mathematics

Article

Joint Discrete Approximation by Shifts of Hurwitz Zeta-Function: The Case of Short Intervals

Antanas Laurinčikas [1,*,†] and Darius Šiaučiūnas [1,2,†]

[1] Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko Str. 24, LT-03225 Vilnius, Lithuania; darius.siauciunas@sa.vu.lt
[2] Institute of Regional Development, Šiauliai Academy, Vilnius University, P. Višinskio Str. 25, LT-76351 Šiauliai, Lithuania
* Correspondence: antanas.laurincikas@mif.vu.lt
† These authors contributed equally to this work.

Abstract

Since 1975, it has been known that the Hurwitz zeta-function has a unique property to approximate by its shifts all analytic functions defined in the strip $\mathfrak{D} = \{s = \sigma + it : 1/2 < \sigma < 1\}$. However, such an approximation causes efficiency problems, and applying short intervals is one of the measures to make that approximation more effective. In this paper, we consider the simultaneous approximation of a tuple of analytic functions in the strip $\mathfrak{D}$ by discrete shifts $(\zeta(s + ikh_1, \alpha_1), \ldots, \zeta(s + ikh_r, \alpha_r))$ with positive $h_1, \ldots, h_r$ of Hurwitz zeta-functions in the interval $[N, N + M]$ with $M = \max_{1 \leqslant j \leqslant r}\left(h_j^{-1}(Nh_j)^{23/70}\right)$. Two cases are considered: 1° the set $\{(h_j \log(m + \alpha_j),\ m \in \mathbb{N}_0,\ j = 1, \ldots, r), 2\pi\}$ is linearly independent over $\mathbb{Q}$; and 2° a general case, where α_j and h_j are arbitrary. In case 1°, we obtain that the set of approximating shifts has a positive lower density (and density) for every tuple of analytic functions. In case 2°, the set of approximated functions forms a certain closed set. For the proof, an approach based on new limit theorems on weakly convergent probability measures in the space of analytic functions in short intervals is applied. The power $\eta = 23/70$ comes from a new mean square estimate for the Hurwitz zeta-function.

Keywords: approximation of analytic functions; Hurwitz zeta-function; Riemann zeta-function; universality; weak convergence of probability measures

MSC: 11M35

check for updates

Received: 14 October 2025
Revised: 9 November 2025
Accepted: 11 November 2025
Published: 14 November 2025

Citation: Laurinčikas, A.; Šiaučiūnas, D. Joint Discrete Approximation by Shifts of Hurwitz Zeta-Function: The Case of Short Intervals. *Mathematics* **2025**, *13*, 3654. https://doi.org/10.3390/math13223654

1. Introduction

Throughout the paper, $s = \sigma + it$ is the main complex variable. We consider the approximation of analytic functions by Hurwitz zeta-functions. Let $\alpha \in (0, 1]$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ was introduced in [1] and, for $\sigma > 1$, is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}.$$

Moreover, $\zeta(s,\alpha)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, and the point $s = 1$ is its simple pole and $\text{Res}_{s=1}\zeta(s,\alpha) = 1$. Clearly, $\zeta(s,1)$ coincides with the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and

$$\zeta\left(s, \tfrac{1}{2}\right) = (2^s - 1)\zeta(s).$$

The latter observations show that the function $\zeta(s,\alpha)$ is an extension of the famous Riemann zeta-function. Unlike $\zeta(s)$, the function $\zeta(s,\alpha)$, except for values $\alpha = 1$ and $\alpha = 1/2$, has no Euler product over prime numbers. Hence, the value distribution of $\zeta(s,\alpha)$ differs from that of $\zeta(s)$. For example, it is well known that $\zeta(s) \neq 0$ for $\sigma > 1$, while $\zeta(s,\alpha)$, where $\alpha \neq 1$, $\alpha \neq 1/2$, has infinitely many zeros in the latter half-plane [2–4].

On the other hand, the function $\zeta(s,\alpha)$ has an indirect connection to the distribution of prime numbers in arithmetic progressions. The main tool for investigations of the asymptotics for

$$\pi(x;a,q) = \sum_{\substack{p \leqslant x \\ p \equiv a (\text{mod}\, q)}} 1, \quad p \text{ is prime number}, \ (a,q) = 1, \ x \to \infty,$$

is Dirichlet L-functions. Let $\chi : \mathbb{N} \to \mathbb{C}$ be a Dirichlet character modulo q (where $\chi(m)$ is periodic with period q, completely multiplicative, $\chi(m) = 0$ if $(m,q) > 1$, and $\chi(m) \neq 0$ for $(m,q) = 1$). The Dirichlet L-function $L(s,\chi)$ with character χ, for $\sigma > 1$, is given by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and has meromorphic continuation to the whole $\mathbb{C}$. From the periodicity of χ, it follows that

$$L(s,\chi) = q^{-s} \sum_{m=1}^{q} \chi(m)\zeta\left(s, \tfrac{m}{q}\right).$$

Thus, properties of $\zeta(s,\alpha)$ with rational α can be applied for investigations of Dirichlet L-functions, and consequently for $\pi(x;a,q)$. Nevertheless, applications of the Hurwitz zeta-function are not limited by the distribution of prime numbers; $\zeta(s,\alpha)$ plays an important role in special function theory, algebraic number theory, probability theory, and even quantum mechanics. The classical theory of $\zeta(s,\alpha)$ can be found in [5–7]. One significant feature of $\zeta(s,\alpha)$ is connected to the approximation of analytic functions by shifts $\zeta(s+i\tau,\alpha)$, $\tau \in \mathbb{R}$. This approximation is of a novel type in function theory, and is called universality: shifts of one and the same function $\zeta(s,\alpha)$ approximate the whole class of analytic functions. The universality of the Riemann zeta-function $\zeta(s)$ was discovered by S.M. Voronin in [8–13]. After Voronin, the universality of $\zeta(s)$ was studied by many authors (see [14–18]). We recall some universality results for $\zeta(s,\alpha)$. For $\mathfrak{D} = \{s \in \mathbb{C} : \sigma \in (1/2,1)\}$, denote by $\mathcal{K}$ the class of compact subsets of the strip $\mathfrak{D}$ with connected complements, and by $H(K)$, $K \in \mathcal{K}$, the set of continuous functions on K that are analytic inside of K. Let $\mathbf{m}_L A$ be the Lebesgue measure of measurable set $A \subset \mathbb{R}$. Then, the following result is known [14–16,18,19].

Proposition 1. *Suppose that α is rational $\neq 1$ or $\neq 1/2$, or a transcendental number, and $K \in \mathcal{K}$, $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathbf{m}_L\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{m}_L\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

If $f(s) \neq 0$ on K, then the proposition remains valid for $\alpha = 1$ or $\alpha = 1/2$ as well.

The case of algebraic irrational α (α is a root of a polynomial $\not\equiv 0$ with rational coefficients) has been considered in [20]. Denote by d the degree of α, and put $\beta = 4 \cdot 27^{-1}(4.45)^{-2}$ and $\gamma = \beta d^{-2}$. Then the universality of $\zeta(s,\alpha)$ with algebraic irrational α is contained in the following statement.

Proposition 2 (see [20]). *Suppose that $\delta \in (0,\gamma)$, $1-\gamma+\delta \leqslant \sigma_0 \leqslant 1$, $s_0 + \sigma_0 + it_0$, and $f(s)$ is a continuous function on the disc $|s-s_0| \leqslant r$, $r > 0$, and analytic inside of that disc. Let $a \in (0,1)$ and $\varepsilon \in (0, |f(s_0)|)$. Then, for all but finitely many $\alpha \in [a,1)$ of degree at most $d_0 - 2\beta/d_0^2 + \delta$ with*

$$d_0 \leqslant \left(\frac{\beta}{1-\sigma_0+\delta}\right)^{1/2},$$

there are $\tau \in [T,2T]$ and $\kappa = \kappa(\varepsilon, f, T) > 0$ such that

$$\max_{|s-s_0|\leqslant\kappa r} |\zeta(s+i\tau,\alpha) - f(s)| < 3\varepsilon.$$

Here, $T = T(\alpha, f, \varepsilon)$ is given explicitly, the set of exceptional α can be described effectively, and κ can be effectively computable as well.

Propositions 1 and 2 are of the so-called continuous type because τ in shifts $\zeta(s+i\tau,\alpha)$ can take arbitrary values in the interval. In parallel to continuous universality theorems for zeta-functions, theorems of discrete universality are studied when τ takes values from certain discrete sets. The first discrete universality theorem for zeta-functions has been obtained by A. Reich. In [21], he proved the discrete universality of Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ of algebraic number fields $\mathbb{K}$ on the approximation of analytic functions by shifts $\zeta_{\mathbb{K}}(s+ikh)$, $k \in \mathbb{N}$, where h is a fixed real number.

For statements of discrete universality theorems, we introduce some notation. Denote by $\#A$ the cardinality of the set A, and, for $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, put

$$M_N(\dots) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \dots\},$$

where in place of dots a condition satisfied for k is to be written. The first discrete universality theorem for $\zeta(s,\alpha)$ has been obtained by B. Bagchi.

Proposition 3 (see [15], Corollary 5.3.7)**.** *Suppose that α is a rational number $\neq 1$, $\neq 1/2$, and $K \in \mathcal{K}$, $f(s) \in H(K)$, $h > 0$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} M_N\left(\sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right) > 0. \tag{1}$$

Discrete universality of $\zeta(s,\alpha)$ with non-rational α involves the set

$$L(\alpha,h,\pi) = \{(\log(m+\alpha) : m \in \mathbb{N}_0), \tfrac{2\pi}{h}\}$$

which can be a multiset.

Proposition 4 (see [22,23])**.** *Let the set $L(\alpha,h,\pi)$ be linearly independent over $\mathbb{Q}$. Then, for every $K \in \mathcal{K}$, $f(s) \in H(K)$ and $\varepsilon > 0$, the inequality (1) is valid. Moreover, the lower limit in (1) can be replaced by the limit for all but at most countably many $\varepsilon > 0$.*

The second assertion of Proposition 4 has been obtained in [23,24]. As was noted in [22], one can take $\alpha = 1/\pi$ and $h \in \mathbb{Q}$ in Proposition 4.

In [25], A. Sourmelidis proved that continuous universality for $\zeta(s,\alpha)$ implies a discrete one with shifts $\zeta(s+ikh,\alpha)$, $h > 0$. Hence, Proposition 1 implies Proposition 3 not only with rational but also with transcendental α. On the other hand, Proposition 4 may be true with algebraic irrational α; however, examples of such α are not known.

Also, a joint universality of Hurwitz zeta-functions is considered. In this case, a tuple $(f_1(s),\dots,f_r(s))$ of analytic functions is approximated simultaneously by shifts $(\zeta(s+i\tau,\alpha_1),\dots,\zeta(s+i\tau,\alpha_r))$ with both continuous and discrete τ. Obviously, for this, the functions $\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)$ must be independent in a certain sense. This independence may be described in terms of parameters $\alpha_1,\dots,\alpha_r$, for example, that $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, i.e., there is no polynomial $p(s_1,\dots,s_r) \not\equiv 0$ with coefficients in $\mathbb{Q}$ such that $p(\alpha_1,\dots,\alpha_r) = 0$. A more general case involves the set $L(\alpha_1,\dots,\alpha_r) = \{(\log(m+\alpha_1) : m \in \mathbb{N}_0),\dots,(\log(m+\alpha_r) : m \in \mathbb{N}_0)\}$. The following statement is known [26].

Proposition 5. *Suppose that the set $L(\alpha_1,\alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1,\dots,r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathbf{m}_L\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

The first joint discrete universality theorem was obtained for Hurwitz zeta-functions with rational parameters. For $q \in \mathbb{N}$, denote the Euler totient function as $\varphi(q)$; let $\chi_1,\dots,\chi_{\varphi(q)}$ be pairwise non-equivalent modulo Dirichlet characters q, and let

$$A \stackrel{\text{def}}{=} \Big(\overline{\chi}_j(a)/\varphi(q)\Big)_{\substack{1\leqslant j\leqslant\varphi(q)\\ 1\leqslant a\leqslant q,\ (a,q)=1}}$$

be the quadratic matrix of order $\varphi(q)$. For some functions $g_a(s)$, $1 \leqslant a \leqslant q$, $(a,q) = 1$, define the matrix

$$\widehat{g} \stackrel{\text{def}}{=} \big(q^{-s}g_a(s)\big)^T_{1\leqslant a\leqslant q\ (a,q)=1}$$

where B^T means the transpose of a matrix B. Then in [27], we find the following result.

Proposition 6. *Suppose that $K \in \mathcal{K}$, for each $1 \leqslant a \leqslant q$, $(a,q) = 1$; let $g_a(s) \in H(K)$, and all components of $A^{-1}\widehat{g}$ be non-vanishing on K. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} M_N\left(\max_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\max_{s\in K}\left|\zeta\left(s+ikh,\frac{a}{q}\right)-g_a(s)\right|<\varepsilon\right)>0.$$

The most general joint discrete universality theorem for Hurwitz zeta-functions uses the set

$$\begin{aligned}&\mathcal{L}(\alpha_,\dots,\alpha_r;h_1,\dots,h_r;\pi)\\&\quad=\{(h_1\log(m+\alpha_1):m\in\mathbb{N}_0),\dots,(h_r\log(m+\alpha_r):m\in\mathbb{N}_0),2\pi\},\quad h_1,\dots,h_r>0.\end{aligned}$$

Proposition 7 (see [28], Theorem 1.7)**.** *Suppose that the set $\mathcal{L}(\alpha_,\dots,\alpha_r;h_1,\dots,h_r;\pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1,\dots,r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$, the inequality*

$$\liminf_{N\to\infty} M_N\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+ikh_j,\alpha_j)-f_j(s)|<\varepsilon\right)>0$$

holds.

Propositions 1 and 3–7 imply that there are infinitely many shifts of the Hurwitz zeta-function approximating a given analytic function or a tuple of analytic functions; however, any concrete shift is not known. In this sense, the mentioned results are ineffective. Proposition 2 has effectivity features because it indicates the explicit interval containing values τ with approximating property.

Another way towards effectivisation of universality for zeta-functions consists of shortening of intervals with approximating values τ. This idea leads to extension of universality theorems for zeta-functions in short intervals. The first result in this direction for the Riemann zeta-function has been given in [29], and improved in [30,31]. We recall that some universality results for Hurwitz zeta-function in short intervals. The main theorem of [32] is stated as follows.

Proposition 8 (see [32], Theorem 4)**.** *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. For $j = 1,\dots,r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{H}\mathbf{m}_L\left\{\tau\in[T,T+H]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}>0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{H}\mathbf{m}_L\left\{\tau\in[T,T+H]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}$$

exists, is explicitly given, and positive for all but at most countably many $\varepsilon > 0$.

The case with $r = 1$ for transcendental α was obtained in [33].

For $N \in \mathbb{N}$ and $M \in \mathbb{N}$, set

$$W_{N,M}(\dots)=\frac{1}{M+1}\#\{N\leqslant k\leqslant N+M:\dots\},$$

where in place of dots a condition satisfied by k is to be written. A version of Proposition 4 in short intervals has been proved in [34].

Proposition 9 (see [34], Theorem 1.5)**.** *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, and $h^{-1}(Nh)^{27/82} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$. Then, for every $K \in \mathcal{K}$, $f(s) \in H(K)$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} W_{N,M}\left(\sup_{s\in K} |\zeta(s + ikh, \alpha) - f(s)| < \varepsilon\right) > 0.$$

Moreover, the lower limit can be replaced by the limit for all but at most countably many $\varepsilon > 0$.

The purpose of this paper is to connect Propositions 8 and 9, i.e., to obtain joint discrete universality for Hurwitz zeta-functions in short intervals.

Theorem 1. *Suppose that the set $\mathcal{L}(\alpha_1, \ldots, \alpha_r; h_1, \ldots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$, and $\max_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{23/70} \leqslant M \leqslant \min_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{1/2}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} W_{N,M}\left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} W_{N,M}\left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon\right)$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Denote by $H(D)$ the space of analytic functions on the strip $\mathfrak{D}$ endowed with the topology of uniform convergence on compacta, and let

$$H^r(\mathfrak{D}) = \underbrace{H(\mathfrak{D}) \times \cdots \times H(\mathfrak{D})}_{r}.$$

$H^r(\mathfrak{D})$ is considered with the product topology.

Theorem 2. *Suppose that the parameter $\alpha_j \in (0,1)$, $\alpha_j \neq 1/2$, and the positive numbers $h_1, \ldots, h_r$ are arbitrary, and $\max_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{23/70} \leqslant M \leqslant \min_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{1/2}$. Then there exists a closed non-empty set $F_{\alpha_1,\ldots,\alpha_r;h_1,\ldots,h_r} \subset H^r(\mathfrak{D})$ such that, for compact sets $K_1, \ldots, K_r \subset \mathfrak{D}$, $(f_1(s), \ldots, f_r(s)) \in F_{\alpha_1,\ldots,\alpha_r;h_1,\ldots,h_r}$ and every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} W_{N,M}\left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} W_{N,M}\left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon\right)$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

We observe that $23/70 < 27/82$.

Remark 1. *We suppose in Theorem 2 that $\alpha_j \neq 1$ and $\alpha_j \neq 1/2$ because, in these cases, $\zeta(s, \alpha_j)$ coincides with the Riemann zeta-function $\zeta(s)$, or differs from it by a simple multiple $2^s - 1$. The function $\zeta(s)$ has the Euler product, and, for studying its universality, another scheme is used. Moreover, the above restriction for α_j, in the case $r = 1$, removes confusion because the universality for $\zeta(s)$ in short intervals is known, and Theorem 2 then becomes meaningless.*

Theorems 1 and 2 will be proved in Section 4. Section 2 is devoted to mean value estimates for Hurwitz zeta-functions in short intervals. In Section 3, we will prove limit theorems on weakly convergent probability measures in the space of analytic functions $H^r(\mathfrak{D})$.

2. Estimates in Short Intervals

Throughout the paper, we will often use the notation $a \ll_\theta b$, $a \in \mathbb{C}$, $b > 0$, which means that there exists a constant $c = c(\theta) > 0$ such that $|a| \leqslant cb$. Thus, $a \ll_\theta b$ is an equivalent of $a = O_\theta(b)$.

Lemma 1. *Suppose that $\alpha \in (0,1) \setminus \{1/2\}$ and $\sigma \in (1/2, 31/52]$ are fixed, and $T^{23/70} \leqslant H \leqslant T^\sigma$. Then, uniformly in H, the estimate*

$$\int_{T-H}^{T+H} |\zeta(\sigma + it, \alpha)|^2 \, dt \ll_{\sigma,\alpha} H$$

holds.

Proof. In the proof of Theorem 2 from [35], the bound of Lemma 1 was obtained for $T^{27/82} \leqslant H \leqslant T^\sigma$ and fixed $\sigma \in (1/2, 7/12)$. For this, the exponent pair $(11/30, 16/30)$ for the estimation of mean squares of Dirichlet polynomials has been applied. Using the exponent pair $(9/26, 7/13)$ in place of $(11/30, 16/30)$ gives Lemma 1. □

Since the present paper is devoted to discrete value distribution problems of Hurwitz zeta-functions, we need a discrete version of Lemma 1. To pass from Lemma 1 to its discrete analogue, we will apply the following Gallagher lemma which connects continuous and discrete mean squares.

Lemma 2 (see [36], Lemma 1.4)**.** *Suppose that $\delta > 0$, $T_0, T \geqslant \delta$, $\mathcal{A}$ is a finite non-empty set, $\mathcal{A} \subset [T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$\mathcal{N}_\delta(\tau) = \sum_{\substack{\tau \in \mathcal{A} \\ |t-\tau|<\delta}} 1, \quad \tau \in \mathcal{A}.$$

Let a complex-valuable function $\mathcal{Z}(t)$ be continuous on the interval $[T_0, T_0 + T]$, and have a continuous derivative inside this interval. Then the inequality

$$\sum_{t \in \mathcal{A}} \mathcal{N}_\delta^{-1}(t) |\mathcal{Z}(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |\mathcal{Z}(t)|^2 \, dt + \left(\int_{T_0}^{T_0+T} |\mathcal{Z}(t)|^2 \, dt \int_{T_0}^{T_0+T} |\mathcal{Z}'(t)|^2 \, dt \right)^{1/2}$$

is valid.

Lemma 3. *Suppose that* $\alpha \in (0,1) \setminus \{1/2\}$, *and fixed* $\sigma \in [1/2, 31/52]$, $h > 0$, $h^{-1}(Nh)^{23/70} \leqslant M \leqslant h^{-1}(Nh)^{1/2}$ *and* $|\tau| \leqslant \log^2(Nh)$. *Then the estimate*

$$\sum_{k=N}^{N+M} |\zeta(\sigma + it + ikh, \alpha)|^2 \ll_{\sigma,\alpha,h} M(1+|t|)$$

is valid.

Proof. We apply Lemma 2, with $\delta = 1$, $T_0 = N - 1/2$, $T = M + 1$, $\mathcal{A} = \{k \in \mathbb{N} : k \in [N, N+M]\}$ and $\mathcal{Z}(\tau) = \zeta(\sigma + ih\tau + it, \alpha)$. Obviously,

$$\mathcal{N}_1(k) = \sum_{\substack{l \in \mathcal{A} \\ |k-l|<1}} 1 = 1.$$

Therefore, in virtue of Lemma 2,

$$\sum_{k=N}^{N+M} |\zeta(\sigma + it + ihk, \alpha)|^2 \ll_h \int_{N-1}^{N+M} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, d\tau + \left(\int_{N-1}^{N+M} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, d\tau \int_{N-1}^{N+M} |\zeta'(\sigma + it + ih\tau, \alpha)|^2 \, d\tau \right)^{1/2}. \quad (2)$$

Clearly, for large N,

$$\int_{N-1}^{N+M} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, dt = \int_{(N-1)+t/h}^{N+M+t/h} |\zeta(\sigma + ih\tau, \alpha)|^2 \, d\tau \ll_h \int_{(N-M)h-|t|}^{(N+M)h+|t|} \zeta(\sigma + i\tau, \alpha)|^2 \, d\tau.$$

We have $Mh + |t| \geqslant (Nh)^{23/70}$ for $M \geqslant h^{-1}(Nh)^{23/70}$, and $Mh + |t| \leqslant (Nh)^{1/2} + \log^2(Nh) \leqslant (Nh)^\sigma$ for $|t| \leqslant \log^2(Nh)$ and large N. Hence, Lemma 1 gives

$$\int_{N-1}^{N+M} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, d\tau \ll_{\sigma,\alpha,h} Mh + |t| \ll_{\sigma,\alpha,h} M(1+|t|). \quad (3)$$

Observe that $(\zeta(\sigma + it + ih\tau, \alpha))'_\tau = ih\zeta'(\sigma + it + ih\tau, \alpha)$. Therefore, a standard application of the Cauchy integral formula and (3) leads to the estimate

$$\int_{N-1}^{N+M} |\zeta'(\sigma + it + ih\tau, \alpha)|^2 \, d\tau \ll_{\sigma,\alpha,h} M(1+|t|).$$

This, together with (3) and (2), yields the estimate of the lemma. □

Let, for brevity, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, and $\underline{\zeta}(s, \underline{\alpha}) = (\zeta(s, \alpha_1), \ldots, \zeta(s, \alpha_r))$. For the investigation of $\underline{\zeta}(s, \underline{\alpha})$, we introduce an auxiliary object. Let $\theta > 1/2$ be a fixed number, and, for $n \in \mathbb{N}$, and $m \in \mathbb{N}_0$,

$$w_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n} \right)^\theta \right\}, \quad j = 1, \ldots, r,$$

where $\exp\{a\} = \mathrm{e}^a$. Define the series

$$\zeta_n(s, \alpha_j) = \sum_{m=1}^{\infty} \frac{w_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,$$

which is absolutely convergent in any half-plane $\sigma \geqslant \sigma_0$ with a finite σ_0. Let $\underline{\zeta}_n(s, \underline{\alpha}) = (\zeta_n(s, \alpha_1), \dots, \zeta(s, \alpha_r))$. Our aim is to replace the investigation of $\underline{\zeta}(s, \underline{\alpha})$ by a simpler one of $\underline{\zeta}_n(s, \underline{\alpha})$. We will show that $\underline{\zeta}(s, \underline{\alpha})$ and $\underline{\zeta}_n(s, \underline{\alpha})$ coincide in the mean. To describe this, we need the metric in the space $H^r(\mathfrak{D})$.

It is well known (see, for example, [37]) that there exists a sequence of compact subsets $\{K_m : m \in \mathbb{N}\} \subset \mathfrak{D}$ such that $K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$,

$$\mathfrak{D} = \bigcup_{m=1}^{\infty} K_m,$$

and every compact set $K \subset \mathfrak{D}$ lies in some K_m. For $g_1, g_2 \in H(\mathfrak{D})$, put

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|}.$$

Then, ρ is a metric in $H(\mathfrak{D})$, which induces its topology of uniform convergence on compact sets.

Now, let $\underline{g}_l = (g_{l1}, \dots, g_{lr}) \in H^r(\mathfrak{D})$, $l = 1, 2$. Then

$$\rho_r(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant m \leqslant r} \rho(g_{1m}, g_{2m})$$

is the metric in $H^r(\mathfrak{D})$, inducing its product topology.

We now state the main lemma of this section. Let $\underline{h} = (h_1, \dots, h_r)$.

Lemma 4. *Suppose that $\alpha_j \in (0,1) \setminus \{1/2\}$ for $j = 1, \dots, r$, and $\max_{1 \leqslant j \leqslant r} h_j^{-1}(Nh_j)^{23/70} \leqslant M \leqslant \min_{1 \leqslant j \leqslant r} h_j^{-1}(Nh_j)^{1/2}$. Then*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \rho_r\Big(\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}), \underline{\zeta}_n(s + ik\underline{h}, \underline{\alpha})\Big) = 0.$$

Proof. The definitions of the metrics ρ_r and ρ show that it suffices to prove the equalities

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s \in K} |\zeta(s + ikh_j, \alpha_j) - \zeta_n(s + ikh_j, \alpha_j)| = 0, \quad j = 1, \dots, r, \tag{4}$$

for every compact set $K \subset \mathfrak{D}$.

We will fix the parameter α, the number $h > 0$, and a compact set $K \subset \mathfrak{D}$, and recall the integral representation for $\zeta_n(s, \alpha)$. Let, as usual, $\Gamma(s)$ stand for the Euler gamma-function, and

$$\kappa_n(s) = \theta^{-1} \Gamma\left(\tfrac{s}{\theta}\right) n^{-s},$$

where θ is the number from the definition of $w_n(m, \alpha)$. Then the integral representation

$$\zeta_n(s, \alpha) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, \alpha) \kappa_n(z) \, \mathrm{d}z \tag{5}$$

holds. It follows easily from the classical Mellin formula

$$e^{-a} = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(z) a^{-z}\, dz, \quad a, b > 0,$$

and the definition of $\kappa_n(s)$.

There exists $1/4 > \delta > 0$ such that $1/2 + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for all $s = \sigma + it \in K$. Now, we choose more precisely $\theta = 1/2 + \delta$, and introduce $\widehat{\theta} = 1/2 + \delta - \sigma$. Clearly, $\widehat{\theta} < 0$ but $\widehat{\theta} > -1/2 + 2\delta$. Therefore, the integrand in (5) has a simple pole $z = 1 - s$ of the function $\zeta(s + z, \alpha)$ with residue $\kappa_n(1 - s)$, and a simple pole $z = 0$ of $\Gamma(z/\theta)$ with residue $\zeta(s, \alpha)$, both of which lie in the strip $(\widehat{\theta}, \theta)$. Hence, taking into account the exponential decreasing of the gamma-function,

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{6}$$

and applying the residue theorem, we obtain, for $s \in K$,

$$\zeta_n(s, \alpha) - \zeta(s, \alpha) = \frac{1}{2\pi i} \int_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} \zeta(s + z, \alpha) \kappa_n(z)\, dz + \kappa_n(1 - s).$$

Hence,

$$\begin{aligned} &\sup_{s\in K} |\zeta(s + ikh, \alpha) - \zeta_n(s + ikh, \alpha)| \\ &\quad \ll \int_{-\infty}^{\infty} \left|\zeta\left(\tfrac{1}{2} + \delta + ikh + i\tau, \alpha\right)\right| \sup_{s\in K} \left|\kappa_n\left(\tfrac{1}{2} + \delta - s + i\tau\right)\right| d\tau \\ &\qquad + \sup_{s\in K} |\kappa_n(1 - s - ikh)|. \end{aligned} \tag{7}$$

For simplification of further estimations, we present some elementary results. For $\sigma \geqslant 1/2$ and $|t| \geqslant 2$ (see, for example, [38]), the estimate

$$\zeta(\sigma + it, \alpha) \ll_{\sigma,\alpha} |t|^{1/2}$$

holds, and, in view of (6), we have, for $s = \sigma + it \in K$,

$$\kappa_n\left(\tfrac{1}{2} + \delta - s + i\tau\right) \ll n^{1/2+\delta-\sigma} \exp\{-\tfrac{c}{\theta}|\tau - t|\} \ll_K n^{-\delta} \exp\{-c_1|\tau|\}, \quad c_1 > 0, \tag{8}$$

and

$$\kappa_n(1 - s - ikh) \ll n^{1-\sigma} \exp\{-\tfrac{c}{\theta}|t + kh|\} \ll_K n^{1/2-2\delta} \exp\{-c_2 kh\}, \quad c_2 > 0.$$

Therefore,

$$\begin{aligned} &\left(\int_{-\infty}^{-\log^2(Nh)} + \int_{\log^2(Nh)}^{\infty} \right) \left|\zeta\left(\tfrac{1}{2} + \delta + ikh + i\tau, \alpha\right)\right| \sup_{s\in K} \left|\kappa_n\left(\tfrac{1}{2} + \delta - s + i\tau\right)\right| d\tau \\ &\qquad \ll_{\alpha,K} n^{-\delta} \left(\int_{-\infty}^{-\log^2(Nh)} + \int_{\log^2(Nh)}^{\infty} \right) (kh + |\tau|)^{1/2} \exp\{-c_1|\tau|\}\, d\tau \\ &\qquad \ll_{\alpha,K} n^{-\delta} \left((kh)^{1/2} + 1\right) \exp\left\{-c_3 \log^2(Nh)\right\}, \quad c_3 > 0, \end{aligned}$$

and

$$\frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}|\kappa_n(1-s-ikh)| \ll_K n^{1/2-2\delta}\frac{1}{M+1}\sum_{k=N}^{N+M}\exp\{-c_2kh\}$$
$$\ll_K n^{1/2-2\delta}\exp\{-\tfrac{c_2}{2}Nh\}\sum_{k=N}^{\infty}\exp\{-\tfrac{c_2}{2}kh\}$$
$$\ll_{h,K} n^{1/2-2\delta}\exp\{-\tfrac{c_2}{2}Nh\}.$$

This and (7) lead to

$$\frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)|$$
$$\ll_{h,K}\int_{-\log^2(Nh)}^{\log^2(Nh)}\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac12+\delta+ikh+i\tau,\alpha\right)\right|\right)\sup_{s\in K}\left|\kappa_n\left(\tfrac12+\delta-s+i\tau\right)\right|\,\mathrm{d}\tau$$
$$+n^{-\delta}\frac{1}{M+1}\sum_{k=N}^{N+M}\left((kh)^{1/2}+1\right)\exp\{-c_3\log^2(Nh)\}+n^{1/2-2\delta}\exp\{-\tfrac{c_2}{2}Nh\}$$
$$\stackrel{\text{def}}{=} A_1+A_2+n^{1/2-2\delta}\exp\{-\tfrac{c_2}{2}Nh\}. \tag{9}$$

It is easily seen that, for large N,

$$A_2 \ll_h n^{-\delta}(Nh)^{1/2}\exp\left\{-\tfrac{c_3}{2}\log^2(Nh)\right\}. \tag{10}$$

For the estimation of A_1, we will apply Lemma 3. Thus, for $\tau \leqslant \log^2(Nh)$,

$$\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac12+\delta+ikh+i\tau,\alpha\right)\right| \leqslant \left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac12+\delta+ikh+i\tau,\alpha\right)\right|^2\right)^{1/2}$$
$$\ll_{\alpha,K,h} (1+|\tau|)^{1/2}.$$

Hence, by (8),

$$A_1 \ll_{\alpha,K,h} n^{-\delta}\int_{-\log^2(Nh)}^{\log^2(Nh)}(1+|\tau|)^{1/2}\exp\{-c_1\tau\}\,\mathrm{d}\tau \ll_{\alpha,K,h} n^{-\delta}. \tag{11}$$

Now, summarising the results (9)–(11), we obtain

$$\frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)|$$
$$\ll_{\alpha,K,h} n^{-\delta}+n^{-\delta}(Nh)^{1/2}\exp\left\{-\tfrac{c_3}{2}\log^2(Nh)\right\}+n^{1/2-2\delta}\exp\{-\tfrac{c_2}{2}Nh\}. \tag{12}$$

We notice that the implied constant in (12) depends on δ; however, this is omitted because δ depends on K.

Taking $N\to\infty$ in (12), and then $n\to\infty$, we obtain (4) with $\alpha_j=\alpha$ and $h_j=h$. The proof of the lemma is complete. □

3. Results on Weak Convergence

This section is devoted to discrete limit theorems on weakly convergent probability measures in the space $H^r(\mathfrak{D})$. Recall the definition of weak convergence. As usual, denote by $\mathcal{B}(\mathcal{X})$ the Borel σ-field of a topological space $\mathcal{X}$. Let Q_n, $n \in \mathbb{N}$, and Q be probability measures on the measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then Q_n converges weakly to Q as $n \to \infty$ ($Q_n \xrightarrow[n\to\infty]{\mathrm{w}} Q$) if, for every real continuous bounded function f on $\mathcal{X}$, the equality

$$\lim_{n\to\infty} \int_{\mathcal{X}} f \, \mathrm{d}Q_n = \int_{\mathcal{X}} f \, \mathrm{d}Q$$

holds. The theory of weak convergence of probability measures is presented in the monograph [39].

In this section, we will deal with the probability measure

$$P_{N,M,\underline{\alpha},\underline{h}}(A) = W_{N,M}\Big(\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}) \in A\Big), \quad A \in \mathcal{B}(H^r(\mathfrak{D})).$$

We will consider the asymptotic behaviour of $P_{N,M,\underline{\alpha},\underline{\zeta}}$ as $N \to \infty$ by using some auxiliary spaces and probability measures on them. We start with analysing probability measures on a certain group. Weak convergence on locally compact groups is developed in [40].

Denote by Ω the Cartesian product of unit circles over $\mathbb{N}_0$, i.e.,

$$\Omega = \prod_{m\in\mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

With the product topology and pointwise multiplication, Ω is a compact Abelian group, which is the product of compact sets (Tikhonov theorem [41,42]). From this, the existence of the probability Haar measure μ follows [43]. Denote by $\underline{\omega} = (\omega(m) : m \in \mathbb{N}_0)$ the elements of Ω.

Introduce one more group

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, again, Ω^r is a compact topological Abelian group, and, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure $\underline{\mu}$ can be defined. Let $\underline{\omega} = (\omega_1, \dots, \omega_r)$, $\omega_j \in \Omega_j$, $j = 1, \dots, r$, be elements of Ω^r. Notice that the Haar measure $\underline{\mu}$ is the product of Haar measures μ_j, $j = 1, \dots, r$, i.e., for $A = A_1 \times \cdots \times A_r \in \mathcal{B}(\Omega^r)$, $A_j \in \Omega_j$, $j = 1, \dots, r$, we have

$$\underline{\mu}(A) = \mu_1(A_1) \cdots \mu_r(A_r).$$

For $A \in \mathcal{B}(\Omega^r)$, define the probability measure

$$Q_{N,M,\underline{\alpha},\underline{h}}(A) = W_{N,M}\bigg(\Big(\Big((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0\Big), \dots, \Big((m+\alpha_r)^{-ikh_r} : m \in \mathbb{N}_0\Big)\Big) \in A\bigg)$$

and consider its weak convergence as $N \to \infty$.

Lemma 5. *Suppose that $\alpha_j \in (0,1) \setminus \{1/2\}$ and $h_j > 0$, where $j = 1, \dots, r$, are arbitrary, and $M \to \infty$ is as $N \to \infty$. Then, on $(\Omega^r, \mathcal{B}(\Omega^r))$, there exists a probability measure $Q_{\underline{\alpha},\underline{h}}$ such that $Q_{N,M,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathrm{w}} Q_{\underline{\alpha},\underline{h}}$.*

Proof. As it is mentioned in [40], for the proof of weak convergence on groups, it is convenient to use Fourier transforms. Since characters of the group Ω^r have the representation

$$\prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}^{*} \omega_j^{l_{jm}}(m),$$

where the star $*$ indicates that the integers $l_{jm} \neq 0$ only for a finite number of $m \in \mathbb{N}$, the Fourier transform $f_{N,M,\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r)$, $\underline{l}_j = (l_{jm} \in \mathbb{Z},\, m \in \mathbb{N}_0)$, $j = 1, \ldots, r$, of the measure $Q_{N,M,\underline{\alpha},\underline{h}}$ is defined by

$$f_{N,M,\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r) = \int_{\Omega^r} \left(\prod_{j=1}^{r} {\prod_{m\in\mathbb{N}_0}}^{*} \omega_j^{l_{jm}}(m) \right) \mathrm{d}Q_{N,M,\underline{\alpha},\underline{h}}.$$

Hence, we find

$$\begin{aligned} f_{N,M,\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r) &= \frac{1}{M+1} \sum_{k=N}^{N+M} \prod_{j=1}^{r} {\prod_{m\in\mathbb{N}_0}}^{*} (m+\alpha_j)^{-il_{jm}kh_j} \\ &= \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\left\{ -ik \sum_{j=1}^{r} h_j {\sum_{m\in\mathbb{N}_0}}^{*} l_{jm} \log(m+\alpha_j) \right\}. \end{aligned}$$

For

$$L = L(\underline{\alpha}, \underline{h}, \underline{l}_1, \ldots, \underline{l}_r) \stackrel{\text{def}}{=} \sum_{j=1}^{r} h_j {\sum_{m\in\mathbb{N}_0}}^{*} l_{jm} \log(m+\alpha_j) = 2\pi v$$

with $v \in \mathbb{Z}$, equality (13) yields

$$f_{N,M,\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r) = 1. \tag{13}$$

Otherwise, we have

$$f_{N,M,\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r) = \frac{\exp\{-iNL\} - \exp\{-i(N+M+1)L\}}{(M+1)(1-\exp\{iL\})} \tag{14}$$

in view of the formula for a sum of geometric progression. Therefore, taking into account (13) and (14), we obtain

$$\lim_{N\to\infty} f_{N,M,\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r) = \begin{cases} 1 & \text{if } L = 2\pi v,\, v \in \mathbb{Z}, \\ 0 & \text{othewrwise} \end{cases} \tag{15}$$

because, in (14), the numerator is bounded and the denominator tends to ∞ as $N \to \infty$. Since the group Ω^r is compact, it is the Lévy group. Hence, the convergence of Fourier transforms implies weak convergence of the corresponding probability measures. Let the probability measure $Q_{\underline{\alpha},\underline{h}}$ on $(\Omega^r, \mathcal{B}(\Omega^r))$ be given by the Fourier transform

$$f_{\underline{\alpha},\underline{h}}(\underline{l}_1, \ldots, \underline{l}_r) = \begin{cases} 1 & \text{if } L = 2\pi v,\, v \in \mathbb{Z}, \\ 0 & \text{othewrwise.} \end{cases} \tag{16}$$

Then, by (15), we have $Q_{N,M,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\text{w}} Q_{\underline{\alpha},\underline{h}}$. □

Lemma 6. *Suppose that the set $\mathcal{L}(\alpha_1, \ldots, \alpha_r; h_1, \ldots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$, and $M \to \infty$ as $N \to \infty$. Then the relation $Q_{N,M,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\text{w}} \underline{\mu}$ holds.*

Proof. The lemma is a corollary of Lemma 5. Actually, since the set $\mathcal{L}(\alpha_1, \ldots, \alpha_r; h_1, \ldots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$, the equality $L = 2\pi v$ holds if and only if $\underline{l}_j = \underline{0}$ for

$j = 1, \dots, r$, and $v = 0$. Let $\underline{0}$ denote a collection consisting from zeros. Thus, in virtue of (16), in this case,

$$f_{\underline{\alpha},\underline{h}}(\underline{l}_1, \dots, \underline{l}_r) = \begin{cases} 1 & \text{if } (\underline{l}_1, \dots, \underline{l}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{othewrwise}, \end{cases}$$

and this proves the lemma because the right-hand side of the latter equality corresponds to the Fourier transform of the Haar measure $\underline{\mu}$ on $(\Omega^r, \mathcal{B}(\Omega^r))$. $\square$

The next limit lemma concerns $\underline{\zeta}_n(s, \underline{\alpha})$. For $A \in \mathcal{B}(H^r(\mathfrak{D}))$, set

$$P_{N,M,n,\underline{\alpha},\underline{h}}(A) = W_{N,M}\Big(\underline{\zeta}_n(s + ik\underline{h}, \underline{\alpha}) \in A\Big).$$

Lemma 7. *Suppose that* $\alpha_j \in (0,1) \setminus \{1/2\}$ *and* $h_j > 0$, $j = 1, \dots, r$, *are arbitrary numbers, and* $M \to \infty$ *as* $N \to \infty$. *Then on* $(H^r(\mathfrak{D}), \mathcal{B}(H^r(\mathfrak{D})))$, *there exists a probability measure* $P_{n,\underline{\alpha},\underline{h}}$ *such that* $P_{N,M,n,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\text{w}} P_{n,\underline{\alpha},\underline{h}}$.

Proof. We will apply the principle of preservation of weak convergence under certain mappings; see 5.1 of [39].

Define

$$\zeta_n(s, \alpha_j, \omega_j) \sum_{m=0}^{\infty} \frac{\omega_j(m) w_n(m, \underline{\alpha})}{(m + \alpha_j)^s}, \quad \omega_j \in \Omega_j, \ j = 1, \dots, r$$

and put

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}) = (\zeta_n(s, \alpha_1, \omega_1), \dots, \zeta_n(s, \alpha_r, \omega_r)).$$

Consider the mapping $v_{n,\underline{\alpha}} : \Omega^r \to H^r(\mathfrak{D})$ given by

$$v_{n,\underline{\alpha}}(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}), \quad \underline{\omega} \in \Omega^r.$$

The series for $\zeta_n(s, \alpha_j, \omega_j)$ are absolutely convergent in any half-plane $\sigma > \sigma_0$; therefore, the mapping $v_{n,\underline{\alpha}}$ is continuous. Moreover, it follows that

$$v_{n,\underline{\alpha}}\Big(\Big((m + \alpha_1)^{-ikh_1} : m \in \mathbb{N}_0\Big), \dots \Big((m + \alpha_r)^{-ikh_r} : m \in \mathbb{N}_0\Big)\Big) = \underline{\zeta}_n(s + ik\underline{h}, \underline{\alpha}),$$

and thus, for $A \in \mathcal{B}(\Omega^r)$,

$$\begin{aligned} &P_{N,M,n,\underline{\alpha},\underline{h}}(A) \\ &\quad = W_{N,M}\Big(\Big(\Big((m + \alpha_1)^{-ikh_1} : m \in \mathbb{N}_0\Big), \dots \Big((m + \alpha_r)^{-ikh_r} : m \in \mathbb{N}_0\Big)\Big) \in v_{n,\underline{\alpha}}^{-1} A\Big) \\ &\quad = Q_{N,M,\underline{\alpha},\underline{h}}(v_{n,\underline{\alpha}}^{-1} A), \end{aligned}$$

where $Q_{N,M,\underline{\alpha},\underline{h}}$ is the measure from Lemma 5, and $v_{n,\underline{\alpha}}^{-1} A$ denotes the preimage of the set A. Since $v_{n,\underline{\alpha}}$ is continuous, it is $(\mathcal{B}(H^r(\mathfrak{D})), \mathcal{B}(\Omega^r))$-measurable [39]. Hence, the limit measure $Q_{\underline{\alpha},\underline{h}}$ in Lemma 5 defines a new measure $Q_{\underline{\alpha},\underline{h}} v_{n,\underline{\alpha}}^{-1}$ in $(H^r(\mathfrak{D}), \mathcal{B}(H^r(\mathfrak{D})))$ given by

$$Q_{\underline{\alpha},\underline{h}} v_{n,\underline{\alpha}}^{-1}(A) = Q_{\underline{\alpha},\underline{h}}(v_{n,\underline{\alpha}}^{-1} A), \quad A \in \mathcal{B}(H^r(\mathfrak{D})).$$

These observations, Lemma 5, and Theorem 5.1 of [39] show that $P_{N,M,n,\underline{\alpha},\underline{h}}$ converges weakly to the measure $Q_{\underline{\alpha},\underline{h}} v_{n,\underline{\alpha}}^{-1}$ as $N \to \infty$. Thus, denoting $P_{n,\underline{\alpha},\underline{h}} = Q_{\underline{\alpha},\underline{h}} v_{n,\underline{\alpha}}^{-1}$, we have $P_{N,M,n\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\text{w}} P_{n,\underline{\alpha},\underline{h}}$. $\square$

Lemma 8. *Suppose that the set $\mathcal{L}(\alpha_1, \ldots, \alpha_r; h_1, \ldots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$ and $M \to \infty$ as $N \to \infty$. Then, the relation $P_{N,M,n,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\text{w}} \underline{\mu v}_{n,\underline{\alpha}}^{-1}$ holds.*

Proof. We repeat the proof of Lemma 7 using Lemma 6 in place of Lemma 5. □

Lemma 7 is a key for the proof of weak convergence for $P_{N,M,\underline{\alpha},\underline{h}}$ in the general case. Additionally, we need one classical result on convergence in distribution. Let X_n, $n \in \mathbb{N}$, and X be $\mathcal{X}$-valued random elements with distributions P_n and P, respectively. We say that X_n converges to X as $n \to \infty$ in distribution ($X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$) if and only if $P_n \xrightarrow[n\to\infty]{\text{w}} P$.

Lemma 9 (see [39], Theorem 4.2)**.** *Suppose that the metric space $(\mathcal{X}, d)$ is separable, $\mathcal{X}$-valued random elements X_{nl} and Y_n, $n, l \in \mathbb{N}$, are defined on the same probability space with a measure ν, and the relations*

$$X_{nl} \xrightarrow[n\to\infty]{\mathcal{D}} X_l, \quad \forall l \in \mathbb{N},$$

and

$$X_l \xrightarrow[n\to\infty]{\mathcal{D}} X$$

hold. If, for every $\delta > 0$,

$$\lim_{l\to\infty} \limsup_{n\to\infty} \nu\{d(X_{nl}, Y_n) \geqslant \delta\} = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Theorem 3. *Suppose that $\alpha_j \in (0,1) \setminus \{1/2\}$ and $h_j > 0$, $j = 1, \ldots, r$, are arbitrary, and $\max_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{23/70} \leqslant M \leqslant \min_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{1/2}$. Then, on $(H^r(\mathfrak{D}), \mathcal{B}(H^r(\mathfrak{D})))$, there exists a probability measure $P_{\underline{\alpha},\underline{h}}$ such that $P_{N,M,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\text{w}} P_{\underline{\alpha},\underline{h}}$.*

Proof. First we will prove that the limit measure $P_{n,\underline{\alpha},\underline{h}}$ is tight, i.e., that, for every $\varepsilon > 0$, there is a compact set $K \subset H^r(\mathfrak{D})$ such that

$$P_{n,\underline{\alpha},\underline{h}}(K) > 1 - \varepsilon \tag{17}$$

for all $n \in \mathbb{N}$. We observe that it suffices to show the tightness for the marginal measures of $P_{n,\underline{\alpha},\underline{h}}$,

$$P_{n,\alpha_j,h_j}(A_j) = P_{n,\underline{\alpha},\underline{h}}\big(\underbrace{H(\mathfrak{D}), \ldots, H(\mathfrak{D})}_{j-1}, A_j, H(\mathfrak{D}), \ldots, H(\mathfrak{D})\big),\ A_j \in \mathcal{B}(H(\mathfrak{D})),\ j = 1, \ldots, r.$$

Actually, if P_{n,α_j,h_j} are tight, then, for $\varepsilon > 0$, there exists compact sets $K_j \subset H(\mathfrak{D})$ such that

$$P_{n,\alpha_j,h_j}(K_j) > 1 - \frac{\varepsilon}{r}$$

for all $n \in \mathbb{N}$. Let $K = K_1 \times \cdots \times K_r$. Then

$$P_{n,\underline{\alpha},\underline{h}}(H^r(\mathfrak{D}) \setminus K) \leqslant \sum_{j=1}^{r} P_{n,\alpha_j,h_j}(H(\mathfrak{D}) \setminus K_j) \leqslant r \cdot \frac{\varepsilon}{r} = \varepsilon$$

for all $n \in \mathbb{N}$. Hence, inequality (17) holds. Thus, it is sufficient to consider $P_{n,\alpha,h}$ with arbitrary $\alpha \in (0,1) \setminus \{1/2\}$ and $h > 0$.

Let K_m be a compact set of $H(\mathfrak{D})$ from the definition of the metric ρ. There exists $\varepsilon > 0$ such that $\sigma \geqslant 1/2 + \varepsilon$ for $s = \sigma + it \in K_m$. Then, by Lemma 3, we have on the hypothesis for M,

$$\sum_{k=N}^{N+M} |\zeta(\sigma + ikh, \alpha)|^2 \ll_{K_m,\alpha,h} M.$$

Hence,

$$\sum_{k=N}^{N+M} |\zeta(\sigma + ikh, \alpha)| \leqslant \left(M \sum_{k=N}^{N+M} |\zeta(\sigma + ikh, \alpha)|^2 \right)^{1/2} \ll_{K_m,\alpha,h} M.$$

This, together with the Cauchy integral formula, yields

$$\sum_{k=N}^{N+M} \sup_{s \in K_m} |\zeta(\sigma + ikh, \alpha)| \ll_{K_m,\alpha,h} M.$$

Therefore, in view of (4),

$$\begin{aligned} \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K_m} |\zeta_n(s+ikh,\alpha)| &\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K_m} |\zeta(s+ikh,\alpha)| \\ &+ \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K_m} |\zeta_n(s+ikh,\alpha) - \zeta(s+ikh,\alpha)| \leqslant C_{K_m,h,\alpha} < \infty. \end{aligned} \tag{18}$$

Suppose that $\eta_{N,M,h}$ is a random variable defined on a certain probability space $(\mathbb{T}, \mathcal{A}, \nu)$ and having the distribution

$$\nu\{\eta_{N,M,h} = kh\} = \frac{1}{M+1}, \quad k = N, N+1, \dots, N+M.$$

Introduce the $H(\mathfrak{D})$-valued random element

$$Y_{N,M,n,\alpha,h} = Y_{N,M,n,\alpha,h}(s) = \zeta_n(s + i\eta_{N,M,n}, \alpha),$$

and denote by $Y_{n,\alpha,h} = Y_{n,\alpha,h}(s)$ the random element having the distribution $P_{n,\alpha,h}$. Then the assertion of Lemma 7 implies

$$Y_{N,M,n,\alpha,h} \xrightarrow[N\to\infty]{\mathcal{D}} Y_{n,\alpha,h}. \tag{19}$$

Since convergence in $H(\mathfrak{D})$ is uniform on compact sets, from this, we get

$$\sup_{s\in K_m} |Y_{N,M,n,\alpha,h}(s)| \xrightarrow[N\to\infty]{\mathcal{D}} \sup_{s\in K_m} |Y_{n,\alpha,h}(s)|. \tag{20}$$

Now, we fix $\varepsilon > 0$, and put $V_m = 2^m C_{K_m,\alpha,h} \varepsilon^{-1}$. Then (19) and (20) give

$$\begin{aligned} \nu\left\{ \sup_{s\in K_m} |Y_{n,\alpha,h}(s)| \geqslant V_m \right\} &= \lim_{N\to\infty} \nu\left\{ \sup_{s\in K_m} |Y_{N,M,n,\alpha,h}(s)| \geqslant V_m \right\} \\ &\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{(M+1)V_m} \sum_{k=N}^{N+M} \sup_{s\in K_m} |\zeta_n(s+ikh,\alpha)| \\ &\leqslant 2^{-m}\varepsilon. \end{aligned} \tag{21}$$

Let

$$K = K_\varepsilon = \left\{ f \in H(\mathfrak{D}) : \sup_{s\in K_m} |f(s)| \leqslant V_m,\ m \in \mathbb{N} \right\}.$$

Then K is a uniformly bounded set in $H(\mathfrak{D})$, and hence it is compact; by (21),

$$\nu\{Y_{n,\alpha,h} \in K\} = 1 - \nu\{Y_{n,\alpha,h} \notin K\} \geqslant 1 - \varepsilon \sum_{m=1}^{\infty} 2^{-m} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, by the definition of $Y_{n,\alpha,h}$,

$$P_{n,\alpha,h}(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, i.e., $P_{n,\alpha,h}$ is tight. □

Now, we continue the direct proof of the theorem preserving the notation for $\eta_{N,M,n,\alpha,\underline{h}}$. Since the measure $P_{n,\underline{\alpha},\underline{h}}$ is tight, by the classical Prokhorov theorem (see [39], Theorem 6.1), it is relatively compact, i.e., every sequence of $\{P_{n,\underline{\alpha},\underline{h}}\}$ contains a subsequence weakly convergent to a certain probability measure on $(H^r(\mathfrak{D}), \mathcal{B}(H^r(\mathfrak{D})))$. Thus, let $\{P_{n_l,\underline{\alpha},\underline{h}}\} \subset \{P_{n,\underline{\alpha},\underline{h}}\}$ be a subsequence such that $P_{N_l,\underline{\alpha},\underline{h}} \xrightarrow[l\to\infty]{\mathrm{w}} P_{\underline{\alpha},\underline{h}}$ with a probability measure on $(H^r(\mathfrak{D}), \mathcal{B}(H^r(\mathfrak{D})))$.

Denoting

$$Y_{n,\underline{\alpha},\underline{h}} = Y_{n,\underline{\alpha},\underline{h}}(s)$$

the $H^r(\mathfrak{D})$-valued random element with distribution $P_{n,\underline{\alpha},\underline{h}}$, we may rewrite this in the form

$$Y_{n_l,\underline{\alpha},\underline{h}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\underline{\alpha},\underline{h}}. \tag{22}$$

Moreover, setting

$$Y_{N,M,n,\underline{\alpha},\underline{h}} = Y_{N,M,n,\underline{\alpha},\underline{h}}(s) = \underline{\zeta}_n(s + i\eta_{N,M,\underline{\alpha},\underline{h}}, \underline{\alpha}),$$

in virtue of Lemma 7, we have

$$Y_{N,M,n,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} Y_{n,\underline{\alpha},\underline{h}}. \tag{23}$$

Introduce one more $H^r(\mathfrak{D})$-valued random element

$$Z_{N,M,\underline{\alpha},\underline{h}} = Z_{N,M,\underline{\alpha},\underline{h}}(s) = \underline{\zeta}(s + i\eta_{N,M,\underline{\alpha},\underline{h}}, \underline{\alpha})$$

Then, application of Lemma 4, for $\varepsilon > 0$, yields

$$\begin{aligned}
&\lim_{l\to\infty} \limsup_{N\to\infty} \nu\{\rho_r(Z_{N,M,\underline{\alpha},\underline{h}}, Y_{N,M,n_l,\underline{\alpha},\underline{h}}) \geqslant \varepsilon\} \\
&\qquad = \lim_{l\to\infty} \limsup_{N\to\infty} W_{N,M}\Big(\rho_r(\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}), \underline{\zeta}_{n_l}(s + iki\underline{h}, \underline{\alpha})) \geqslant \varepsilon\Big) \\
&\qquad \leqslant \lim_{l\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(M+1)} \sum_{k=N}^{N+M} \rho_r(\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}), \underline{\zeta}_{n_l}(s + iki\underline{h}, \underline{\alpha})) = 0.
\end{aligned}$$

Therefore, the latter equality, (22), and (23) show that Lemma 9 is applicable for the random elements $Z_{N,M,\underline{\alpha},\underline{h}}$, $Y_{N,M,n_l,\underline{\alpha},\underline{h}}$, and $Y_{\underline{\alpha},\underline{h}}$, which corresponds to the measure $P_{\underline{\alpha},\underline{h}}$. This leads to the relation

$$Z_{N,M,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\alpha},\underline{h}},$$

and the definition of $Z_{N,M,\underline{\alpha},\underline{h}}$ proves the theorem.

The case of Theorem 3 with the linearly independent over $\mathbb{Q}$ set $\mathcal{L}(\alpha_1,\dots,\alpha_r;h_1,\dots,h_r;\pi)$ is simpler, and we have a full answer. On the probability space $(\Omega^r,\mathcal{B}(\Omega^r),\underline{\mu})$, define the $H^r(\mathfrak{D})$-valued random element

$$\underline{\zeta}(s,\underline{\omega},\underline{\alpha})=(\zeta(s,\omega_1,\alpha_1),\dots,\zeta(s,\omega_r,\alpha_r)),$$

where

$$\zeta(s,\omega_j,\alpha_j)=\sum_{m=1}^{\infty}\frac{\omega_j(m)}{(m+\alpha_j)^s},\quad j=1,\dots,r,$$

and denote by $P_{\underline{\zeta},\underline{\alpha}}$ its distribution. Thus,

$$P_{\underline{\zeta},\underline{\alpha}}(A)=\underline{\mu}\{\underline{\omega}\in\Omega^r:\zeta(s,\underline{\omega},\alpha)\in A\},\quad A\in\mathcal{B}(H^r(\mathfrak{D})).$$

Theorem 4. *Suppose that the set $\mathcal{L}(\alpha_1,\dots,\alpha_r;h_1,\dots,h_r;\pi)$ is linearly independent over $\mathbb{Q}$, and $\max_{1\leqslant j\leqslant r}h_j^{-1}(Nh_j)^{23/70}\leqslant M\leqslant\min_{1\leqslant j\leqslant r}h_j^{-1}(Nh_j)^{1/2}$. Then $P_{N,M,\underline{\alpha},\underline{h}}\xrightarrow[N\to\infty]{\mathrm{w}}P_{\underline{\zeta},\underline{\alpha}}$.*

Proof. Denote by $Q_{n,\underline{\alpha}}$ the limit measure in Lemma 8, i.e., $Q_{n,\underline{\alpha}}=\underline{\mu}v_{n,\underline{\alpha}}^{-1}$. This measure is independent of M and $\underline{h}$. In other words, we have the same situation as considered in [28]; see proofs of Lemma 2.6 and Theorem 2.1. Thus, we have the relation

$$Q_{n,\underline{\alpha}}\xrightarrow[n\to\infty]{\mathrm{w}}P_{\underline{\zeta},\underline{\alpha}}. \tag{24}$$

Let $Y_{n,\underline{\alpha}}$ be the $H^r(\mathfrak{D})$-valued random element with distribution $Q_{n,\underline{\alpha}}$. Then, by Lemma 8, the relation

$$Y_{N,M,n,\underline{\alpha},\underline{h}}\xrightarrow[N\to\infty]{\mathcal{D}}Y_{n,\underline{\alpha}} \tag{25}$$

holds. Moreover, Lemma 4 yields

$$\lim_{n\to\infty}\limsup_{N\to\infty}\nu\left\{\rho_r(Z_{N,M,\underline{\alpha},\underline{h}},Y_{N,M,n,\underline{\alpha},\underline{h}})\geqslant\varepsilon\right\}=0 \tag{26}$$

for every $\varepsilon>0$. Here, we have used the notations of the proof of Theorem 3. Now, Lemma 9 and (24)–(26) give the assumption of the theorem. □

4. Proofs of Universality Theorems

In this section, we will apply Theorems 3 and 4 for the proof of universality. Also, the Mergelyan theorem on approximation of analytic functions will be useful.

Lemma 10 (see [44–46])**.** *Suppose that $K\subset\mathbb{C}$ is a compact set with connected complement; $u(s)$ is a continuous function on K and is analytic inside of K. Then, for any $\varepsilon>0$, there is a polynomial $q_{\varepsilon,u,K}(s)$ such that*

$$\sup_{s\in K}|u(s)-q_{\varepsilon,u,K}(s)|<\varepsilon.$$

The proof of Theorem 1 involves the support $S_{\underline{\zeta},\underline{\alpha}}$ of the measure $P_{\underline{\zeta},\underline{\alpha}}$. We recall that $S_{\underline{\zeta},\underline{\alpha}}$ is a minimal closed subset of $H^r(\mathfrak{D})$ such that $P_{\underline{\zeta},\underline{\alpha}}(S_{\underline{\zeta},\underline{\alpha}})=1$. The set $S_{\underline{\zeta},\underline{\alpha}}$ consists of all elements $\underline{u}\in H^r(\mathfrak{D})$ with a property that $P_{\underline{\zeta},\underline{\alpha}}(\mathcal{G}_{\underline{u}})>0$ for every open neighbourhood $\mathcal{G}_{\underline{u}}$ of $\underline{u}$.

Lemma 11 (see [28], Lemma 3.1)**.** *Suppose that the set $\mathcal{L}(\alpha_1,\dots,\alpha_r;h_1,\dots,h_r:\pi)$ is linearly independent over $\mathbb{Q}$. Then $S_{\underline{\zeta},\underline{\alpha}}=H^r(\mathfrak{D})$.*

Proof of Theorem 1. By Lemma 10, there exist polynomials $q_1(s), \ldots, q_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-q_j(s)|<\frac{\varepsilon}{2}. \tag{27}$$

Introduce the set $G_\varepsilon \subset H^r(\mathfrak{D})$ by

$$G_\varepsilon=\left\{(u_1,\ldots,u_r)\in H^r(\mathfrak{D}):\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|u_j(s)-q_j(s)|<\frac{\varepsilon}{2}\right\}.$$

Then, G_ε is an open neighbourhood of the set $(q_1(s), \ldots, q_r(s))$. However, $(q_1(s), \ldots, q_r(s)) \in H^r(\mathfrak{D})$. Thus, in view of Lemma 11, the set G_ε is an open neighbourhood of an element of the support of the measure $P_{\underline{\zeta},\underline{\alpha}}$. Then, by the support property,

$$P_{\underline{\zeta},\underline{\alpha}}(G_\varepsilon)>0. \tag{28}$$

Taking into account inequality (27), we find that, for $(u_1, \ldots, u_r) \in G_\varepsilon$,

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|u_j(s)-f_j(s)|<\varepsilon.$$

Hence, it follows that

$$G_\varepsilon\subset\widehat{G}_\varepsilon\stackrel{\text{def}}{=}\left\{(u_1,\ldots,u_r)\in H^r(\mathfrak{D}):\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-u_j(s)|<\varepsilon\right\}.$$

Therefore, in virtue of (28),

$$P_{\underline{\zeta},\underline{\alpha}}(\widehat{G}_\varepsilon)>0. \tag{29}$$

Now, applying Theorem 4 and the equivalent of weak convergence of probability measures in terms of open sets ([39], Theorem 2.1), by (29) we have

$$\liminf_{N\to\infty}P_{N,M,\underline{\alpha},\underline{h}}(\widehat{G}_\varepsilon)\geqslant P_{\underline{\zeta},\underline{\alpha}}(\widehat{G}_\varepsilon)>0.$$

This, and the definitions of $\widehat{G}_\varepsilon$ and $P_{N,M,\underline{\alpha},\underline{h}}$ show that

$$\liminf_{N\to\infty}W_{N,M}\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+ikh_j,\alpha_j)-f_j(s)|<\varepsilon\right)>0,$$

i.e., the first assertion of the theorem is proved.

The boundary $\partial\widehat{G}_\varepsilon$ of $\widehat{G}_\varepsilon$ lies in the union of the sets in which, for at least one j,

$$\sup_{s\in K_j}|u_j(s)-f_j(s)|=\varepsilon,$$

and, for the other j,

$$\sup_{s\in K_j}|u_j(s)-f_j(s)|<\varepsilon.$$

Therefore, $\partial\widehat{G}_{\varepsilon_1}\cap\partial\widehat{G}_{\varepsilon_2}=\varnothing$ for different positive ε_1 and ε_2. From this remark, it follows $P_{\underline{\zeta},\underline{\alpha}}(\partial\widehat{G}_\varepsilon)\neq 0$ for at most countably many $\varepsilon>0$, in other words, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta},\underline{\alpha}}$ ($P_{\underline{\zeta},\underline{\alpha}}(\partial\widehat{G}_\varepsilon)=0$) for all but at most countably many $\varepsilon>0$. Applying

Theorem 4 and the equivalent of weak convergence of probability measures in terms of continuity sets ([39], Theorem 2.1), we determine that the limit

$$\lim_{N\to\infty} P_{N,M,\underline{\alpha},\underline{h}}(\widehat{G}_\varepsilon) = P_{\underline{\zeta},\underline{\alpha}}(\widehat{G}_\varepsilon)$$

exists for all but at most countably many $\varepsilon > 0$. This, (29), and the definitions of $P_{N,M,\underline{\alpha},\underline{h}}$ and $\widehat{G}_\varepsilon$ prove the second statement of the theorem. □

Proof of Theorem 2. Let $P_{\underline{\alpha},\underline{h}}$ be the limit measure in Theorem 3. Denote by $F_{\underline{\alpha},\underline{h}}$ the support of the measure $P_{\underline{\alpha},\underline{h}}$. Then $F_{\underline{\alpha},\underline{h}}$ is a non-empty closed subset of $H^r(\mathfrak{D})$. For arbitrary compact set $K_1, \dots, K_r$ of $\mathfrak{D}$, and $(f_1(s), \dots, f_r(s)) \in F_{\underline{\alpha},\underline{h}}$, define

$$\mathcal{G}_\varepsilon = \left\{ (u_1, \dots, u_r) \in H^r(\mathfrak{D}) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |u_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then $\mathcal{G}_\varepsilon$ is an open neighbourhood of the element $(f_1(s), \dots, f_r(s))$ of the support of the measure $P_{\underline{\alpha},\underline{h}}$. Hence,

$$P_{\underline{\alpha},\underline{h}}(\mathcal{G}_\varepsilon) > 0. \tag{30}$$

Therefore, Theorem 3, and the equivalent of weak convergence of probability measures in terms of open sets yield

$$\liminf_{N\to\infty} P_{N,M,\underline{\alpha},\underline{h}}(\mathcal{G}_\varepsilon) \geqslant P_{\underline{\alpha},\underline{h}}(\mathcal{G}_\varepsilon) > 0,$$

and this proves the first statement of the theorem.

For the proof of the second statement of the theorem, we repeat arguments used in the proof of Theorem 1. The set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\alpha},\underline{h}}$ for all but at most countably many $\varepsilon > 0$. Hence, using Theorem 3 and the equivalent of weak convergence in terms of continuity sets, we find by (30) that the limit

$$\lim_{N\to\infty} P_{M,N,\underline{\alpha},\underline{h}}(\mathcal{G}_\varepsilon)$$

exists and is positive for all but at most countably many $\varepsilon > 0$. This proves the second assertion of the theorem. □

5. Conclusions

In the paper, we obtained joint approximation theorems of analytic functions by discrete shifts $(\zeta(s + ikh_1, \alpha_1), \dots, \zeta(s + ikh_r, \alpha_r))$, $k \in \mathbb{N}$, $h_j > 0$ for $j = 1, \dots, r$, of Hurwitz zeta-functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$. If the set $\{(h_1 \log(m + \alpha_1) : m \in \mathbb{N}_0), \dots, (h_r \log(m + \alpha_r) : m \in \mathbb{N}_0), 2\pi\}$ is linearly independent over $\mathbb{Q}$, then every tuple $(f_1(s), \dots, f_r(s))$ of analytic functions on the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ is approximated by the above shifts, and the set of approximating shifts has a positive lower density in the interval $[N, N + M]$ with $\max_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{23/70} \leqslant M \leqslant \min_{1\leqslant j\leqslant r} h_j^{-1}(Nh_j)^{1/2}$. This improves Proposition 7, where the interval of length N was considered. In the general case, the above shifts also preserve a good approximation property: they approximate a certain closed set of analytic functions. The used method is based on new mean square estimates for the Hurwitz zeta-function and multidimensional probabilistic limit theorem in short intervals. Note that the general case earlier was not considered. A big problem arises in identifying the set of approximated analytic functions. Also, the bounds for M should be extended.

Author Contributions: Conceptualisation, A.L. and D.Š.; methodology, A.L. and D.Š.; investigation, A.L. and D.Š.; writing—original draft preparation, A.L. and D.Š. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors thank the referees for useful remarks and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hurwitz, A. Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{D}{n}\right) \frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahlen binärer quadratischer Formen auftreten. *Z. Math. Phys.* **1882**, *27*, 86–101.
2. Davenport, H.; Heilbronn, H. On the zeros of certain Dirichlet series. *J. Lond. Math. Soc.* **1936**, *11*, 181–185. [CrossRef]
3. Cassels, J.W.S. Footnote to a note of Davenport and Heilbronn. *J. Lond. Math. Soc.* **1961**, *36*, 177–184. [CrossRef]
4. Davenport, H. *Multiplicative Number Theory*; Springer: New York, NY, USA, 1980.
5. Apostol, T.M. *Introduction to Analytic Number Theory*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1976.
6. Ivič, A. *The Riemann Zeta-Function*; John Wiley & Sons: New York, NY, USA, 1985.
7. Karatsuba, A.A.; Nathanson, M.B. *Basic Analytic Number Theory*; Springer: Berlin/Heidelberg, Germany, 1993.
8. Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.* **1975**, *9*, 443–453. [CrossRef]
9. Voronin, S.M. A theorem on the distribution of values of the Riemann zeta-function. *Sov. Math., Dokl.* **1975**, *16*, 410.
10. Voronin, S.M. The functional independence of Dirichlet *L*-functions. *Acta Arith.* **1975**, *27*, 493–503. (In Russian)
11. Voronin, S.M. Analytic Properties of Arithmetic Objects. Doctoral Thesis, V.A. Steklov Mathematical Institute, Moscow, Russia, 1977.
12. Voronin, S.M. *Selected Works: Mathematics*; Karatsubaed, A.A., Ed.; Moscow State Technical Univ. Press: Moscow, Russia, 2006.
13. Karatsuba, A.A.; Voronin, S.M. *The Riemann Zeta-Function*; Walter de Gruiter: Berlin, Germany; New York, NY, USA, 1992.
14. Gonek, S.M. Analytic Properties of Zeta and *L*-Functions. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA, 1979.
15. Bagchi, B. The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series. Ph.D. Thesis, Indian Statistical Institute, Calcutta, India, 1981.
16. Steuding, J. *Value-Distribution of L-Functions*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007; Volume 1877.
17. Kowalski, E. *An Introduction to Probabilistic Number Theory*; Cambridge University Press: Cambridge, UK, 2021.
18. Matsumoto, K. A survey on the theory of universality for zeta and *L*-functions. In *Number Theory: Plowing and Starring Through High Wave Forms, Proceedings of the 7th China-Japan Seminar (Fukuoka 2013), Fukuoka, Japan, 28 October–1 November 2013*; Series on Number Theory and Its Applications; Kaneko, M., Kanemitsu, S., Liu, J., Eds.; World Scientific Publishing Co.: New Jersey, NJ, USA; London, UK; Singapore; Bejing, China; Shanghai, China; Hong Kong, China; Taipei, China; Chennai, India, 2015; pp. 95–144.
19. Laurinčikas, A.; Garunkštis, R. *The Lerch Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2002.
20. Sourmelidis, A.; Steuding, J. On the value distribution of Hurwitz zeta-function with algebraic irrational parameter. *Constr. Approx.* **2022**, *55*, 829–860. [CrossRef]
21. Reich, A. Werteverteilung von Zetafunktionen. *Arch. Math.* **1980**, *34*, 440–451. [CrossRef]
22. Laurinčikas, A. A discrete universality theorem for the Hurwitz zeta-function. *J. Number Theory* **2014**, *143*, 232–247. [CrossRef]
23. Meška, L. Modified Universality Theorems for the Riemann and Hurwitz Zeta-Functions. Ph.D. Thesis, Vilnius University, Vilnius, Lithuania, 2017.
24. Mauclaire, J.-L. Universality of the Riemann zeta function: Two remarks. *Ann. Univ. Sci. Budap. Rolando Eötvös Sect. Comput.* **2013**, *39*, 311–319. [CrossRef]
25. Sourmelidis, A. Continuous and discrete universality of zeta-functions: Two sides of the same coin? *Proc. Amer. Math. Soc.* **2025**, *153*, 1435–1445. [CrossRef]
26. Laurinčikas, A. The joint universality of Hurwitz zeta-functions. *Šiauliai Math. Semin.* **2008**, *3*, 169–187. [CrossRef]
27. Sander, J.; Steuding, J. Joint universality for Euler products of Dirichlet *L*-functions. *Analysis* **2006**, *26*, 295–312. [CrossRef]
28. Laurinčikas, A. A general joint discrete universality theorem for Hurwitz zeta-functions. *J. Number Theory* **2015**, *154*, 44–62. [CrossRef]
29. Laurinčikas, A. Universality of the Riemann zeta-function in short intervals. *J. Number Theory* **2019**, *204*, 279–295. [CrossRef]

30. Andersson, J.; Garunkštis, R.; Kačinskaitė, R.; Nakai, K.; Pańkowski, Ł.; Sourmelidis, A.; Steuding, R.; Steuding, J.; Wananiyakul, S. Notes on universality in short intervals and exponential shifts. *Lith. Math. J.* **2024**, *64*, 125–137. [CrossRef]
31. Lee, Y.; Pańkowski, L. Universality of the zeta function in short intervals. *Ramanujan J.* **2025**, *68*, 5. [CrossRef]
32. Laurinčikas, A.; Šiaučiūnas, D. Joint approximation of analytic functions by the shifts of Hurwitz zeta-functions in short intervals. *Open Math.* **2025**, *23*, 20250173. [CrossRef]
33. Laurinčikas, A. Universality of the Hurwitz zeta-function in short intervals. *Bol. Soc. Mat. Mex.* **2025**, *31*, 17. [CrossRef]
34. Laurinčikas, A.; Šiaučiūnas, D. Approximation of analytic functions by discrete shifts of the Hurwitz zeta-function in short intervals. *Math. Slovaca* 2025, *submitted*.
35. Laurinčikas, A.; Šiaučiūnas, D. The mean square of the Hurwitz zeta-function in short intervals. *Axioms* **2024**, *13*, 510. [CrossRef]
36. Montgomery, H.L. *Topics in Multiplicative Number Theory*; Lecture Notes Math; Springer: Heidelberg/Berlin, Germany; New York, NY, USA, 1971; Volume 227.
37. Conway, J.B. *Functions of One Complex Variable*; Springer: New York, NY, USA, 1973.
38. Prachar, K. *Primzahlverteilung*; Springer: Berlin/Göttingen/Heidelberg, Germany, 1957. (In Germany)
39. Billingsley, P. *Convergence of Probability Measures*, 2nd ed.; John Wiley & Sons: New York, NY, USA, 1999.
40. Heyer, H. *Probability Measures on Locally Compact Groups*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1977.
41. Tychonoff, A. Über einen Funktionenraum. *Math. Ann.* **1935**, *111*, 762–766. [CrossRef]
42. Wright, D.G. Tychonoff's theorem. *Proc. Amer. Math. Soc.* **1994**, *120*, 985–987.
43. Rudin, W. *Functional Analysis*; Mc Graw-Hill: New York, NY, USA, 1973.
44. Mergelyan, S.N. Uniform approximations of functions of a complex variable. *Uspehi Matem. Nauk (N.S.) 7* **1952**, *2*, 31–122.
45. Mergelyan, S.N. Uniform approximations to functions of a complex variable. In *American Mathematical Society Translations*; No. 101; American Mathematical Society: Providence, RI, USA, 1954.
46. Walsh, J.L. *Interpolation and Approximation by Rational Functions in the Complex Domain*; Math. Soc. Colloq. Publ.; American Mathematical Society: Providence, RI, USA, 1960; Volume 20.