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Joint Discrete Approximation by the Riemann and Hurwitz Zeta Functions in Short Intervals

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Abstract

In this paper, we prove the theorems on the simultaneous approximation of a pair of analytic functions by discrete shifts $(\zeta(s+ikh_1), \zeta(s+ikh_2,\alpha))$, $h_1 > 0$, $h_2 > 0$ of the Riemann zeta function $\zeta(s)$ and Hurwitz zeta function $\zeta(s,\alpha)$. The lower density and density of the above approximating shifts are considered in short intervals $[N, N+M]$ as $N \to \infty$ with $M = o(N)$. If the set $\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m+\alpha) : m \in \mathbb{N}_0), 2\pi\}$ is linearly independent over $\mathbb{Q}$, the class of approximated pairs is explicitly given. If α and h_1, h_2 are arbitrary, then it is known that the set of approximated pairs is a certain non-empty closed subset of $\mathbb{H}^2(\Delta)$, where $\mathbb{H}(\Delta)$ is the space of analytic functions on the strip $\Delta = \{s \in \mathbb{C} : 1/2 < \operatorname{Re} s < 1\}$. For the proof, limit theorems on weakly convergent probability measures in the space $H^2(\Delta)$ are applied.

Keywords: approximation of analytic functions; Hurwitz zeta function; Riemann zeta function; universality; weak convergence of probability measures

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1. Introduction

One of the most important problems of function theory is approximation of analytic functions. The brilliant Mergelyan theorem completed a series of works on approximation of analytic functions by polynomials. The Mergelyan theorem states [1,2] (see also [3]) that every continuous function $g(s)$, $s = \sigma + it$, on a compact set $K \subset \mathbb{C}$ with connected complement that is analytic inside of K can be approximated by a polynomial. This means that, for every $\varepsilon > 0$, there exists a polynomial $p_{\varepsilon,g}(s)$ such that

$$\sup_{s \in K} \left| g(s) - p_{\varepsilon,g}(s) \right| < \varepsilon.$$

However, since 1975 it has been known that, for approximation of analytic functions, an another class of functions can be applied, and this class is the zeta (or L) functions, widely used in analytic number theory and having applications in other natural sciences. The classical zeta functions usually are defined in a certain half-plane by Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > \sigma_0,$$

with coefficients a_m having some arithmetical sense and meromorphically continued to the left of the half-plane $\sigma > \sigma_0$. The most important among them is the Riemann zeta function $\zeta(s)$, for $\sigma > 1$, given by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

or, equivalently, by the Euler product

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

where $\mathbb{P}$ is the set of all prime numbers, which is analytically continued to the entire complex plane, except for the point $s = 1$ which is a simple pole with residue 1. Value distribution of the function $\zeta(s)$ is continuously receciving attention from mathematicians. One cause of this is the Riemann hypothesis (RH) [4], which states that all zeros of $\zeta(s)$ in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$ are on the line $\sigma = 1/2$. The RH is mentioned in the list of Hilbert problems [5,6] and remains among the most important seven Millennium problems of mathematics [7].

Let us return to approximation. In [8], see also [9–12], Voronin proved a theorem on approximation of analytic functions by shifts $\zeta(s + i\tau)$: let $0 < r < 1/4$, $f(s)$ be a continuous non-vanishing function on $|s| \leqslant r$ and analytic on $|s| < r$. Then, for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \leqslant r} \left|\zeta\left(\tfrac{3}{4} + s + i\tau\right) - f(s)\right| < \varepsilon.$$

Thus, in the approximation sense, the function $\zeta(s)$ is universal: it shifts approximately an entire class of analytic functions. This is the main difference from Mergelyan's theorem in which, for every analytic function, a new polynomial is constructed.

Voronin theorem has an improved version [13–17]. Let $\Delta = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Let $\mathcal{K}$ denote the class of compact subsets of the region Δ with connected complements, and let $\mathcal{H}_0(K)$, $K \in \mathcal{K}$ be the set of continuous non-vanishing functions on K that are analytic inside of K. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the following statement on universality of $\zeta(s)$ is true:

Suppose that $K \in \mathcal{K}$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon > 0$ [18].

Thus, the initial form of Voronin's theorem is extended in two directions: the approximation on discs is replaced by that on general compact sets of the class $\mathcal{K}$, and the set of approximating shifts is infinite because it has a positive lower density.

Now, introduce one more zeta function. Let $\alpha \in (0, 1]$ be a fixed parameter. The Hurwitz zeta function $\zeta(s, \alpha)$ is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1,$$

and as $\zeta(s)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Different from $\zeta(s)$, the function $\zeta(s, \alpha)$ for all $\alpha \notin \{1/2, 1\}$ has no representation by a product over primes.

Properties of the Hurwitz zeta function, including approximation of analytic functions, depend on the parameter α, in contrast with $\zeta(s)$ satisfying the symmetric functional equation

$$\xi(1-s) = \xi(s),$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, and $\Gamma(s)$ is the gamma-function, the function $\zeta(s,\alpha)$, for $\alpha \in (0,1]$ and $\sigma < 0$, has the equation

$$\zeta(s,\alpha) = 2(2\pi)^{s-1}\Gamma(1-s)\left(\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\cos 2\pi m\alpha}{m^{1-s}} + \cos\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\sin 2\pi m\alpha}{m^{1-s}}\right).$$

On the other hand, the shifts $\zeta(s+i\tau,\alpha)$ approximate a wider class of analytic functions than $\mathcal{H}_0(K)$. Let $\mathcal{H}(K)$, $K \in \mathcal{K}$ denote the extension of $\mathcal{H}_0(K)$ including functions having zeros on K. Then, the following statement is known [13–15,19]:

Suppose that the parameter α is transcendental or rational lying in $(0,1) \setminus \{1/2\}$, and $K \in \mathcal{K}$, $f(s) \in \mathcal{H}(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

Note that the cases $\alpha = 1$ and $\alpha = 1/2$ are contained in previous statement for $\zeta(s)$.

The case of algebraic irrational α is exceptional, and universality of $\zeta(s,\alpha)$ by using discs was solved in [20] with some exceptions related to a degree of α. The result on approximation is very deep but sufficiently complicated to state it.

Universality of zeta functions has a lot of theoretical and practical applications [16], including the important independence property of zeta functions [11,19,21,22].

It is an interesting problem to consider a simultaneous approximation of a pair of analytic functions by shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$. This problem was proposed and solved for transcendental α by H. Mishou in [23].

Suppose that the parameter α is transcendental, $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in \mathcal{H}_0(K_1)$, $f_2(s) \in \mathcal{H}(K_2)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1}|\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2}|\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

The latter result was improved in [24], where the approximation of a pair (f_1, f_2) by $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$ was given in the so-called short intervals, i.e., in the intervals $[T, T+H]$ with $H = o(T)$ as $T \to \infty$. More precisely, the main result of [24] states

Suppose that the parameter α is transcendental, $T^{27/82} \leqslant H \leqslant T^{1/2}$, $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in \mathcal{H}_0(K)$, $f_2(s) \in \mathcal{H}(K_2)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty}\frac{1}{H}\text{meas}\Big\{\tau \in [T,T+H] : \sup_{s\in K_1}|\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2}|\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

Moreover, "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

This result has a certain advantage against that for the interval $[0,T]$ of length T because in short intervals it is easier to detect τ with approximating shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$.

Approximation of analytic functions by shifts $\zeta(s+i\tau)$ in short intervals was introduced in [25] and improved in [26]. An analogical problem for $\zeta(s,\alpha)$ was treated in [27], and a more precise result than cited above was established.

All above-mentioned theorems on approximation of analytic functions are of continuous type because τ in shifts $\zeta(s+i\tau)$ and $\zeta(s+i\tau,\alpha)$ can take arbitrary values in the interval. There exists another type of approximation theorem for zeta functions when τ in approximating shifts takes values from a certain discrete set. Such a type was proposed by A. Reich in [28] and is more convenient for practical applications because a discrete set lying in the interval is narrower than the whole interval.

The study of discrete universality theorems for $\zeta(s)$ and $\zeta(s,\alpha)$ was continued in [13]. Let $\#A$ denote the cardinality of a set $A \subset \mathbb{R}$, and let, for $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$C_N(\dots) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \dots\},$$

where, in place of dots, a condition satisfying k is to be written. In what follows, we will deal with approximation of analytic functions by discrete shifts $\zeta(s+ikh)$ and $\zeta(s+ikh,\alpha)$ with fixed $h>0$ and $k \in \mathbb{N}_0$.

First recall some known results.

Theorem 1 (see [13,28])**.** *Suppose that $K \in \mathcal{K}$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $h>0$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty} C_N\left(\sup_{s\in K}|\zeta(s+ikh) - f(s)| < \varepsilon\right) > 0.$$

We notice that in [28] Theorem 1 was proved for more general Dedekind zeta function $\zeta_{\mathbb{K}}(s)$ of algebraic number fields. For $\mathbb{K} = \mathbb{Q}$, this gives Theorem 1.

The case of the function $\zeta(s,\alpha)$ is more complicated, and results depend on arithmetic of α.

Theorem 2 (see [13,29])**.** *Suppose that α is rational $\alpha \neq 1$, $\alpha \neq 1/2$, and $K \in \mathcal{K}$, $f(s) \in \mathcal{H}(K)$. Then, for every $h>0$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty} C_N\left(\sup_{s\in K}|\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right) > 0. \tag{1}$$

For other α, introduce the set

$$L(\alpha,h,\pi) = \left\{(\log(m+\alpha) : m \in \mathbb{N}_0), \tfrac{2\pi}{h}\right\}, \quad h>0,$$

which can be a multiset.

Theorem 3 (see [30])**.** *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$, $K \in \mathcal{K}$ and $f(s) \in \mathcal{H}(K)$. Then, for every $\varepsilon>0$, inequality (1) is true. Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon>0$.*

The arithmetic of α in the theorem is defined by the linear independence of the set $L(\alpha,h,\pi)$. This is implied by the algebraic independence over $\mathbb{Q}$ of the numbers α and $\exp\{2\pi/h\}$. For example, we can take $\alpha = 1/\pi$ and $h = 2$ or $\alpha = 2^{-\sqrt[3]{2}}$ and $h = 2\pi/(\sqrt[3]{4}\log 2)$ [30].

The most general discrete universality theorem for $\zeta(s)$ and $\zeta(s,\alpha)$ has been obtained in [31]. Define the set

$$L(\mathbb{P};\alpha,h_1,h_2,\pi)=\{(h_1\log p: p\in\mathbb{P}),(h_2\log(m+\alpha): m\in\mathbb{N}_0),2\pi\},\quad h_1>0,h_2>0.$$

Theorem 4 (see [31])**.** *Suppose that the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in\mathcal{H}_0(K_1)$, $f_2(s)\in\mathcal{H}(K_2)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty} C_N\left\{\sup_{s\in K_1}|\zeta(s+ikh_1)-f_1(s)|<\varepsilon,\sup_{s\in K_2}|\zeta(s+ikh_2,\alpha)-f_2(s)|<\varepsilon\right\}>0.$$

A paper [32] is devoted to a connection between continuous and discrete universalities.

The aim of this paper is a joint discrete approximation of analytic functions by shifts $(\zeta(s+ikh_1),\zeta(s+ikh_2,\alpha))$ in short intervals. For brevity, we use the notation

$$C_{N,M}(\dots)=\frac{1}{M+1}\#\{N\leqslant k\leqslant N+M:\dots\},\quad N,M\in\mathbb{N}_0,$$

where, in place of dots, we write a condition satisfied by k.

Theorem 5. *Suppose that $h^{-1}(Nh)^{27/82}\leqslant M\leqslant h^{-1}(Nh)^{1/2}$, $K\in\mathcal{K}$, and $f(s)\in\mathcal{H}_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty} C_{N,M}\left(\sup_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon\right)>0.$$

Theorem 5 with $h^{-1}(NH)^{1/3}(\log Nh)^{26/15}\leqslant M\leqslant h^{-1}Nh$ was obtained in [33]. For this, the mean square estimate

$$\int_{T-H}^{T+H}|\zeta(\sigma+it)|^2\,\mathrm{d}t\ll_\sigma H \tag{2}$$

with $T^{1/3}(\log T)^{26/15}\leqslant H\leqslant T$ and $\sigma>1/2$ was applied, which follows from Lemma 2, using the exponential pair $(4/11,6/11)$. However, application of the exponential pair $(11/30,16/30)$ [34] leads to (2) with $T^{27/82}\leqslant H\leqslant T$, and this implies the hypothesis for M in Theorem 5.

Now, we state the main results of the paper, i.e., joint theorems on approximation of a pair of analytic functions in short intervals by discrete shifts of the Riemann and Hurwitz zeta-functions.

Theorem 6. *Suppose that the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$,*

$$\max\left(h_1^{-1}(Nh_1)^{27/82},h_2^{-1}(Nh_2)^{27/82}\right)\leqslant M\leqslant\min\left(h_1^{-1}(Nh_1)^{1/2},h_2^{-1}(Nh_2)^{1/2}\right),$$

$K_1,K_2\in\mathcal{K}$, and $f_1(s)\in\mathcal{H}_0(K_1)$, $f_2(s)\in\mathcal{H}(K_2)$. Then, for every $\varepsilon>0$,

$$\liminf_{N\to\infty} C_{N,M}\left(\sup_{s\in K_1}|\zeta(s+ikh_1)-f_1(s)|<\varepsilon,\sup_{s\in K_2}|\zeta(s+ikh_2,\alpha)-f_2(s)|<\varepsilon\right)>0. \tag{3}$$

Moreover, "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon>0$.

The arithmetic of α is contained in the linear independence of $L(\mathbb{P};\alpha,h_1,h_2,\pi)$. For example, this is true if h_1 and h_2 are rational numbers, and α and e^π are algebraically independent over $\mathbb{Q}$ [31]. For example, we can take $\alpha = 1/\pi$. We have no information on algebraic irrational α.

Let $\mathbb{H}(\Delta)$ denote the space of analytic functions on the strip Δ equipped with the topology of uniform convergence on compacta. Moreover, let $\mathbb{H}^2(\Delta) = \mathbb{H}(\Delta) \times \mathbb{H}(\Delta)$. For arbitrary α, the following statement is valid.

Theorem 7. *Suppose that the parameter $\alpha \in (0,1) \setminus \{1/2\}$ and*

$$\max\left(h_1^{-1}(Nh_1)^{27/82}, h_2^{-1}(Nh_2)^{27/82}\right) \leqslant M \leqslant \min\left(h_1^{-1}(Nh_1)^{1/2}, h_2^{-1}(Nh_2)^{1/2}\right).$$

Then, there exists a non-empty closed set $F_{\alpha,h_1,h_2} \subset \mathbb{H}^2(\Delta)$ such that, for compact sets $K_1, K_2 \subset \Delta$, $(f_1(s), f_2(s)) \in F_{\alpha,h_1,h_2}$ and $\varepsilon > 0$, inequality (3) holds. Moreover, "$\liminf$" in (3) can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

For the proof of Theorems 6 and 7, we will apply a probabilistic approach based on weakly convergent probability measures in the space $\mathbb{H}^2(\Delta)$. This paper is organized as follows: Section 2 is devoted to approximating in the mean $(\zeta(s), \zeta(s,\alpha))$ in short intervals by a certain pair $(\zeta_n(s), \zeta_n(s,\alpha))$ of absolutely convergent Dirichlet series. In Section 3, we will prove joint limit theorems on weak convergence in those intervals. Theorems 6 and 7 will be proved in Section 4.

2. Discrete Estimates in the Mean

We start with mean square estimates for the functions $\zeta(s)$ and $\zeta(s,\alpha)$ in short intervals. We recall the Gallagher lemma that connects continuous and discrete mean squares of some functions.

Lemma 1 (see [35])**.** *Let $\delta > 0$, $T_0, T \geqslant \delta$, $\mathcal{A}$ be a finite non-empty set of the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and, for $\tau \in \mathcal{A}$,*

$$\mathcal{N}_\delta(\tau) = \sum_{\substack{t \in \mathcal{A} \\ |\tau - t| < \delta}} 1.$$

Suppose that a complex-valued function $S(t)$ is continuous in $I \overset{\text{def}}{=} [T_0, T_0 + T]$ and has a continuous derivative inside I. Then,

$$\sum_{t \in \mathcal{A}} \mathcal{N}_\delta^{-1}(t) |S(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(t)|^2 \, \mathrm{d}t + \left(\int_{T_0}^{T_0+T} |S(t)|^2 \, \mathrm{d}t \int_{T_0}^{T_0+T} |S'(t)|^2 \, \mathrm{d}t \right)^{1/2}.$$

Two next known lemmas are devoted to the mean square of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ in short intervals. We start with a general result for the Riemann zeta function.

Lemma 2 (see Theorem 7.1 of [34])**.** *Let (κ, λ) be an exponential pair and $1/2 < \sigma < 1$ be fixed. Then, for $T^{(\kappa+\lambda+1-2\sigma)/2(\kappa+1)} (\log T)^{(2+\kappa)/(\kappa+1)} \leqslant H \leqslant T$, $1 + \lambda - \kappa \geqslant 2\sigma$, we have, uniformly in H,*

$$\int_{T-H}^{T+H} |\zeta(\sigma + it)|^2 \, \mathrm{d}t \ll H.$$

Proof of Lemma 2 is based on the approximate functional equation for $\zeta(s)$ and mean square estimates for Dirichlet polynomials appearing in it. For this, the theory of exponential pairs is involved, and some results for the divisor function were used.

Lemma 3. *Suppose that* $1/2 < \sigma \leqslant 7/12$ *is fixed, and* $T^{27/82} \leqslant H \leqslant T$*. Then, uniformly in* H*,*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll_\sigma H.$$

Proof of the lemma follows from Lemma 2 with application of the exponential pair $(11/30, 16/30)$. A slightly better result can be given by using the pair $(9/26, 7/13)$ [36]. In this case, $T^{23/70} < H \leqslant T$; however, $1/2 < \sigma \leqslant 8/13$.

The number $27/82$ has an interesting history: it appears in the estimation of the number of zeros of $\zeta(s)$ lying on the critical line $\sigma = 1/2$ (Selberg hypothesis). Let $N_0(T)$ denote the number of zeros $1/2 + i\beta$ of $\zeta(s)$ with $0 < \beta \leqslant T$. In [37], it was obtained that, for any $\varepsilon > 0$ and $H = T^{27/82+\varepsilon}$, there exists $c = c(\varepsilon) > 0$ such that

$$N_0(T+H) - N_0(T) \geqslant cH \log T, \quad T \geqslant T_0(\varepsilon) > 0.$$

Therefore, we prefer the pair $(11/30, 16/30)$.

Lemma 4 (see [38])**.** *Suppose that* $\alpha \in (0,1) \setminus \{1/2\}$ *and* $1/2 < \sigma \leqslant 7/12$ *are fixed, and* $T^{27/82} \leqslant H \leqslant T^\sigma$*. Then, uniformly in* H*,*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it,\alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} H.$$

The proof of the lemma, as of Lemma 3, uses the functional equation for $\zeta(s,\alpha)$ and follows that of Lemma 2 using specific properties of $\zeta(s,\alpha)$; for example, we are limited by the upper bound T^σ for H.

Using Lemmas 1, 3, and 4 leads to discrete mean square estimates in short intervals for $\zeta(s)$ and $\zeta(s,\alpha)$.

Lemma 5. *Suppose that* $1/2 < \sigma \leqslant 7/12$ *is fixed,* $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$ *and* $|t| \leqslant \log^2 Nh$*. Then, uniformly in* M*,*

$$\sum_{k=N}^{N+M} |\zeta(\sigma+it+ikh)|^2 \ll_{\sigma,h} M(1+|t|).$$

Proof. Apply Lemma 1 with $\delta = 1$, $T_0 = N - 1/2$, $T = M + 1/2$, $\mathcal{A} = \{k \in \mathbb{N} : k \in [N, N+M]\}$ and $S(\tau) = \zeta(\sigma + it + i\tau)$. Then, clearly, $N_\delta(k) = 1$. Hence, in view of Lemma 1,

$$\sum_{k=N}^{N+M} |\zeta(\sigma+it+ikh)|^2 \ll \int_{N-1/2}^{N+M+1/2} |\zeta(\sigma+it+ih\tau)|^2 \, \mathrm{d}\tau + \left(\int_{N-1/2}^{N+M+1/2} |\zeta(\sigma+it+ih\tau)|^2 \, \mathrm{d}\tau \int_{N-1/2}^{N+M+1/2} |\zeta'(\sigma+it+ih\tau)|^2 \, \mathrm{d}\tau \right)^{1/2}. \tag{4}$$

We have

$$\int_{N-1/2}^{N+M+1/2} |\zeta(\sigma+it+ih\tau)|^2 \, \mathrm{d}\tau \ll_h \int_{(N-1/2)h-|t|}^{(N+M+1/2)h+|t|} |\zeta(\sigma+iu)|^2 \, \mathrm{d}u. \tag{5}$$

From the hypothesis of the lemma, $(M+1/2)h+|t| \geqslant Mh+|t| \geqslant (Nh)^{27/82}$ and $(M+1/2)h|t| \leqslant (Nh)^{1/2}+h/2+\log^2 Mh \leqslant (Nh)^{\sigma}$. Therefore, Lemmas 3 and (5) give

$$\int\limits_{N-1/2}^{N+M+1/2} |\zeta(\sigma+it+ih\tau)|^2\,\mathrm{d}\tau \ll_h Mh+\frac{h}{2}+|t| \ll_h M(1+|t|). \tag{6}$$

Application of the Cauchy integral formula and (6) leads to the bound

$$\int\limits_{N-1/2}^{N+M+1/2} |\zeta'(\sigma+it+ih\tau)|^2\,\mathrm{d}\tau \ll_h M(1+|t|).$$

Thus, this, (4), and (6) prove the lemma. □

Lemma 6. *Suppose that* $\alpha \in (0,1)\setminus\{1/2\}$ *and* $1/2<\sigma\leqslant 7/12$ *are fixed,* $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$, *and* $|t|\leqslant \log^2 Nh$. *Then, uniformly in* M,

$$\sum_{k=N}^{N+M} |\zeta(\sigma+it+ikh,\alpha)|^2 \ll_{\sigma,h} M(1+|t|).$$

Proof. We repeat the proof of Lemma 5 and use Lemma 4 in place of Lemma 3. □

Now, we introduce two absolutely convergent Dirichlet series. Let $\theta>1/2$ be a fixed number, and, for $n\in\mathbb{N}$,

$$w_n(m)=\exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\}, \quad m\in\mathbb{N},$$

and

$$w_n(m,\alpha)=\exp\left\{-\left(\frac{m+\alpha}{n}\right)^{\theta}\right\}, \quad m\in\mathbb{N}_0.$$

Here and in what follows, $\exp\{a\}$ means e^a. Using $w_n(m)$ and $w_n(m,\alpha)$, define two series

$$\zeta_n(s)=\sum_{m=1}^{\infty}\frac{w_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s,\alpha)=\sum_{m=0}^{\infty}\frac{w_n(m,\alpha)}{(m+\alpha)^s}.$$

In virtue of the exponential decrease with respect to m of the coefficients $w_n(m)$ and $w_n(m,\alpha)$, the latter series are absolutely convergent in any half-plane $\sigma>\sigma_a$ with finite σ_a. Moreover, for $\zeta_n(s)$ [13] and $\zeta_n(s,\alpha)$ [19], the following integral representations

$$\zeta_n(s)=\frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z)l_n(z)\,\mathrm{d}z \quad \text{and} \quad \zeta_n(s,\alpha)=\frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha)l_n(z)\,\mathrm{d}z \tag{7}$$

with $l_n(z)=\theta^{-1}\Gamma(\theta^{-1}z)n^z$ are valid.

We will approximate $\zeta(s)$ and $\zeta(s,\alpha)$ by $\zeta_n(s)$ and $\zeta_n(s,\alpha)$, respectively. For this, we recall the metric in $\mathbb{H}^2(\Delta)$. First we deal with $\mathbb{H}(\Delta)$. It is known [39] that there exists a sequence $\{K_j\}\subset\Delta$ of compact embedded sets such that

$$\Delta=\bigcup_{j=1}^{\infty}K_j,$$

and every compact set $K \subset \Delta$ is in some K_j. For example, we can take closed rectangles with edges parallel to the axis. Then,

$$d(f_1, f_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s \in K_j} |f_1(s) - f_2(s)|}{1 + \sup_{s \in K_j} |f_1(s) - f_2(s)|}, \quad f_1, f_2 \in \mathbb{H}(\Delta),$$

is the metric in $\mathbb{H}(\Delta)$ that induces the topology of uniform convergence on compacta. Now, for $\underline{f}_k = (f_{k1}, f_{k2})$, $k = 1, 2$ taking

$$d_2(\underline{f}_1, \underline{f}_2) = \max(d(f_{11}, f_{12}), d(f_{21}, f_{22})),$$

we obtain the metric in $\mathbb{H}^2(\Delta)$ inducing its product topology.

Now, we are ready to state an important lemma for approximation of $(\zeta(s), \zeta(s, \alpha))$ by $(\zeta_n(s), \zeta_n(s, \alpha))$ in the mean. Let $\underline{h} = (h_1, h_2)$, and

$$\underline{\zeta}(s, \alpha) = (\zeta(s), \zeta(s, \alpha))$$

and

$$\underline{\zeta}_n(s, \alpha) = (\zeta_n(s), \zeta_n(s, \alpha)).$$

Moreover,

$$\underline{\zeta}(s + ik\underline{h}, \alpha) = (\zeta(s + ikh_1), \zeta(s + ikh_2, \alpha))$$

and

$$\underline{\zeta}_n(s + ik\underline{h}, \alpha) = (\zeta_n(s + ikh_1), \zeta_n(s + ikh_2, \alpha)).$$

Lemma 7. *Suppose that $\alpha \in (0, 1) \setminus \{1/2\}$, $h_1 > 0$, $h_2 > 0$ and*

$$\max\left(h_1^{-1}(Nh_1)^{27/82}, h_2^{-1}(Nh_2)^{27/82}\right) \leqslant M \leqslant \min\left(h_1^{-1}(Nh_1)^{1/2}, h_2^{-1}(Nh_2)^{1/2}\right).$$

Then, the equality

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{M+1} \sum_{k=N}^{N+M} d_2\left(\underline{\zeta}(s + ik\underline{h}, \alpha), \underline{\zeta}_n(s + ik\underline{h}, \alpha)\right) = 0$$

holds.

Proof. The definitions of the metrics d_2 and d imply that it suffices to prove the following equalities

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s \in K} |\zeta(s + ikh_1) - \zeta_n(s + ikh_1)| = 0 \tag{8}$$

for $h_1^{-1}(Nh_1)^{27/82} \leqslant M \leqslant h_1^{-1}(Nh_1)^{1/2}$, and

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s \in K} |\zeta(s + ikh_2, \alpha) - \zeta_n(s + ikh_2, \alpha)| = 0 \tag{9}$$

for $h_2^{-1}(Nh_2)^{27/82} \leqslant M \leqslant h_2^{-1}(Nh_2)^{1/2}$ with arbitrary compact set $K \subset \Delta$.

We start with equality (8). A plan of the proof is the following: Using the integral representation (7) and the residue theorem, we find the integral representation for the difference $\zeta_n(s + ikh) - \zeta(s + ikh)$. Summing this differences over $k \in [N, N + H]$, we obtain the integral representation for the sum of the left-hand side in (8) and estimate it by using Lemma 5. More precisely, take a fixed compact set $K \subset \Delta$. Since K is a closed bounded set, there exists $\varepsilon = \varepsilon_K > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for all

$s = \sigma + it \in K$. In the definition of $w_n(m)$, the number $\theta > 1/2$. Now, we take $\theta = 1/2 + \varepsilon$. Define $\theta_1 = 1/2 + \varepsilon - \sigma$. Then, we have $\theta_1 < 0$ and $\theta_1 \geqslant 1/2 - \varepsilon$. This and the integral representation (7) of $\zeta_n(s)$ show that the function $\zeta(s+z)l_n(z)$ has a simple pole at the point $z = 0$ (pole of $l_n(z)$) and a simple pole at the point $z = 1 - s$ (pole of $\zeta(s+z)$) lying in the strip $\theta_1 \leqslant \operatorname{Re} z \leqslant \theta$. Therefore, from the residue theorem and (7), it follows that

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int\limits_{\theta_1 - i\infty}^{\theta_1 + i\infty} \zeta(s+z) l_n(z)\, \mathrm{d}z + l_n(1-s)$$

for all $s \in K$. Hence

$$\zeta_n(s + ikh_1) - \zeta(s + ikh_1) \ll \int\limits_{-\infty}^{\infty} \left|\zeta\left(\tfrac{1}{2} + \varepsilon + it + ikh_1 + i\tau\right)\right| \sup_{s \in K}\left|l_n\left(\tfrac{1}{2} + \varepsilon - \sigma + i\tau\right)\right| \mathrm{d}\tau \\ + |l_n(1 - s - ikh_1)|.$$

Therefore, after shifting $t + \tau$ to τ, we get

$$\sup_{s \in K}|\zeta(s + ikh_1) - \zeta_n(s + ikh_1)| \ll \int\limits_{-\infty}^{\infty} \left|\zeta\left(\tfrac{1}{2} + \varepsilon + ikh_1 + i\tau\right)\right| \sup_{s \in K}\left|l_n\left(\tfrac{1}{2} + \varepsilon - s + i\tau\right)\right| \mathrm{d}\tau \\ + \sup_{s \in K} |l_n(1 - s - ikh_1)|. \tag{10}$$

Taking into account the uniform bound in σ in every finite interval,

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{11}$$

we find, for $s \in K$,

$$l_n\left(\tfrac{1}{2} + \varepsilon - s + i\tau\right) \ll_\varepsilon n^{1/2 + \varepsilon - \sigma} \exp\left\{-\tfrac{c}{\theta}|\tau - t|\right\} \ll_K n^{-\varepsilon} \exp\{-c_1|\tau|\}, \quad c_1 > 0. \tag{12}$$

This, together with

$$\zeta(\sigma + it) \ll_\sigma \sqrt{t}, \quad \sigma \geqslant \frac{1}{2}, \ |t| \geqslant 2,$$

yields

$$\left(\int\limits_{-\infty}^{-\log^2 Nh_1} + \int\limits_{\log^2 Nh_1}^{\infty}\right) \left|\zeta\left(\tfrac{1}{2} + \varepsilon + ikh_1 + i\tau\right)\right| \sup_{s \in K}\left|l_n\left(\tfrac{1}{2} + \varepsilon - s + i\tau\right)\right| \mathrm{d}\tau \\ \ll_{\varepsilon, K} n^{-\varepsilon} \left(\int\limits_{-\infty}^{-\log^2 Nh_1} + \int\limits_{\log^2 Nh_1}^{\infty}\right) \left(\sqrt{kh_1} + \sqrt{|\tau|}\right) \exp\{-c_1|\tau|\}\, \mathrm{d}\tau \\ \ll_{K, h_1} n^{-\varepsilon}\left(1 + \sqrt{kh_1}\right) \exp\left\{-c_2 \log^2 Nh_1\right\}, \quad c_2 > 0.$$

Therefore, in view of (10),

$$
\begin{aligned}
Z &\stackrel{\text{def}}{=} \frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}|\zeta(s+ikh_1)-\zeta_n(s+ikh_1)| \\
&\ll_{K,h_1} \int\limits_{-\log^2 Nh_1}^{\log^2 Nh_1}\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac{1}{2}+\varepsilon+ikh_1+i\tau\right)\right|\right)\sup_{s\in K}\left|l_n\left(\tfrac{1}{2}+\varepsilon-s+i\tau\right)\right|\mathrm{d}\tau \\
&\quad+\frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}|l_n(1-s-ikh_1)|+n^{-\varepsilon}\exp\left\{-c_2\log^2 Nh_1\right\}\frac{1}{M+1}\sum_{k=N}^{N+M}\left(1+\sqrt{kh_1}\right) \\
&\stackrel{\text{def}}{=} J+S_1+S_2.
\end{aligned} \tag{13}
$$

It is easily seen that

$$S_2 \ll n^{-\varepsilon}\exp\left\{-c_2\log^2 Nh_1\right\}\sqrt{2Nh_1}. \tag{14}$$

The Cauchy–Schwarz inequality and Lemma 5 give

$$
\begin{aligned}
\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac{1}{2}+\varepsilon+ikh_1+i\tau\right)\right| &\leqslant \frac{1}{M+1}\sum_{k=N}^{N+M}\left(\left|\zeta\left(\tfrac{1}{2}+\varepsilon++ikh_1+i\tau\right)\right|^2\right)^{1/2} \\
&\ll_{\varepsilon,h_1} (1+|\tau|)^{1/2}
\end{aligned}
$$

for $|\tau|\leqslant\log^2 Nh_1$. Therefore, by (12),

$$J \ll_{K,h_1} n^{-\varepsilon}\int\limits_{-\log^2 Nh_1}^{\log^2 Nh_1}(1+|\tau|)^{1/2}\exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{K,h_1} n^{-\varepsilon}. \tag{15}$$

Moreover, in view of (11), we obtain, for $s\in K$,

$$l_n(1-s-ikh_1) \ll_K n^{1-\sigma}\exp\{-\tfrac{c}{\theta}|t+kh_1|\} \ll_{K,h_1} n^{1/2-\varepsilon}\exp\{-c_3kh_1\},\quad c_3>0.$$

Hence,

$$S_1 \ll_{K,h_1} n^{1/2-\varepsilon}\frac{1}{M+1}\sum_{k=N}^{N+M}\exp\{-c_2kh_1\} \ll_{K,h_1} n^{1/2-\varepsilon}\exp\{-c_3kh_1\}.$$

Therefore, this together with (13)–(15) yields

$$Z \ll_{K,h_1} n^{-\varepsilon}+n^{1/2-\varepsilon}\exp\{-c_3Nh_1\}+n^{-\varepsilon}\exp\{-c_2\log^2 Nh_1\}\sqrt{2Nh_1}.$$

Thus,

$$\lim_{n\to\infty}\limsup_{N\to\infty} Z=0,$$

and the proof of (8) is complete.

The proof of (9) is similar to that of (8) and uses Lemma 6 in place of Lemma 5. □

3. Discrete Probabilistic Statements

This section is devoted to limit theorems in the sense of weakly convergent probability measures in the space $\mathbb{H}^2(\Delta)$. For convenience, we recall the main terminology.

Let $\mathcal{B}(\mathbb{X})$ denote the Borel σ field of the topological space $\mathbb{X}$. Let Q and Q_n, $n\in\mathbb{N}$, be probability measures on the measurable space $(\mathbb{X},\mathcal{B}(\mathbb{X}))$. Then, by the definition, Q_n

converges weakly to Q as $n \to \infty$ or shortly $Q_n \xrightarrow[n\to\infty]{\text{w}} Q$, if, for all real bounded continuous function g on $\mathbb{X}$,

$$\lim_{n\to\infty} \int\limits_{\mathbb{X}} g \, \mathrm{d}Q_n = \int\limits_{\mathbb{X}} g \, \mathrm{d}Q.$$

It is convenient to use equivalents of weak convergence of probability measures in terms of some classes of sets. For the proofs of Theorems 6 and 7, we will apply the following well-known statement [40]:

1° $Q_n \xrightarrow[n\to\infty]{\text{w}} Q \Longleftrightarrow$ For every open set $G \subset \mathbb{X}$,

$$\liminf_{n\to\infty} Q_n(G) \geqslant Q(G).$$

2° Let ∂A denote the boundary of a set A. A set $A \in \mathcal{B}(\mathbb{X})$ is a continuity set of Q if $Q(\partial A) = 0$. $Q_n \xrightarrow[n\to\infty]{\text{w}} Q \Longleftrightarrow$ for every continuity set A of Q,

$$\lim_{n\to\infty} Q_n(A) = Q(A).$$

The main probability measure of this section is

$$P_{N,M,h_1,h_2,\alpha}(A) = C_{N,M}\Big(\underline{\zeta}(s + ik\underline{h}, \alpha) \in A\Big), \quad A \in \mathcal{B}(\mathbb{H}^2(\Delta)).$$

Before a theorem on weak convergence of $P_{N,M,h_1,h_2,\alpha}$ as $N \to \infty$, we will prove some auxiliary results.

Introduce two sets given by Cartesian products. We set

$$\mathbb{T}_1 = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\} \qquad \text{and} \qquad \mathbb{T}_2 = \prod_{m\in\mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

The set $\mathbb{T}_1$ consists of all functions $\mathfrak{t}_1 : \mathbb{P} \mapsto \{s \in \mathbb{C} : |s| = 1\}$, and $\mathbb{T}_2$ is the set of all functions $\mathfrak{t}_2 : \mathbb{N}_0 \mapsto \{s \in \mathbb{C} : |s| = 1\}$. With the product topology and pointwise multiplication, $\mathbb{T}_1$ and $\mathbb{T}_2$ are compact topological groups. Define one more set

$$\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2.$$

Then, $\mathbb{T}$ is again a compact topological group by the classical Tikhonov theorem. On compact groups, the probability Haar measures can be defined. Let μ_1, μ_2, and μ denote the Haar measures on $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}$, respectively. Observe that μ is the product of the Haar measures μ_1 and μ_2 ($\mu = \mu_1 \times \mu_2$, i.e., $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ for all $A_1 \in \mathcal{B}(\mathbb{T}_1)$, $A_2 \in \mathcal{B}(\mathbb{T}_2)$). Moreover, μ is invariant with respect to shifts of elements of $\mathbb{T}$.

For $A \in \mathcal{B}(\mathbb{T})$, define

$$Q_{N,M,h_1,h_2,\alpha}(A) = C_{N,M}\Big(\Big(p^{-ikh_1} : p \in \mathbb{P}\Big), \Big((m+\alpha)^{-ikh_2} : m \in \mathbb{N}_0\Big) \in A\Big).$$

Lemma 8. *Suppose that $\alpha \in (0,1) \setminus \{1/2\}$ and $h_1 > 0$, $h_2 > 0$ are arbitrary fixed numbers, and $M \to \infty$ as $N \to \infty$. Then, on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, there exists a probability measure $Q_{h_1,h_2,\alpha}$ such that $Q_{N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} Q_{h_1,h_2,\alpha}$.*

Proof. We apply the Fourier transform method. Denote by $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2)$ the elements of $\mathbb{T}$, where $\mathfrak{t}_1 = (\mathfrak{t}_{1p} : p \in \mathbb{P})$ and $\mathfrak{t}_2 = (\mathfrak{t}_{2m} : m \in \mathbb{N}_0)$ are elements of $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Then, it is well known that the characters χ of the group $\mathbb{T}$ have the representation

$$\chi(\mathfrak{t}) = \prod_{p\in\mathbb{P}}^{*} \mathfrak{t}_{1p}^{l_{1p}} \prod_{m\in\mathbb{N}_0}^{*} \mathfrak{t}_{2m}^{l_{2m}},$$

where the stars show that only a finite number of integers l_{1p} and l_{2m} are distinct from zero. Hence, it follows that the Fourier transform $\mathcal{F}_{N,M,h_1,h_2,\alpha}(l_1, l_2)$, $l_1 = (l_{1p} : l_{1p} \in \mathbb{Z}, p \in \mathbb{P})$, $l_2 = (l_{2m} : l_{2m} \in \mathbb{Z}, m \in \mathbb{N}_0)$, is given by

$$\mathcal{F}_{N,M,h_1,h_2,\alpha}(l_1, l_2) = \frac{1}{M+1} \int_{\mathbb{T}} \left(\prod_{p\in\mathbb{P}}^{*} \mathfrak{t}_{1p}^{l_{1p}} \prod_{m\in\mathbb{N}_0}^{*} \mathfrak{t}_{2m}^{l_{2m}} \right) \mathrm{d}Q_{N,M,h_1,h_2,\alpha}.$$

Therefore, by the definition of $Q_{N,M,h_1,h_2,\alpha}$, we have

$$\begin{aligned} \mathcal{F}_{N,M,h_1,h_2,\alpha}(l_1, l_2) &= \frac{1}{M+1} \sum_{k=N}^{N+M} \left(\prod_{p\in\mathbb{P}}^{*} p^{-ikh_1 l_{1p}} \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha)^{-ikh_2 l_{2m}} \right) \\ &= \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\left\{ -ik \left(h_1 \sum_{p\in\mathbb{P}}^{*} l_{1p} \log p + h_2 \sum_{m\in\mathbb{N}_0}^{*} l_{2m} \log(m+\alpha) \right) \right\}. \end{aligned} \tag{16}$$

Let

$$A = A(\mathbb{P}; \alpha, h_1, h_2, l_1, l_2) \stackrel{\text{def}}{=} h_1 \sum_{p\in\mathbb{P}}^{*} l_{1p} \log p + h_2 \sum_{m\in\mathbb{N}_0}^{*} l_{2m} \log(m+\alpha).$$

In the case

$$A = 2\pi r, \quad r \in \mathbb{Z},$$

(16) implies

$$\mathcal{F}_{N,M,h_1,h_2,\alpha}(l_1, l_2) = 1. \tag{17}$$

In the opposite case, it follows that

$$\mathcal{F}_{N,M,h_1,h_2,\alpha}(l_1, l_2) = \frac{\exp\{-iNA\} - \exp\{-i(N+M+1)A\}}{(M+1)(1-\exp\{-iA\})}.$$

This, together with (16) and (17), shows that

$$\lim_{N\to\infty} \mathcal{F}_{N,M,h_1,h_2,\alpha}(l_1, l_2) = \begin{cases} 1 & \text{if } A = 2\pi r \text{ with } r \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the group $\mathbb{T}$ is compact, $Q_{N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} Q_{h_1,h_2,\alpha}$, where $Q_{h_1,h_2,\alpha}$ is the probability measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ with the Fourier transform

$$\mathcal{F}_{h_1,h_2,\alpha}(l_1, l_2) = \begin{cases} 1 & \text{if } A = 2\pi r \text{ with } r \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 9. *Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and $M \to \infty$ as $N \to \infty$. Then, $Q_{N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} \mu$.*

Proof. The lemma is a corollary of Lemma 8. If the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, then $A(\mathbb{P},\alpha,h_1,h_2,l_1,l_2) = 2\pi r$ if and only if $l_1 = \underline{0}$, $l_2 = \underline{0}$ and $r = 0$. Thus,

$$\mathcal{F}_{h_1,h_2,\alpha}(l_1,l_2) = \begin{cases} 1 & \text{if } (l_1,l_2) = (\underline{0},\underline{0}), \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the limit measure $Q_{h_1,h_2,\alpha} = \mu$. □

Now, we are ready to consider weak convergence for some probability measures in $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$. First we extend the function $\mathfrak{t}_1$ to the set $\mathbb{N}$ using the formula

$$\mathfrak{t}_1(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \mathfrak{t}_1^l(p), \quad m \in \mathbb{N}.$$

Now, define $v_{n,\alpha} : \mathbb{T} \mapsto \mathbb{H}^2(\Delta)$ by the formula

$$v_{n,\alpha}(\mathfrak{t}) = \underline{\zeta}_n(s,\mathfrak{t},\alpha),$$

where

$$\underline{\zeta}_n(s,\mathfrak{t},\alpha) = (\zeta_n(a,\mathfrak{t}_1), \zeta_n(s,\mathfrak{t}_2,\alpha)),$$

and

$$\zeta_n(s,\mathfrak{t}_1) = \sum_{m+1}^{\infty} \frac{\mathfrak{t}_1(m) w_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s,\mathfrak{t}_2,\alpha) = \sum_{m=0}^{\infty} \frac{\mathfrak{t}_2(m) w_n(m,\alpha)}{(m+\alpha)^s}.$$

Since $|\mathfrak{t}_1(m)| = |\mathfrak{t}_2(m)| = 1$, the series for $\zeta_n(s,\mathfrak{t}_1)$ and $\zeta_n(s,\mathfrak{t}_2,\alpha)$ are absolutely convergent for $\sigma > \sigma_a$, as for $\zeta_n(s)$ and $\zeta_n(s,\alpha)$. From this, the continuity of $v_{n,\alpha}(\mathfrak{t})$ follows. Thus, the map $v_{n,\alpha}(\mathfrak{t})$ is $(\mathcal{B}(\mathbb{T}), \mathcal{B}(\mathbb{H}^2(\Delta)))$-measurable. Therefore, the probability measure $Q_{h_1,h_2,\alpha}$ of Lemma 8 implies, on $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$, the unique probability measure $\widehat{Q}_{n,h_1,h_2,\alpha}$ given by

$$\widehat{Q}_{n,h_1,h_2,\alpha}(A) = Q_{h_1,h_2,\alpha} v_{n,\alpha}^{-1}(A) = Q_{h_1,h_2,\alpha}(v_{n,\alpha}^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{H}^2(\Delta)).$$

For $A \in \mathcal{B}(\mathbb{H}^2(\Delta))$, set

$$\widehat{Q}_{n,N,M,h_1,h_2,\alpha}(A) = C_{N,M}\Big(\underline{\zeta}_n(s+ik\underline{h},\alpha) \in A\Big).$$

Lemma 10. *Suppose that $\alpha \in (0,1) \setminus \{1/2\}$ and h_1, h_2 are arbitrary positive numbers, and $M \to \infty$ as $N \to \infty$. Then, $\widehat{Q}_{n,N,M,h_1 h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} \widehat{Q}_{n,h_1,h_2,\alpha}$.*

Proof. The definitions of the measures $\widehat{Q}_{n,N,M,h_1,h_2,\alpha}$ and $\widehat{Q}_{n,h_1,h_2,\alpha}$ and the map $v_{n,\alpha}$ show that, for $A \in \mathcal{B}(\mathbb{H}^2(\Delta))$,

$$\begin{aligned} v_{n,\alpha}\Big(\Big(p^{-ikh_1} : p \in \mathbb{P}\Big), \Big((m+\alpha)^{-ikh_2} : m \in \mathbb{N}_0\Big)\Big) \\ = \left(\sum_{m=1}^{\infty} \frac{m^{-ikh_1} w_n(m)}{m^s}, \sum_{m=0}^{\infty} \frac{(m+\alpha)^{-ikh_2} w_n(m,\alpha)}{(m+\alpha)^s}\right) \\ = (\zeta_n(s+ikh_1), \zeta_n(s+ikh_2,\alpha)) = \underline{\zeta}_n(s+ik\underline{h},\alpha); \end{aligned}$$

thus,

$$\widehat{Q}_{n,N,M,h_1,h_2,\alpha} = Q_{h_1,h_2,\alpha} v_{n,\alpha}^{-1}. \tag{18}$$

Since the map $v_{n,\alpha}$ is continuous, it is possible to apply Theorem 5.1 of [40] for preservation of weak convergence under mappings. Therefore, Lemma 8 and (18) give the relation

$$\widehat{Q}_{n,N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} Q_{h_1,h_2,\alpha} v_{n,\alpha}^{-1} = \widehat{Q}_{n,h_1,h_2,\alpha}.$$

□

From Lemmas 9 and 10, the next corollary follows.

Corollary 1. *Suppose that $\alpha \in (0,1) \setminus \{1/2\}$ and h_1, h_2 are arbitrary positive numbers, the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, and $M \to \infty$ as $N \to \infty$. Then, $\widehat{Q}_{n,N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} \mu v_{n,\alpha}^{-1}$.*

The measure $\widehat{Q}_{n,h_1,h_2,\alpha}$ is very important for the future. At least, we need the weak convergence for some subsequent $\widehat{Q}_{n_r,h_1,h_2,\alpha}$ with $n_r \to \infty$ as $r \to \infty$. For this, we will utilize the notation of tightness.

Lemma 11. *The probability measure $\widehat{Q}_{n,h_1,h_2,\alpha}$ is tight; i.e., for every $\varepsilon > 0$, there is a compact set $K \subset \mathbb{H}^2(\Delta)$ such that*

$$\widehat{Q}_{n,h_1,h_2,\alpha}(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

Proof. Let $\widehat{Q}^{(1)}_{n,h_1}$ and $\widehat{Q}^{(2)}_{n,h_2,\alpha}$ be marginal measures of $\widehat{Q}_{n,h_1,h_2,\alpha}$, i.e.,

$$\widehat{Q}^{(1)}_{n,h_1}(A) = \widehat{Q}_{n,h_1,h_2,\alpha}(A \times \mathbb{H}(\Delta)), \quad A \in \mathcal{B}(\mathbb{H}(\Delta)),$$

and

$$\widehat{Q}^{(2)}_{n,h_2,\alpha}(A) = \widehat{Q}_{n,h_1,h_2,\alpha}(\mathbb{H}(\Delta) \times A), \quad A \in \mathcal{B}(\mathbb{H}(\Delta)).$$

Moreover, from Lemma 10, we have the relations

$$\widehat{Q}^{(1)}_{n,N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} \widehat{Q}^{(1)}_{n,h_1,h_2,\alpha},$$

and

$$\widehat{Q}^{(2)}_{n,N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} \widehat{Q}^{(2)}_{n,h_1,h_2,\alpha},$$

where

$$\widehat{Q}^{(1)}_{n,N,M,h_1,h_2,\alpha}(A) = \widehat{Q}_{n,N,M,h_1,h_2,\alpha}(A \times \mathbb{H}(\Delta)), \quad A \in \mathcal{B}(\mathbb{H}(\Delta)),$$

and

$$\widehat{Q}^{(2)}_{n,N,M,h_1,h_2,\alpha}(A) = \widehat{Q}_{n,N,M,h_1,h_2,\alpha}(\mathbb{H}(\Delta) \times A), \quad A \in \mathcal{B}(\mathbb{H}(\Delta)).$$

Using Lemma 5, the Cauchy integral formula, and (8), we find that, for a compact set K_j from the definition of the metric d,

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K_j} |\zeta_n(s+ikh_1)| \\
&\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \left(\sum_{k=N}^{N+M} \sup_{s\in K_j} |\zeta(s+ikh_1)| + \sum_{k=N}^{N+M} \sup_{s\in K_j} |\zeta(s+ikh_1) - \zeta_n(s+ikh_1)| \right) \\
&\leqslant C_{1j} < \infty.
\end{aligned} \tag{19}$$

Similarly, from Lemma 6 and (9), we obtain, for $K_j \in \mathbb{H}(\Delta)$,

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K_j} |\zeta_n(s+ikh_2,\alpha)| \leqslant C_{2j} < \infty. \tag{20}$$

On a certain probability space $(\Omega, \mathcal{A}, \nu)$, define the random variable η_{N,M,h_1} with the distribution

$$\nu\{\eta_{N,M,h_1} = kh_1\} = \frac{1}{M+1}, \quad k = N, \dots, N+M.$$

Suppose that the $\mathbb{H}(\Delta)$-valued random element X_{n,N,M,h_1} is given by

$$X_{n,N,M,h_1} = X_{n,N,M,h_1}(s) = \zeta_n(s + i\eta_{N,M,h_1}).$$

Then, by virtue of Lemma 10,

$$X_{n,N,M,h_1} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,h_1},$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution, and X_{n,h_1} has the distribution $\widehat{Q}^{(1)}_{n,h_1}$. Hence, from the topology of $\mathbb{H}(\Delta)$,

$$\sup_{s\in K_j} \left| X_{n,N,M,h_1}(s) \right| \xrightarrow[N\to\infty]{\mathcal{D}} \sup_{s\in K_j} \left| X_{n,h_1}(s) \right|. \tag{21}$$

Fix $\varepsilon > 0$, and take $R_j = 2^j \varepsilon^{-1} C_{1j}$. Then, (19) and (21) imply

$$\begin{aligned} \nu\left\{ \sup_{s\in K_j} \left| X_{n,h_1}(s) \right| \geqslant R_j \right\} &= \lim_{N\to\infty} \nu\left\{ \sup_{s\in K_j} \left| X_{n,N,M,h_1}(s) \right| \geqslant R_j \right\} \\ &\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K_j} |\zeta_n(s+ikh_1)| \leqslant \frac{\varepsilon}{2^j}. \end{aligned} \tag{22}$$

Define

$$K = K_\varepsilon = \left\{ g \in \mathbb{H}(\Delta) : \sup_{s\in K_j} |g(s)| \leqslant R_j, j \in \mathbb{N} \right\}.$$

Then, K_ε is a compact set of the space $\mathbb{H}(\Delta)$, and, from (22),

$$\nu\{X_{n,h_1} \in K_\varepsilon\} = 1 - \nu\{X_{n,h_1} \notin K_\varepsilon\} \geqslant 1 - \varepsilon \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This shows that $\widehat{Q}^{(1)}_{n,h_1}$ is tight.

From the same arguments, using (20) and the random variable η_{N,M,h_2} with distribution

$$\nu\{\eta_{N,M,h_2} = kh_2\} = \frac{1}{M+1}, \quad k = N, \dots, N+M,$$

we obtain that the measure $\widehat{Q}^{(2)}_{n,h_2,\alpha}$ is tight.

The tightness of $\widehat{Q}^{(1)}_{n,h_1}$ and $\widehat{Q}^{(2)}_{n,h_2,\alpha}$ implies that, for every $\varepsilon > 0$, there exist compact sets $K_1, K_2 \subset \mathbb{H}(\Delta)$ such that, for all $n \in \mathbb{N}$,

$$\widehat{Q}^{(1)}_{n,h_1}(\mathbb{H}(\Delta) \setminus K_1) < \frac{\varepsilon}{2} \quad \text{and} \quad \widehat{Q}^{(2)}_{n,h_2}(\mathbb{H}(\Delta) \setminus K_2) < \frac{\varepsilon}{2}.$$

Let $K = K_1 \times K_2$. Then, K is a compact set in $\mathbb{H}^2(\Delta)$. Moreover,

$$\widehat{Q}_{n,h_1,h_2,\alpha}(\mathbb{H}^2(\Delta) \setminus K) = \widehat{Q}_{n,h_1,h_2,\alpha}\left(\bigcup_{l=1}^{2}((\mathbb{H}(\Delta) \setminus K_l) \times \mathbb{H}(\Delta))\right)$$
$$\leqslant \widehat{Q}^{(1)}_{n,h_1}(\mathbb{H}(\Delta) \setminus K_1) + \widehat{Q}^{(2)}_{n,h_2,\alpha}(\mathbb{H}(\Delta) \setminus K_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \in \mathbb{N}$. Hence,

$$\widehat{Q}_{n,h_1,h_2,\alpha}(K) = 1 - \widehat{Q}_{n,h_1,h_2,\alpha}(\mathbb{H}^2(\Delta) \setminus K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$, and the proof of the lemma is complete. □

Now, we are ready to prove a limit theorem for the measure $P_{N,M,h_1,h_2,\alpha}$ as $N \to \infty$. For this, the following lemma plays an important role.

Lemma 12 (see [40], Theorem 4.2)**.** *Suppose that $(\mathbb{X}, \rho)$ is a separable metric space, and the $\mathbb{X}$-valued random elements η_{nm} and ξ_n, $m, n \in \mathbb{N}$, are defined on the same probability space as measure ν. If*

$$\eta_{nm} \xrightarrow[n\to\infty]{\mathcal{D}} \eta_m, \qquad \eta_m \xrightarrow[m\to\infty]{\mathcal{D}} \eta,$$

and, for every $\delta > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \nu\{\rho(\eta_{nm}, \xi_n) \geqslant \delta\} = 0,$$

then $\xi_n \xrightarrow[n\to\infty]{\mathcal{D}} \eta$.

The main result in this section is the following statement.

Theorem 8. *Suppose that $\alpha \in (0,1) \setminus \{1/2\}$, $h_1 > 0$, $h_2 > 0$ are arbitrary fixed numbers, and*

$$\max\left(h_1^{-1}(Nh_1)^{27/82}, h_2^{-1}(Nh_2)^{27/82}\right) \leqslant M \leqslant \min\left(h_1^{-1}(Nh_1)^{1/2}, h_2^{-1}(Nh_2)^{1/2}\right).$$

Then, on $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$, there exists a probability measure $P_{h_1,h_2,\alpha}$ such that $P_{N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} P_{h_1,h_2,\alpha}$.

Proof. For the proof, we apply a scheme of the statement of Lemma 11. This is possible due to Lemmas 7 and 10.

On a certain probability space with measure ν, define the two-dimensional random variable $\underline{\eta}_{N,M}$ with the distribution

$$\nu\left\{\underline{\eta}_{N,M} = (kh_1, kh_2)\right\} = \frac{1}{M+1}, \quad k = N, \dots, N+M.$$

Moreover, let the $\mathbb{H}^2(\Delta)$-valued random element $X_{n,N,M}$ be given by

$$X_{n,N,M} = X_{n,N,M,h_1,h_2,\alpha}(s) = \underline{\zeta}_n(s + i\underline{\eta}_{N,M}, \alpha),$$

and $\widehat{Q}_{n,h_1,h_2,\alpha}$ is the distribution of $X_n = X_{n,h_1,h_2}(s)$. Then, in view of Lemma 10, the relation

$$X_{n,N,M} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{23}$$

Since, from Lemma 11, the measure $\widehat{Q}_{n,h_1,h_2,\alpha}$ is tight, from Prokhorov theorem (see [40], Theorem 6.1), it is relatively compact. This means that there is a subsequent n_r such that

$\widehat{Q}_{n_r,h_1,h_2,\alpha}$ converges weakly to a certain probability measure $Q_{h_1,h_2,\alpha}$ on $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$ as $r \to \infty$. This fact can be written in the form

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} Q_{h_1,h_2,\alpha}. \tag{24}$$

Introduce one more $\mathbb{H}^2(\Delta)$-valued random element:

$$Y_{N,M} = Y_{N,M,h_1,h_2,\alpha}(s) = \underline{\zeta}(s + i\underline{\eta}_{N,M}, \alpha).$$

Then, an application of Lemma 7 leads, for $\varepsilon > 0$, to

$$\begin{aligned}
&\lim_{r\to\infty} \limsup_{N\to\infty} \nu\{d_2(Y_{N,M}, X_{n_r,N,M}) \geqslant \varepsilon\} \\
&\quad = \lim_{r\to\infty} \limsup_{N\to\infty} C_{N,M}\left\{d_2\left(\underline{\zeta}(s + ik\underline{h}, \alpha), \underline{\zeta}_r(s + ik\underline{h}, \alpha)\right) \geqslant \varepsilon\right\} \\
&\quad \leqslant \lim_{r\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(M+1)} \sum_{k=N}^{N+M} d_2\left(\underline{\zeta}(s + ik\underline{h}, \alpha), \underline{\zeta}_{n_r}(s + ik\underline{h}, \alpha)\right) = 0.
\end{aligned}$$

This equality, together with relations (23) and (24), allows us to apply Lemma 12. Thus, we have

$$Y_{N,M} \xrightarrow[N\to\infty]{\mathcal{D}} Q_{h_1,h_2,\alpha},$$

and this is an equivalent of the statement of the theorem. □

Now, on the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$, define the $\mathbb{H}^2(\Delta)$-valued random element $\underline{\zeta}(s, \mathfrak{t}, \alpha)$ by

$$\underline{\zeta}(s, \mathfrak{t}, \alpha) = (\zeta(s, \mathfrak{t}_1), \zeta(s, \mathfrak{t}_2, \alpha))$$

where

$$\zeta(s, \mathfrak{t}_1) = \sum_{m=1}^{\infty} \frac{\mathfrak{t}_1(m)}{m^s} \qquad \text{and} \qquad \zeta(s, \mathfrak{t}_2, \alpha) = \sum_{m=0}^{\infty} \frac{\mathfrak{t}_2(m)}{(m+\alpha)^s}.$$

Notice that the latter series, for almost all $\mathfrak{t}_1$ with respect to μ_1 and $\mathfrak{t}_2$ with respect μ_2, respectively, converge uniformly on compact subsets of the strip Δ, and define the $\mathbb{H}(\Delta)$-valued random elements [13,19]. Let $P_{\underline{\zeta}}$ denote the distribution of the random element $\underline{\zeta}(s, \mathfrak{t}, \alpha)$, i.e., $P_{\underline{\zeta}}$ is a probability measure on $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$ given by

$$P_{\underline{\zeta}}(A) = \mu\left\{\mathfrak{t} \in \mathbb{T} : \underline{\zeta}(s, \mathfrak{t}, \alpha) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{H}^2(\Delta)).$$

Theorem 9. *Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and*

$$\max\left(h_1^{-1}(Nh_1)^{27/82}, h_2^{-1}(Nh_2)^{27/82}\right) \leqslant M \leqslant \min\left(h_1^{-1}(Nh_1)^{1/2}, h_2^{-1}(Nh_2)^{1/2}\right).$$

Then $P_{N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} P_{\underline{\zeta}}$.

Proof. For $A \in \mathcal{B}(\mathbb{H}^2(\Delta))$, define

$$P_N(A) = P_{N,h_1,h_2,\alpha}(A) = C_N\left(\underline{\zeta}(s + ik\underline{h}, \alpha) \in A\right).$$

For the proof of Theorem 4 in [31], the weak convergence of P_N as $N \to \infty$ was obtained ([31] Theorem 4). During this process, it was obtained that P_N as $n \to \infty$, and, in our notation, $\widehat{Q}_{n,h_1,h_2,\alpha}$ as $n \to \infty$ has the same limit measure. This measure is $P_{\underline{\zeta}}$. Since $\widehat{Q}_{n,h_1,h_2,\alpha}$ is independent on M, the above remarks show that the measure $Q_{h_1,h_2,\alpha}$ in the proof of Theorem 8 coincides with $P_{\underline{\zeta}}$. Thus, it remains to repeat the proof of Theorem 8 with $P_{\underline{\zeta}}$ intead of $Q_{h_1,h_2,\alpha}$. □

4. Proofs of Approximation Theorems

Proofs of Theorems 6 and 7 are based on limit Theorems 8 and 9, respectively. Moreover, we use a notion of the support of probability measures on the space $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$. Since the space $\mathbb{H}^2(\Delta)$ is separable, the support of a probability measure P on $(\mathbb{H}^2(\Delta), \mathcal{B}(\mathbb{H}^2(\Delta)))$ is a minimal closed set $S_P \subset \mathbb{H}^2(\Delta)$ such that $P(S_P) = 1$. The set S_P consists of all elements $\underline{g} \in \mathbb{H}^2(\Delta)$ such that $P(G_{\underline{g}}) > 0$ for every open neighborhood $G_{\underline{g}}$ of $\underline{g}$.

Lemma 13 (see [31], Lemma 11)**.** *The support of measure $P_{\underline{\zeta}}$ is the set $S \times \mathbb{H}(\Delta)$, where $S = \{g \in \mathbb{H}(\Delta) : g(s) \neq 0$ on D, or $g(s) \equiv 0\}$.*

For the proof of Theorem 6, we also apply the Mergelyan theorem for the approximation of analytic functions by polynomials, which was mentioned in the Introduction. For convenience, we state this theorem.

Lemma 14 (see [1,2])**.** *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complements, and $g(s)$ is a continuous on K function that is analytic inside of K. Then, for every $\varepsilon > 0$, there is a polynomial $p_{\varepsilon,g}(s)$ such that*

$$\sup_{s\in K}\left|g(s) - p_{\varepsilon,g}(s)\right| < \varepsilon.$$

Proof of Theorem 6. Since $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in \mathcal{H}_0(K)$, $f_2(s) \in \mathcal{H}(K)$, Lemma 14 can be applied. Thus, there are polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1}\left|f_1(s) - e^{p_1(s)}\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{s\in K_2}|f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{25}$$

Consider the set

$$\mathcal{G}_\varepsilon = \left\{(g_1 g_2) \in \mathbb{H}^2(\Delta) : \sup_{s\in K_1}\left|f_1(s) - e^{p_1(s)}\right| < \frac{\varepsilon}{2}, \sup_{s\in K_2}|f_2(s) - p_2(s)| < \frac{\varepsilon}{2}\right\}.$$

In view of Lemma 13, $\left(e^{p_1(s)}, p_2(s)\right)$ is an element of the support of the measure $P_{\underline{\zeta}}$. Thus, $\mathcal{G}_\varepsilon$ is an open neighborhood of an element of the support of $P_{\underline{\zeta}}$; hence,

$$P_{\underline{\zeta}}(\mathcal{G}_\varepsilon) > 0. \tag{26}$$

Introduce one more set:

$$\widehat{\mathcal{G}}_\varepsilon = \left\{(g_1, g_2) \in \mathbb{H}^2(\Delta) : \sup_{s\in K_1}|g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2}|g_2(s) - f_2(s)| < \varepsilon\right\}.$$

Then, $\mathcal{G}_\varepsilon \subset \widehat{\mathcal{G}}_\varepsilon$. Actually, let $(g_1, g_2) \in \mathcal{G}_\varepsilon$. Then, taking into account (25), we have

$$\sup_{s\in K_1}|g_1(s) - f_1(s)| \leqslant \sup_{s\in K_1}\left|g_1(s) - e^{p_1(s)}\right| + \sup_{s\in K_1}\left|f_1(s) - e^{p_1(s)}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and

$$\sup_{s\in K_2}|g_2(s) - f_2(s)| \leqslant \sup_{s\in K_2}|g_2(s) - p_2(s)| + \sup_{s\in K_2}|f_2(s) - p_2(s)| < \varepsilon.$$

This shows that $(g_1, g_2) \in \widehat{\mathcal{G}}_\varepsilon$. The inclusion $\mathcal{G}_\varepsilon \subset \widehat{\mathcal{G}}_\varepsilon$ and (26) imply the inequality

$$P_{\underline{\zeta}}(\widehat{\mathcal{G}}_\varepsilon) > 0. \tag{27}$$

Thus, the application of Theorem 9 for the open set $\widehat{\mathcal{G}}_\varepsilon$ yields

$$\liminf_{N\to\infty} P_{N,M,h_1,h_2,\alpha}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_{\underline{\zeta}}(\widehat{\mathcal{G}}_\varepsilon) > 0,$$

and the definitions of $P_{N,M,h_1,h_2,\alpha}$ and $\widehat{\mathcal{G}}_\varepsilon$ lead to the inequality

$$\liminf_{N\to\infty} C_{N,M}\left(\sup_{s\in K_1}|f_1(s)-\zeta(s+ikh_1)|<\varepsilon, \sup_{s\in K_2}|f_2(s)-\zeta(s+ikh_2,\alpha)|<\varepsilon\right) > 0.$$

For the proof of the second statement of the theorem, we apply Theorem 9 in terms of continuity sets. The boundary of $\partial\widehat{\mathcal{G}}_\varepsilon$ of the set $\widehat{\mathcal{G}}_\varepsilon$ lies in the set

$$\begin{aligned}&\left\{(g_1,g_2)\in\mathbb{H}^2(\Delta): \sup_{s\in K_1}|g_1(s)-f_1(s)|=\varepsilon, \sup_{s\in K_2}|g_2(s)-f_2(s)|=\varepsilon\right\}\\ &\quad\bigcup\left\{(g_1,g_2)\in\mathbb{H}^2(\Delta): \sup_{s\in K_1}|g_1(s)-f_1(s)|<\varepsilon, \sup_{s\in K_2}|g_2(s)-f_2(s)|=\varepsilon\right\}\\ &\quad\bigcup\left\{(g_1,g_2)\in\mathbb{H}^2(\Delta): \sup_{s\in K_1}|g_1(s)-f_1(s)|=\varepsilon, \sup_{s\in K_2}|g_2(s)-f_2(s)|<\varepsilon\right\}.\end{aligned}$$

Therefore, $\partial\widehat{\mathcal{G}}_{\varepsilon_1}$ and $\partial\widehat{\mathcal{G}}_{\varepsilon_2}$ do not intersect for $\varepsilon_1 \neq \varepsilon_2$. From this remark, it follows that the set $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Hence, from Theorem 9 in terms of continuity sets, the limit

$$\lim_{n\to\infty} C_{N,M}\left(\sup_{s\in K_1}|f_1(s)-\zeta(s+ikh_1)|<\varepsilon, \sup_{s\in K_2}|f_2(s)-\zeta(s+ikh_2,\alpha)|<\varepsilon\right) = P_{\underline{\zeta}}(\widehat{\mathcal{G}}_\varepsilon)$$

exists and, from (26), is positive for all but at most countably many $\varepsilon > 0$. The proof is complete. □

Proof of Theorem 7. From Theorem 8, the relation $P_{N,M,h_1,h_2,\alpha} \xrightarrow[N\to\infty]{\text{w}} P_{h_1,h_2,\alpha}$ holds. Let $F_{h_1,h_2,\alpha}$ denote the support of the limit measure $P_{h_1,h_2,\alpha}$. Then, $F_{h_1,h_2,\alpha}$ is a non-empty closed subset of $H^2(\Delta)$. Let $\widehat{\mathcal{G}}_\varepsilon$ be the set from the proof of Theorem 6. Since $(f_1, f_2) \in F_{h_1,h_2,\alpha}$, the set $\widehat{\mathcal{G}}_\varepsilon$ is an open neighborhood of the support of the measure $P_{h_1,h_2,\alpha}$. Therefore, from a support property,

$$P_{h_1,h_2,\alpha}(\widehat{\mathcal{G}}_\varepsilon) > 0.$$

Hence, from Theorem 8 in terms of open sets,

$$\liminf_{N\to\infty} P_{N,M,h_1,h_2,\alpha}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_{h_1,h_2,\alpha}(\widehat{\mathcal{G}}_\varepsilon) > 0.$$

This inequality proves the first statement of the theorem.

The second statement of the theorem follows from the same lines as that of Theorem 6 by using Theorem 8 in terms of continuity sets and the fact that $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_{h_1,h_2,\alpha}$ for all but possibly at most countably many $\varepsilon > 0$. The theorem is proven. □

5. Conclusions

We obtained theorems on the approximation of a pair of analytic functions defined on a strip $\Delta = \{s \in \mathbb{C} : 1/2 < \text{Re}s < 1\}$ by the Riemann and Hurwitz zeta functions using discrete shifts $s + ikh_1$ and $s + ikh_2$, $k \in \mathbb{N}$, $h_1 > 0$, $h_2 > 0$, respectively, in short intervals. We discussed the lower density (and density) of approximating shifts in the interval $[N, N + M]$ with

$$\max\Big(h_1^{-1}(Nh_1)^{27/82}, h_2^{-1}(Nh_2)^{27/82}\Big) \leqslant M \leqslant \min\Big(h_1^{-1}(Nh_1)^{1/2}, h_2^{-1}(Nh_2)^{1/2}\Big)$$

as $N \to \infty$. Two cases were examined. Let $\mathbb{H}^2(\Delta)$ be two-dimensional space of analytic functions on the strip D. For arbitrary parameter α of the Hurwitz zeta function and arbitrary positive h_1 and h_2, we obtained only the existence of a certain subset of $\mathbb{H}^2(\mathcal{D})$, the functions of which were approximated by the above shifts. In the case, when the multiset $\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m + \alpha) : m \in \mathbb{N}_0), 2\pi\}$ is linearly independent over the field of rational numbers, the set of approximated functions coincides with the set $\{g \in \mathbb{H}(\Delta) : g(s) \neq 0 \text{ or } g(s) \equiv 0\} \times \mathbb{H}(\Delta)$.

Proofs of the results are closely connected to mean square estimates for the Riemann and Hurwitz zeta functions in short intervals. The problem remains to obtain the above results with a smaller lower bound for M. Having in mind the discrete universality for Riemann zeta function with arbitrary h_1 and the universality of Hurwitz zeta function with arbitrary parameter, we believe that the set of approximated functions may be identified in the general case as well.

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References

1. Mergelyan, S.N. Uniform approximations of functions of a complex variable. *Uspehi Matem. Nauk* **1952**, *2*, 31–122.
2. Mergelyan, S.N. Uniform approximations to functions of a complex variable. In *American Mathematical Society Translations*; no. 101; American Mathematical Society: Providence, RI, USA, 1954.
3. Walsh, J.L. *Interpolation and Approximation by Rational Functions in the Complex Domain*; American Mathematical Society Colloquium Publications; American Mathematical Society: Providence, RI, USA, 1960; Volume 20.
4. Riemann, B. Über die Anzahl der Primzahlen unterhalb einer gegebenen Grösse. *Monatsber. Preuss. Akad. Wiss. Berl.* **1859**, 671–680.
5. Hilbert, D. Mathematische Probleme. *Arch. Math. Phys.* **1901**, *1*, 44–63; 213–317.
6. Hilbert, D. Mathematical problems. *Bull. Amer. Math. Soc.* **1902**, *8*, 437–479. [CrossRef]
7. The Millennium Prize Problems. Available online: https://www.claymath.org/millennium-problems/ (accessed on 10 July 2025).
8. Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.* **1975**, *9*, 443–453. [CrossRef]
9. Voronin, S.M. A theorem on the distribution of values of the Riemann zeta-function. *Sov. Math. Dokl.* **1975**, *16*, 410.
10. Voronin, S.M. Analytic Properties of Arithmetic Objects. Ph.D. Thesis, V.A. Steklov Mathematical Institute, Moscow, Russia, 1977.
11. Karatsuba, A.A.; Voronin, S.M. *The Riemann Zeta-Function*; Walter de Gruiter: Berlin, Germany; New York, NY, USA, 1992.
12. Voronin, S.M. *Selected Works: Mathematics*; Karatsuba, A.A., Ed.; Moscow State Technical University Press: Moscow, Russia, 2006.
13. Bagchi, B. The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series. Ph.D. Thesis, Indian Statistical Institute, Calcutta, India, 1981.
14. Gonek, S.M. Analytic Properties of Zeta and L-Functions. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA, 1979.
15. Steuding, J. *Value-Distribution of L-Functions*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007; Volume 1877.

16. Matsumoto, K. A survey on the theory of universality for zeta and L-functions. In *Number Theory: Plowing and Starring Through High Wave Forms, Proceedings of the 7th China-Japan Seminar (Fukuoka 2013), Fukuoka, Japan, 28 October–1 November 2013*; Series on Number Theory and Its Applications; Kaneko, M., Kanemitsu, S., Liu, J., Eds.; World Scientific Publishing Co.: Singapore, 2015; pp. 95–144.
17. Kowalski, E. *An Introduction to Probabilistic Number Theory*; Cambridge University Press: Cambridge, UK, 2021.
18. Mauclaire, J.-L. Universality of the Riemann zeta function: Two remarks. *Ann. Univ. Sci. Budap. Rolando Eötvös Sect. Comput.* **2013**, *39*, 311–319. [CrossRef]
19. Laurinčikas, A.; Garunkštis, R. *The Lerch Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2002.
20. Sourmelidis, A.; Steuding, J. On the value distribution of Hurwitz zeta-function with algebraic irrational parameter. *Constr. Approx.* **2022**, *55*, 829–860. [CrossRef]
21. Voronin, S.M. The differential independence of ζ-functions. *DAN SSSR* **1973**, *209*, 1264–1266. (In Russian)
22. Voronin, S.M. On the functional independence of Dirichlet L-functions. *Acta Arith.* **1975**, *27*, 443–453. (In Russian)
23. Mishou, H. The joint value-distribution of the Riemann zeta function and Hurwitz zeta functions. *Lith. Math. J.* **2007**, *47*, 32–47. [CrossRef]
24. Laurinčikas, A. Joint approximation by the Riemann and Hurwitz zeta-functions in short intervals. *Symmetry* **2024**, *16*, 1707. [CrossRef]
25. Laurinčikas, A. Universality of the Riemann zeta-function in short intervals. *J. Number Theory* **2019**, *204*, 279–295. [CrossRef]
26. Andersson, J.; Garunkštis, R.; Kačinskaitė, R.; Nakai, K.; Pańkowski, Ł.; Sourmelidis, A.; Steuding, R.; Steuding, J.; Wananiyakul, S. Notes on universality in short intervals and exponential shifts. *Lith. Math. J.* **2024**, *64*, 125–137. [CrossRef]
27. Laurinčikas, A. Universality of the Hurwitz zeta-function in short intervals. *Bol. Soc. Mat. Mex.* **2025**, *31*, 17. [CrossRef]
28. Reich, A. Werteverteilung von Zetafunktionen. *Arch. Math.* **1980**, *34*, 440–451. [CrossRef]
29. Sander, J.; Steuding, J. Joint universality for Euler products of Dirichlet L-functions. *Analysis* **2006**, *26*, 295–312. [CrossRef]
30. Laurinčikas, A. A discrete universality theorem for the Hurwitz zeta-function. *J. Number Theory* **2014**, *143*, 232–247. [CrossRef]
31. Buivydas, E.; Laurinčikas, A. A generalized joint discrete universality theorem for Riemann and Hurwitz zeta-functions. *Lith. Math. J.* **2015**, *55*, 193–206. [CrossRef]
32. Sourmelidis, A. Continuous and discrete universality of zeta-functions: Two sides of the same coin? *Proc. Amer. Math. Soc.* **2025**, *153*, 1435–1445. [CrossRef]
33. Laurinčikas, A. Discrete universality of the Riemann zeta-function in short intervals. *Appl. Anal. Discrete Math.* **2020**, *14*, 382–405. [CrossRef]
34. Ivič, A. *The Riemann Zeta-Function*; John Wiley & Sons: New York, NY, USA, 1985.
35. Montgomery, H.L. *Topics in Multiplicative Number Theory*; Lecture Notes Math; Springer: Heidelberg/Berlin, Germany, 1971; Volume 227.
36. Heath-Brown, D.R. A new kth derivative estimate for exponential sums via Vinogradov's mean value. *Proc. Steklov Inst. Math.* **2017**, *296*, 88–103. [CrossRef]
37. Karatsuba, A.A. On the zeros of the function $\zeta(s)$ in the neighborhood of the critical line. *Math. USSR Izv.* **1986**, *26*, 307–313. [CrossRef]
38. Laurinčikas, A.; Šiaučiūnas, D. The mean square of the Hurwitz zeta-function in short intervals. *Axioms* **2024**, *13*, 510. [CrossRef]
39. Conway, J.B. *Functions of One Complex Variable*; Springer: New York, NY, USA, 1973.
40. Billingsley, P. *Convergence of Probability Measures*, 2nd ed.; John Wiley & Sons: New York, NY, USA, 1999.