mathematics

MDPI

Article

Classification of Three-Dimensional Contact Metric Manifolds with Almost-Generalized $\mathcal{Z}$-Solitons

Shahroud Azami [1,*], Mehdi Jafari [2] and Daniele Ettore Otera [3,*]

[1] Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
[2] Department of Mathematics, Payame Noor University, Tehran, Iran; m.jafarii@pnu.ac.ir
[3] Institute of Data Science and Digital Technologies, Faculty of Mathematics and Informatics, Vilnius University, LT-08412 Vilnius, Lithuania
* Correspondence: azami@sci.ikiu.ac.ir (S.A.); daniele.otera@mif.vu.lt (D.E.O.); Tel.: +98-28-33901321 (S.A.); Fax: +98-28-33780083 (S.A.)

Abstract

This work investigates the classification of three-dimensional complete contact metric manifolds that are non-Sasakian and satisfy the relation $Q\xi = \sigma\xi$, focusing on those that support an almost-generalized $\mathcal{Z}$-soliton. In the scenario where σ is constant, we prove that if a generalized $\mathcal{Z}$-soliton $(M^n, g, \delta, \eta, V, \mu, \Lambda)$ satisfies the condition $g(V, \xi) = 0$, then M^n must be either an Einstein manifold or locally isometric to the Lie group $E(1,1)$. Comparable classifications are obtained for (κ, μ, ϑ)-contact metric manifolds. Furthermore, we explore situations in which the potential vector field aligns with the Reeb vector field. We then provide the corresponding structural characterizations.

Keywords: generalized $\mathcal{Z}$-solitons; Sasakian manifold; Lie group; contact metric structure; isometry

MSC: 53C15; 53C20; 53C25

check for updates

Academic Editor: Adara M. Blaga

Received: 18 October 2025
Revised: 16 November 2025
Accepted: 19 November 2025
Published: 24 November 2025

Citation: Azami, S.; Jafari, M.; Otera, D.E. Classification of Three-Dimensional Contact Metric Manifolds with Almost-Generalized $\mathcal{Z}$-Solitons. *Mathematics* **2025**, *13*, 3765. https://doi.org/10.3390/math13233765

1. Introduction

In mathematics and theoretical physics, geometric evolution equations, especially the Ricci flow and the Yamabe flow, have emerged as central objects of investigation, revealing deep connections between curvature dynamics and the underlying manifold structure. The foundation of $\mathcal{Z}$-tensors, initially presented in [1], serves to characterize weakly $\mathcal{Z}$-symmetric geometries via a specific class of symmetric $(0,2)$-tensors. A tensor $\mathcal{Z}$ qualifies as a $\mathcal{Z}$-tensor [2] when it admits the following representation:

$$\mathcal{Z}(U_1, U_2) = \mathrm{Ric}(U_1, U_2) + \delta\ g(U_1, U_2),$$

where g denotes the metric, Ric represents the Ricci tensor, and δ is a smooth scalar function on the manifold.

Building upon this framework, Pandey [2] proposed the notion of generalized $\mathcal{Z}$-tensor (GZT), formulated as

$$\mathcal{Z}(U_1, U_2) = \mathrm{Ric}(U_1, U_2) + \delta\ g(U_1, U_2) + \mu\,\eta(U_1)\eta(U_2),$$

with δ and μ being scalar fields, and η a 1-form determined by $\eta(W) = g(W, \xi)$ for some vector field ξ. This extension effectively encompasses a wider spectrum of curvature structures and establishes connections with physical models, especially in the realm of general relativity (GR).

Consider a complete pseudo-Riemannian manifold (M, g) of dimension n. According to [3], such a manifold is called an *almost-generalized $\mathcal{Z}$-soliton* (AGZS) when there exists a septuple $(M^n, g, \delta, \eta, V, \mu, \Lambda)$ where the vector field V and scalar function Λ fulfill the relation

$$\mathcal{Z} + \frac{1}{2}\mathcal{L}_V g + \Lambda g = 0, \tag{1}$$

with $\mathcal{L}_V g$ denoting the Lie derivative of the metric along V. In this context, V represents the solitonic potential. When Λ takes a constant value, the configuration is known as a generalized $\mathcal{Z}$-soliton (GZS). Based on the value of Λ, these solitons are categorized as shrinking ($\Lambda < 0$), steady ($\Lambda = 0$), or expanding ($\Lambda > 0$). Motivated by the role of Ricci solitons in geometric evolution and the interest in generalized curvature tensors ($\mathcal{Z}$-tensors), we study AGZS on three-dimensional contact metric manifolds to probe rigidity and classification phenomena in a setting rich enough to produce nontrivial examples but of dimension small enough to allow for complete classification.

In the special case where $V = \nabla h$ for some smooth function h, Equation (1) reduces to:

$$\mathcal{Z} + \nabla\nabla h + \Lambda g = 0,$$

which describes a gradient AGZS. Various specialized forms of (1) emerge under particular conditions on δ and μ. For instance, setting $\mu = 0$ produces an almost-$\mathcal{Z}$-soliton; when both $\delta = 0$ and $\mu = 0$, the structure simplifies to an almost-Ricci soliton. The case with only $\mu = 0$ corresponds to an almost-η-Ricci soliton. When $\delta = kr$ (where r represents the scalar curvature and k is a constant), we obtain the almost-η-Ricci–Bourguignon soliton, which further reduces to an almost-Ricci–Bourguignon soliton when $\mu = 0$. Additionally, the assignment $\delta = -\left(p + \frac{1}{n}\right)$ with $\mu = 0$ characterizes the AGZS as an almost-conformal Ricci soliton.

Previous research has examined weak symmetry conditions across diverse geometrical and physical contexts [4]. Notable progress includes investigations of pseudo $\mathcal{Z}$-symmetric manifolds and spacetimes by Mantica and Molinari [5,6], along with examinations of weakly cyclic $\mathcal{Z}$-symmetric configurations by De et al. [7]. In addition, K. De and U. C. De [8] have studied generalized $\mathcal{Z}$-recurrent spacetimes in connection with $f(R, T)$-gravity theories.

Catino and Mazzieri [9] presented a comprehensive categorization of gradient shrinking Schouten solitons on three-dimensional manifolds, demonstrating that such manifolds are isometrically equivalent to either quotient spaces of S^3, the Euclidean space $\mathbb{R}^3$, or the product manifold $\mathbb{R} \times S^2$. The study of Ricci solitons in the setting of three-dimensional normal almost-contact metric manifolds was carried out in [10], whereas Kim [11] addressed the classification problem specifically for Ricci solitons on Kenmotsu manifolds. More recently, the authors in [12] examined Ricci ρ-solitons within the context of η-Einstein almost-Kenmotsu manifolds.

In a related line of research, Koufogiorgos [13] analyzed three-dimensional contact metric manifolds (3D CMM) with condition $Q\xi = \sigma\xi$ where Q is the Ricci operator and σ is a constant. This approach encompasses several cases: Sasakian manifolds (with $\sigma = 2$) and certain non-Sasakian manifolds fulfilling the commutation relation $Q\phi = \phi Q$ (for $\sigma < 2$). Employing Milnor's theorem [14] concerning Lie groups endowed with left-invariant metrics, Koufogiorgos demonstrated that for $\sigma \neq 2$, these manifolds are locally isometric to one of the following Lie groups: $SU(2)$, $SO(3)$, $SL(2, \mathbb{R})$, $E(2)$, $E(1, 1)$, or $O(1, 2)$. In [15] Venkatesha et al. studied three-dimensional complete contact Riemannian manifolds with $Q\varphi = \varphi Q$ which admit quasi Yamabe soliton. Also, Khatri and Singh [16] investigated three-dimensional complete contact Riemannian manifolds with $Q\xi = \sigma\xi$ which admit Ricci–Bourguignon soliton.

This paper investigates AGZSs on three-dimensional contact metric manifolds where ξ is an eigenvector of Q. Building on Koufogiorgos' results [13], we provide classifications for these solitons. Unlike previous works (see [13] and the Milnor classification), our work provides a complete classification of three-dimensional non-Sasakian contact metric manifolds with $Q\xi = \sigma\xi$ that admit almost-generalized $\mathcal{Z}$-solitons, covering both the cases where the potential vector field is collinear with ξ and where it is orthogonal to ξ. In particular, we obtain new non-Ricci examples and obstructions in the $\mathcal{Z}$-tensor setting.

Our results contribute to the geometric analysis by clarifying the existence and rigidity of soliton-like structures in contact geometry, provide explicit left-invariant models useful in mathematical physics and contact topology, and offer insights for further study of flows adapted to the $\mathcal{Z}$-tensor.

Koufogiorgos' classification [13] and Milnor's analysis [14] concern three-dimensional contact-metric manifolds under Ricci-based conditions and left-invariant metrics. The present paper extends those classifications to the almost-generalized $\mathcal{Z}$-soliton setting, including the parameters δ and μ, and treating almost (non-gradient) solitons where the potential vector field V need not to be a gradient. Concretely, the new cases covered are as follows:

(i) Nonzero μ producing new existence obstructions;
(ii) Almost solitons with non-gradient potentials;
(iii) Mixed configurations where V is collinear or orthogonal to ξ in the presence of δ, μ terms.

When $\delta = \mu = 0$ and $V = 0$, our results recover the classical Koufogiorgos–Milnor classifications.

While the present study is devoted to the analytical classification of soliton solutions, several numerical approaches have been proposed in the literature for investigating fractional and nonlinear soliton equations [17–19].

The paper is organized as follows: Section 2 reviews necessary preliminaries. In Section 3, we analyze contact metric manifolds satisfying $Q\xi = \sigma\xi$ that supports an AGZS, addressing scenarios where the potential vector field is either parallel or perpendicular to the Reeb vector field. Section 4 generalizes these findings to the broader class of (κ, μ, ϑ)-contact metric manifolds.

2. Preliminaries

A smooth, connected manifold M of dimension $(2n+1)$ is called an *almost-contact manifold* if it is equipped with a Reeb vector field ξ, a $(1,1)$-tensor field ϕ, and a 1-form η such that the following conditions hold:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0.$$

This is equivalent to a reduction of the tangent bundle's structure group to $U(n) \times 1$ [20,21]. When equipped with a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y, it becomes an *almost-contact metric manifold* [22,23]. If additionally:

$$d\eta(X, Y) = g(X, \phi Y),$$

it is a *contact metric manifold*. The operator $h = \frac{1}{2}\mathcal{L}_\xi \phi$ is symmetric, self-adjoint, and satisfies tr$h = 0$, $h\xi = 0$, and $h\phi = -\phi h$.

A contact metric manifold is *normal* if the almost-complex structure on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable [22]. This is equivalent to

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

A normal contact metric manifold is *Sasakian*, when it is characterized by

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for all $X, Y \in TM$. A contact metric manifold is said to be a K-contact manifold precisely when $h = 0$, which is equivalent to stating that the Reeb vector field ξ is Killing. In the special case of dimension three, this condition automatically ensures that the manifold is Sasakian.

Let M be a 3D CMM. Define $U \subset M$ as the region where $h \neq 0$, and let $U' \subset M$ correspond to the neighborhoods around all the points $p \in M$ where $h = 0$. Since $U \cup U'$ is dense in M, one can select, at any $p \in U \cup U'$, a local orthonormal frame $\{e, \phi e, \xi\}$. On U, this frame satisfies $he = \lambda e$ and $h\phi e = -\lambda \phi e$, with λ a strictly positive function. For points in U', the three-dimensional manifold M^3 is Sasakian. In what follows, we assume U is non-empty and adopt the frame $\{e, \phi e, \xi\}$ as the ϕ-adapted basis throughout U.

For clarity, we summarize the geometric interpretation of the main parameters used in the paper. The parameter σ is the eigenvalue of the Ricci operator Q in the Reeb direction ($Q\xi = \sigma\xi$), measuring the Ricci curvature along ξ. The constants such as μ arise from the equality $h = (1/2)\mathcal{L}_\xi \phi$, and quantify the deviation from the Sasakian condition ($h = 0$), whereas the coefficients appearing in the $\mathcal{Z}$-tensor determine the linear combination of curvature quantities entering in the definition of $\mathcal{Z}$, influencing both rigidity and the existence of solitons. Roughly speaking, these parameters capture the geometric deformation from the Sasakian model and control curvature and torsion features governing the almost-generalized $\mathcal{Z}$-soliton structure.

From reference [16], we have the following lemma.

Lemma 1. *On U, the Levi-Civita connection satisfies*

$$\begin{aligned}
&\nabla_\xi e = a\phi e, && \nabla_\xi \phi e = -ae, && \nabla_\xi \xi = 0,\\
&\nabla_e \xi = -(1+\lambda)\phi e, && \nabla_e e = b\phi e, && \nabla_e \phi e = -be + (1+\lambda)\xi,\\
&\nabla_{\phi e} \xi = (1-\lambda)e, && \nabla_{\phi e} \phi e = ce, && \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi,
\end{aligned}$$

where a, b, c are smooth functions with

$$2\lambda b = \phi e(\lambda) + g(Qe, \xi), \tag{2}$$

$$2\lambda c = e(\lambda) + g(Q\phi e, \xi). \tag{3}$$

The components of the Ricci operator are determined as follows

$$\begin{aligned}
Qe &= \Big(-1 - 2\lambda a + \frac{r}{2} + \lambda^2\Big)e + \xi(\lambda)\phi e + g(Qe, \xi)\xi,\\
Q\phi e &= \xi(\lambda)e + \Big(\frac{r}{2} + 2\lambda a + \lambda^2 - 1\Big)\phi e + g(Q\phi e, \xi)\xi,\\
Q\xi &= g(Qe, \xi)e + g(Q\phi e, \xi)\phi e + 2(1 - \lambda^2)\xi,
\end{aligned} \tag{4}$$

and the associated scalar curvature is given by

$$r = 2\Big(1 - \lambda^2 - b^2 - c^2 + 2a + e(c) + \phi e(b)\Big). \tag{5}$$

From Lemma 1, the Lie brackets are

$$\begin{aligned} [e, \phi e] &= -be + c\phi e + 2\xi, \\ [e, \xi] &= -(a + \lambda + 1)\phi e, \\ [\phi e, \xi] &= (a - \lambda + 1)e, \end{aligned} \tag{6}$$

and the Jacobi identity yields

$$\begin{aligned} b(a + \lambda + 1) - \xi(c) - \phi e(\lambda) - \phi e(a) &= 0, \\ c(a - \lambda + 1) + \xi(b) + e(\lambda) - e(a) &= 0. \end{aligned} \tag{7}$$

3. Three-Dimensional Contact Metric Manifolds (3D CMM) with $Q\xi = \sigma\xi$

Theorem 1. *Let (M, ϕ, ξ, η, g) denote a non-Sasakian 3D CMM satisfying $Q\xi = \sigma\xi$, where σ remains constant along ξ. If M supports an AGZS whose potential vector field is aligned with ξ, then M is a η-Einstein manifold.*

Remark 1. *When $V = f\xi$, the soliton equation with $\xi(\sigma) = 0$ gives $f\lambda = 0$, so V vanishes on the non-Sasakian set. If $V \perp \xi$, constancy of its transverse components forces either $V = 0$ or a left-invariant $E(1,1)$ structure.*

Proof. Given $Q\xi = \sigma\xi$ with σ constant along ξ, we have $g(Qe, \xi) = g(Q\phi e, \xi) = 0$, and $\sigma = 2(1 - \lambda^2)$. Let $V = f\xi$ for some function f. The soliton equation gives

$$\mathrm{Ric}(X,Y) + \frac{1}{2}((Xf)\eta(Y) + (Yf)\eta(X) - 2fg(\phi hX, Y)) + (\Lambda + \delta)g(X,Y) + \mu\eta(X)\eta(Y) = 0. \tag{8}$$

Setting $X = Y = e$ and using (4) yields

$$\frac{r}{2} - \delta - \Lambda - 2a\lambda - 1 + \lambda^2 = 0. \tag{9}$$

By setting $Y = X = \phi e$ in (8), we obtain

$$\lambda^2 + 2a\lambda - 1 - \Lambda + \frac{r}{2} - \delta = 0. \tag{10}$$

By taking the difference between Equations (9) and (10), we immediately deduce that $a = 0$. Next, substituting $X = e$ and $Y = \phi e$ into (8) leads to $\xi(\lambda) = f\lambda$. Given that $\xi(\sigma) = 0$, it follows that $\xi(\lambda) = 0$, which implies $f = 0$. Consequently, the potential vector field V vanishes, and the manifold M is η-Einstein. □

In Theorem 1 if we assume that $\mu = 0$ then we get the following result.

Corollary 1. *Consider a non-Sasakian contact metric manifold (M, ϕ, ξ, η, g) satisfying $Q\xi = \sigma\xi$, where σ remains constant along ξ. If M supports an almost-$\mathcal{Z}$-soliton (AZS) whose potential vector field is parallel to ξ, then M is an Einstein manifold.*

Theorem 2. *Suppose (M, ϕ, η, g, ξ) is a complete non-Sasakian 3D CMM for which $Q\xi = \sigma\xi$ with constant σ, and suppose μ is constant. If M admits an AGZS whose potential vector field is orthogonal to ξ everywhere, then M is either Einstein or locally isometric to the Lie group $E(1,1)$.*

Proof. With $Q\xi = \sigma\xi$ and σ constant, from (4) we have that $g(Qe, \xi) = g(Q\phi e, \xi) = 0$ and $\sigma = 2(1 - \lambda^2)$. Since σ is a constant, we can conclude that λ is a constant too. For

$V \perp \xi$, the soliton equations fix the connection coefficients in a ϕ-frame; under constant parameters these become constant, giving locally left-invariant geometry. By Milnor's and Koufogiorgos' results, this corresponds to the solvable Lie group $E(1,1)$. By [13], $Q = \alpha I + \beta\eta \otimes \xi + \gamma h$ with $\alpha = \frac{1}{2}(r-2k)$, $\beta = \frac{1}{2}(6k-r)$, $\gamma = -\alpha$, $k = \frac{1}{2}\mathrm{Tr}l$, and r, $\lambda = \sqrt{1-k}$, $a = \alpha/2$ are constants. Since λ is a constant, Equations (2) and (3) yield $c = b = 0$. The vector field V is orthogonal to ξ, and then there are smooth functions f_1 and f_2 such that $V = f_1 e + f_2 \phi e$. The soliton Equation (1) gives

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2\mathrm{Ric}(X,Y) + 2(\delta + \Lambda)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{11}$$

Setting $Y = X = e$ in (11) and applying (5) and Lemma 1, we deduce

$$e(f_1) - 1 - 2a\lambda + \frac{r}{2} + \delta + \lambda^2 + \lambda = 0. \tag{12}$$

Putting $Y = X = \phi e$ in (11) and applying (5) and Lemma 1, we conclude

$$2a\lambda + \frac{r}{2} - 1 + \phi e(f_2) + \delta + \lambda^2 + \lambda = 0. \tag{13}$$

For $Y = X = \xi$ in (11), we have

$$\sigma = -\Lambda - \delta - \mu. \tag{14}$$

Since σ and μ are constants, we deduce that $\Lambda + \delta$ is a constant. Equations (12) and (13) imply, respectively, that $e(f_1)$ and $\phi e(f_2)$ are constants too. Setting $X = \xi$, $Y = e$ in (11) leads to

$$\xi(f_1) - af_2 + (1+\lambda)f_2 = 0. \tag{15}$$

Equation (11) for $X = \xi$, $Y = \phi e$ implies that

$$af_1 + \xi(f_2) + (\lambda - 1)f_1 = 0. \tag{16}$$

Similarly, Equation (11) for $X = e$, $Y = \phi e$ yields

$$e(f_2) + \phi e(f_1) = 0. \tag{17}$$

Differentiating Equation (15) with respect to e and using (17) gives

$$e(\xi(f_1)) = -(a - 1 - \lambda)\phi e(f_1).$$

Operating the second Lie bracket relation (6) over f_1, we find $\phi e(f_1) = 0$, so (17) gives $e(f_2) = 0$. Applying the first Lie bracket relation (6) over f_1 and f_2 gives $\xi(f_1) = \xi(f_2) = 0$. Then (15) and (16) become

$$(1 + \lambda - a)f_2 = 0, \quad (a + \lambda - 1)f_1 = 0.$$

If $1 + \lambda - a = 0$, then the last equations imply $f_1 = 0$. Operating the second Lie bracket relation (6) over f_2 provides $\phi e(f_2) = 0$. Thus f_2 is a constant. Equations (13) and (14) lead to $a = 0$ and $\lambda = -1$, which is a contradiction. If $a + \lambda - 1 = 0$, then $f_2 = 0$. Applying the third Lie bracket relation (6) over f_1 provides $(1-\lambda)e(f_1) = 0$. If $\lambda = 1$, then $a = 0$ and the Lie brackets become

$$[e, \phi e] = 2\xi, \quad [e, \xi] = -2\phi e, \quad [\phi e, \xi] = 0,$$

which by [13] implies that M is locally isometric to $E(1,1)$. If $\lambda \neq 1$, then $e(f_1) = 0$, so f_1 is a constant. Equations (12) and (13) provide $a = 0$ and $\lambda = 1$, which is a contradiction. $\square$

Remark 2. *The hypotheses $\xi(\sigma) = 0$ (i.e., σ constant along the Reeb flow) and μ constant are standard in three-dimensional contact-metric classification problems and are adopted here to target locally homogeneous and ξ-invariant models. Geometrically, $\xi(\sigma) = 0$ means that the Ricci-eigenvalue in the Reeb direction is invariant along the Reeb flow, a natural condition when seeking left-invariant or locally homogeneous examples. Constant μ likewise enforces uniformity of the $\eta \otimes \eta$-component of the Z-tensor and simplifies the reduction to Lie-group models (see [13]).*

Corollary 2. *Suppose (M, ϕ, η, g, ξ) is a complete non-Sasakian 3D CMM for which $Q\xi = \sigma\xi$ with constant σ. If the manifold admits an almost-$\mathcal{Z}$-soliton (AZS) whose potential vector field is everywhere orthogonal to ξ, then M is either Einstein or locally isometric to the Lie group $E(1,1)$.*

Theorem 3. *Consider a non-Sasakian 3D CMM (M, ϕ, ξ, η, g) with $Q\xi = \sigma\xi$, where σ and the scalar curvature are constant. If M supports a gradient AGZS characterized by constants $\Lambda + \delta$ and μ, then M is necessarily Einstein or locally equivalent to $E(1,1)$.*

Proof. Given $Q\xi = \sigma\xi$ with σ constant, we have $g(Qe, \xi) = g(Q\phi e, \xi) = 0$ and $\sigma = 2(1 - \lambda^2)$, so λ is constant and $b = c = 0$. Let $Df = f_1 e + f_2 \phi e + f_3 \xi$ for some smooth functions f_1, f_2, and f_3. The gradient AGZS equation is

$$\nabla_X \nabla f + QX + (\Lambda + \delta)X + \mu\eta(X)\xi = 0. \tag{18}$$

Assigning $X = \xi$ in (18) and applying Lemma 1 together with (4), we deduce that

$$\xi(f_1) - af_2 = 0, \ af_1 + \xi(f_2) = 0, \ \sigma + \xi(f_3) + \Lambda + \delta + \mu = 0. \tag{19}$$

Then $\xi(f_3)$ is constant. Equation (18) for $X = e$ yields

$$\begin{aligned} e(f_1) + \frac{r}{2} - 1 + \lambda^2 - 2a\lambda + \Lambda + \delta &= 0, \\ e(f_2) - (1+\lambda)f_3 &= 0, \\ (1+\lambda)f_2 + e(f_3) &= 0. \end{aligned} \tag{20}$$

Equation (18) for $X = \phi e$ implies that

$$\begin{aligned} \phi e(f_1) + (1-\lambda)f_3 &= 0, \\ \phi e(f_2) + \frac{r}{2} - 1 + \lambda^2 + 2a\lambda + \Lambda + \delta &= 0, \\ (\lambda - 1)f_1 + \phi e(f_3) &= 0. \end{aligned} \tag{21}$$

Applying the second Lie bracket relation over f_3 and using Equations (19) and (20), we obtain $(\lambda^2 + 2a\lambda - 1)f_1 = 0$. If $f_1 = 0$ and $\lambda^2 + 2a\lambda - 1 \neq 0$, Equation (19) implies that $af_2 = 0$. If $f_2 = 0$ then Equations (20) and (21) lead to $a = 0$. The second equation of (20) yields $(1+\lambda)f_3 = 0$. Then $f_3 = 0$, in this case, $V = 0$ and the manifold is η-Einstein. If $f_2 \neq 0$ and $a = 0$, the first equation of (21) implies that $(1+\lambda)f_3 = 0$ and $f_3 = 0$. The third equation of (20) yields $\lambda = -1$, which is a contradiction.

Now, we assume that $f_1 \neq 0$ and $\lambda^2 + 2a\lambda - 1 = 0$. Operating the second term of Lie bracket (6) over f_1, and using Lemma 1 and (4), we obtain $ae(f_2) = -(a + \lambda + 1)\phi e(f_1)$. In addition, operating the third term of Lie bracket (6) over f_2 provides $a\phi e(f_1) = -(a - \lambda + 1)e(f_2)$. Then $(\lambda^2 - 2a - 1)\phi e f_1 = 0$. When $\lambda^2 - 2a\lambda - 1 = 0$ then $a = 0$ and $\lambda = 1$. In this case, Equation (6) becomes

$$[e, \phi e] = 2\xi, \ \ [e, \xi] = -2\phi e, \ \ [\phi e, \xi] = 0,$$

which, by [13], implies that M is locally isometric to $E(1,1)$. When $\phi e f_1 = 0$ and $\lambda^2 - 2a\lambda - 1 \neq 0$, Equation (21) leads to $f_3 = 0$ or $\lambda = 1$. Equation (21) in its third component yields $f_3 \neq 0$ together with $\lambda = 1$. Repeating the argument of above shows again that M is locally equivalent to $E(1,1)$. Hence, the proof is finished. □

4. (κ, μ, ϑ)-Contact Metric Manifolds

A contact metric 3-manifold (M, ϕ, ξ, η, g) is referred to as a $(\kappa, \varrho, \vartheta)$-contact metric manifold (see [24]) if

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \varrho(\eta(Y)hX - \eta(X)hY) + \vartheta(\eta(Y)\phi hX - \eta(X)\phi hY),$$

where R is the Riemann curvature and $\kappa, \varrho, \vartheta$ are smooth functions. If $\vartheta = 0$, it is a generalized (κ, ϱ)-contact metric manifold.

Since $Q\xi = 2\kappa\xi$, we have, $g(Qe, \xi) = g(Q\phi e, \xi) = 0$, $\kappa = 1 - \lambda^2$ [25], $\xi(\lambda) = \lambda\vartheta$, and $\varrho = -2a$. The following relations hold [24]:

$$e(\kappa) - \lambda e(\varrho) - \lambda\phi e(\vartheta) = 0, \quad -\phi e(\kappa) - \lambda\phi e(\varrho) + \lambda e(\vartheta) = 0. \tag{22}$$

We have the following lemma from [26].

Lemma 2 ([26]). *For a $(\kappa, \varrho, \vartheta)$-contact metric manifold,*

$$\xi(r) = 2\xi(\kappa) = -4(1-\kappa)\vartheta.$$

We now focus on three-dimensional $(\kappa, \varrho, \vartheta)$-contact metric manifolds that admit an AGZS, leading to the following results.

Theorem 4. *Let (M, ϕ, ξ, η, g) be a $(\kappa, \varrho, \vartheta)$-contact metric manifold with $\kappa < 1$ and μ be a constant. If M admits an AGZS with the potential vector field $V = f\xi$ for some smooth function f, then M is η-Einstein.*

Proof. Setting $X = Y = e$ and $X = Y = \phi e$ in the Equation (8) gives $\frac{r}{2} - \kappa + \varrho\lambda + \Lambda + \delta = 0$ and $\frac{r}{2} - \kappa - \varrho\lambda + \Lambda + \delta = 0$, respectively. Combining these equations gives $\varrho = 0$. Setting $X = e$, $Y = \phi e$ in Equation (8) gives $\vartheta = f$. For $X = Y = \xi$ in (8) we have,

$$2\kappa + \xi(f) + \Lambda + \delta + \mu = 0. \tag{23}$$

Combining these equations yields $r = 6\kappa + 2\xi(f) + 2\mu$. Differentiating along ξ and using Lemma 2 gives $\xi(2\kappa + \xi(f) + \mu) = 0$.

Now, differentiating (23) along ξ again, and applying the previous relation, we obtain $\xi(\Lambda + \delta) = 0$. Taking the trace of (1) gives $r + 3(\lambda + \delta) - \mu = 0$. Differentiation along ξ, combined with the constancy of μ, leads to $\xi(r) = 0$, which implies $(1-\kappa)\vartheta = 0$.

If $\vartheta = 0$, then $f = 0$, meaning that M is η-Einstein. On the other hand, if $\kappa = 1$, one would obtain $\lambda^2 = 1 - \kappa = 0$, which leads to a contradiction. This ends the proof. □

Corollary 3. *Let (M, ϕ, ξ, η, g) be a generalized (κ, ϱ)-contact metric manifold with $\kappa < 1$. If M admits an AGZS with $V = f\xi$ and μ is a constant, then M is η- Einstein.*

Theorem 5. *Let (M, ϕ, ξ, η, g) be a $(\kappa, \varrho, \vartheta)$-contact metric manifold with $\kappa < 1$, ϱ constant along ξ, and constant scalar curvature. If M admits an AGZS with potential vector field orthogonal to ξ and $\Lambda + \delta$ is a constant, then M is Einstein or locally isometric to $E(1,1)$.*

Proof. Let $V = f_1 e + f_2 \phi e$. for some smooth functions f_1, f_2. Setting $X = Y = \xi$ in (11) gives $2\kappa = -(\Lambda + \delta + \mu) = 2(1 - \lambda^2)$. For $X = Y = e$ and $X = Y = \phi e$ in (11) it follows that

$$e(f_1) - bf_2 + \frac{r}{2} - \kappa + \lambda\varrho + \Lambda + \delta = 0, \tag{24}$$

$$\phi e(f_2) - cf_1 + \frac{r}{2} - \kappa - \lambda\varrho + \Lambda + \delta = 0. \tag{25}$$

Combining the above equations we deduce

$$e(f_1) - \phi e(f_2) - bf_2 + cf_1 + 2\lambda\varrho = 0. \tag{26}$$

Setting $X = \xi, Y = e$ and $X = \xi, Y = \phi e$ in (11) we obtain

$$\xi(f_1) - af_2 + (1 + \lambda)f_2 = 0, \;\; af_1 + \xi(f_2) + (\lambda - 1)f_1 = 0. \tag{27}$$

Taking the covariant derivative of $2\kappa = -(\Lambda + \delta + \mu) = 2(1 - \lambda^2)$ along ξ and using Lemma 2, we get

$$\xi(r) = (1 - \kappa)\vartheta = 0.$$

If $\kappa = 1$, then Theorem 2 applies.

Now, assume $\kappa \neq 1$ and $\vartheta = 0$, which leads to $\xi(\lambda) = 0$. Consequently, Equation (22) reduce to

$$e(\varrho) = \frac{e(\kappa)}{\lambda} = \frac{e(1 - \lambda^2)}{\lambda} = -4c\lambda, \qquad \varphi e(\varrho) = -\frac{\phi e(\kappa)}{\lambda} = 4b\lambda. \tag{28}$$

By invoking the second relation in (6) for ϱ and combining it with the preceding equations, we obtain

$$\xi(c) = -b(a + \lambda + 1). \tag{29}$$

Equation $\kappa = 1 - \lambda^2$ yields $\phi e(\kappa) = -2\lambda\phi e(\lambda)$. From (28) and $\varrho = -2a$, we have $\varphi e(\lambda) = -\varphi e(a)$, which in Equation (7) gives

$$\xi(c) = b(a + \lambda + 1). \tag{30}$$

Combining (29) and (30) yields

$$b(a + \lambda + 1) = 0. \tag{31}$$

Similarly, operating the third term of (6) over ϱ, we get

$$\xi(b) = c(a - \lambda + 1). \tag{32}$$

From (28), we have $e(a) = e(\lambda)$, which in Equation (7) gives

$$\xi(b) = -c(a - \lambda + 1).$$

Combining the previous relation with (32) yields

$$c(a - \lambda + 1) = 0. \tag{33}$$

We now proceed by analyzing the following possibilities:

Case I: When $b = c = 0$, Equations (2) and (3) indicate λ remains constant, which in turn ensures that κ becomes constant. Hence, the conclusion of Theorem 2 can be directly applied.

Case II: If both b and c are non-zero, then from (31) and (33) we have $a+\lambda+1=0$ and $a-\lambda+1=0$, leading to $\lambda=0$, which is a contradiction.

Case III: For $b=0$ and $c\neq 0$, Equation (33) gives $a+1=\lambda$. Then (27) implies that $\xi(f_2)=-2(\lambda-1)f_1$, while (28) implies $\varphi e(\varrho)=\varphi e(\lambda)=\varphi e(\kappa)=0$.

Substituting $X=e$ and $Y=\varphi e$ into (11) yields

$$e(f_2)+\varphi e(f_1)+cf_2=0.$$

Differentiating (25) along ξ and using $\xi(c)=0$ from (30), we obtain $\xi(\varphi e(f_2))=c\,\xi(f_1)$. Applying the third part of (6) to f_2 gives

$$\xi(\varphi e(f_2))=-2(\lambda-1)\varphi e(f_1).$$

Combining these expressions and using the first term of (27) with $a=\lambda-1$, we get

$$(\lambda-1)\varphi e(f_1)-cf_2=0. \tag{34}$$

Differentiating (26) along ξ gives $\xi(e(f_1))=0$, and applying the second part of (6) to f_1 yields

$$cf_2+(\lambda+1)\varphi e(f_1)=0. \tag{35}$$

Combining (34) and (35) leads to $\varphi e(f_1)=f_2=0$. Consequently,

$$(\lambda-1)f_1=0,$$

which leads to $\lambda=1$. Plugging this value into (3) results in $c=0$, which is a contradiction.

Case IV: When $b\neq 0$ and $c=0$, Equation (31) yields $a+\lambda+1=0$. Additionally, from (30) and (32), we find $\xi(b)=\xi(c)=0$, which, together with (3), implies that $e(\lambda)=e(a)=e(\varrho)=e(\kappa)=0$.

Taking the covariant derivative of (24) in the direction of ξ gives $\xi(e(f_1))=b\,\xi(f_2)$. Under the condition $a+\lambda+1=0$, Equation (27) reduces to

$$\xi(f_1)=-2(\lambda+1)f_2,\quad \xi(f_2)=2f_1. \tag{36}$$

Applying the second part of (6) to f_1 and using the above relations leads to

$$-2(\lambda+1)e(f_2)=2bf_1. \tag{37}$$

Differentiating (25) along ξ gives $\xi(\varphi e(f_2))=0$. Using this in the third part of (6) applied to f_2 yields

$$\varphi e(f_1)+\lambda e(f_2)=0. \tag{38}$$

Equation (28) for $X=e$ and $Y=\phi e$ gives

$$e(f_2)+\varphi e(f_1)+b\,e(f_1)=0. \tag{39}$$

By combining the relations (37)–(39) one gets $\varphi e(f_1)=e(f_2)=0$, and then (39) implies $bf_1=0$, hence $f_1=0$. Consequently, (36) reduces to

$$(\lambda+1)f_2=0,$$

forcing $f_2=0$, which is a contradiction. This completes the proof. $\square$

Remark 3. *The manifold becomes Einstein or locally isometric to a specific Lie group as a direct consequence of the imposed curvature and soliton conditions. In fact, the algebraic reduction of the almost-generalized $\mathcal{Z}$-soliton equation and Milnor's classification of three-dimensional Lie algebras show that nonzero structure constants correspond either to $E(1,1)$, for potentials orthogonal to ξ, to η-Einstein manifolds, or to Einstein structures for aligned or trivial potentials (respectively). This property emerges from the geometric constraints rather than being assumed a priori.*

Examples

We include three brief examples to illustrate the main phenomena:

- Einstein example with $V = 0$: This is the simplest case where the potential vanishes, leading directly to an Einstein metric as expected.
- Explicit left-invariant $E(1,1)$ model: The orthogonal potential satisfies the bracket relations used in the proofs, demonstrating the construction explicitly.
- Non-Ricci example with $\mu \neq 0$: Only the trivial potential arises in this case, showing that non-Ricci conditions severely restrict the form of V.

5. Conclusions

In this paper, we have investigated three-dimensional $(\kappa, \varrho, \vartheta)$-contact metric manifolds admitting AGZS. Our results show that when the potential vector field is parallel to ξ, the manifold is necessarily η-Einstein, with $\vartheta = 0$ being a necessary condition for compatibility. For potential vector fields orthogonal to ξ, in manifolds with $\kappa < 1$, ϱ constant along ξ, and constant scalar curvature, the manifold is either locally isometric to $E(1,1)$ or Einstein. The analysis of the coefficients a, b, c further indicates that κ and λ must be constant for nontrivial AGZS to exist, highlighting the significant role of these parameters in the geometry. Overall, our results emphasize that the existence of AGZS imposes strong geometric constraints on the underlying contact metric structure, providing a framework for identifying manifolds with special soliton properties.

The main limitations of our work can be summarized as follows: (i) the assumption $Q\xi = \sigma\xi$ which is used in several results, (ii) we focus on manifolds of dimension three, and (iii) some theorems require constancy of some parameters, such as σ or μ. Future work may relax these assumptions, construct additional explicit examples, extend the classification to higher dimensions, and investigate flows adapted to the $\mathcal{Z}$-tensor framework.

Note also that the curvature-operator assumption, $Q\xi = \sigma\,\xi$, together with the $\mathcal{Z}$-tensor constraints, is essential for our classification. It reduces the almost-generalized $\mathcal{Z}$-soliton equation to algebraic relations in a ϕ-adapted orthonormal frame, and ensures vanishing of certain Lie-derivative and torsion components, maintaining compatibility with the contact structure. Partial relaxations are possible if $Q\xi$ is invariant along the Reeb flow ($\mathcal{L}_\xi Q = 0$), which may lead to locally homogeneous but non-Einstein examples. Removing this assumption entirely would require very different analytical approaches and could, in principle, produce new, non-homogeneous soliton families.

Author Contributions: Conceptualization, S.A. and M.J.; Methodology, S.A. and M.J.; Software, S.A. and M.J.; Validation, S.A. and D.E.O.; Formal analysis, M.J.; Investigation, S.A.; Writing—original draft, M.J.; Writing—review and editing, S.A.; Supervision, D.E.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding authors.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Mantica, C.A.; Molinari, L.G. Weakly Z-symmetric manifolds. *Acta Math. Hung.* **2012**, *135*, 80–96. [CrossRef]
2. Pandey, P. On weakly cyclic generalized Z-symmetric manifolds. *Natl. Acad. Sci. Lett.* **2020**, *43*, 347–350. [CrossRef]
3. Azami, S.; De, U.C. Generalized $\mathcal{Z}$-solitons on magneto-fluid spacetimes in $f(r)$-Gravity. *Int. J. Theor. Phys.* **2025**, *64*, 33. [CrossRef]
4. Chaturvedi, B.B.; Pandey, P. Study on Special Type of a Weakly Symmetric Kahler Manifold. *Differ.-Geom.-Dyn. Syst.* **2015**, *17*, 32–37.
5. Mantica, C.A.; Suh, Y.J. Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensor. *Int. J. Geom. Methods Mod. Phys.* **2012**, *9*, 1250004. [CrossRef]
6. Mantica, C.A.; Suh, Y.J. Pseudo Z symmetric spacetimes. *J. Math. Phys.* **2014**, *55*, 042502. [CrossRef]
7. De, U.C.; Mantica, C.A.; Molinari, L.G.; Suh, Y.J. On weakly cyclic Z symmetry manifolds. *Acta Math. Hunar.* **2016**, *149*, 462–477. [CrossRef]
8. De, K.; De, U.C. Investigation of generalized Z-recurrent spacetimes and $F(R,T)$-gravity. *Adv. Appl. Clifford Algebr.* **2021**, *31*, 38. [CrossRef]
9. Catino, G.; Mazzieri, L. Gradient Einstein solitons. *Nonlinear Anal.* **2016**, *132*, 66–94. [CrossRef]
10. De, U.C.; Turan, M.; Yildiz, A.; De, A. Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds. *Publ. Math. Debr.* **2012**, *80*, 127–142. [CrossRef]
11. Cho, J.T. Almost contact 3-manifolds and Ricci solitons. *Int. J. Geom. Methods Mod. Phys.* **2013**, *10*, 1220022. [CrossRef]
12. Azami, S.; Fasihi-Ramandi, G. Ricci ρ-solitons on 3-dimensional η-Einstein Almost Kenmotsu manifolds. *Commun. Korean Math. Soc.* **2020**, *35*, 613–623.
13. Koufogiorgos, T. On a class of contact Riemannian 3-manifolds. *Results Math.* **1995**, *27*, 51–62. [CrossRef]
14. Milnor, J. Curvature of left invariant metrics on Lie groups. *Adv. Math.* **1976**, *21*, 293–329. [CrossRef]
15. Venkatesha, V.; Kumara, H.A. Quasi Yamabe solitons on 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$. *Commun. Math.* **2022**, *30*, 191–199.
16. Khatri, M.; Singh, J.P. Ricci–Bourguignon soliton on three-dimensional contact metric manifolds. *Mediterr. J. Math.* **2024**, *21*, 70. [CrossRef]
17. Shakeel, K.; Baleanu, D.; Abbas, M.; Yousif, M.A.; Mohammed, P.O.; Abdullah, F.A.; Abdeljawad, T. Advanced fractional soliton solutions of the Joseph-Egri equation via Tanh-Coth and Jacobi function methods. *Sci. Rep.* **2025**, *15*, 35717. [CrossRef] [PubMed]
18. Yousif, M.A.; Baleanu, D.; Abdelwahed, M.; Azzo, S.; Mohammed, P. Finite difference β-fractional approach for solving the time-fractional FitzHugh-Nagumo equation. *Alex. Eng. J.* **2025**, *125*, 127–132. [CrossRef]
19. Yousif, M.A.; Hamasalh, F.K. The fractional non-polynomial spline method: Precision and modeling improvements. *Math. Comput. Simul.* **2024**, *218*, 512–525. [CrossRef]
20. Sasaki, S. On differentiable manifolds with certain structures which are closely related to almost contact structure I. *Tohoku Math. J.* **1960**, *12*, 459–476. [CrossRef]
21. Sasaki, S.; Hatakeyama, Y. On differentiable manifolds with certain structures which are closely related to almost contact structure II. *Tohoku Math. J.* **1961**, *13*, 281–294. [CrossRef]
22. Blair, D.E. *Riemannian Geometry of Contact and Symplectic Manifolds*; Progress in Mathematics; Birkhäuser: Boston, MA, USA, 2010.
23. Blair, D.E. Two remarks on contact metric structures. *Tohoku Math. J.* **1977**, *29*, 319–324. [CrossRef]
24. Koufogiorgos, T.; Markellous, M.; Papantoniou, V.J. The harmonicity of the Reeb vector field on contact metric 3-manifolds. *Pac. J. Math.* **2008**, *234*, 325–344. [CrossRef]
25. Koufogiorgos, T.; Tsichlias, C. On the existence of a new class of contact metric manifolds. *Can. Math. Bull.* **2000**, *43*, 440–447. [CrossRef]
26. Chen, X. Three dimensional contact metric manifolds with Cotton solitons. *Hiroshima Math. J.* **2021**, *51*, 275–299. [CrossRef]