrm{Aut}(S).$$

7 teorema. *Tegu p – pirminis skaičius ir $k\geqslant 2$. Tarkime, kad (p^k,p^k,c) yra C-trejetas, indukuotas skaičių kūnų K ir L. Be to, laikykime, kad tiek* $\mathrm{Gal}(K_{\mathcal{G}}/\mathbb{Q})$, *tiek* $\mathrm{Gal}(L_{\mathcal{G}}/\mathbb{Q})$ *yra beveik pirminės grupės. Tuomet $c=p^k s$ ir galioja vienas iš žemiau pateiktų atvejų:*

(i) $s \in \{1,\ p^k-1,\ p^k\}$;

(ii) $p=2,\ k \geqslant 4,\ s \in \{2^{k-1}-1,\ 2^{k-1}\}$;

(iii) $p=k=3,\ s \in \{10,16\}$;

(iv) $p^k = \frac{q^d-1}{q-1},\ s \in \left\{\frac{q^{d-1}-1}{q-1},\ q^{d-1}\right\},\ q$ – *pirminio skaičiaus laipsnis,* $d \geqslant 3$.

7 teoremos įrodymas remiasi klasikiniais Guralniko (Guralnick) rezultatais apie pirmines grupes [24] bei 2024 metais užbaigtais Džounso (Jones) ir Sezero (Sezer) darbais apie tranzityvias grupes [27]. Atkreipiame dėmesį, jog išskyrus atvejį $p=k=2$, 7 teoremos nepakanka norint nustatyti visus C-trejetus, kurių forma yra (p^k,p^k,c). Dabartinė C-trejetų klasifikacija yra užbaigta trejetams (a,b,c), tenkinantiems sąlygas $a \leqslant b \leqslant c$ ir $b \leqslant 9$ [5, 6, 7]. Taigi, publikacijoje A2 nagrinėjamo klausimo kontekste, kol kas pilnai ištirti tik trejetai, kurių forma yra $(2^2,2^2,c)$, $(2^3,2^3,c)$ arba $(3^2,3^2,c)$. Sujungus Drungilo ir Maciulevičiaus [7] įrodytus teiginius su 7 teorema gauname štai tokį tvirtinimą:

8 teorema. *Trejetas* $(2^4,2^4,c)$ *yra C-trejetas tada ir tik tada, kai*

$$c = 2^4 \cdot s \quad ir \quad s \in \{1,\ldots,16\} \setminus \{11,13\}.$$

Publikacijos A3 rezultatai

1970 m. Izakas (Isaacs) [26] įrodė, kad jei algebrinių skaičių α ir β laipsniai yra tarpusavyje pirminiai, tai sumos $\alpha+\beta$ laipsnis yra didžiausias įmanomas. Vadinasi, šiuo atveju nelygybėje

$$\deg(\alpha+\beta) \leqslant \deg(\alpha)\cdot\deg(\beta)$$

pasiekiamas lygybės atvejis. Šis rezultatas ypač naudingas tyrinėjant S-trejetus. Iš tikrųjų, jei a ir b yra tarpusavyje pirminiai, tai gauname, kad trejetas (a,b,c) yra S-trejetas tada ir tik tada, kai $c=ab$.

Kita vertus, nesunku sukonstruoti pavyzdžių, parodančių, kad analogiškas teiginys jau nebėra teisingas P-trejetams. Kitaip tariant, net jei α ir β laipsniai yra tarpusavyje pirminiai, sandaugos $\alpha\cdot\beta$ laipsnis vis tiek gali būti griežtai mažesnis už α ir β laipsnių sandaugą.

Su tam tikromis prielaidomis algebrinių skaičių laipsniams, publikacijoje A3 rastos dvi naujos sąlygos, kurios užtikrina, kad nelygybėje

$$\deg(\alpha\cdot\beta)\leqslant\deg(\alpha)\cdot\deg(\beta)$$

pasiekiamas lygybės atvejis.

9 teorema. *Tegu p – nelyginis pirminis skaičius ir m — bet koks natūralusis skaičius, nesidalijantis iš p. Be to, laikykime, kad α ir β yra algebriniai skaičiai, kurių laipsniai atitinkamai lygūs m ir p. Jei β minimalusis polinomas nėra x^p+c su tam tikru $c\in\mathbb{Q}$, tai*

$$\deg(\alpha\cdot\beta)=\deg(\alpha)\cdot\deg(\beta)=mp.$$

Pavyzdžiui, imkime

$$\alpha=\sqrt{\sqrt{2}+2}\quad\text{ir}\quad\beta=1+\sqrt[3]{2}+\sqrt[3]{4}.$$

Su kompiuterine algebros sistema SageMath[41] galima patikrinti, kad α ir β minimalieji polinomai atitinkamai yra

$$x^4-4x^2+2\quad\text{ir}\quad x^3-3x^2-3x-1.$$

Taigi, $\deg(\alpha)=4$ ir $\deg(\beta)=3$. Matome, kad visos 9 teoremos sąlygos yra išpildytos. Vadinasi, šiuo atveju $\alpha\cdot\beta$ laipsnis lygus $3\cdot4=12$. Šie pastebėjimai papildo Veintraubo (Weintraub) [42] tyrimus apie primityviuosius plėtinio $\mathbb{Q}(\alpha,\beta)$ elementus. Iš tiesų, jei algebriniai skaičiai α ir β tenkina 9 teoremos sąlygas, tai galioja lygybė $\mathbb{Q}(\alpha,\beta)=\mathbb{Q}(\alpha\cdot\beta)$.

Remiantis 9 teorema išvedamas dar vienas kriterijus, kuris kaip parodysime netrukus, gali būti tiesiogiai pritaikytas P-trejetų analizėje.

10 teorema. *Tegu p – nelyginis pirminis skaičius ir m – bet koks natūralusis skaičius, nesidalijantis nei iš p, nei iš $p-1$. Be to, laikykime, kad α ir β yra algebriniai skaičiai, kurių laipsniai atitinkamai lygūs m ir p. Tuomet*

$$\deg(\alpha\cdot\beta)=\deg(\alpha)\cdot\deg(\beta)=mp.$$

11 išvada. *Tegu p – nelyginis pirminis skaičius ir a – bet koks natūralusis skaičius, nesidalijantis nei iš p, nei iš $p-1$. Tuomet trejetas (a,p,c) yra P-trejetas tada ir tik tada, kai $c=ap$.*

11 išvada leidžia pilnai ištirti tam tikras galimų P-trejetų klases ir taip ateityje būti pritaikyta tolimesnei P-trejetų klasifikacijai, kuri 2023 m. buvo pradėta Maciulevičiaus [35]. Šiuo metu yra nustatyti visi P-trejetai (a,b,c), tenkinantys sąlygas $a \leqslant b \leqslant c$ ir $b \leqslant 7$ [13].

Publikacijos A4 rezultatai

Laikykime, kad α algebrinis skaičius, kurio laipsnis lygus d. Šioje publikacijoje tyrinėjamos bendresnės netrivialių adityviųjų sąryšių

$$\alpha_i + \alpha_j + \alpha_k = 0 \quad \text{ir} \quad \alpha_i + \alpha_j - \alpha_k = 0$$

formos, čia $\alpha_i, \alpha_j, \alpha_k$ žymi tris skirtingus α algebrinius jungtinius. Netrivialūs sąryšiai dažniausiai nagrinėjami remiantis atitinkamų Galua grupių analize. Iš tiesų, kiekvienam algebriniam skaičiui α galime priskirti Galua grupę G imdami α minimaliojo polinomo Galua grupę. Publikacijoje A4 paskelbta teorema aprašo visus įmanomus netrivialius adityviuosius sąryšius tarp trijų α algebrinių jungtinių, kai α laipsnis neviršija 8. Teoremos tvirtinime vartojami grupių žymėjimai sutampa su tais, kurie naudojami tranzityvių grupių skaitmeniniame kataloge [40].

12 teorema. *Tarkime, kad α – algebrinis skaičius, kurio laipsnis d neviršija 8. Tegu G – α minimaliojo polinomo Galua grupė ir a,b,c – nenuliniai sveikieji koeficientai. Jei trys skirtingi α algebriniai jungtiniai $\alpha_i, \alpha_j, \alpha_k$ tenkina adityvųjį sąryšį*

$$a\alpha_i + b\alpha_j + c\alpha_k = 0,$$

tai d, G ir a,b,c (arba jų perstata) išpildo vieną iš šių sąlygų:

(i) $d = 3,\ G \in \{C_3,\ S_3\}$ *ir* $a = b = c$;

(ii) $d = 6,\ G \in \{C_6,\ S_3,\ D_6,\ C_3 \times S_3,\ S_3^2;\ C_3^2 \rtimes C_4,\ S_3 \wr C_2\}$ *ir* $a = b = c$;

(iii) $d = 6,\ G \in \{C_6,\ S_3,\ D_6\}$ *ir* $a = b = -c$;

(iv) $d = 6,\ G = S_3$ *ir* $a^2 + b^2 - ab = c^2$;

(v) $d = 8,\ G = D_4$ *ir* $a^2 + b^2 = c^2$.

Gautas rezultatas papildo Lalando (Lalande) [32] ir Girstmajerio [21] įžvalgas bei pratęsia Dubicko ir Jankausko [11] darbus, kuriuose tyrinėjami netrivialūs sąryšiai tarp mažo laipsnio algebrinių jungtinių. Pažymėtina, kad 12 teoremos (iv) ir (v) dalyse nurodyti netrivialūs sąryšiai susiformuoja tik ypač išskirtinėmis sąlygomis. Iš teoremos tvirtinimo taip pat išplaukia, kad jei $\alpha_i, \alpha_j, \alpha_k$ yra aštunto laipsnio algebriniai jungtiniai, tenkinantys lygybę

$$a\alpha_i + b\alpha_j + c\alpha_k = 0, \tag{2.17}$$

tai būtinai $a^2 + b^2 = c^2$. Be to, teoremos įrodyme pademonstruojama, kaip su kiekvienu lygties $a^2 + b^2 = c^2$ sveikuoju sprendiniu sudaryti algebrinius skaičius $\alpha_i, \alpha_j, \alpha_k$, tenkinančius (2.17) sąryšį. Pavyzdžiui, imkime lygties $a^2 + b^2 = c^2$ sveikąjį sprendinį $(3, 4, 5)$. Su kompiuterine algebros programa SageMath [41] galima patikrinti, kad neredukuojamo virš $\mathbb{Q}$ polinomo

$$p(x) = x^8 + 580608x^4 + 1074954240000$$

Galua grupė yra izomorfiška grupei D_4 ir trys skirtingos $p(x)$ šaknys $\alpha_i, \alpha_j, \alpha_k$ tenkina sąryšį

$$3\alpha_i + 4\alpha_j + 5\alpha_k = 0.$$

Publikacijos A5 rezultatai

Pastebėkime, kad jei tarp α algebrinių jungtinių egzistuoja netrivialus adityvusis sąryšis

$$\alpha_i + \alpha_j = 0,$$

tai α minimalusis polinomas yra formos $p(x) = f(x^2)$. Iš to seka, kad šiuo atveju α laipsnis turi būti dalus iš 2. Dubickas ir Jankauskas [11] parodė, jog kiekvienam d, kuris dalijasi iš 3, galima sudaryti d-tojo laipsnio algebrinį skaičių α, kurio algebriniai jungtiniai tenkina netrivialų adityvųjį sąryšį

$$\alpha_i + \alpha_j + \alpha_k = 0. \tag{2.18}$$

Natūraliai kyla klausimas, ar (2.18) sąryšis gali egzistuoti tik tarp algebrinių jungtinių, kurių laipsnis d dalus iš 3? Dubickas ir Smitas [14] nustatė, kad atsakymas į šį klausimą yra neigiamas, pateikdami kontrpavyzdį su $d = 20$. Vis dar neišsiaiškinta, ar $d = 20$ yra mažiausio laipsnio

kontrpavyzdys. Ankstesnių rezultatų pakanka nustatyti, kad minimalus toks d tenkina sąlygą $10 \leqslant d \leqslant 20$. Publikacijoje A5 įrodomas teiginys, leidžiantis atmesti reikšmes $d = 10$ ir $d = 14$.

13 teorema. *Tegu α – algebrinis skaičius, kurio laipsnis lygus $2p$, čia $p \geqslant 5$ yra pirminis skaičius. Tuomet su bet kuriais trimis α algebriniais jungtiniais $\alpha_i, \alpha_j, \alpha_k$ visada turėsime*

$$\alpha_i + \alpha_j + \alpha_k \neq 0.$$

Sujungę klasikinį Kurbatovo (Kurbatov) [30] rezultatą su 13 teorema gauname štai tokį tvirtinimą:

14 išvada. *Laikykime, kad α yra mažiausio laipsnio, nedalaus iš 3, algebrinis skaičius, kurio trys algebriniai jungtiniai $\alpha_i, \alpha_j, \alpha_k$ tenkina netrivialų adityvųjį sąryšį*

$$\alpha_i + \alpha_j + \alpha_k = 0.$$

Pažymėkime α laipsnį d. Tuomet $d = 16$ arba $d = 20$.

Publikacijos A6 rezultatai

Dar 1957 metais Kurbatovas [30] parodė, kad jei α yra pirminio laipsnio algebrinis skaičius, tai netrivialūs adityvieji sąryšiai tarp α algebrinių jungtinių neegzistuoja. Kita vertus, su kiekvienu sudėtiniu d galima rasti tokį d-tojo laipsnio algebrinį skaičių α, kad tarp jo jungtinių egzistuotų netrivialus adityvusis sąryšis. Visgi nei su vienu sudėtiniu d iki šiol nebuvo aprašyta visų netrivialių adityviųjų sąryšių struktūra. Publikacijoje A6 pratęsiami Kitaokos (Kitaoka) [29] rezultatai, leidžiantys išspręsti atvejį $d = 4$.

15 teorema. *Tarkime, kad α – ketvirto laipsnio algebrinis skaičius, kurio skirtingi algebriniai jungtiniai $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ tenkina netrivialų adityvųjį sąryšį*

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 = 0,$$

čia $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. Pažymėkime

$$r = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.$$

Jei $r \neq 0$, tai (iki α jungtinių pernumeravimo) galioja lygybė

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0,$$

o jei $r = 0$, tai (iki α jungtinių pernumeravimo) galioja lygybės

$$\alpha_1 + \alpha_2 = 0 \quad ir \quad \alpha_3 + \alpha_4 = 0.$$

15 teorema lemia, kad visų netrivialių adityviųjų sąryšių struktūra tarp ketvirto laipsnio algebrinių jungtinių gali būti aprašyta trimis tiesinėmis formomis su sveikaisiais koeficientais. Iš tikrųjų, imkime tris tiesines formas

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= x_1 + x_2 - x_3 - x_4, \\ f_2(x_1, x_2, x_3, x_4) &= x_1 + x_2, \\ f_3(x_1, x_2, x_3, x_4) &= x_3 + x_4. \end{aligned}$$

Jeigu

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 = 0$$

ir tarp sveikųjų k_1, k_2, k_3, k_4 bent du yra skirtingi (netrivialaus adityviojo sąryšio apibrėžimas), tai visada turėsime

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in \sum_{j \in S} \mathbb{Z} f_j(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

čia poaibis $S \subseteq \{1, 2, 3\}$ parinktas taip, kad $f_j(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$ kiekvienam $j \in S$.

Iš tikrųjų, remiantis 15 teorema galima parodyti, kad jei visų α algebrinių jungtinių suma $r \neq 0$, tai

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in \mathbb{Z} f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

čia

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0.$$

Tuo tarpu, jei visų α algebrinių jungtinių suma $r = 0$, tai

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in \mathbb{Z} f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) + \mathbb{Z} f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

čia

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha_1+\alpha_2=0,\quad f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha_3+\alpha_4=0.$$

Pavyzdžiui, remiantis suformuluotu teiginiu nesunku įrodyti, kad nėra tokio ketvirto laipsnio algebrinio skaičiaus α, kurio algebriniai jungtiniai tenkintų netrivialų adityvųjį sąryšį

$$\alpha_1+2\alpha_2+3\alpha_3+\alpha_4=0. \tag{2.19}$$

Toliau parodome, kad analogiškas teiginys atveju $d=6$ nėra teisingas. Kitaip tariant, neįmanoma visų netrivialiųjų adityviųjų sąryšių tarp šešto laipsnio algebrinių jungtinių aprašyti baigtiniu tiesinių formų su sveikaisiais koeficientais sąrašu.

16 teorema. *Tegu m – natūralusis skaičius. Nagrinėkime tokią m tiesinių formų aibę*

$$f_i(x_1,\dots,x_6)=\sum_{j=1}^{6}u_{ij}x_j,$$

$i=1,\dots,m$, kad su bet kuriuo i bent du iš koeficientų $u_{i1},\dots,u_{i6}\in\mathbb{Z}$ būtų skirtingi. Tuomet egzistuoja be galo daug šešto laipsnio algebrinių skaičių α, kurių skirtingi algebriniai jungtiniai $\alpha_1,\dots,\alpha_6$ tenkina netrivialų adityvųjį sąryšį

$$k_1\alpha_1+k_2\alpha_2+k_3\alpha_3=0\quad \textit{su tam tikrais}\quad k_1,k_2,k_3\in\mathbb{Z},$$

tačiau

$$k_1\alpha_1+k_2\alpha_2+k_3\alpha_3\notin\sum_{j\in S}\mathbb{Z}f_j(\alpha_1,\dots,\alpha_6)$$

kad ir kokia būtų aibė $S\subseteq\{1,\dots,m\}$, kai $f_j(\alpha_1,\dots,\alpha_6)=0$ kiekvienam $j\in S$.

17 teorema. *Tegu m – natūralusis skaičius. Nagrinėkime tokią m tiesinių formų aibę*

$$f_i(x_1,\dots,x_6)=\sum_{j=1}^{6}u_{ij}x_j,$$

$i=1,\dots,m$, kad su bet kuriuo i bent du iš koeficientų $u_{i1},\dots,u_{i6}\in\mathbb{Z}$ būtų skirtingi. Tuomet egzistuoja be galo daug šešto laipsnio algebrinių skaičių α, kurių skirtingi algebriniai jungtiniai $\alpha_1,\dots,\alpha_6$ tenkina netrivialų

adityvųjį sąryšį

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0 \quad \textit{su tam tikrais} \quad k_1, k_2, k_3 \in \mathbb{Z},$$

tačiau su bet kuriuo tokiu poaibiu $S \subseteq \{1, \ldots, m\}$, *kad* $f_j(\alpha_1, \ldots, \alpha_6) = 0$ *kiekvienam* $j \in S$, *turėsime*

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \notin \sum_{j \in S} \mathbb{Z} f_j(\alpha_1, \ldots, \alpha_6).$$

Paskutinis disertacijoje pateiktas rezultatas, papildantis Girstmajerio darbus [19], parodo, kad kiekvienas adityvusis sąryšis turi atitinkamą multiplikatyvųjį sąryšį.

18 teorema. *Tarkime, kad algebrinio skaičiaus* α, *kurio laipsnis lygus* d, *skirtingi algebriniai jungtiniai* $\alpha_1, \ldots, \alpha_d$ *su tam tikrais sveikaisiais koeficientais* $k_1, \ldots, k_d$ *tenkina netrivialų adityvųjį sąryšį*

$$k_1\alpha_1 + \cdots + k_d\alpha_d = 0.$$

Tuomet galima sudaryti algebrinį skaičių β, *kurio laipsnis lygus* d, *o skirtingi* β *algebriniai jungtiniai* $\beta_1, \ldots, \beta_d$ *tenkina netrivialų multiplikatyvųjį sąryšį*

$$\beta_1^{k_1} \ldots \beta_d^{k_d} = 1.$$

Pavyzdžiui, iš publikacijos A4 apžvalgos žinome, kad egzistuoja aštunto laipsnio algebrinis skaičius α, kurio skirtingi algebriniai jungtiniai $\alpha_i, \alpha_j, \alpha_k$ tenkina netrivialų adityvųjį sąryšį

$$3\alpha_i + 4\alpha_j + 5\alpha_k = 0.$$

18 teoremos įrodyme pademonstruojama, kaip iš skaičiaus α sudaryti aštunto laipsnio algebrinį skaičių β, kurio skirtingi algebriniai jungtiniai $\beta_i, \beta_j, \beta_k$ tenkintų netrivialų multiplikatyvųjį sąryšį

$$\beta_i^3 \cdot \beta_j^4 \cdot \beta_k^5 = 1.$$

Pažymėtina, jog iš Drmotos (Drmota) ir Skalbos (Skałba) darbų [3] išplaukia, kad atvirkštinis teiginys 18 teoremai nėra teisingas.

Išvados

Apibendrinant, pagrindiniai disertacijos rezultatai yra šie:

- nustatyti visi C-trejetai, S-trejetai ir P-trejetai, kurių forma yra (p,p,c), čia p – fiksuotas pirminis skaičius;
- atrasti nauji C-trejetai, kurių forma yra (p^k,p^k,c), čia p – fiksuotas pirminis skaičius ir $k \geqslant 2$;
- atrastos naujos sąlygos, kurias išpildžius, dviejų algebrinių skaičių sandauga įgyja didžiausią įmanomą laipsnį;
- nustatyti visi netrivialūs adityvieji sąryšiai tarp trijų algebrinio skaičiaus α algebrinių jungtinių, kai α laipsnis neviršija 8;
- įrodyta, kad jei algebrinio skaičiaus α laipsnis lygus $2p$, čia $p \geqslant 5$ yra pirminis skaičius, tai bet kurių trijų α algebrinių jungtinių suma nelygi 0;
- aprašyta netrivialių adityviųjų sąryšių tarp algebrinio skaičiaus α algebrinių jungtinių struktūra, kai α laipsnis lygus 4;
- parodyta, kaip iš kiekvieno netrivialaus adityviojo sąryšio sudaryti atitinkamą multiplikatyvųjį sąryšį.

Acknowledgements

I would like to express special gratitude to the following people from the mathematical community (in chronological order), who had the greatest influence throughout my mathematical journey.

- My school teachers, Leonas Narkevičius and Rita Bareišienė, for encouraging my interest in mathematics throughout the teenage years.
- Professor Romualdas Kašuba, for suggesting, in his own elegant way, that pursuing an interesting mathematical problem could be a way of life that suits me.
- Professor Paulius Drungilas, for his helpful advice and guidance in the early years of my mathematical studies.
- My colleague Lukas Maciulevičius, for introducing me to mathematical research and for many stimulating discussions about mathematics and its teaching.
- Professor Artūras Dubickas, my academic supervisor, for guiding me through the dissertation years, for showing me firsthand what it means to live the life of a mathematician, and for providing lasting inspiration to continue my mathematical journey in the years to come.

Dissemination of results

The results of this thesis were presented in the following conferences.

- *International scientific conference dedicated to the 160th anniversary of Prof. Dr. Herman Minkowski*, June 20–22, 2024, Kaunas, Lithuania.
- *International Conference on Probability Theory and Number Theory*, September 16–20, 2024, Palanga, Lithuania.

Curriculum Vitae

Name: Paulius
Surname: Virbalas
Place of birth: Lithuania

Education

2015: Bachelor's Degree in Economics,
University of Amsterdam (the Netherlands).

2019: Master's Degree in Mathematics,
Vilnius University (Lithuania).

Work experience

2017 – present: Mathematics Teacher,
Kaunas Saulės Gymnasium.

2022 – present: Junior Assistant,
Vilnius University, Faculty of Mathematics and Informatics.

Copies of Articles

Article 1

A1

Compositum of Two Number Fields of Prime Degree

P. Virbalas

New York Journal of Mathematics **29** (2023), 171–192.

Available online at `https://nyjm.albany.edu/j/2023/29-6.html`

Article 2

A2

Compositum of Two Number Fields of Prime Power Degree

P. Virbalas

Lithuanian Mathematical Journal **65** (2025), 445–454.

Available online at https://doi.org/10.1007/s10986-025-09683-1

Article 3

A3

Degree of the Product of Two Algebraic Numbers One of Which is of Prime Degree

P. Virbalas

Mathematics **11** (2023), Paper No. 1485.

Available online at `https://doi.org/10.3390/math11061485`

Article 4

A4

Linear Relations Between Three Conjugate Algebraic Numbers of Low Degree

P. Virbalas

Journal of the Korean Mathematical Society **62** (2025), 253–284.

Available online at `https://doi.org/10.4134/JKMS.j240002`

Article 5

A5

Linear Relations Between Three Algebraic Conjugates of Degree Twice a Prime

P. Virbalas

Glasnik Matematički Serija III (accepted for publication)

Forthcoming articles are available at the journal website:
`https://web.math.pmf.unizg.hr/glasnik/forthcoming.html`

Article 6

A6

Additive and Multiplicative Relations with Algebraic Conjugates

A. Dubickas and P. Virbalas

Revista de la Unión Matemática Argentina (accepted for publication)

Available online at https://doi.org/10.33044/revuma.5106

NOTES

NOTES

Vilniaus universiteto leidykla
Saulėtekio al. 9, III rūmai, LT-10222 Vilnius
El. p. info@leidykla.vu.lt, www.leidykla.vu.lt
bookshop.vu.lt, journals.vu.lt
Tiražas 20 egz.