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On Self-Approximation of the Riemann Zeta Function in Short Intervals

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Abstract

The Riemann hypothesis (RH) says that all zeros of the Riemann zeta function $\zeta(s)$, $s = \sigma + it$, in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$ lie on the line $\sigma = 1/2$. There are many equivalents of RH in various terms. In this paper, we propose equivalents of RH in terms of self-approximation, i.e., of the approximation of $\zeta(s)$ by $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, in the interval $\tau \in [T, T + U]$ with $T^\eta \leqslant U \leqslant T$, $\eta = 1273/4033$. We show that the RH is equivalent to the positivity of lower density and (with some exception for the accuracy of approximation) the density of the set of approximating shifts $\zeta(s + i\tau)$. For the proof, a probabilistic approach and mean square estimates for $\zeta(s)$ in short intervals are applied.

Keywords: equivalent of the Riemann hypothesis; limit theorem; non-trivial zeros; Riemann hypothesis; Riemann zeta function; universality; weak convergence of probability measures; zero-free region

1. Introduction and Results

Denote by $\mathbb{P}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all prime, positive integer, real and complex numbers, respectively, and let $s = \sigma + it$ be a complex variable. The Riemann zeta function $\zeta(s)$ is defined for $\sigma > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \tag{1}$$

or equivalently, by the Euler product

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

From (1), it follows that $\zeta(s)$ is an analytic function in the half-plane $\sigma > 1$. Moreover, $\zeta(s)$ has the functional equation of the symmetric form

$$\eta(s) = \eta(1 - s), \tag{3}$$

where

$$\eta(s) = \zeta(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right),$$

and $\Gamma(s)$ denotes the Euler gamma function and has meromorphic continuation to the entire complex plane with the unique simple pole at the point $s = 1$ with $\text{Res}_{s=1}\zeta(s) = 1$.

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The interest of $\zeta(s)$ was already observed by L. Euler; however, the most outstanding merits in the investigation of $\zeta(s)$ belongs to B. Riemann. He proved [1] that the functional Equation (3) continued analytically $\zeta(s)$ and proposed a way to apply $\zeta(s)$ for studying the distribution of prime numbers, i.e., for the asymptotic formula for

$$\pi(x) = \sum_{\substack{p \leqslant x \\ p \in \mathbb{P}}} 1, \quad x \to \infty.$$

Riemann's method was based on the location of some zeros of the function $\zeta(s)$.

From (2), it follows easily that $\zeta(s) \neq 0$ for $\sigma > 1$. Equation (3) implies that $\zeta(s) = 0$ for $s = -2m$, $m \in \mathbb{N}$, which are poles of $\Gamma(s/2)$, and $\zeta(s) \neq 0$ if $\sigma \leqslant 0$, $t \neq 0$. The points $s = -2m$ are called trivial zeros of $\zeta(s)$. Thus, it remains the so-called critical strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. Riemann knew that $\zeta(s)$ has infinitely many zeros in the critical strip. More precisely, he affirmed that, for the number $N(T)$ of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \beta < 1, 0 \leqslant \gamma \leqslant T$, the asymptotic formula

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \quad T \to \infty,$$

is valid. Riemann was right: The above formula was obtained by H. von Mangoldt in [2]. The zeros of $\zeta(s)$ lying in the critical strip are called non-trivial. However, the most interesting and important of Riemann's conjecture claims that all non-trivial zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$. This conjecture is named the Riemann hypothesis (RH). The RH is one of the seven most important Millenium mathematical problems [3].

Let

$$\Lambda(m) = \begin{cases} \log p & \text{if } m = p^n, n \in \mathbb{P}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\psi(x) = \sum_{m \leqslant x} \Lambda(m).$$

The Riemann method for the investigation of $\pi(x)$ is based on the following formula:

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^2), \quad x > 1, \ x \neq p^m, \tag{4}$$

where the summation runs over non-trivial zeros of $\zeta(s)$, which was stated in [1] without proof. It is easily seen that estimations for $\psi(x) - x$ lead to bounds for $\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u}$. In general, the Riemann idea was very good; however, for the estimation of $\psi(x) - x$, a truncated version of (4) is needed, and this was carried out independently by J. Hadamard [4] and C.J. de la Vallée Poussin [5–7]. They proved that there is an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for

$$\sigma \geqslant 1 - \frac{c}{\log(|t| + 2)}. \tag{5}$$

Throughout the paper, we will use the notation $\ll_\theta$, which is synonymous with $O_\theta(\dots)$, with the implied constant depending on θ.

From (5), one has

$$\psi(x) - x \ll x \exp\left\{-c_1 (\log x)^{1/2}\right\}, \quad c_1 > 0,$$

and the latter bound implies

$$\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u} \ll x \exp\Big\{-c_2(\log x)^{1/2}\Big\}, \quad c_2 > 0. \tag{6}$$

Here and throughout, we use the notation $\exp\{a\} = \mathrm{e}^a$.

The improvement of (6) depends on the extension of the zero-free region for $\zeta(s)$. The best known result asserts (see, for example, [8–10]) that there exists a constant $C > 0$ such that $\zeta(s) \neq 0$ for

$$\sigma \geqslant 1 - \frac{C}{\log^{2/3}|t|(\log\log|t|)^{1/3}}, \quad t \geqslant 3. \tag{7}$$

It is indicated in [10] that the latter result belongs to H.-E. Richert (unpublished). This is the interesting problem with respect to the estimation of the constant in (7). The last known results are the following: K. Ford [11] proved (7) with $C = 1/57.54$ and, for sufficiently large t, with $C = 1/49.13$. P.P. Nielsen [12] replaced the latter value with $C = 1/49.08$. The best result in the field were reported by M.J. Mossinghoff, T.S. Trudgian, and A. Yang [13]. They obtained $\zeta(s) \neq 0$ in the region

$$\sigma \geqslant 1 - \frac{1}{55.241\log^{2/3}|t|(\log\log|t|)^{1/3}}, \quad t \geqslant 3,$$

and, for sufficiently large $|t|$,

$$\sigma \geqslant 1 - \frac{1}{48.1588\log^{2/3}|t|(\log\log|t|)^{1/3}}.$$

These results show how deep is the problem.

Using a truncated formula for $\psi(x)$, it can be obtained (see, for example, [8]) that RH is equivalent to the estimate

$$\psi(x) - x \ll x^{1/2}\log^2 x. \tag{8}$$

Let

$$\psi_1(x) \stackrel{\text{def}}{=} \sum_{m\leqslant x} \Lambda(m)(\log m)^{-1} = \pi(x) + \sum_{\substack{m\leqslant x\\ m=p^k\\ k\geqslant 2}} \Lambda(m)(\log m)^{-1}.$$

Then,

$$\psi_1(x) - \pi(x) \ll x^{1/2}\log x. \tag{9}$$

Clearly, from the definitions of $\psi(x)$ and $\psi_1(x)$, we have

$$\psi_1(x) = \int_2^x \frac{1}{\log u}\,\mathrm{d}\psi(u).$$

Thus, in view of (8),

$$\psi_1(x) = \psi(x)(\log x)^{-1} + \int_2^x \psi(u)\frac{\mathrm{d}u}{u\log^2 u} = \int_2^x (\log u)^{-2}\,\mathrm{d}u + \frac{x}{\log x} + O\Big(x^{1/2}\log x\Big).$$

This, together with (9), gives

$$\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u} \ll x^{1/2}\log x, \tag{10}$$

and the RH implies (10). It turns out that (10) also implies RH. In consequence, estimate (10) is an equivalent of the RH. This was obtained by N.H. von Koch in [14].

At the moment, many equivalents of the Riemann hypothesis in various terms are known (see [15,16]). We focus on equivalents connected to the approximation properties of $\zeta(s)$ of some class of analytic functions. This property is called the universality of $\zeta(s)$, and this was reported by S.M. Voronin in [17] (see also [18–21]). Let $r \in (0, 1/4)$ be a fixed number. Voronin proved that, for every non-vanishing continuous function $g(s)$ on the disc $|s| \leqslant r$ that is analytic in $|s| < r$ and for any $\varepsilon > 0$, there is a number $\tau = \tau_{\varepsilon,g} \in \mathbb{R}$ satisfying the following:

$$\max_{|s|\leqslant r}\left|g(s) - \zeta\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

The latter interesting result has been observed by the mathematical community and stated in a more general form. Denote by $\mathfrak{m}_{\mathfrak{L}}A$ the Lebesgue measure of a measurable set A of real numbers. Let $\mathscr{D} = \{s \in \mathbb{C} : \sigma \in (1/2, 1)\}$.

Theorem 1 (see [22], Corollary 5.3.6; see also [23–25])**.** *Suppose that $K \subset \mathscr{D}$ is a compact set with a connected complement and $g(s)$ is a non-vanishing continuous function on K and analytic inside of K. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{m}_{\mathfrak{L}}\left\{\tau \in [0, T] : \sup_{s\in K} |g(s) - \zeta(s + i\tau)| < \varepsilon\right\} > 0.$$

The inequality of Theorem 1 implies that there are infinitely many τ such that $\zeta(s + i\tau)$ approximates a given analytic function $g(s)$.

Theorem 1 is modified in terms of the density of approximating shifts.

Theorem 2 (see [26,27])**.** *Let K and $g(s)$ be as in Theorem 1. Then, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mathfrak{m}_{\mathfrak{L}}\left\{\tau \in [0, T] : \sup_{s\in K} |g(s) - \zeta(s + i\tau)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

In a certain sense, Theorem 2 is stronger than Theorem 1; however, the exceptional set of values of ε is not explicitly defined.

It turned out that the RH is equivalent to self-approximation by shifts $\zeta(s + i\tau)$. This was carried out by B. Bagchi in [22,28]. Let $\mathcal{K}$ denote the class of compact subsets of the strip $\mathscr{D}$ with connected complements, and let $H_0(K)$ with $K \in \mathcal{K}$ be the class of non-vanishing continuous functions on K that are analytic inside of K.

Theorem 3 (see [22,28])**.** *The RH is equivalent to the statement that, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{m}_{\mathfrak{L}}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s) - \zeta(s + i\tau)| < \varepsilon\right\} > 0.$$

In [24], Theorem 3 has been extended to the region $\sigma > \theta$ with $\theta \geqslant 1/2$.

Theorem 4 (see [24])**.** *The function $\zeta(s) \neq 0$ for $\sigma > \theta$, $\theta \geqslant 1/2$, if and only if, for any $\varepsilon > 0$ and z with $\theta < \mathrm{Re}z < 1$, and for any $0 < r < \min\{\mathrm{Re}z - \theta, 1 - \mathrm{Re}z\}$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{m}_{\mathfrak{L}}\left\{\tau \in [0, T] : \max_{|s-z|\leqslant r} |\zeta(s) - \zeta(s + i\tau)| < \varepsilon\right\} > 0.$$

In [29], Theorem 3 has been stated in terms of density.

Theorem 5. *The RH is equivalent to the statement that, for any $K \in \mathcal{K}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s) - \zeta(s + i\tau)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorems 3 and 5 remain valid for some generalized shifts $\zeta(s + i\varphi(\tau))$ with certain $\varphi(\tau)$. In [30], the Gram function t_τ has been applied. Let $\vartheta(t)$ be the increment of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment between the points $s = 1/2$ and $s = 1/2 + it$. Since the function $\vartheta(t)$ is increasing for $t \geqslant 6.2898\ldots$, the equation

$$\vartheta(t) = (\tau - 1)\pi, \quad \tau \geqslant 0,$$

has an unique solution t_τ that is called the Gram function. The theory of the function t_τ is given in [31–33]. The points t_m, $m \in \mathbb{N}$, have been introduced and studied by J.-P. Gram [34] for the investigation of imaginary parts of non-trivial zeros of $\zeta(s)$.

Theorem 6 (see [30]). *The RH is equivalent to the statement that, for any $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s) - \zeta(s + it_\tau)| < \varepsilon \right\} > 0.$$

An analogue of Theorem 5 with shifts $\zeta(s + it_\tau)$ is valid as well.

Theorem 7 (see [30]). *The RH is equivalent to the statement that, for any $K \in \mathcal{K}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s) - \zeta(s + it_\tau)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Paper [35] is devoted to discrete versions of Theorems 6 and 7. In this case, the shifts $\zeta(s + it_k)$ with Gram points t_k are used.

Proofs of Theorems 5–7 utilize the corresponding universality theorems for $\zeta(s)$ and weakly convergent probability measures in the space of analytic functions.

Universality theorems for $\zeta(s)$ formulated using a notion of density or lower density are more effective when they are considered in short intervals, i.e., if the length of the interval is $o(T)$ as $T \to \infty$. Thus, the density of approximating shifts $\zeta(s + i\tau)$ is considered in the interval $[T, T + U]$, with $U = o(T)$ taken as small as possible. For brevity, denote $\eta = 1273/4033$. The strongest universality result for $\zeta(s)$ in short intervals has been obtained in [36], with U satisfying $T^\eta \leqslant U \leqslant T$. The method of [36] is different from the classical method of [8], and it is based on a result of J. Bourgain and N. Watt [37] where

$$\frac{1}{2U} \int_{T-U}^{T+U} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \mathrm{d}t \ll \log T$$

for $U = T^{\eta+\varepsilon}$, $\forall\varepsilon > 0$. From this, the constant η is obtained. We believe that the decrease of η is a very difficult problem of analytic number theory. Under RH, it was obtained in [36] that $\zeta(s)$ is universal for U, satisfying $\exp\{(\log T)^{1-\varepsilon}\} \leqslant U \leqslant T$ with every $\varepsilon > 0$.

However, the best lower bound for U, $U = (\log T)^B$, in the case of discs K with B depending on K was given in [38].

The purpose of this paper is to give versions of Theorems 3 and 5 in terms of short intervals. We will establish the following statements.

Theorem 8. *The RH is equivalent to the statement that, for $T^\eta \leqslant U \leqslant T$, any $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [T, T+U] : \sup_{s\in K} |\zeta(s) - \zeta(s+i\tau)| < \varepsilon \right\} > 0.$$

Theorem 9. *The RH is true if and only if, for $T^\eta \leqslant U \leqslant T$ and any $K \in \mathcal{K}$, the limit*

$$\lim_{T\to\infty} \frac{1}{U} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [T, T+U] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

For the proof of Theorems 8 and 9, a probabilistic method based on weakly convergent probability measures in short intervals in the space of analytic functions will be applied. We devote Section 2 to this. In the proof of Proposition 1, we omit some details that were used several times by various authors. The constant η comes from [36,37], where it was involved in the mean square estimates of the Riemann zeta function. We conjecture that η can decrease to T^ε, $\forall \varepsilon > 0$. However, this requires of new ideas.

2. Probabilistic Results

The idea of applying statistical methods to the characterisation of the chaotic behaviour of the Riemann zeta function was formulated by H. Bohr at the beginning of the 20th century [39]. This was realized in the joint works with B. Jessen on the density for some sets of values of $\zeta(s)$. Denote by $\mathfrak{m}_{\mathfrak{J}}$ the Jordan measure on $\mathbb{R}$, and let $\mathcal{R}$ be the rectangle with edges parallel to the axes. Then, in [40], it was proved that, for fixed $\sigma > 1$, the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathfrak{m}_{\mathfrak{J}} \{t \in [0, T] : \log \zeta(\sigma + it) \in \mathcal{R}\}$$

exists. In [41], the latter theorem with some modifications was extended to the half-plane $\sigma > 1/2$. The Bohr–Jessen results were developed in the papers of B. Jessen and A. Wintner, V. Borchsenius and B. Jessen, and A. Selberg.

Later, in the middle of the 20th century, the theory of the weak convergence of probability measures was formulated and developed. The created theory created the conditions for formulating Bohr–Jessen-type theorems in terms of the weak convergence of probability measures. For a topological space $\mathscr{X}$, let $\mathcal{B}(\mathscr{X})$ stand for its σ-field. On $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$, define probability measures P and P_n, $n \in \mathbb{N}$. By definition, P_n converges weakly to P as $n \to \infty$, or, shortly, $P_n \xrightarrow[n\to\infty]{\text{w}} P$, if, for any real bounded continuous function u on $\mathscr{X}$,

$$\int_{\mathscr{X}} u \, \mathrm{d}P_n \xrightarrow[n\to\infty]{} \int_{\mathscr{X}} u \, \mathrm{d}P.$$

Using weak convergence, the above-mentioned Bohr–Jessen theorem can be restated in the following form: on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there is a probability measure P_σ such that, for $\sigma > 1/2$, the measure

$$\frac{1}{T} \mathfrak{m}_{\mathfrak{L}} \{t \in [0, T] : \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$ (see, for example, [24,42] and a survey paper [43]).

B. Bagchi made a significant impact on probabilistic function theory in his thesis [22]. He introduced and obtained limit theorems on weakly convergent probability measures in the space of analytic functions, and he applied them for the proof of the universality of various zeta functions, including $\zeta(s)$. Developments of the Bagchi method were continued in [24,25].

Recall that $\mathscr{D} = \{s \in \mathbb{C} : \sigma \in (1/2, 1)\}$. Define the space $\mathscr{H}(\mathscr{D})$ of analytic functions on $\mathscr{D}$ equipped with the topology of uniform convergence on compact sets. For $A \in \mathcal{B}(\mathscr{H}(\mathscr{D}))$, set the following:

$$P_{T,U}(A) = \frac{1}{U}\mathfrak{m}_{\mathfrak{L}}\{\tau \in [T, T+U] : \zeta(s+i\tau) \in A\}.$$

The weak convergence of the measure $P_{T,U}$ as $T \to \infty$ is the main ingredient for the proof of Theorems 8 and 9.

Introduce the set

$$\Omega = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\},$$

i.e., Ω is the infinite Cartesian product of unit circles. On Ω, the operation of pairwise multiplication and product topology can be defined. This makes Ω a compact Abelian topological group and ensures the existence of the probability Haar measure $\mathfrak{m}_{\mathfrak{H}}$ on $(\Omega, \mathcal{B}(\Omega))$. Hence, we have the probability space $(\Omega, \mathcal{B}(\Omega), \mathfrak{m}_{\mathfrak{H}})$. Let $\omega = (\omega(p) : p \in \mathbb{P})$ denote the elements of Ω. Now, on the space $(\Omega, \mathcal{B}(\Omega), \mathfrak{m}_{\mathfrak{H}})$, define the $\mathscr{H}(\mathscr{D})$-valued random element $\zeta(s, \omega)$ by

$$\zeta(s,\omega) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

This infinite Euler product, for almost all $\omega \in \Omega$, converges uniformly on compact sets of the strip $\mathscr{D}$, and it defines the $\mathscr{H}(\mathscr{D})$-valued random element ([23], Theorem 5.1.7). Let P_ζ stand for the distribution of $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = \mathfrak{m}_{\mathfrak{H}}\{\omega \in \Omega : \zeta(s,\omega) \in A\}, \quad A \in \mathcal{B}(\mathscr{H}(\mathscr{D})).$$

Suppose that $T_1 = T_1(T) \to \infty$ as $T \to \infty$ and, for $T_1 \leqslant U \leqslant T$, the mean square estimate

$$\int_{T-U}^{T+U} |\zeta(\sigma+it)|^2 \, dt \ll_\sigma U \tag{11}$$

holds uniformly in U for $1/2 < \sigma \leqslant \sigma_0 < 1$ with some σ_0.

Proposition 1. *Suppose that $T_1 \leqslant U \leqslant T$ satisfies (11). Then, $P_{T,U} \xrightarrow[T\to\infty]{\text{w}} P_\zeta$.*

Proof. We apply standard arguments. First, using the Fourier transform method leads to the relation

$$P^{\Omega}_{T,U} \xrightarrow[T\to\infty]{\text{w}} \mathfrak{m}_{\mathfrak{H}}, \tag{12}$$

where, for $A \in \mathcal{B}(\Omega)$,

$$P^{\Omega}_{T,U}(A) = \frac{1}{U}\mathfrak{m}_{\mathfrak{L}}\Big\{\tau \in [T, T+U] : \Big(p^{-i\tau} : p \in \mathbb{P}\Big) \in A\Big\}.$$

Further, introduce the absolutely convergent Dirichlet series

$$\zeta_n(s) \sum_{m=1}^{\infty} \frac{w_n(m)}{m^s}, \quad n \in \mathbb{N},$$

with

$$w_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\}, \quad \theta > \frac{1}{2},$$

and consider the probability measure

$$P_{T,U,n}(A) = \frac{1}{U}\mathfrak{m}_{\mathfrak{L}}\{\tau \in [T, T+U] : \zeta_n(s+i\tau) \in A\}, \quad A \in \mathcal{B}(\mathscr{H}(\mathscr{D})).$$

Using the mapping $v_n : \Omega \to \mathscr{H}(\mathscr{D})$ given by

$$v_n(\omega) = \sum_{m=1}^{\infty} \frac{\omega(m) w_n(m)}{m^s}, \quad \omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \; m \in \mathbb{N},$$

we have that

$$v_n\left(p^{-i\tau} : p \in \mathbb{P}\right) = \zeta_n(s+i\tau),$$

Therefore, $P_{T,U,n} = P^{\Omega}_{T,U} v_n^{-1}$, where $P^{\Omega}_{T,U} v_n^{-1}(A) = P^{\Omega}_{T,U}(v_n^{-1}A)$, with $A \in \mathcal{B}(\mathscr{H}(\mathscr{D}))$. With this remark, the continuity of v_n and (12) implies the relation

$$P_{T,U,n} \xrightarrow[T\to\infty]{\text{w}} P_n \stackrel{\text{def}}{=} \mathfrak{m}_{\mathfrak{H}} u_n^{-1}. \tag{13}$$

It remains to pass from $P_{T,U,n}$ to $P_{T,U}$. For this, the bound (11) plays a crucial role. For $\zeta_n(s)$, the following integral representation

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z)\kappa_n(z)\,\mathrm{d}z, \quad \kappa_n(z) = \frac{1}{\theta}\Gamma\left(\frac{z}{\theta}\right)n^z, \tag{14}$$

is valid [23]. We will prove that, for every compact set $K \subset \mathscr{D}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T}^{T+U} \sup_{s\in K} |\zeta_n(s+i\tau) - \zeta(s+i\tau)|\,\mathrm{d}\tau = 0. \tag{15}$$

Let $K \subset \mathscr{D}$ be a fixed compact set. Then, K lies in some strip $1/2 + 2\delta \leqslant \sigma \leqslant 1-\delta$, $\delta > 0$. Take $\theta = 1/2 + \delta$ and $\theta_1 = 1/2 + \delta - \sigma$. Then, the integrand in (14), in the strip $\theta_1 \leqslant \mathrm{Re} z \leqslant \theta$, only has simple poles at the points $z = 0$ (a pole of $\Gamma(s/\theta)$) and $z = 1-s$ (a pole of $\zeta(s+z)$). Therefore, the representation (14), residue theorem, and the well-known estimates

$$\zeta(\sigma + it) \ll t^{1/2-\sigma}\log t, \quad t \geqslant t_0,$$

and

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad |t| \geqslant t_0,$$

lead, for all $s \in K$, to

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty} \zeta(s+z)\kappa_n(z)\,\mathrm{d}z + \kappa_n(1-s).$$

Hence, the above estimates for the functions $\zeta(s)$ and $\Gamma(s)$ yield

$$\begin{aligned}
&\tfrac{1}{U}\int\limits_{T}^{T+U}\sup_{s\in K}|\zeta_n(s+i\tau)-\zeta(s+i\tau)|\,\mathrm{d}\tau\\
&\ll_K \int\limits_{-\log^2 T}^{\log^2 T}\left(\tfrac{1}{U}\int\limits_{T}^{T+U}\left|\zeta\left(\tfrac{1}{2}+\delta+iu+i\tau\right)\right|^2\mathrm{d}\tau\right)^{1/2}\sup_{s\in K}\left|\kappa_n\left(\tfrac{1}{2}+\delta-s+iu\right)\right|\mathrm{d}u\\
&\quad+\tfrac{1}{U}\int\limits_{T}^{T+U}\sup_{s\in K}|\kappa_n(1-s-i\tau)|\,\mathrm{d}\tau+n^{-\delta}\exp\left\{-c\log^2 T\right\}\tfrac{1}{U}\int\limits_{T}^{T+U}|\tau|^{1/2}\,\mathrm{d}\tau\\
&\stackrel{\text{def}}{=}A_1+A_2+A_3.
\end{aligned}\tag{16}$$

Here and throughout, c is a positive constant that is not always the same. Clearly,

$$\frac{1}{U}\int\limits_{T}^{T+U}\left|\zeta\left(\frac{1}{2}+\delta+iu+i\tau\right)\right|^2\mathrm{d}\tau\leqslant\frac{1}{U}\int\limits_{T-U-|u|}^{T+U+|u|}\left|\zeta\left(\frac{1}{2}+\delta+i\tau\right)\right|^2\mathrm{d}\tau\ll 1+|u|$$

in view of (11) if $U+|u|\leqslant T$; otherwise, it is

$$\ll\frac{1}{U}\int\limits_{-2(U+|u|)}^{2(U+|u|)}\left|\zeta\left(\frac{1}{2}+\delta+i\tau\right)\right|^2\mathrm{d}\tau\ll_K 1+|u|.$$

Thus,

$$A_1\ll_K\int\limits_{-\log^2 T}^{\log^2 T}(1+|u|)^{1/2}n^{1/2-\sigma+\delta}\exp\{-c|t-u|\}\,\mathrm{d}u\ll_K n^{-\delta}.\tag{17}$$

Similarly,

$$A_2\ll_K n^{1-\sigma}\int\limits_{T}^{T+U}\exp\{-c|\tau|\}\,\mathrm{d}\tau\ll_K n^{-2\delta+1/2}\exp\{-cT\},$$

and

$$A_3\ll_K n^{-\delta}T^{1/2}\exp\{-c\log^2 T\}.$$

This, combined with (16) and (17), implies (15).

Let d be the metric in the space $\mathscr{H}(\mathscr{D})$ that induces the topology of uniform convergence on compact sets, i.e., for $g_1,g_2\in\mathscr{H}(\mathscr{D})$,

$$d(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|},$$

where $\{K_l:l\in\mathbb{N}\}\subset\mathscr{D}$ is a sequence of embedded compact set such that $\mathscr{D}=\bigcup\limits_{l=1}^{\infty}K_l$, and any compact set $K\subset\mathscr{D}$ is in some K_l. Now, (15) shows that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int\limits_{T}^{T+U}d(\zeta_n(s+i\tau),\zeta(s+i\tau))\,\mathrm{d}\tau=0.\tag{18}$$

We observe that the probability measure P_n in (13) is independent on U, and it is the same as in the case of the measures

$$P_{T,n}(A)\stackrel{\text{def}}{=}\frac{1}{T}\mathfrak{m}_{\mathfrak{L}}\{\tau\in[0,T]:\zeta_n(s+i\tau)\in A\}$$

and

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T}\mathfrak{m}_{\mathfrak{L}}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}, \quad A \in \mathcal{B}(\mathscr{H}(\mathscr{D})),$$

are discussed in [22,23]. Therefore, by Theorem 5.1.8 of [23], we have the asymptotic relation

$$P_n \xrightarrow[n\to\infty]{\text{w}} P_\zeta.$$

This, combined with (13), (18), and Theorem 4.2 of [44] via the standard method, proves Proposition 1. □

We also need certain information on the measure P_ζ. More precisely, we need the explicit form of the support of P_ζ, i.e., a closed minimal set $S_{P_\zeta} \subset \mathscr{H}(\mathscr{D})$ satisfying $P_\zeta(S_{P_\zeta}) = 1$. Observe that $g \in S_{P_\zeta}$ if and only if, for every neighbourhood $\mathcal{G}$ of g, the inequality $P_\zeta(\mathcal{G}) > 0$ holds.

Lemma 1 (see [23], Lemma 6.5.5)**.** *The set $S = \{g \in \mathscr{H}(\mathscr{D}) : g(s) \neq 0$ on $\mathscr{D}$, or $g(s) \equiv 0\}$ is the support of P_ζ.*

Lemma 2. *Suppose that $T^\eta \leqslant U \leqslant T$ and $1/2 < \sigma \leqslant 1$ is fixed. Then, uniformly in U,*

$$\int_T^{T+U} |\zeta(\sigma+it)|^2\,\mathrm{d}t \ll_\sigma U.$$

Here, the implied constant depends on σ but is independent on U.

Proof. The lemma is proved in [36]; see Lemmas 1 and 2. □

Lemma 3. *Suppose that the RH is true, and $\exp\{(\log T)^{1-\delta}\} \leqslant U \leqslant T$ with arbitrary fixed $\delta > 0$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U}\mathfrak{m}_{\mathfrak{L}}\left\{\tau \in [T, T+U] : \sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0. \tag{19}$$

Moreover, the lower limit can be replaced by the limit for all but at most countably many $\varepsilon > 0$.

Proof. Inequality (19) has been obtained in [36], Theorem 4. For the proof of the second assertion of the lemma, we use a result from [45]. Suppose that

$$\frac{1}{2} + \frac{2c_1}{\log\log T} \leqslant \sigma \leqslant 1-\delta, \quad \delta > 0,$$

and $\exp\{(\log T)^{2-2\sigma}\} \leqslant U \leqslant T$ for $T \geqslant T_0$. Then, the RH implies

$$\frac{1}{U}\int_T^{T+U} |\zeta(\sigma+it)|^2\,\mathrm{d}t - \zeta(2\sigma) \ll_{\delta,\sigma} \exp\left\{-c_2\frac{(\log T)^{2-2\sigma}}{\log\log T}\right\}$$

with $c_2 = c_2(c_1) > 0$. Therefore, in the interval $\exp\{(\log T)^{1-\gamma}\} \leqslant U \leqslant T$ with $\gamma > 0$, Proposition 1 is applicable. Thus, we have

$$P_{T,U} \xrightarrow[T\to\infty]{\text{w}} P_\zeta. \tag{20}$$

Consider $P_\zeta(\mathcal{G}_\varepsilon)$ with the set

$$\mathcal{G}_\varepsilon = \mathcal{G}_{\varepsilon,f} = \left\{ g \in \mathscr{H}(\mathscr{D}) : \sup_{s\in K} |f(s) - g(s)| < \varepsilon \right\}.$$

The boundary $\partial\mathcal{G}_\varepsilon$ of the set $\mathcal{G}_\varepsilon$ lies in the set

$$\left\{ g \in \mathscr{H}(\mathscr{D}) : \sup_{s\in K} |f(s) - g(s)| = \varepsilon \right\}.$$

Therefore, the intersection of the boundaries $\partial\mathcal{G}_{\varepsilon_1}$ and $\partial\mathcal{G}_{\varepsilon_2}$ is empty for positive $\varepsilon_1 \neq \varepsilon_2$. Moreover, $\partial\mathcal{G}_\varepsilon$ is a closed set; therefore, $\partial\mathcal{G}_\varepsilon \in \mathcal{B}(\mathscr{H}(\mathscr{D}))$. Hence, it follows that $P_\zeta(\partial\mathcal{G}_\varepsilon) \neq 0$ for at most countably many $\varepsilon > 0$. Actually, for every $k \in \mathbb{N} \setminus \{1\}$, there are at most $k-1$ values of $\varepsilon > 0$ such that $P_\zeta(\partial\mathcal{G}_\varepsilon) > 1/k$. Moreover,

$$\{\varepsilon > 0 : P_\zeta(\partial\mathcal{G}_\varepsilon) \neq 0\} \subset \bigcup_{k=2}^{\infty} \left\{ \varepsilon : P_\zeta(\partial\mathcal{G}_\varepsilon) > \frac{1}{k} \right\}.$$

This shows that $\{\varepsilon > 0 : P_\zeta(\partial\mathcal{G}_\varepsilon) \neq 0)\}$ is at most a countable set.

Now, we deal with the continuity sets A of the measure P_ζ, i.e., $P_\zeta(\partial A) = 0$. By the above remark, we have that $\mathcal{G}_\varepsilon$ is a continuity set of P_ζ for all but at most countably many $\varepsilon > 0$. In view of (20) and the equivalent of weak convergence in terms of continuity sets, we find

$$\lim_{T\to\infty} P_{T,U}(\mathcal{G}_\varepsilon) = P_\zeta(\mathcal{G}_\varepsilon) \tag{21}$$

for all but at most countably many $\varepsilon > 0$.

It remains to prove that $P_\zeta(\mathcal{G}_\varepsilon) > 0$. If $f(s) \in S$, then, in virtue of Lemma 1, $P_\zeta(\mathcal{G}_\varepsilon) > 0$ because $\mathcal{G}_\varepsilon$ is a neighbourhood of an element of the support of the measure P_ζ. For example, we may take a polynomial $p(s)$ and consider the set $\mathcal{G}_{\varepsilon/2,p}$. Then,

$$P_\zeta(\mathcal{G}_{\varepsilon/2,p}) > 0. \tag{22}$$

Moreover, using the Mergelyan theorem [46,47], we may choose the polynomial $p(s)$ satisfying

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}.$$

Then, it is easily seen that $\mathcal{G}_{\varepsilon/2,p} \subset \mathcal{G}_{\varepsilon,f}$. Thus, by (22), we have $P_\zeta(\mathcal{G}_{\varepsilon,f}) > 0$. This, in addition to (21) and the definition of $P_{T,U}$ and $\mathcal{G}_\varepsilon$, completes the proof. □

3. Proof of Theorems 8 and 9

Proof of Theorem 8. *Necessity.* If the RH holds, then $\zeta(s) \neq 0$ and it is analytic in $\mathscr{D}$. Hence, for every $K \in \mathcal{K}$, $\zeta(s)$ is continuous and non-vanishing on K, and it is analytic inside K. In other words, $\zeta(s) \in H_0(K)$. Thus, by the first statement of Lemma 3, for any $K \in \mathcal{K}$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{U} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [T, T+U] : \sup_{s\in K} |\zeta(s) - \zeta(s+i\tau)| < \varepsilon \right\} > 0. \tag{23}$$

Sufficiency. Suppose, on the contrary, that (23) holds. However, the RH is not true. Then, $\zeta(s)$ has zeros in $\mathscr{D}$; thus, $\zeta(s) \notin S$. Therefore, by Lemma 1, $\zeta(s)$ is not an element of the support of the measure P_ζ. Consequently, there is an open neighbourhood $\mathcal{G}$ of $\zeta(s)$ satisfying $P_\zeta(\mathcal{G}) = 0$. Therefore, there are $K \in \mathcal{K}$ and $\varepsilon > 0$ such that, for the set

$$\mathcal{G}_{K,\varepsilon} = \left\{ g \in \mathscr{H}(\mathscr{D}) : \sup_{s \in K} |\zeta(s) - g(s)| < \varepsilon \right\},$$

the equality

$$P_\zeta(\mathcal{G}_{K,\varepsilon}) = 0 \tag{24}$$

holds. Lemma 2 and Proposition 1 imply that, for every U, $T^\eta \leqslant U \leqslant T$,

$$P_{T,U} \xrightarrow[T\to\infty]{\text{w}} P_\zeta. \tag{25}$$

Similarly to the case of Lemma 3, we deduce that $\mathcal{G}_{K,\varepsilon}$ is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Hence, using continuity sets, we find, in virtue of (25), that

$$\lim_{T\to\infty} P_{T,U}(\mathcal{G}_{K,\varepsilon}) = P_\zeta(\mathcal{G}_{K,\varepsilon})$$

for all but at most countably many $\varepsilon > 0$. This and (24) show that

$$\lim_{T\to\infty} P_{T,U}(\mathcal{G}_{K,\delta}) = 0$$

for all but at most countably many $0 < \delta \leqslant \varepsilon$. Therefore, there exists $\delta > 0$ satisfying

$$\lim_{T\to\infty} \frac{1}{U} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [T, T+U] : \sup_{s \in K} |\zeta(s+i\tau) - \zeta(s)| < \delta \right\} = 0,$$

which contradicts inequality (23). This contradiction implies the RH. The theorem is proved. □

Proof of Theorem 9. *Necessity.* Suppose that RH holds. Then, as in the proof of Theorem 8, we have $\zeta(s) \in H_0(K)$ for all $K \in \mathcal{K}$, and by the second statement of Lemma 3, the limit

$$\lim_{T\to\infty} \frac{1}{U} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [T, T+U] : \sup_{s \in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon \right\} \tag{26}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Sufficiency. Suppose that the limit (26) exists and is positive for all but at most countably many $\varepsilon > 0$. However, the RH is not true. Then, as in the proof of Theorem 8, we find that then there is $\varepsilon > 0$ such that

$$\liminf_{T\to\infty} \frac{1}{U} \mathfrak{m}_{\mathfrak{L}} \left\{ \tau \in [T, T+U] : \sup_{s \in K} |\zeta(s+i\tau) - \zeta(s)| < \delta \right\} = 0$$

for all but at most countably many $0 < \delta \leqslant \varepsilon$, and this contradicts the positivity of (26). The contradiction proves that the RH is true. The theorem is proved. □

4. Conclusions

In this paper, we show that the Riemann hypothesis on non-trivial zeros of the Riemann zeta function is equivalent to the self-approximation of $\zeta(s)$ by shifts $\zeta(s+i\tau)$ in short intervals $[T, T+U]$ with $T^{1273/4033} \leqslant U \leqslant T$. This is closely connected to the universality of $\zeta(s)$ in short intervals. The constant $1273/4033$ was introduced in [37] and applied in [36]. The results of this paper extend the known ones for $U = T$ (Theorems 3 and 5). The proof applies the probabilistic approach. Future research will aim at decreasing the lower bound of U. We expect that it can be reduced until T^{ε}, $\forall\varepsilon > 0$.

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