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# Asymptotic Formulas for the Haezendonck–Goovaerts Risk Measure of Sums with Consistently Varying Increments

**Jonas Šiaulys *, Mantas Dirma, Neda Nakliuda and Luca Zanardelli**

Institute of Mathematics, Vilnius University, Naugarduko Str. 24, LT-03225 Vilnius, Lithuania; mantasdirma96@gmail.com (M.D.); neda.nakliuda@mif.vu.lt (N.N.); luca.zanardelli@mif.stud.vu.lt (L.Z.)

* Correspondence: jonas.siaulys@mif.vu.lt

**Abstract**

The Haezendonck–Goovaerts (HG) risk measure defined on Orlicz spaces via the so-called normalised Young function is a direct generalisation of the Expected Shortfall risk measure. The HG measure is known to be a coherent one, thus making it more robust than some of the alternatives, such as Value-at-Risk, for aggregating and comparing risks, and at the same time more flexible for capital allocation problems, risk premium estimation, solvency assessment, and stress testing in insurance and finance. As random risk in practical applications is often assessed in a portfolio setting—a collection of insurance policies or financial assets, like stocks or bonds—we examine the situation in which the total portfolio risk is expressed as the sum of individual random risks. For this, we consider the sum $S_n^{(\xi)} = \xi_1 + \ldots + \xi_n$ of possibly dependent and non-identically distributed real-valued random variables $\xi_1, \ldots, \xi_n$ with consistently varying distributions. Assuming that the collection $\{\xi_1, \ldots, \xi_n\}$ follows the dependence structure, similar to the asymptotic independence, we obtain the asymptotic estimations of the HG risk measure for the sum $S_n^{(\xi)}$ when the confidence level tends to 1. The formulas presented in our work show that in the case where a portfolio of random losses contains consistently varying losses and the others are asymptotically negligible, it is sufficient for risk assessment to consider only the tails of those dominant losses.

**Keywords:** sum of random variables; asymptotic independence; tail-moment; heavy tail; consistently varying distribution; Haezendonck–Goovaerts risk measure

**MSC:** 91B05; 60F10; 91G15

## 1. Introduction

Risk measurement plays a central role in modern actuarial science, financial regulation, and quantitative risk management. As markets become increasingly complex and uncertain, the limitations of classical risk measures, such as variance, Value-at-Risk (VaR), and Expected Shortfall (ES), have become more apparent. In particular, widely used regulatory metrics often fail to adequately capture tail risk, lack desirable coherence properties, or provide insufficient flexibility to model risk preferences. For instance, it is well known that the VaR measure in general does not satisfy the sub-additivity axiom, and, as a result, fails to obey the diversification principle. Additionally, as it is defined as a certain percentile of the loss distribution, it does not account for the severity of losses beyond that percentile [1]. Therefore, it may underestimate risk in market stress situations under extreme asset price

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fluctuations [2]. These challenges have motivated the search for more robust and theoretically sound alternatives. One such alternative is the so-called Haezendonck–Goovaerts (HG) risk measure, which is defined in Section 1.6. The HG risk measure, like other risk measures, is a method to assess the amount of capital we should hold to protect ourselves against impending losses, taking into account the level of risk aversion. The HG risk measure is actually a family of risk measures that depend on the so-called Young function $\varphi$. Choosing another Young function yields a different version of this risk measure. The HG risk measure derived from the principles of premium calculation in [3] unifies economic intuition with strong mathematical foundations. According to the results of [4,5], the HG risk measure has many properties necessary for risk measurements. In particular, this measure belongs to the class of coherent risk measures in the sense of Artzner et al. [1], as it is monotonic, translation-invariant, sub-additive, and homogeneous, and, as a result, is a monetary and convex risk measure as in [6,7]. It is well known that the simplest case of HG risk measure is the Expected Shortfall (ES) (or, synonymously, Conditional Value-at-Risk or Tail Value-at-Risk or Conditional Tail Expectation, considered in detail in [8]. The ES risk measure can be obtained from the HG risk measure using the Young function $\varphi(t) = t$. Therefore, the HG risk measure is a direct generalisation of the ES risk measure.

The HG risk measure is particularly useful in actuarial science and financial risk management, where capturing heavy-tailed and extreme losses is essential. Its foundation in Orlicz functions allows flexible modelling of risk aversion and tail behavior, making it well suited for premium calculation, solvency assessment, and capital allocation. In particular, the flexibility to select the Young function, which characterises the behaviour of the measure, gives more control over the degree to which extreme losses influence the risk assessment. It is important to note that, despite being more theoretically grounded than some of the alternatives, the HG risk measure might be more challenging to implement in practice.

Tang and Yang in [9] provided a formula for calculating the HG risk measure when the Young function is a power function, see Theorem 3 below. However, in the same article, the authors mention that in the general case, an analytical formula for calculating the HG risk measure does not exist, since the calculation of the values of this measure is related to finding quantiles. In this case, the authors of the article [9] suggest using asymptotic formulas by writing one of the formulas of this form for random risk with a regularly varying tail, see Theorem 4 below.

We observe that a random risk is often expressed as a sum of random variables when it represents the aggregate loss of multiple individual risks. This occurs, for example, when modelling total claims in an insurance portfolio, where each claim amount is a random variable. It also arises in finance when a portfolio's total loss is composed of the losses of individual assets. More generally, any situation involving accumulated, component-wise, or event-based contributions—such as operational losses, credit defaults, or catastrophe events—naturally leads to a representation as a sum of random variables. In this paper, we consider the case where the random risk is written as a sum of random variables that are somehow dependent. We consider the case where, among the summable random variables, there is a subset of variables with consistently varying tails, and the tails of the other random variables are vanishing with respect to the latter. The main results of the work are formulated in Theorems 6 and 7.

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### 1.1. Preliminaries

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$, and let $\{\xi_1, \ldots, \xi_n\}$ be a collection of possibly dependent and non-identically distributed random variables (r.v.'s). Denote

$$S_n^{(\xi)} := \xi_1 + \ldots + \xi_n. \tag{1}$$

In the main results of the paper, we assume that random summands have consistently varying distributions. Before a more detailed discussion on heavy-tailed distributions, we first define the support of any distribution or distribution function (d.f.). For a d.f. $F : \mathbb{R} \to [0,1]$, the support of $F$ is defined as

$$\begin{aligned} \operatorname{supp} F &= \big\{x : F(x+\delta) - F(x-\delta) > 0 \text{ for all } \delta > 0\big\} \\ &= \big\{x : \overline{F}(x-\delta) - \overline{F}(x+\delta) > 0 \text{ for all } \delta > 0\big\}, \end{aligned}$$

where $\overline{F} = 1 - F$ denotes the tail function (t.f.). If $\operatorname{supp} F \subset [0, \infty)$, then we say that the distribution (or d.f.) $F$ is supported on $\mathbb{R}^+$. If $\operatorname{supp} F \not\subset [0, \infty)$, then we say that the distribution (or d.f.) $F$ is supported on $\mathbb{R}$.

### 1.2. Heavy-Tailed Distributions

In this subsection, we discuss the class of heavy-tailed distributions and the most popular and closest subclasses of this class of distributions.

- *A d.f. $F$ supported on $\mathbb{R}$ is said to be heavy-tailed, written as $F \in \mathscr{H}$, if for all $\delta > 0$, we have*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\delta x} \mathrm{d}F(x) = \infty.$$

Due to the alternative expectation formula (see, e.g., [10,11]), we have

$$\int_{[0,\infty)} \mathrm{e}^{\delta x} \mathrm{d}F(x) = 1 + \delta \int_0^{\infty} \mathrm{e}^{\delta x} \overline{F}(x)\, \mathrm{d}x$$

implying that

$$\begin{aligned} &F \in \mathscr{H} \;\Rightarrow\; \overline{F}(x) > 0,\ x \in \mathbb{R}; \\ &F \in \mathscr{H} \;\Leftrightarrow\; \limsup_{x\to\infty} \mathrm{e}^{\delta x}\, \overline{F}(x) = \infty \;\text{ for each }\; \delta > 0. \end{aligned}$$

- *A d.f. $F$ supported on $\mathbb{R}$ is said to be regularly varying with index $\alpha \geqslant 0$, written as $F \in \mathscr{R}_\alpha$, if for any $y > 0$, we have*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

It is easy to verify that the Pareto law with d.f.

$$F(x) = \left(1 - \frac{1}{(x+1)^{\alpha}}\right) \mathbb{I}_{[0,\infty)]}(x)$$

belongs to the class $\mathscr{R}_\alpha$ for any positive $\alpha$.

- *A d.f. $F$ supported on $\mathbb{R}$ is said to be extended regularly varying with indices $0 \leqslant \alpha \leqslant \beta$, written as $F \in \mathscr{ERV}_{\{\alpha,\beta\}}$, if for any $y > 1$, we have*

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha}.$$

Due to the results of [12], the distribution with t.f.

$$\overline{F}(x) = \exp\{-\lfloor \log x \rfloor - (\log x - \lfloor \log x \rfloor)^2\},\ x \geqslant 1,$$

belongs to the class $\bigcup\limits_{0\leqslant\alpha\leqslant\beta} \mathscr{ERV}_{\{\alpha,\beta\}}$.

- *A d.f. F supported on $\mathbb{R}$ is said to be consistently varying, written as $F \in \mathscr{C}$, if*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

It can be shown (see [13,14]) that the distribution with t.f.

$$\overline{F}(x) = p^{\lfloor \log_2 x \rfloor}\left(2 - p - (1-p)\frac{x}{2^{\lfloor \log_2 x \rfloor}}\right),\ x \geqslant 1,$$

belongs to the subclass $\mathscr{C} \setminus \bigcup\limits_{\alpha\geqslant 0} \mathscr{R}_\alpha$ for any parameter $p \in (0,1)$.

- *A d.f. F supported on $\mathbb{R}$ is said to be dominatedly varying, written as $F \in \mathscr{D}$, if for any $y \in (0,1)$, we have*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

The presented definition implies that the generalized Peter–Paul distribution with d.f.

$$F(x) = 1 - b^{-a\lfloor \log_b x \rfloor},\ x \geqslant 1,$$

belongs to the subclass $\mathscr{D} \setminus \mathscr{C}$ for all possible parameters $a > 0, b > 1$, see [15].

It is well known (see, for instance, [13]) that

$$\mathscr{R} = \bigcup_{\alpha\geqslant 0} \mathscr{R}_\alpha \subset \bigcup_{0\leqslant\alpha\leqslant\beta} \mathscr{ERV}_{\{\alpha,\beta\}} \subset \mathscr{C} \subset \mathscr{D} \subset \mathscr{H}.$$

The following two indices are important to determine whether the distribution $F$ belongs to the aforementioned classes. The first index is the so-called upper Matuszewska index (see, e.g., [16] (Section 2), [12,17]), defined as

$$J_F^+ = \inf_{y>1}\left\{-\frac{1}{\log y}\log \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\right\} = -\lim_{y\to\infty}\frac{1}{\log y}\log \liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}.$$

The other index, the so-called $L$-index, is defined as

$$L_F = \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

For instance, this index is used in the main results of [18–20]. The definitions of the aforementioned heavy-tailed distribution classes imply that

$$\begin{aligned} &F \in \mathscr{D} \;\Leftrightarrow\; J_F^+ < \infty \;\Leftrightarrow\; L_F > 0; \\ &F \in \mathscr{C} \;\Leftrightarrow\; L_F = 1 \;\Rightarrow\; J_F^+ < \infty; \\ &F \in \mathscr{R}_\alpha \;\Rightarrow\; L_F = 1,\ J_F^+ = \alpha. \end{aligned}$$

The classes of distributions $\mathscr{R}$ and $\mathscr{D}$ have been extensively used in real analysis and various spheres of probability (see, e.g., [16,21–24]). The class of consistently varying distributions $\mathscr{C}$ was introduced as a generalization of the class $\mathscr{R}$ in [25] and was referred to as "intermediate regular variation". The concept of consistent variation has been applied in various studies within the context of applied probability, including queueing systems and ruin theory (see, e.g., [12,13,26–32]).

### 1.3. Asymptotic Relations

In this subsection, we introduce the notation used throughout the paper. For two positive functions $f$ and $g$, we write the following:

$$\begin{aligned} f(x) \underset{x\to\infty}{\lesssim} g(x) \quad &\text{if} \quad \limsup_{x\to\infty} \frac{f(x)}{g(x)} \leqslant 1; \\ f(x) = O\big(g(x)\big) \quad &\text{if} \quad \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty; \\ f(x) \asymp g(x) \quad &\text{if} \quad f(x) = O\big(g(x)\big) \text{ and } g(x) = O\big(f(x)\big); \\ f(x) \underset{x\to\infty}{\sim} g(x) \quad &\text{if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1. \end{aligned}$$

In addition, we use the standard notation for indicators:

$$\mathbb{I}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A; \end{cases}$$

and for positive and negative parts of any real $x$,

$$x^+ := \max\{0, x\}, \;\; x^- := \max\{0, -x\}.$$

### 1.4. Quasi-Asymptotic Independence

In this paper, we suppose that the collection $\{\xi_1, \ldots, \xi_n\}$ consists of pairwise quasi-asymptotically independent r.v.'s. This dependence structure, which is a direct generalization of independence, was introduced by Chen and Yuen [28] and has been considered in [33–39], among others.

- *Real-valued r.v.'s $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, with distributions supported on $\mathbb{R}$ are called pairwise quasi-asymptotically independent* (pQAI), *if for all pairs of indices $k, l \in \{1, \ldots, n\}$, $k \neq l$, it holds that*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x,\, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x,\, \xi_l^- > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = 0.$$

The following statement is Theorem 3.1 in [28]. It provides an asymptotic result for the tail probabilities of sums of pQAI r.v.'s with distributions from the class $\mathscr{C}$.

**Theorem 1.** *Let $\{\xi_1, \ldots, \xi_n\}$ be a collection of real-valued* pQAI *r.v.'s, such that $F_{\xi_k} \in \mathscr{C}$ for all $k \in \{1, \ldots, n\}$. Then, for the random sum defined in (1), we have*

$$\mathbb{P}\big(S_n^{(\xi)} > x\big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x).$$

The following statement directly generalises Theorem 1. The paper [40] (see Theorem 4 and the remark below the theorem) gives its detailed proof.

**Theorem 2.** *Let $\{\xi_1, \ldots, \xi_n\}$ be a collection of real-valued* pQAI *r.v.'s, such that $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \ldots, n\}$, and for other indices $k \neq L$ suppose $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}(|\xi_k| > x) = o(\overline{F}_{\xi_L}(x))$. If $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}(\xi_k^+)^\alpha < \infty$, then for each $\beta \in [0, \alpha]$,*

$$\mathbb{E}\Big( (S_n^{(\xi)})^\beta \mathbb{I}_{\{S_n^{(\xi)} > x\}} \Big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big( \xi_k^\beta \mathbb{I}_{\{\xi_k > x\}} \Big), \tag{2}$$

*and for each $\beta \in (0, \alpha]$,*

$$\mathbb{E}\bigg( \Big( (S_n^{(\xi)} - x)^+ \Big)^\beta \bigg) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big( (\xi_k - x)^+ \Big)^\beta, \tag{3}$$

*where $\mathcal{I}_n \subseteq \{1, \ldots, n\}$ is a subset of indices such that $F_{\xi_k} \in \mathscr{C}$ for $k \in \mathcal{I}_n$.*

### 1.5. Positively Decreasing Distribution

In this paper, we will use another concept to describe the monotonicity of a function. This concept can be applied to a fairly broad class of functions, see [41,42]. However, we will only consider the relevant subclass of d.f.'s considered in [16,43,44] among others.

- *A d.f. $F$ supported on $\mathbb{R}$ is said to have a positively decreasing tail, written as $F \in \mathcal{PD}$, if for any fixed $y \in (0, 1)$*
$$\liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} > 1,$$
*or equivalently*
$$\limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < 1$$
*for any $y > 1$.*

We observe that

$$\mathscr{R}_+ := \bigcup_{\alpha>0} \mathscr{R}_\alpha \subset \mathcal{PD}; \quad \mathscr{R}_0 \cap \mathcal{PD} = \emptyset; \quad \mathscr{C} \cap \mathcal{PD} \neq \emptyset.$$

Other properties of distributions from the class $\mathcal{PD}$ are presented in [43].

### 1.6. Haezendonck–Goovaerts Risk Measure

To give a precise definition of the HG risk measure, let us suppose that $X$ is a real-valued r.v. with a d.f. $F_X$ representing a random risk.

- *A function $\varphi$ defined on $\mathbb{R}$ is said to be a normalised Young function if $\varphi$ is nonnegative and convex on the interval $[0, \infty)$ and such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(\infty) = \infty$.*

The Orlicz space $\mathbb{L}^\varphi$ and the Orlicz heart $\mathbb{L}_0^\varphi$ of real-valued r.v.'s $X$ associated with the function $\varphi$ are defined as

$$\mathbb{L}^\varphi = \Big\{ X : \mathbb{E}(\varphi(cX)) < \infty \text{ for some } c > 0 \Big\}, \quad \mathbb{L}_0^\varphi = \Big\{ X : \mathbb{E}(\varphi(cX)) < \infty \text{ for all } c > 0 \Big\}.$$

It is obvious that $\mathbb{L}^\varphi = \mathbb{L}_0^\varphi$ if
$$\limsup_{x\to\infty} \frac{\varphi(2x)}{\varphi(x)} < \infty.$$

- *Let $\varphi$ be a Young function, $q \in (0, 1)$, and $X \in \mathbb{L}_0^\varphi$. The Haezendonck–Goovaerts (HG) risk measure for variable $X$ is defined as*

$$HG_q(X) = \inf_{x\in\mathbb{R}} \big( x + h(x, q) \big), \tag{4}$$

*where $h = h(x,q)$ is a solution of the equation*

$$\mathbb{E}\varphi\left(\frac{(X-x)^+}{h}\right) = 1-q \tag{5}$$

*if $\overline{F}_X(x) > 0$, and $h(x,q) = 0$ if $\overline{F}_X(x) = 0$.*

According to Proposition 3(b,d) from [5], the minimiser $(x_*, h_*)$ to the Equation (4) exists for all $q \in (0,1)$, and it is unique if the function $\varphi$ is strictly convex. The HG risk measure has received much attention in insurance and finance. The risk measure was first introduced by Haezendonck–Goovaerts in [3]. For more results on HG risk measures, see, e.g., [5,9,45,46] and references therein. As it is remarked in [9], an analytic expression for the risk measure $HG_q(X)$ is not possible in general because an explicit solution of Equation (5) is generally not available. However, in the case of the power function $\varphi(t) = t^\varkappa$, $\varkappa \geqslant 1$, the analytic expressions and asymptotic formulas can be derived for certain distributions. Below, we present two statements from the paper [9] (see Equality (1.3), Theorem 2.1, and Theorem 4.1).

**Theorem 3.** *Consider the power Young function $\varphi(t) = t^\varkappa$ with $\varkappa \geqslant 1$.*

(i) *If $q \in (0,1)$, $\varkappa = 1$, and $\mathbb{E}X^+ < \infty$, then*

$$HG_q(X) = \overleftarrow{F}_X(q) + \frac{\mathbb{E}\Big(X - \overleftarrow{F}_X(q)\Big)^+}{1-q}, \tag{6}$$

*where $\overleftarrow{F}_X(q) = \inf\{x \in \mathbb{R} : F_X(x) \geqslant q\} = \inf\{x \in \mathbb{R} : \overline{F}_X(x) \leqslant 1-q\}$ is the quantile function of r.v. $X$.*

(ii) *If $q \in (0,1)$, $\varkappa > 1$, $\mathbb{P}\big(X = \overleftarrow{F}_X(1)\big) = 0$, and $\mathbb{E}(X^+)^\varkappa < \infty$, then*

$$HG_q(X) = x + \left(\frac{\mathbb{E}\big((X-x)^+\big)^\varkappa}{1-q}\right)^{1/\varkappa}, \tag{7}$$

*where $x = x(q) \in \Big(-\infty, \overleftarrow{F}_X(1)\Big)$ is the unique solution of the equation*

$$\frac{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}\Big)^\varkappa}{\Big(\mathbb{E}\big((X-x)^+\big)^\varkappa\Big)^{\varkappa-1}} = 1-q. \tag{8}$$

**Theorem 4.** *Consider the power Young function $\varphi(t) = t^\varkappa$ with $\varkappa \geqslant 1$, and let $X$ be the real-valued r.v. such that $F_X \in \mathscr{R}_\alpha$ for some $\alpha > \varkappa$. Then,*

$$HG_q(X) \underset{q\uparrow 1}{\sim} \frac{\alpha(\alpha-\varkappa)^{(\varkappa/\alpha)-1}}{\varkappa^{(\varkappa-1)/\alpha}} \big(B(\alpha-\varkappa,\varkappa)\big)^{1/\alpha}\, \overleftarrow{F}_X(q),$$

*where*

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}\mathrm{d}y, \;\; a>0, b>0,$$

*is the beta function.*

In this paper, we obtain asymptotic formulas similar to those in Theorem 4 of HG risk measure for distributions from the class $\mathscr{C}$. More precisely, we derive asymptotic formulas for the HG risk measure when the risk variable $X$ represents a sum of consistently varying risks $S_n^{(\xi)}$. The main results of this paper are based on general Formulas (7), (8) and asymptotic relations (2), (3).

Note that the obtained asymptotic formulas can be applied to a special-form set of random variables $\{\theta_1 X_1, \ldots, \theta_n X_n\}$, where $\{X_1, \ldots, X_n\}$ are real-valued primary r.v.'s, and $\{\theta_1, \ldots, \theta_n\}$ are nonnegative random weights. In such a case, the sum (1) becomes the randomly weighted sum

$$S_n^{(\theta X)} = \sum_{k=1}^{n} \theta_k X_k.$$

In actuarial applications, this sum usually represents the present value of the total future net loss of the insurance company during the first $n$ time periods. In this case, the real-valued r.v. $X_k$ represents a net loss of an insurance company during the $k$-th time period, and the random weight $\theta_k$ is a stochastic discount factor from time $k$ to time 0. A financial interpretation of the sum $S_n^{(\theta X)}$ is based on the construction of the portfolio. Each investment portfolio consists of $n$ lines of risks with random losses $X_k$ and random weights $\theta_k$ describing the investment environment. In such a case, the sum $S_n^{(\theta X)}$ represents the total amount of future losses potentially incurred by the investment portfolio. The problems related to the asymptotic behavior of the sum $S_n^{(\theta X)}$ were considered in [37,47–61], and asymptotic properties of distributions of sums $S_n^{(\theta X)}$ related to the asymptotic behavior of risk measures, similar to the ES risk measure, were considered in [62–67].

The transition from the asymptotic properties of the sum $S_n^{(\xi)}$ to the analogous properties of the sum $S_n^{(\theta X)}$ is ensured by the product-convolution properties reviewed in detail in Chapter 5 of [44]. We only recall that for two independent r.v.'s. $X$ and $Y$, the distribution of the product $XY$ is described by the product-convolution of the d.f.'s. $F_X \otimes F_Y$. Below, we formulate a particular statement on the product-convolution closure (see Proposition 5.2(iii), Proposition 5.3(iii), Proposition 5.4(i) in [44]).

**Theorem 5.** *Let $F$ be a distribution on $\mathbb{R}$ and $G$ be a distribution such that $G(0-) = 0$ and $G(0) < 1$.*

(i) *If $F \in \mathscr{R}_\alpha$ and $\overline{G}(yx) = o\big(\overline{F}(x)\big)$ for some $y > 0$, then $F \otimes G \in \mathscr{R}_\alpha$;*
(ii) *If $F \in \mathscr{C}$ and $\overline{G}(yx) = o\big(\overline{F}(x)\big)$ for some $y > 0$, then $F \otimes G \in \mathscr{C}$;*
(iii) *If $F \in \mathscr{D}$, then $F \otimes G \in \mathscr{D}$.*

The rest of the paper is organised as follows. In Section 2, we provide a formulation of the main results. Section 3 deals with the auxiliary results that are essentially related to the properties of quantile functions. Section 4 presents the proof of the asymptotic formulas for the HG risk measure. Section 5 deals with the analysis of the particular example illustrating the accuracy of the derived asymptotic formula. Finally, Section 6 concludes.

## 2. Main Results

In this section, we state two theorems on the asymptotic behavior of the HG risk measure in the case where that measure is determined for the sum of distributions from the class $\mathscr{C}$. The first theorem is derived for the sum of simple r.v.'s, and the second theorem is derived for the sum of weighted r.v.'s.

**Theorem 6.** *Consider the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$, and let $\{\xi_1, \ldots, \xi_n\}$ be a collection of real-valued* pQAI *r.v.'s.*

(i) *Let* $\varkappa = 1$, $\max_{1\leqslant k\leqslant n} \mathbb{E}\big(\xi_k^+\big) < \infty$, $F_{\xi_L} \in \mathscr{C}$ *for some* $L \in \{1,\ldots,n\}$, *and for other indices* $k \neq L$ *let* $F_{\xi_k} \in \mathscr{C}$ *or* $\mathbb{P}\big(|\xi_k| > x\big) = o\big(\overline{F}_{\xi_L}(x)\big)$. *If* $F_{\xi_k} \in \mathcal{PD}$ *for all* $k \in \mathcal{I}_n = \big\{k \in \{1,\ldots,n\} : F_{\xi_k} \in \mathscr{C}\big\}$, *then*

$$HG_q\Big(S_n^{(\xi)}\Big) \underset{q\uparrow 1}{\sim} \overleftarrow{H}_n(q) + \frac{1}{1-q}\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - \overleftarrow{H}_n(q)\Big)^+,$$

*where* $\overleftarrow{H}_n$ *is the quantile function of the d.f.*

$$H_n := \max\Big\{0, 1 - \sum_{k\in\mathcal{I}_n}\overline{F}_{\xi_k}\Big\}.$$

(ii) *Let* $\varkappa \geqslant 2$, $\max_{1\leqslant k\leqslant n} \mathbb{E}\big(\xi_k^+\big)^{\varkappa} < \infty$, $F_{\xi_L} \in \mathscr{C}$ *for some* $L \in \{1,\ldots,n\}$, *and for other indices* $k \neq L$ *let* $F_{\xi_k} \in \mathscr{C}$ *or* $\mathbb{P}\big(|\xi_k| > x\big) = o\big(\overline{F}_{\xi_L}(x)\big)$. *Let, in addition,*

$$\left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^{\varkappa}\left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1} < 1 \tag{9}$$

*for some* $y > 1$, *where* $\mathcal{I}_n = \big\{k \in \{1,\ldots,\} : F_{\xi_k} \in \mathscr{C}\big\}$. *Then,*

$$HG_q\Big(S_n^{(\xi)}\Big) \underset{q\uparrow 1}{\sim} \widehat{x}(q) + \left(\frac{1}{1-q}\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa},$$

*where* $\widehat{x} = \widehat{x}(q)$ *is the unique solution to the equation*

$$\frac{\left(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}} = 1-q.$$

As mentioned earlier, our second main theorem is on the asymptotics of the HG risk measure for a weighted sum. We remark only that in this theorem, we suppose that not some, but all distributions of primary r.v.'s belong to the class $\mathscr{C}$.

**Theorem 7.** *Consider the same power Young function as in Theorem 6. Let* $\{X_1,\ldots,X_n\}$ *be a collection of real-valued pQAI r.v.'s with corresponding d.f.'s* $\{F_{X_1},\ldots,F_{X_n}\}$ *belonging to class* $\mathscr{C}$. *Let* $\{\theta_1,\ldots,\theta_n\}$ *be another collection of arbitrary dependent, nonnegative, and nondegenerate at zero r.v.'s. Suppose that collections* $\{X_1,\ldots,X_n\}$ *and* $\{\theta_1,\ldots,\theta_n\}$ *are independent and that* $\max_{1\leqslant k\leqslant n}\{\mathbb{E}\theta_k^p\}$ *is finite for some* $p > \max_{1\leqslant k\leqslant n} J^+_{F_{X_k}}$.

(i) *If* $\varkappa = 1$, $\max_{1\leqslant k\leqslant n} \mathbb{E}\big(X_k^+\big) < \infty$, *and* $F_{X_k} \in \mathcal{PD}$ *for all* $k \in \{1,\ldots,n\}$, *then,*

$$HG_q\Big(S_n^{(\theta X)}\Big) \underset{q\uparrow 1}{\sim} \overleftarrow{H}^*_n(q) + \frac{1}{1-q}\sum_{k=1}^{n}\mathbb{E}\Big(\theta_k X_k - \overleftarrow{H}^*_n(q)\Big)^+,$$

*where* $\overleftarrow{H}^*_n$ *is the quantile function of the d.f.*

$$H^*_n := \max\Big\{0, 1 - \sum_{k=1}^{n}\overline{F}_{\theta_k X_k}\Big\}.$$

(ii) *Let* $\varkappa \geqslant 2$ *and* $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}(X_k^+)^{\varkappa} < \infty$. *Let, in addition,*

$$\left(\max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\overline{F}_{X_k}(yx)}{\overline{F}_{X_k}(x)}\right)^{\varkappa} \left(\max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_k}(yx)}\right)^{\varkappa-1} < 1$$

*for some* $y > 1$. *Then,*

$$HG_q\left(S_n^{(\theta X)}\right) \underset{q\uparrow 1}{\sim} x^*(q) + \left(\frac{1}{1-q}\sum_{k=1}^{n} \mathbb{E}\Big(\big(\theta_k X_k - x^*(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa},$$

*where* $x^* = x^*(q)$ *is the unique solution to the equation*

$$\frac{\left(\sum_{k=1}^{n} \mathbb{E}\Big(\big(\theta_k X_k - x^*\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum_{k=1}^{n} \mathbb{E}\Big(\big(\theta_k X_k - x^*\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}} = 1 - q.$$

## 3. Auxiliary Statements

### *3.1. Some Properties of Quantile Functions*

In this subsection, we formulate and prove several auxiliary results on useful properties of the quantile functions. Analogs of many of the statements formulated below can be found in articles [41,42,68,69]. For completeness, we present their complete proofs.

**Lemma 1.** *Let* $F$ *be a d.f. from the class* $\mathscr{C}$. *Then,*

$$F\left(\overleftarrow{F}(q)\right) \underset{q\uparrow 1}{\sim} q, \quad \text{and} \quad \overline{F}\left(\overleftarrow{\overline{F}}(p)\right) \underset{p\downarrow 0}{\sim} p$$

*where*

$$\overleftarrow{F}(q) = \inf\{x : F(x) \geqslant q\}, q \in (0,1),$$

*is the quantile function for* $F$, *and*

$$\overleftarrow{\overline{F}}(p) = \inf\{x : \overline{F}(x) \leqslant p\}, p \in (0,1),$$

*is the quantile function for t.f.* $\overline{F}$.

**Proof.** Let $q \in (0,1)$. Since d.f. is right-continuous, the set $\{x : F(x) \geqslant q\} := \mathcal{A}_q$ is closed. Hence, $\overleftarrow{F}(q) \in \mathcal{A}_q$, which implies that

$$F\left(\overleftarrow{F}(q)\right) \geqslant q \tag{10}$$

for any $q \in (0,1)$. Therefore,

$$F\left(\overleftarrow{F}(q)\right) \underset{q\uparrow 1}{\gtrsim} q. \tag{11}$$

For the first asymptotic relation of the lemma, it remains to prove that

$$\limsup_{q\uparrow 1} \frac{F\left(\overleftarrow{F}(q)\right)}{q} \leqslant 1. \tag{12}$$

Suppose, on the contrary, that

$$\limsup_{q\uparrow 1} \frac{F\Big(\overset{\leftarrow}{F}(q)\Big)}{q} > 1.$$

Then, there exist $\Delta > 0$ and the sequence $q_n \uparrow 1$ such that

$$\lim_{n\to\infty} \frac{F\Big(\overset{\leftarrow}{F}(q_n)\Big)}{q_n} = 1+\Delta,$$

implying that $F\Big(\overset{\leftarrow}{F}(q_n)\Big) > (1+\Delta/2)q_n$ for large $n$. Since $F(x) \leqslant 1$ for all $x$, we have that

$$q_n < \big(1+\Delta/2\big)^{-1}$$

for large $n$. This is impossible because $q_n \uparrow 1$. The obtained contradiction proves estimate (12). Estimates (11) and (12) imply the first asymptotic relation of the lemma.

Now, consider the second relation of the lemma. Let $p > 0$. Due to inequality (10), we have

$$\begin{aligned}\overline{F}\Big(\overset{\leftarrow}{\overline{F}}(p)\Big) &= 1 - F\big(\inf\{x : \overline{F}(x) \leqslant p\}\big) = 1 - F\big(\inf\{x : F(x) \geqslant 1-p\}\big)\\ &= 1 - F\Big(\overset{\leftarrow}{F}(1-p)\Big) \leqslant p, \end{aligned} \tag{13}$$

implying that

$$\overline{F}\Big(\overset{\leftarrow}{\overline{F}}(p)\Big) \underset{p\downarrow 0}{\lesssim} p. \tag{14}$$

It remains to prove that

$$\liminf_{p\downarrow 0} \frac{\overline{F}\Big(\overset{\leftarrow}{\overline{F}}(p)\Big)}{p} \geqslant 1. \tag{15}$$

Suppose, on the contrary, that

$$\liminf_{p\downarrow 0} \frac{\overline{F}\Big(\overset{\leftarrow}{\overline{F}}(p)\Big)}{p} < 1.$$

From this, it follows that there exist $\Delta \in (0,1)$ and a sequence $p_n \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{\overline{F}\Big(\overset{\leftarrow}{\overline{F}}(p_n)\Big)}{p_n} = 1-\Delta.$$

Hence,

$$\overline{F}\Big(\overset{\leftarrow}{\overline{F}}(p_n)\Big) = \overline{F}\big(\inf\{x : \overline{F}(x) \leqslant p_n\}\big) < (1-\Delta/2)p_n,$$

or, equivalently,

$$F\big(\inf\{x : \overline{F}(x) \leqslant p_n\}\big) = F\big(\inf\{x : F(x) \geqslant 1-p_n\big) > 1 - p_n + p_n(\Delta/2)$$

for large $n$.

Let us temporarily denote

$$x_n = \inf\{x : F(x) \geqslant 1 - p_n\}.$$

Since $\overline{F}(x) > 0$ for all $x$, the sequence $x_n$ is nondecreasing, and $x_n \uparrow \infty$. In addition, the d.f. $F$ is right-continuous so that $F(x_n) \geqslant 1 - p_n$. For fixed $n$, there are two possible cases: $F(x_n) = 1 - p_n$ or $F(x_n) > 1 - p_n$. In the first case, we get

$$1 - p_n > 1 - p_n + p_n(\Delta/2),$$

which is impossible due to the positivity of $\Delta$. Consequently, the d.f. $F$ has a jump at each point $x_n$, from which it follows that

$$F\big((1-\varepsilon_n)x_n\big) < 1 - p_n \text{ and } F(x_n) > (1-p_n) + p_n(\Delta/2),$$

or, equivalently,

$$\overline{F}\big((1-\varepsilon_n)x_n\big) > p_n \text{ and } \overline{F}(x_n) < p_n\big(1 - \Delta/2\big)$$

for some sequence $\varepsilon_n \downarrow 0$. These estimates imply that

$$\lim_{n\to\infty} \frac{\overline{F}(((1-\varepsilon_n)x_n)}{\overline{F}(x_n)} \geqslant \frac{2}{2-\Delta} > 1$$

for some sequences $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$. The derived relation contradicts the condition $F \in \mathscr{C}$, because it should be

$$\lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \frac{\overline{F}\big((1-\varepsilon)x\big)}{\overline{F}(x)} \leqslant 1.$$

by the definition of the class $\mathscr{C}$ presented in Section 1.2.

The obtained contradiction proves the validity of inequality (15), which, together with estimate (14), proves the second asymptotic relation of the lemma. □

**Lemma 2.** *Let $F$ be a d.f. from the class $\mathcal{PD}$. Then,*

$$\overleftarrow{\overline{F}}\,\big(\overline{F}(x)\big) \underset{x\to\infty}{\sim} x, \quad \text{and} \quad \overleftarrow{F}\,\big(F(x)\big) \underset{x\to\infty}{\sim} x,$$

*where the quantile functions $\overleftarrow{\overline{F}}$ and $\overleftarrow{F}$ are defined in Lemma 1.*

**Proof.** Consider the first relation. By the definition of the quantile function $\overleftarrow{\overline{F}}$, we have

$$\overleftarrow{\overline{F}}\,\big(\overline{F}(x)\big) = \inf\{y : \overline{F}(y) \leqslant \overline{F}(x)\}.$$

Since the function $\overline{F}$ is nonincreasing, that is $\overline{F}(y) \leqslant \overline{F}(x)$ for $y \geqslant x$, we have that $\overleftarrow{\overline{F}}\,\big(\overline{F}(x)\big) \leqslant x$, implying

$$\overleftarrow{\overline{F}}\,\big(\overline{F}(x)\big) \underset{x\to\infty}{\lesssim} x. \tag{16}$$

It remains to prove that

$$\overleftarrow{\overline{F}}\left(\overline{F}(x)\right) \underset{x\to\infty}{\gtrsim} x. \tag{17}$$

Suppose, on the contrary, that

$$\liminf_{x\to\infty} \frac{\overleftarrow{\overline{F}}\left(\overline{F}(x)\right)}{x} < 1.$$

In such a case, there exist $\Delta \in (0,1)$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overleftarrow{\overline{F}}\left(\overline{F}(x_n)\right)}{x_n} = 1 - \Delta.$$

Hence,

$$\overleftarrow{\overline{F}}\left(\overline{F}(x_n)\right) = \inf\{y : \overline{F}(y) \leqslant \overline{F}(x_n)\} \leqslant (1 - 2\Delta/3)x_n$$

for sufficiently large $n$, say $n \geqslant N_\Delta$. This relation implies that

$$\overline{F}\left(\left(1 - \frac{\Delta}{2}\right)x_n\right) \leqslant \overline{F}(x_n)$$

Since the function $\overline{F}$ is nonincreasing, it must be

$$\overline{F}\left(\left(1 - \frac{\Delta}{2}\right)x_n\right) = \overline{F}(x_n)$$

for $n \geqslant N_\Delta$. Therefore,

$$\lim_{n\to\infty} \frac{\overline{F}\left(\left(1 - \frac{\Delta}{2}\right)x_n\right)}{\overline{F}(x_n)} = 1,$$

which contradicts the definition of $F \in \mathcal{PD}$ presented in Section 1.5. Consequently, the asymptotic relation (17) holds. Relations (16) and (17) imply the first one of the lemma.

The second relation of the lemma follows from the equalities

$$\begin{aligned}
\lim_{x\to\infty} \frac{\overleftarrow{F}\left(F(x)\right)}{x} &= \lim_{x\to\infty} \frac{1}{x}\inf\{y : F(y) \geqslant F(x)\} \\
&= \lim_{x\to\infty} \frac{1}{x}\inf\{y : \overline{F}(y) \leqslant \overline{F}(x)\} \\
&= \lim_{x\to\infty} \frac{\overleftarrow{\overline{F}}\left(\overline{F}(x)\right)}{x} = 1.
\end{aligned}$$

The lemma is now proved. □

**Lemma 3.** *Let d.f. $F \in \mathcal{PD}$. Then, for any two positive functions $a(x)$ and $b(x)$ such that $a(x) \underset{x\to\infty}{\sim} b(x)$ and $a(x) \underset{x\to\infty}{\to} 0$, we have that*

$$\overleftarrow{\overline{F}}\left(a(x)\right) \underset{x\to\infty}{\sim} \overleftarrow{\overline{F}}\left(b(x)\right)$$

*and for any positive functions $\hat{a}(p)$, $\hat{b}(p)$ such that $\hat{a}(p) \underset{p\downarrow 0}{\sim} \hat{b}(p)$ and $\hat{a}(p) \underset{p\downarrow 0}{\to} 0$, we have*

$$\overleftarrow{\overline{F}}\left(\hat{a}(p)\right) \underset{p\downarrow 0}{\sim} \overleftarrow{\overline{F}}\left(\hat{b}(p)\right).$$

**Proof.** It suffices to prove the first equivalence relation because the second relation follows from the first one by noting that

$$\lim_{p\downarrow 0}\frac{\overleftarrow{\overline{F}}\left(\hat{a}(p)\right)}{\overleftarrow{\overline{F}}\left(\hat{b}(p)\right)}=\lim_{x\to\infty}\frac{\overleftarrow{\overline{F}}\left(\hat{a}\left(\frac{1}{x}\right)\right)}{\overleftarrow{\overline{F}}\left(\hat{b}\left(\frac{1}{x}\right)\right)}.$$

Consider the first relation. Firstly, we will show that

$$\liminf_{x\to\infty}\frac{\overleftarrow{\overline{F}}(a(x))}{\overleftarrow{\overline{F}}(b(x))}\geqslant 1. \tag{18}$$

We will proceed with the proof by contradiction again. Assume that

$$\liminf_{x\to\infty}\frac{\overleftarrow{\overline{F}}(a(x))}{\overleftarrow{\overline{F}}(b(x))}< 1.$$

This means that there exist $\Delta > 0$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty}\frac{\overleftarrow{\overline{F}}(a(x_n))}{\overleftarrow{\overline{F}}(b(x_n))}=1-\Delta.$$

It follows that

$$\frac{\overleftarrow{\overline{F}}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}<\overleftarrow{\overline{F}}(b(x_n))$$

for large enough $n$. By the definition of $\overleftarrow{\overline{F}}$, we have

$$\overline{F}\left(\frac{\overleftarrow{\overline{F}}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}\right)>b(x_n).$$

This relation, together with (13), implies

$$\frac{\overline{F}\left(\left(1-\frac{\Delta}{2}\right)\frac{\overleftarrow{\overline{F}}(a(x_n))}{(1-\frac{\Delta}{2})}\right)}{\overline{F}\left(\frac{\overleftarrow{\overline{F}}(a(x_n))}{(1-\frac{\Delta}{2})}\right)}=\frac{\overline{F}\left(\overleftarrow{\overline{F}}\left(a(x_n)\right)\right)}{\overline{F}\left(\frac{\overleftarrow{\overline{F}}(a(x_n))}{(1-\frac{\Delta}{2})}\right)}\leqslant\frac{a(x_n)}{b(x_n)}\underset{n\to\infty}{\to}1,$$

since $a(x)\underset{x\to\infty}{\sim}b(x)$. Thus,

$$\lim_{n\to\infty}\frac{\overline{F}\left(\left(1-\frac{\Delta}{2}\right)\frac{\overleftarrow{\overline{F}}(a(x_n))}{(1-\frac{\Delta}{2})}\right)}{\overline{F}\left(\frac{\overleftarrow{\overline{F}}(a(x_n))}{(1-\frac{\Delta}{2})}\right)}\leqslant 1$$

for some sequence $\overleftarrow{\overline{F}}(a(x_n))/(1-\Delta/2)\underset{n\to\infty}{\to}\infty$. This contradicts the definition of $F\in\mathcal{PD}$ in Section 1.2.

It remains to prove that

$$\limsup_{x\to\infty} \frac{\overleftarrow{\overline{F}}(a(x))}{\overleftarrow{\overline{F}}(b(x))} \leqslant 1.$$

Suppose, on the contrary, that

$$\limsup_{x\to\infty} \frac{\overleftarrow{\overline{F}}(a(x))}{\overleftarrow{\overline{F}}(b(x))} > 1.$$

It follows that there exist $\Delta > 0$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overleftarrow{\overline{F}}(a(x_n))}{\overleftarrow{\overline{F}}(b(x_n))} = 1 + \Delta.$$

Consequently,

$$\overleftarrow{\overline{F}}(a(x_n)) > \left(1 + \frac{\Delta}{2}\right) \overleftarrow{\overline{F}}(b(x_n))$$

for large $n$. By the definition of $\overleftarrow{\overline{F}}$, it follows that

$$a(x_n) < \overline{F}\left(\left(1 + \frac{\Delta}{2}\right) \overleftarrow{\overline{F}}(b(x_n))\right).$$

This relation, together with (13), implies

$$\frac{\overline{F}\left(\left(1 + \frac{\Delta}{2}\right) \overleftarrow{\overline{F}}(b(x_n))\right)}{\overline{F}\left(\overleftarrow{\overline{F}}(b(x_n))\right)} \geqslant \frac{a(x_n)}{b(x_n)} \underset{n\to\infty}{\to} 1,$$

since $a(x) \underset{x\to\infty}{\sim} b(x)$. Thus,

$$\lim_{n\to\infty} \frac{\overline{F}\left(\left(1 + \frac{\Delta}{2}\right) \overleftarrow{\overline{F}}\,(b(x_n))\right)}{\overline{F}\left(\overleftarrow{\overline{F}}\,(b(x_n))\right)} \geqslant 1$$

for some sequence $\overleftarrow{\overline{F}}(b(x_n)) \underset{n\to\infty}{\to} \infty$. This contradicts the definition of $F \in \mathcal{PD}$ in Section 1.2. The lemma is now proved. □

**Lemma 4.** *Let $F$ and $G$ be two d.f.'s such that $F \in \mathscr{C} \cap \mathcal{PD}$, and $\overline{G}(x) \underset{x\to\infty}{\sim} c\overline{F}(x)$ for some constant $c > 0$. Then,*

$$\overleftarrow{G}(1-p) \underset{p\downarrow 0}{\sim} \overleftarrow{\overline{F}}\left(1 - \frac{p}{c}\right).$$

**Proof.** The condition $F \in \mathcal{PD}$ and Lemmas 2 and 3 imply that

$$x \underset{x\to\infty}{\sim} \overleftarrow{\overline{F}}\left(\overline{F}(x)\right) \underset{x\to\infty}{\sim} \overleftarrow{\overline{F}}\left(\frac{\overline{G}(x)}{c}\right). \tag{19}$$

The conditions $F \in \mathscr{C} \cap \mathcal{PD}$ and $\overline{G}(x) \underset{x\to\infty}{\sim} c\overline{F}(x)$ imply that $G \in \mathscr{C} \cap \mathcal{PD}$. Hence, $\overline{G}(x) > 0$ for all $x \in \mathbb{R}$, and therefore, $\overleftarrow{\overline{G}}(p) \to \infty$ as $p \downarrow 0$. By taking $x = \overleftarrow{\overline{G}}(p)$ in (19), we get

$$\overleftarrow{\overline{G}}(p) \underset{p\downarrow 0}{\sim} \overleftarrow{\overline{F}}\left(\frac{1}{c}\overline{G}\Big(\overleftarrow{\overline{G}}(p)\Big)\right).$$

By Lemma 1, we have

$$\overline{G}\Big(\overleftarrow{\overline{G}}(p)\Big) \underset{p\downarrow 0}{\sim} p.$$

Hence, from Lemma 3, we derive that

$$\overleftarrow{\overline{G}}(p) \underset{p\downarrow 0}{\sim} \overleftarrow{\overline{F}}\left(\frac{p}{c}\right),$$

or, equivalently,

$$\overleftarrow{G}(1-p) \underset{p\downarrow 0}{\sim} \overleftarrow{F}\left(1-\frac{p}{c}\right).$$

The lemma is proved. □

*3.2. Properties of the Special Function in Equation (8)*

Let $X$ be an r.v. such that $\mathbb{E}(X^+)^\varkappa < \infty$ and $\overline{F}_X(x) = \mathbb{P}(X > x) > 0$ for all $x \in \mathbb{R}$. Let

$$\gamma_X(x) = \frac{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\Big)^{\varkappa-1}}, \quad x \in \mathbb{R}, \tag{20}$$

be a well-defined function from Equation (8). In this subsection, we consider properties of the function $\gamma_X$ and its components. The first such statement is proved in [9] (Lemma 2.1).

**Lemma 5.** *Let $X$ be an r.v. such that $\mathbb{E}(X^+)^\varkappa < \infty$ and $\overline{F}_X(x) > 0$ for all real $x$. If $\varkappa > 1$, then function $g(x) := \mathbb{E}\big((X-x)^+\big)^\varkappa$ is continuously differentiable with*

$$g'(x) = -\varkappa\,\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}, \quad x \in \mathbb{R}.$$

*If $\varkappa = 1$, then for each $x \in \mathbb{R}$, the derivative from the right $g'_+(x) = -\overline{F}(x)$ and the derivative from the left $g'_-(x) = -\overline{F}(x-0)$.*

Other statements directly concern the properties of the function $\gamma_X$.

**Lemma 6.** *Let $X$ be an r.v. such that $\mathbb{E}(X^+)^\varkappa < \infty$ and $\overline{F}_X(x) > 0$ for all $x \in \mathbb{R}$. Let $\gamma_X$ be the function defined in Equation (20). If $\varkappa > 1$, then*

$$\lim_{x\to-\infty} \gamma_X(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} \gamma_X(x) = 0.$$

**Proof.** While the main steps of the proof are outlined in [9] (see the proof of Theorem 2.1), we provide the complete proof here in more detail. Since $F_X(x) > 0$, we have that

$$\mathbb{E}\big((X-x)^+\big)^\varkappa = \varkappa \int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u > 0$$

for any $x \in \mathbb{R}$. By using the classical Hölder's inequality, we get

$$\gamma_X(x) = \frac{\Big(\mathbb{E}\Big(\big((X-x)^+\big)^{\varkappa-1}\mathbb{1}_{(x,\infty)}(X)\Big)\Big)^{\varkappa}}{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\Big)^{\varkappa-1}}$$

$$\leqslant \frac{1}{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\Big)^{\varkappa-1}}\left\{\Big(\mathbb{E}\big((X-x)^+\big)^{(\varkappa-1)\frac{\varkappa}{\varkappa-1}}\Big)^{\frac{\varkappa-1}{\varkappa}}\Big(\mathbb{E}\Big(\mathbb{1}_{(x,\infty)}(X)\Big)^{\varkappa}\Big)^{\frac{1}{\varkappa}}\right\}^{\varkappa}$$

$$= \overline{F}_X(x),$$

which implies the second equality of the lemma.

Let us consider the first equality. It is obvious that

$$\lim_{x\to-\infty}\gamma_X(x) = \lim_{x\to\infty}\frac{\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa}\right)^{\varkappa-1}}. \tag{21}$$

The classical $c_r$-inequality and inequality $(a+b)^+ \leqslant a^+ + b^+$ provided for any real numbers $a, b$ imply that for positive $x$

$$\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1} \leqslant \mathbb{E}\left(\frac{X^+ + x}{x}\right)^{\varkappa-1} \leqslant c_{\varkappa}\left(\frac{1}{x^{\varkappa-1}}\mathbb{E}(X^+)^{\varkappa-1} + 1\right) < \infty,$$

where

$$c_{\varkappa} = \begin{cases} 1 & \text{if} \quad 1 < \varkappa \leqslant 2, \\ 2^{\varkappa-2} & \text{if} \quad \varkappa > 2. \end{cases}$$

Hence,

$$\lim_{x\to\infty}\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1}\right)^{\varkappa} = \left(\mathbb{E}\lim_{x\to\infty}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1}\right)^{\varkappa} = 1$$

by the dominated convergence theorem.
Similarly,

$$\lim_{x\to\infty}\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa}\right)^{\varkappa-1} = 1.$$

Now, we can derive the first equality of the lemma by substituting the last two relations into equality (21). The lemma is proved. □

**Lemma 7.** *Let $X$ be an r.v. such that $\mathbb{E}(X^+)^{\varkappa} < \infty$ and $\overline{F}_X(x) > 0$ for $x \in \mathbb{R}$. If $\varkappa \geqslant 2$, then the function $\gamma_X$ is continuous and strongly decreasing on $\mathbb{R}$.*

**Proof.** At first, let us consider the case $\varkappa > 2$. By denoting $\eta_x = (X-x)^+$, we have

$$\gamma_X(x) = \frac{(\mathbb{E}\eta_x^{\varkappa-1})^{\varkappa}}{(\mathbb{E}\eta_x^{\varkappa})^{\varkappa-1}}.$$

Due to Lemma 5, we get

$$\gamma_X'(x) = -\varkappa(\varkappa-1)\frac{(\mathbb{E}\eta_x^{\varkappa-1})^{\varkappa-1}}{(\mathbb{E}\eta_x^{\varkappa})^{\varkappa}}\Big(\mathbb{E}\eta_x^{\varkappa-2}\mathbb{E}\eta_x^{\varkappa} - \big(\mathbb{E}\eta_x^{\varkappa-1}\big)^2\Big), \; x \in \mathbb{R}.$$

The classical Cauchy–Schwarz inequality and condition $\overline{F}_X(x) > 0,\ x \in \mathbb{R}$, imply that

$$\mathbb{E}\eta_x^{\varkappa-1} = \mathbb{E}\eta_x^{\frac{\varkappa-2}{2}}\eta_x^{\frac{\varkappa}{2}} < \left(\mathbb{E}\eta_x^{\varkappa-2}\right)^{\frac{1}{2}}\left(\mathbb{E}\eta_x^{\varkappa}\right)^{\frac{1}{2}}.$$

Therefore, $\gamma_X'(x) < 0$ for $x \in \mathbb{R}$ implying the assertion of the lemma in the case $\varkappa > 2$.

Now, let us suppose that $\varkappa = 2$. In this case,

$$\gamma_X(x) = \frac{\left(\mathbb{E}\eta_x\right)^2}{\mathbb{E}\eta_x^2}.$$

Using Lemma 5, we obtain the derivative from the right

$$\{\gamma_X\}_+'(x) = -2\,\frac{\mathbb{E}\eta_x}{\left(\mathbb{E}\eta_x^2\right)^2}\left(\overline{F}(x)\mathbb{E}\eta_x^2 - \left(\mathbb{E}\eta_x\right)^2\right)$$

is negative for any $x \in \mathbb{R}$ due to the Hölder inequality

$$\mathbb{E}\eta_x = \mathbb{E}\,\eta_x\mathbb{1}_{(x,\infty)}(X) < \left(\mathbb{E}\,\eta_x^2\right)^{\frac{1}{2}}\left(\mathbb{E}\mathbb{1}_{(x,\infty)}(X)\right)^{\frac{1}{2}} = \left(\mathbb{E}\,\eta_x^2\right)^{\frac{1}{2}}\left(\overline{F}(x)\right)^{\frac{1}{2}},$$

where the strictness of the inequality follows from the condition $\overline{F}_X(x) > 0, x \in \mathbb{R}$. In the same way, we get that the derivative from the left $\{\gamma_X\}_-'(x)$ is also negative for any real $x$ because

$$\mathbb{E}\eta_x\mathbb{1}_{(x,\infty)}(X) = \mathbb{E}\eta_x\mathbb{1}_{[x,\infty)}(X),\ x \in \mathbb{R}.$$

The derived properties of the function $\gamma_X$ imply that this function is continuous and strongly decreasing on $\mathbb{R}$. The lemma is proved. □

**Lemma 8.** *Let $X$ be an r.v. such that $\mathbb{E}(X^+)^{\varkappa} < \infty$ and $F_X \in \mathscr{C}$. If $\varkappa > 1$, then*

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\gamma_X(xy)}{\gamma_X(x)} \leqslant 1.$$

**Proof.** If $\mathbb{E}(X^+)^{\varkappa}$ is finite, then

$$\mathbb{E}\big((X-x)^+\big)^p = p\int_x^\infty (u-x)^{p-1}\mathbb{P}(X>u)\mathrm{d}u$$

for any $0 < p \leqslant \varkappa$. Therefore, for all positive $x$ and $y$

$$\begin{aligned}\frac{\gamma_X(xy)}{\gamma_X(x)} &= \left(\frac{\mathbb{E}\big((X-xy)^+\big)^{\varkappa-1}}{\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}}\right)^{\varkappa}\left(\frac{\mathbb{E}\big((X-x)^+\big)^{\varkappa}}{\mathbb{E}\big((X-yx)^+\big)^{\varkappa}}\right)^{\varkappa-1}\\ &= \left(\frac{\int_{xy}^\infty (u-xy)^{\varkappa-2}\overline{F}_X(u)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_X(u)\mathrm{d}u}\right)^{\varkappa}\left(\frac{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u}{\int_{xy}^\infty (u-xy)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u}\right)^{\varkappa-1}\\ &= \left(\frac{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_X(uy)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_X(u)\mathrm{d}u}\right)^{\varkappa}\left(\frac{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(uy)\mathrm{d}u}\right)^{\varkappa-1}\\ &\leqslant \left(\sup_{u\geqslant x}\frac{\overline{F}_X(uy)}{\overline{F}_X(u)}\right)^{\varkappa}\left(\sup_{u\geqslant x}\frac{\overline{F}_X(u)}{\overline{F}_X(uy)}\right)^{\varkappa-1}.\end{aligned} \tag{22}$$

If $y \in (0,1)$, then

$$\limsup_{x\to\infty}\frac{\gamma_X(xy)}{\gamma_X(x)} \leqslant \left(\limsup_{x\to\infty}\sup_{u\geqslant x}\frac{\overline{F}_X(uy)}{\overline{F}_X(u)}\right)^{\varkappa}.$$

Since d.f. $F_X \in \mathscr{C}$, the last inequality implies the statement of the lemma. □

**Lemma 9.** *Let $X$ be an r.v. with d.f. $F_X$ such that $\mathbb{E}(X^+)^{\varkappa} < \infty$ and*

$$\left(\limsup_{x\to\infty} \frac{\overline{F}_X(yx)}{\overline{F}_X(x)}\right)^{\varkappa} \left(\limsup_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(yx)}\right)^{\varkappa-1} < 1 \tag{23}$$

*for some $\varkappa > 1$ and some $y > 1$. Then,*

$$\limsup_{x\to\infty} \frac{\gamma_X(xy)}{\gamma_X(x)} < 1.$$

*for some $y > 1$.*

**Remark 1.** *Since $\overline{F}_X(x) \geqslant \overline{F}_X(vx)$ for any $v > 1$, condition (23) implies that $F_X \in \mathcal{PD}$. If $F_X \in \mathscr{R}_\alpha$ for some $\alpha > 0$, then the condition (23) holds for all $\varkappa > 1$. If $F_X \in \mathscr{ERV}_{\{\alpha,\beta\}} \subset \mathscr{C}$ with some $0 < \alpha < \beta < \infty$, then condition (23) holds in the case $1 < \varkappa < \frac{\beta}{\beta-\alpha}$.*

**Proof of the lemma.** Let $\varkappa > 1$ and $y > 1$ be such that condition (23) holds. By estimate (22), we obtain

$$\begin{aligned}\limsup_{x\to\infty} \frac{\gamma_X(xy)}{\gamma_X(x)} &\leqslant \limsup_{x\to\infty}\left(\sup_{u\geqslant x} \frac{\overline{F}_X(uy)}{\overline{F}_X(u)}\right)^{\varkappa}\left(\sup_{u\geqslant x} \frac{\overline{F}_X(u)}{\overline{F}_X(uy)}\right)^{\varkappa-1} \\ &= \left(\limsup_{x\to\infty} \frac{\overline{F}_X(xy)}{\overline{F}_X(x)}\right)^{\varkappa}\left(\limsup_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(xy)}\right)^{\varkappa-1}.\end{aligned}$$

Now, the lemma follows from condition (23). The lemma is proved. □

**Corollary 1.** *Let $\varkappa \geqslant 2$ and let $X$ be an r.v. with d.f. $F_X$ satisfying condition (23). Then, the function $\gamma_X$ defined in equality (20) is the t.f. of d.f. $1-\gamma_X$ belonging to the class $\mathscr{C} \cap \mathcal{PD}$.*

The corollary immediately follows from Lemmas 6–9.

### *3.3. Some Closure Properties*

In this subsection, we present some closure properties of d.f.'s with respect to the dependence structure and the product convolution. We use the results of this subsection to derive Theorem 7. The proof of the first such lemma can be found in [18] (Lemma 4).

**Lemma 10.** *Let $X_1, X_2$ be QAI r.v.'s with d.f. $F_{X_1} \in \mathscr{D}$, $F_{X_2} \in \mathscr{D}$. Let $\theta_1, \theta_2$ be two nonnegative nondegenerate at zero r.v.'s such that vectors $(X_1, X_2)$ and $(\theta_1, \theta_2)$ are independent. If $\max\{\theta_1^p, \theta_2^p\} < \infty$ for some $p > \max\{J^+_{F_{X_1}}, J^+_{F_{X_2}}\}$ then the r.v.'s $\theta_1 X_1$ and $\theta_2 X_2$ are QAI.*

The second lemma on the closure of class $\mathscr{C}$ with respect to the product convolution is proved in [70] (Lemma 2.5).

**Lemma 11.** *Let $X$ and $\theta$ be independent r.v.'s, and let d.f. $F_X$ belong to the class $\mathscr{C}$. If $\theta$ is a nonnegative, nondegenerate at zero r.v. such that $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_X}$, then d.f. $F_{\theta X}$ of the product $\theta X$ also belongs to the class $\mathscr{C}$.*

The next lemma is on the closure of class $\mathcal{PD}$.

**Lemma 12.** *Let $X$ be an r.v. with d.f. $F_X \in \mathcal{PD} \cap \mathscr{D}$, and let $\theta$ be an independent of $X$, nonnegative, and nondegenerate at zero r.v. such that $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_X}$. Then, the product d.f. $F_{\theta X}$ belongs to the class $\mathcal{PD}$.*

**Proof.** Fix $y > 1$ such that

$$\limsup_{x\to\infty} \frac{\overline{F}_X(xy)}{\overline{F}_X(x)} < 1,$$

and let $a \in (0,1)$ and $b > 0$ be such that $\mathbb{P}(b > 0) > 0$. If $x$ is sufficiently large, then

$$\begin{aligned}
\frac{\overline{F}_{\theta X}(xy)}{\overline{F}_{\theta X}(x)} &= \frac{\mathbb{P}(\theta X > xy, \theta \leqslant x^a)}{\mathbb{P}(\theta X > x)} + \frac{\mathbb{P}(\theta X > xy, \theta > x^a)}{\mathbb{P}(\theta X > x)} \\
&\leqslant \frac{\mathbb{P}(\theta X > xy, \theta \leqslant x^a)}{\mathbb{P}(\theta X > x, \theta \leqslant x^a)} + \frac{\mathbb{P}(\theta > x^a)}{\mathbb{P}(\theta X > x, \theta > b)} \\
&\leqslant \frac{\mathbb{P}(\theta X > xy, 0 < \theta \leqslant x^a)}{\mathbb{P}(\theta X > x, 0 < \theta \leqslant x^a)} + \frac{\mathbb{E}\theta^p}{x^{ap}\mathbb{P}(\theta > b)\mathbb{P}(X > x/b)} \\
&\leqslant \sup_{0 < u \leqslant x^a} \frac{\overline{F}_X\left(\frac{yx}{u}\right)}{\overline{F}_X\left(\frac{x}{u}\right)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{x^{ap}\overline{F}_X(x)} \frac{\overline{F}_X(x)}{\overline{F}\left(\frac{x}{b}\right)} \\
&= \sup_{z \geqslant x^{1-a}} \frac{\overline{F}_X(zy)}{\overline{F}_X(z)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{x^{ap}\overline{F}_X(x)} \frac{\overline{F}_X(x)}{\overline{F}\left(\frac{x}{b}\right)}.
\end{aligned} \tag{24}$$

Since $F_X \in \mathscr{D}$, it is obvious that $\overline{F}_X = O\big(\overline{F}_X(x/b)\big)$. Moreover, $\lim\limits_{x\to\infty} x^q \overline{F}_X(x) = \infty$ for $q > J^+_{F_X}$; see, e.g., Lemma 3.5 in [54]. If $p > J^+_{F_X}$, then there exists $a \in (0,1)$ such that $ap > J^+_{F_X}$. For this particular $a$, the second term on the right side of (24) tends to zero as $x$ tends to infinity. Consequently, for this particular $a \in (0,1)$

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(xy)}{\overline{F}_{\theta X}(x)} \leqslant \limsup_{x\to\infty} \sup_{z \geqslant x^{1-a}} \frac{\overline{F}_X(zy)}{\overline{F}_X(z)} = \limsup_{x\to\infty} \frac{\overline{F}_X(xy)}{\overline{F}_X(x)} < 1. \tag{25}$$

The lemma is proved. We remark only that a similar statement, but for a positive random weight, is proved in Theorem 4.1 of [71]. □

**Lemma 13.** *Let $X$ be an r.v. with d.f. $F_X \in \mathscr{D}$, and let $\theta$ be an independent of $X$, nonnegative, and nondegenerate at zero r.v. such that $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_X}$. Then, for all $y > 0$*

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(yx)}{\overline{F}_{\theta X}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_X(yx)}{\overline{F}_X(x)} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(x)}{\overline{F}_{\theta X}(yx)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(yx)}.$$

**Proof.** According to (25)

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(yx)}{\overline{F}_{\theta X}(x)} \leqslant \limsup_{x\to\infty} \sup_{z \geqslant x^{1-a}} \frac{\overline{F}_X(yz)}{\overline{F}_X(z)} = \limsup_{x\to\infty} \frac{\overline{F}_X(yx)}{\overline{F}_X(x)}$$

with some special $a \in (0,1)$. Hence, the first inequality of the lemma holds. Similarly to (24), we obtain

$$\frac{\overline{F}_{\theta X}(x)}{\overline{F}_{\theta X}(yx)} \leqslant \sup_{z \geqslant x^{1-a}} \frac{\overline{F}(z)}{\overline{F}(yz)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{(yx)^{ap}\overline{F}_X(x)} \frac{\overline{F}_X(x)}{\overline{F}_X(yx/b)}$$

for large $x$, $b > 0$ such that $\mathbb{P}(\theta > b) > 0$, and for $a \in (0,1)$ such that $ap > J^{+}_{F_X}$. The same arguments as in the derivation of (25) imply the second inequality of the lemma. The lemma is proved. $\square$

## 4. Proof of the Main Results

**Proof of Theorem 6.**

• Let us begin with part (i) of the theorem. By relation (2) of Theorem 2, we have

$$\mathbb{E}\Big(\big(S_n^{(\xi)}\big)^{\beta}\mathbb{I}_{\{S_n^{(\xi)}>x\}}\Big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k^{\beta}\,\mathbb{I}_{\{\xi_k>x\}}\Big), \tag{26}$$

for any $\beta \in [0,1]$. In the particular case, if $\beta = 0$, we get

$$\overline{F}_{S_n^{(\xi)}}(x) = \mathbb{P}(S_n^{(\xi)} > x) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \overline{F}_{\xi_k}(x) \underset{x\to\infty}{\sim} \overline{H}_n(x). \tag{27}$$

We observe that $\overline{F}_{\xi_k}(x) > 0$ for each $k \in \mathcal{I}_n$ if $x$ is sufficiently large. Hence, according to the min-max inequality

$$\min\left\{\frac{a_1}{b_1},\ldots\frac{a_m}{b_m}\right\} \leqslant \frac{a_1+\ldots+a_m}{b_1+\ldots+b_m} \leqslant \left\{\frac{a_1}{b_1},\ldots\frac{a_m}{b_m}\right\}, \tag{28}$$

provided if $a_i \geqslant 0$, and $b_i > 0$ for $i \in \{1,\ldots,m\}$, we get that

$$\min_{k\in\mathcal{I}_n}\frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \leqslant \frac{\overline{H}_n(yx)}{\overline{H}_n(x)} \leqslant \max_{k\in\mathcal{I}_n}\frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \tag{29}$$

for each $y \in (0,1]$ and sufficiently large $x$. This double inequality shows that the d.f. $H_n$, together with the d.f. $F_{S_n^{(\xi)}}$, belongs to the class $\mathscr{C} \cap \mathcal{PD}$. Hence, by Lemma 4, we have

$$\overleftarrow{F}_{S_n^{(\xi)}}(q) \underset{q\uparrow 1}{\sim} \overleftarrow{H}_n(q). \tag{30}$$

For $q \in (0,1)$, by equality (6) and the min-max inequality (28), we derive that

$$\begin{aligned}
&\frac{HG_q\big(S_n^{(\xi)}\big)}{\overleftarrow{H}_n(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - \overleftarrow{H}_n(q)\Big)^{+}} \\
&\leqslant \max\left\{\frac{\overleftarrow{F}_{S_n^{(\xi)}}(q)}{\overleftarrow{H}_n(q)}, \frac{\mathbb{E}\Big(S_n^{(\xi)} - \overleftarrow{F}_{S_n^{(\xi)}}(q)\Big)^{+}}{\mathbb{E}\Big(S_n^{(\xi)} - \overleftarrow{H}_n(q)\Big)^{+}}\, \frac{\mathbb{E}\Big(S_n^{(\xi)} - \overleftarrow{H}_n(q)\Big)^{+}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - \overleftarrow{H}_n(q)\Big)^{+}}\right\} \\
&\leqslant \frac{\overleftarrow{F}_{S_n^{(\xi)}}(q)}{\overleftarrow{H}_n(q)} \max\left\{1, \frac{\mathbb{E}\Big(S_n^{(\xi)} - \overleftarrow{H}_n(q)\Big)^{+}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - \overleftarrow{H}_n(q)\Big)^{+}} \sup_{u\geqslant \overleftarrow{H}_n(q)} \frac{\mathbb{P}\Big(S_n^{(\xi)} > u\,\overleftarrow{F}_{S_n^{(\xi)}}(q)/\overleftarrow{H}_n(q)\Big)}{\mathbb{P}\big(S_n^{(\xi)} > u\big)}\right\}
\end{aligned} \tag{31}$$

because, according to the alternative expectation formula,

$$\frac{\mathbb{E}(\eta - x_1)^{+}}{\mathbb{E}(\eta - x_2)^{+}} = \frac{\int\limits_{x_1}^{\infty}\mathbb{P}(\eta > u)\mathrm{d}u}{\int\limits_{x_2}^{\infty}\mathbb{P}(\eta > u)\mathrm{d}u} \leqslant \frac{x_1}{x_2}\sup_{u\geqslant x_2}\frac{\mathbb{P}(\eta > ux_1/x_2)}{\mathbb{P}(\eta > u)}$$

for all positive $x_1, x_2$, and every r.v. $\eta$ such that $\mathbb{P}(\eta > x) > 0$, $x \in \mathbb{R}$. Now, let $\varepsilon \in (0, 1/2)$. By relations (30) and (31), we get that

$$\frac{HG_q\left(S_n^{(\xi)}\right)}{\overleftarrow{H}_n(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+}$$
$$\leqslant (1+\varepsilon)\max\left\{1, \frac{\mathbb{E}\left(S_n^{(\xi)} - \overleftarrow{H}_n(q)\right)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+}\sup_{u\geqslant \overleftarrow{H}_n(q)}\frac{\mathbb{P}(S_n^{(\xi)} > u(1-\varepsilon))}{\mathbb{P}(S_n^{(\xi)} > u)}\right\}$$

for all $q$ sufficiently close to the unit from the left. Since $\overleftarrow{H}_n(q) \to \infty$ if $q \uparrow 1$, we get from the last estimate

$$\limsup_{q\uparrow 1}\frac{HG_q\left(S_n^{(\xi)}\right)}{\overleftarrow{H}_n(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+}$$
$$\leqslant (1+\varepsilon)\max\left\{1, \limsup_{q\uparrow 1}\frac{\mathbb{E}\left(S_n^{(\xi)} - \overleftarrow{H}_n(q)\right)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+}\limsup_{q\uparrow 1}\sup_{u\geqslant \overleftarrow{H}_n(q)}\frac{\mathbb{P}(S_n^{(\xi)} > u(1-\varepsilon))}{\mathbb{P}(S_n^{(\xi)} > u)}\right\}$$
$$= (1+\varepsilon)\max\left\{1, \limsup_{x\to\infty}\frac{\mathbb{E}(S_n^{(\xi)} - x)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}(\xi_k - x)^+}\limsup_{x\to\infty}\sup_{u\geqslant x}\frac{\mathbb{P}(S_n^{(\xi)} > u(1-\varepsilon))}{\mathbb{P}(S_n^{(\xi)} > u)}\right\}.$$

The fact that $F_{S_n^{(\xi)}} \in \mathscr{C}$, relation (3) of Theorem 2, and the arbitrariness of $\varepsilon \in (0, 1/2)$ imply that

$$\limsup_{q\uparrow 1}\frac{HG_q\left(S_n^{(\xi)}\right)}{\overleftarrow{H}_n(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+} \leqslant 1. \tag{32}$$

In a similar way, we can obtain

$$\liminf_{q\uparrow 1}\frac{HG_q\left(S_n^{(\xi)}\right)}{\overleftarrow{H}_n(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+}$$
$$\geqslant (1-\varepsilon)\min\left\{1, \liminf_{x\to\infty}\frac{\mathbb{E}(S_n^{(\xi)} - x)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}(\xi_k - x)^+}\liminf_{x\to\infty}\inf_{u\geqslant x}\frac{\mathbb{P}(S_n^{(\xi)} > u(1+\varepsilon))}{\mathbb{P}(S_n^{(\xi)} > u)}\right\}$$

for all $\varepsilon \in (0, 1/2)$, and by similar arguments, we can derive that

$$\liminf_{q\uparrow 1}\frac{HG_q\left(S_n^{(\xi)}\right)}{\overleftarrow{H}_n(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\left(\xi_k - \overleftarrow{H}_n(q)\right)^+} \geqslant 1.$$

The last estimate, jointly with inequality (32), finishes the proof of the first part of Theorem 6.

• Now, suppose $\varkappa \geqslant 2$, and all conditions of part (ii) of Theorem 6 are satisfied. Due to Theorem 3

$$HG_q(S_n^{(\xi)}) = \widetilde{x} + \left( \frac{\mathbb{E}\Big( \big(S_n^{(\xi)} - \widetilde{x}\big)^+ \Big)^{\varkappa}}{1-q} \right)^{1/\varkappa}, \tag{33}$$

where $\widetilde{x} = \widetilde{x}(q)$ is the unique solution of the equation

$$\frac{\left( \mathbb{E}\Big( \big(S_n^{(\xi)} - \widetilde{x}\big)^+ \Big)^{\varkappa-1} \right)^{\varkappa}}{\left( \mathbb{E}\Big( \big(S_n^{(\xi)} - \widetilde{x}\big)^+ \Big)^{\varkappa} \right)^{\varkappa-1}} = \gamma_{S_n^{(\xi)}}(\widetilde{x}) = 1-q.$$

Let $H_n$ be the d.f. defined in part (i) of Theorem 6. Due to the asymptotic relation (27), the upper bound in (29), and the conditions of the theorem, we have that the d.f.'s $F_{S_n^{(\xi)}}$ and $H_n$ both belong to the class $\mathscr{C}$. In addition, according to relation (27), the min-max inequality (28), and condition (9), we get that

$$\begin{aligned}
&\left( \limsup_{x\to\infty} \frac{\overline{F}_{S_n^{(\xi)}}(yx)}{\overline{F}_{S_n^{(\xi)}}(x)} \right)^{\varkappa} \left( \limsup_{x\to\infty} \frac{\overline{F}_{S_n^{(\xi)}}(x)}{\overline{F}_{S_n^{(\xi)}}(yx)} \right)^{\varkappa-1} \\
&= \left( \limsup_{x\to\infty} \frac{\overline{H}_n(yx)}{\overline{H}_n(x)} \right)^{\varkappa} \left( \limsup_{x\to\infty} \frac{\overline{H}_n(x)}{\overline{H}_n(yx)} \right)^{\varkappa-1} \\
&\leqslant \left( \max_{k\in\mathcal{I}_n} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)} \right)^{\varkappa} \left( \max_{k\in\mathcal{I}_n} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)} \right)^{\varkappa-1} < 1
\end{aligned} \tag{34}$$

for some $y > 1$. Therefore, Corollary 1 implies that the d.f. $1-\gamma_{S_n^{(\xi)}}$ belongs to the class $\mathscr{C} \cap \mathcal{PD}$.

For sufficiently large $x$, by the alternative expectation formula, we have that

$$\begin{aligned}
\frac{\left( \sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big((\xi_k - x)^+\big)^{\varkappa-1} \right)^{\varkappa}}{\left( \sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big((\xi_k - x)^+\big)^{\varkappa} \right)^{\varkappa-1}} &= \frac{\left( (\varkappa-1) \int\limits_x^{\infty} (u-x)^{\varkappa-2} \overline{H}_n(u) \mathrm{d}u \right)^{\varkappa}}{\left( \varkappa \int\limits_x^{\infty} (u-x)^{\varkappa-1} \overline{H}_n(u) \mathrm{d}u \right)^{\varkappa-1}} \\
&= \frac{\left( \mathbb{E}\big((\zeta_n - x)^+\big)^{\varkappa-1} \right)^{\varkappa}}{\left( \mathbb{E}\big((\zeta_n - x)^+\big)^{\varkappa} \right)^{\varkappa-1}} := \gamma_{\zeta_n}(x),
\end{aligned}$$

where $\zeta_n$ is an r.v. with d.f. $H_n$. Due to the estimate (34) d.f. $F_{\zeta_n} = H_n$ satisfies condition (23) which, according to Corollary 1, implies that the d.f. $1-\gamma_{\zeta_n}$ belongs to the class $\mathscr{C} \cap \mathcal{PD}$ together with the d.f. $1-\gamma_{S_n^{(\xi)}}$.

On the other hand, relation (3) of Theorem 2 implies that

$$\gamma_{S_n^{(\xi)}}(x) \underset{x\to\infty}{\sim} \gamma_{\zeta_n}(x).$$

Consequently, all conditions of Lemma 4 are satisfied with $c = 1$ and d.f.'s $1-\gamma_{S_n^{(\xi)}}, 1-\gamma_{\zeta_n}$. According to this lemma, we get that

$$\widetilde{x}(q) \underset{q\uparrow 1}{\sim} \widehat{x}(q). \tag{35}$$

Now, we continue the proof of this part in the same way as in the proof of part (i). For any $q \in (0,1)$, by relation (33) and the min-max inequality (28), we get

$$\frac{HG_q\big(S_n^{(\xi)}\big)}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}}$$
$$\leqslant \max\left\{\frac{\widetilde{x}(q)}{\widehat{x}(q)}, \left(\frac{\mathbb{E}\Big(\big(S_n^{(\xi)}-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}}\right)^{1/\varkappa}\right\}$$
$$= \max\left\{\frac{\widetilde{x}(q)}{\widehat{x}(q)}, \left(\frac{\mathbb{E}\Big(\big(S_n^{(\xi)}-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}\frac{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}}\right)^{1/\varkappa}\right\}.$$

If $q$ is sufficiently close to the unit from the left, then

$$\frac{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}} = \frac{\int\limits_{\widetilde{x}(q)}^{\infty}(u-\widetilde{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}{\int\limits_{\widehat{x}(q)}^{\infty}(u-\widehat{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}$$
$$= \left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}\right)^{\varkappa}\frac{\int\limits_{\widehat{x}(q)}^{\infty}(u-\widehat{x}(q))^{\varkappa-1}\overline{H}_n\Big(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\Big)\mathrm{d}u}{\int\limits_{\widehat{x}(q)}^{\infty}(u-\widehat{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}$$
$$\leqslant \left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}\right)^{\varkappa}\sup_{u\geqslant\widehat{x}(q)}\frac{\overline{H}_n\Big(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\Big)}{\overline{H}_n(u)}.$$

Consequently, for $q$ sufficiently close to the unit from the left, we have

$$\frac{HG_q\big(S_n^{(\xi)}\big)}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}}$$
$$\leqslant \frac{\widetilde{x}(q)}{\widehat{x}(q)}\max\left\{1, \left(\frac{\mathbb{E}\Big(\big(S_n^{(\xi)}-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}\sup_{u\geqslant\widehat{x}(q)}\frac{\overline{H}_n\Big(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\Big)}{\overline{H}_n(u)}\right)^{1/\varkappa}\right\}.$$

Since $\widetilde{x}(q)\underset{q\uparrow 1}{\to}\infty$, $\widetilde{x}(q)\underset{q\uparrow 1}{\sim}\widehat{x}(q)$ and the d.f. $H_n$ belongs to the class $\mathscr{C}$, the last estimate and the asymptotic relation (3) of Theorem 2 imply that

$$\limsup_{q\uparrow 1}\frac{HG_q\big(S_n^{(\xi)}\big)}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}}\leqslant 1.$$

Similarly, we can derive that

$$\liminf_{q\uparrow 1} \frac{HG_q\big(S_n^{(\xi)}\big)}{\widehat{x}(q) + \left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}} \geqslant 1.$$

This finishes the proof of the second part of the theorem. □

**Proof of Theorem 7.**
• We begin with case $\varkappa = 1$. According to the conditions and Lemma 10, a collection of r.v.'s $\{\theta_1 X_1, \ldots, \theta_n X_n\}$ follows the pQAI dependence structure. By Lemma 11, we have that the d.f.'s $F_{\theta_k X_k}$ belong to the class $\mathscr{C}$ for all $k \in \{1,2,\ldots,n\}$. Since $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}X_k^+ < \infty$ and $p > \max\limits_{1\leqslant k\leqslant n} J^+_{F_{X_k}}$, by Lemma 3.5 of [54], we get that $\mathbb{E}\,\theta_k X_x^+ < \infty$ for $k \in \{1,2,\ldots,n\}$, because

$$\sup\left\{v : \int\limits_{[0,\infty)} x^v \mathrm{d}F(x) < \infty\right\} \leqslant J_F^+$$

for every d.f. $F$ from the class $\mathscr{D}$ . Finally, by Lemma 12, we derive that $F_{\theta_k X_k} \in \mathcal{PD}$ for all $k \in \{1,2,\ldots,n\}$. Hence, the collection of r.v.'s $\{\theta_1 X_1, \ldots, \theta_n X_n\}$ satisfies all conditions of part (i) of Theorem 6. This implies the statement of Theorem 7(i).

• Now, let us consider part (ii) of the theorem. If $\varkappa \geqslant 2$, Theorem 7(ii) follows from Theorem 6(ii) by completely analogous reasoning as in the first part of the proof, only instead of Lemma 12, we need to use Lemma 13. □

## 5. Illustrative Example

In this section, we present an example showing how Theorem 6 can be applied to the evaluation of the HG risk measure when the risk level is close to unity and the leading risk has a consistently varying distribution.

**Example 1.** *Let us consider collection of independent r.v.'s $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ with the following d.f.'s:*

$$F_1(x) = \left(1 - \frac{1}{(x+1)^3}\right)\mathbb{I}_{[0,\infty)}(x),\;\; F_2(x) = \left(1 - \frac{1}{2(x+1)^3}\right)\mathbb{I}_{[0,\infty)}(x),$$
$$F_3(x) = F_4(x) = \left(1 - \mathrm{e}^{-x}\right)\mathbb{I}_{[0,\infty)}(x).$$

*We derive asymptotic formulas for $HG_q$ risk measures in the cases of the power Young functions $\varphi(t) = t^{\varkappa}$ with $\varkappa = 1$ and $\varkappa = 2$.*

It is clear that the collection of r.v.'s satisfies the conditions of Theorem 6 with $\mathcal{I}_4 = \{1,2\}$. In the case $\varkappa = 1$, we have

$$HG_q\big(S_4^{(\xi)}\big) \underset{q\uparrow 1}{\sim} \overleftarrow{H}_4(q) + \frac{1}{1-q}\Big(\mathbb{E}\big(\xi_1 - \overleftarrow{H}_4(q)\big) + \mathbb{E}\big(\xi_2 - \overleftarrow{H}_4(q)\big)\Big),$$

where $\overleftarrow{H}_4$ is the quantile function of the d.f.

$$H_4 = \max\{0, 1 - \overline{F}_1 - \overline{F}_2\}.$$

For large $x$

$$H_4(x) = 1 - \frac{3}{2}\frac{1}{(1+x)^3}.$$

Therefore,

$$\overleftarrow{H}_4(q) = \frac{1}{\sqrt[3]{\frac{2}{3}(1-q)}} - 1$$

for $q$ sufficiently close to unity. For any sufficiently large $x$,

$$\mathbb{E}(\xi_1 - x)^+ = \frac{1}{2(x+1)^2}, \quad \mathbb{E}(\xi_2 - x)^+ = \frac{1}{4(x+1)^2}.$$

Therefore, in the case $\varkappa = 1$

$$\begin{aligned} HG_q\big(S_4^{(\xi)}\big) &\underset{q\uparrow 1}{\sim} \frac{1}{\sqrt[3]{\frac{2}{3}(1-q)}} - 1 + \frac{1}{1-q}\,\frac{3}{4}\left(\sqrt[3]{\frac{2}{3}(1-q)}\right)^2 \\ &= \frac{1}{\sqrt[3]{1-q}}\left(\frac{3}{2}\right)^{\frac{4}{3}} - 1. \end{aligned}$$

The graph of this function, together with the values of $HG_q\big(S_4^{(\xi)}\big)$ obtained using the Monte-Carlo (MC) method, can be seen in Figure 1. The Monte-Carlo simulations were performed using statistical software **R** (v4.0.3); the graphs provided in this section were generated in **R** using ggplot2 and tikzDevice libraries. .

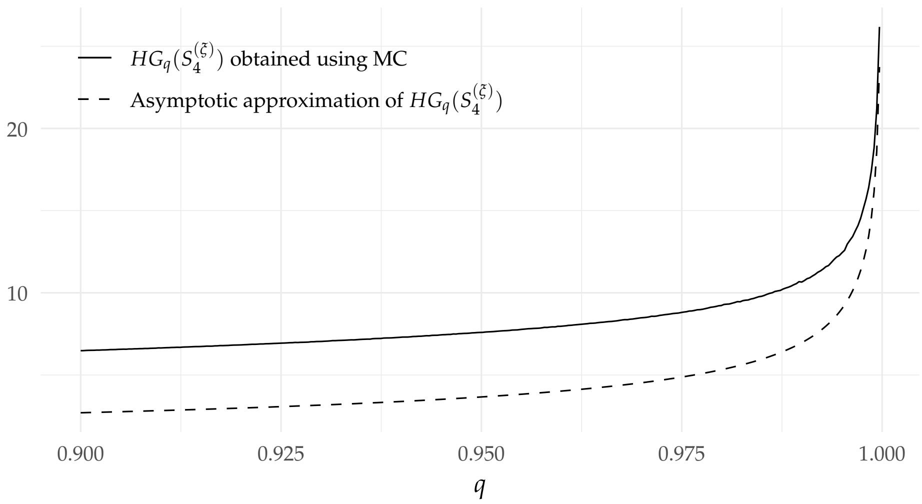


**Figure 1.** Simulated and asymptotically approximated values of $HG_q(S_4^{(\xi)})$ in the case $\varkappa = 1$.

If $\varkappa = 2$, then

$$HG_q\big(S_4^{(\xi)}\big) \underset{q\uparrow 1}{\sim} \widehat{x}(q) + \frac{1}{\sqrt{1-q}}\Big(\mathbb{E}\big((\xi_1 - \widehat{x}(q))^+\big)^2 + \mathbb{E}\big((\xi_2 - \widehat{x}(q))^+\big)^2\Big)^{1/2},$$

where $\widehat{x}(q)$ is the solution of the equation

$$\frac{\mathbb{E}(\xi_1 - x)^+ + \mathbb{E}(\xi_2 - x)^+}{\mathbb{E}\big((\xi_1 - x)^+\big)^2 + \mathbb{E}\big((\xi_2 - x)^+\big)^2} = 1 - q.$$

In the case under consideration,

$$\begin{aligned} \mathbb{E}(\xi_1 - x)^+ &= \frac{1}{2(x+1)^2}, \quad & \mathbb{E}(\xi_2 - x)^+ &= \frac{1}{4(x+1)^2}, \\ \mathbb{E}\big((\xi_1 - x^+\big)^2 &= \frac{1}{x+1}, \quad & \mathbb{E}\big((\xi_2 - x^+\big)^2 &= \frac{1}{2(x+1)}. \end{aligned}$$

Hence,

$$\widehat{x}(q) = \sqrt[3]{\frac{3}{8(1-q)}} - 1$$

and

$$HG_q\big(S_4^{(\xi)}\big) \underset{q\uparrow 1}{\sim} \frac{3}{2}\sqrt[3]{\frac{3}{1-q}} - 1.$$

The graph of this function, together with the values of $HG_q\big(S_4^{(\xi)}\big)$ obtained using the Monte-Carlo method, can be seen in Figure 2.

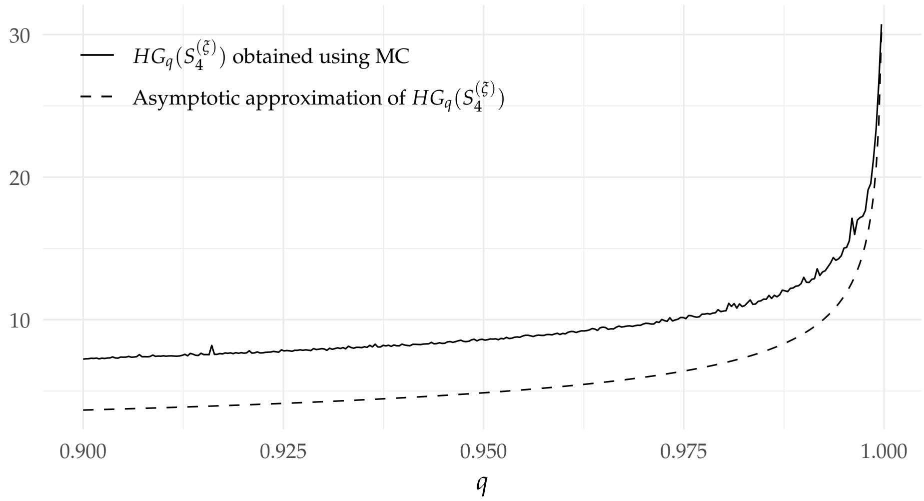


**Figure 2.** Simulated and asymptotically approximated values of $HG_q(S_4^{(\xi)})$ in the case $\varkappa = 2$.

## 6. Concluding Remarks

The study of risk measures is a relatively popular topic in probability theory, as risk measures are applied in numerous fields where uncertainty and potential losses must be quantified, including banking and finance, insurance, investment management, and energy markets. Acceptable properties of risk measures are listed, for example, in the work [1]. The book [72] discusses classical mathematical problems related to risk measures. The mathematical problems of risk measures are still being intensively studied today. Let us mention a few works. Distortion risk measures are considered, e.g., in [73,74]. The duality of risk measures is considered in [75]. Risk measures with special properties are analysed, e.g., in [76–78]. In addition, we note that our results complement those of the papers [79–82], which also utilise higher-order moments in the construction of risk measures.

As already mentioned, the HG risk measure is a direct generalisation of the ES risk measure by introducing a governing Young function. Other generalisations of the ES measure are obtained by replacing quantiles with generalised quantiles or expectiles. Such risk measures are described and studied in [83–85].

In our paper, we study the properties of the HG risk measure, which was first defined in [3]. The main result of our work is given in Theorem 6. It follows from this result that, to find the asymptotics of the HG risk measure for the sum of random risks, it is sufficient to determine the appropriate characteristics of the sum of dominant risks. From Theorem 6, we see that such a result holds when the distribution functions of the dominant risks are consistently varying and satisfy additional requirements related to having positively decreasing tails. In addition, Theorem 6 implies that in order to find the asymptotics of the HG risk measure for the sum of random risks, there is no need to calculate complex convolutions. It is enough to construct the distribution function from the sum of the distribution functions of the selected group. The next main theorem of the work, Theorem 7, is actually a consequence of Theorem 6. This theorem demonstrates that the procedure for determining the asymptotics of the HG risk measure also applies in cases where a set of random variables with random weights is considered.

Possibly similar results on the asymptotic behavior of the HG risk measure can be obtained for a wider class of distributions $\mathscr{D}$, i.e., for distributions with dominatedly varying d.f.'s. However, in this case, the asymptotic formulas should have additional constants resulting in less strict asymptotic bounds, or additional constraints should be imposed on the d.f.'s of random risks to preserve exact asymptotic relations. In the near future, we hope to determine how the asymptotic formulas of the HG risk measure change when the distribution class $\mathscr{C}$ is replaced by the wider distribution class $\mathscr{D}$.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| HG | Haezendonck–Goovaerts risk measure |
| $HG_q$ | Haezendonck–Goovaerts risk measure with level $q$ |
| ES | Expected Shortfall (particular case of HG risk measure) |
| r.v. | random variable |
| d.f. | distribution function |
| t.f. | tail function |
| pQAI | (pairwise) quasi-asymptotically independent (random variables) |
| $\mathcal{PD}$ | positively decreasing (distribution function) |
| MC | Monte-Carlo (method) |

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