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Square Root of a Multivector of Clifford Algebras in 3D: A Game with Signs

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Abstract

An algorithm is presented to extract the square root from a multivector (MV) in real Clifford algebras $Cl_{p,q}$, where $n = p + q \leq 3$, in radicals. It is shown that in $Cl_{3,0}$, $Cl_{1,2}$, and $Cl_{0,3}$ algebras, there are up to four isolated square roots in a case of the most general (generic) MV. The algebra $Cl_{2,1}$ is an exception and, there, the MV can have up to 16 isolated roots. In addition, a continuum of roots has been found in all Clifford algebras except $p + q = 1$. Examples which clarify computations are provided to illustrate the properties of roots in all $n = 3$ algebras. The results may be useful in solving nonlinear equations, like for example, the Clifford–Riccati equation.

Keywords: square root of multivector; Clifford algebra; geometric algebra; computer-aided theory

MSC: 15A18; 15A66

1. Introduction

The square root has a long history. Solution by radicals of the cubic equation was first published in 1545 by G. Cardano. Simultaneously, a concept of the square root of a negative number was developed [1]. In 1872, A. Cayley was the first to carry over the square root to matrices [2]. In the recent book by N. J. Higham [3], where an extensive literature is presented on nonlinear functions of matrices, two sections are devoted to matrix square roots. In the context of Clifford algebra (CA), the main attention up till now was concentrated on the square roots of quaternions [4,5], or their derivatives such as coquaternions (also called split quaternions) or nectarines [5–7]. The square root of biquaternions (complex quaternions) was considered in [8]. Quaternions and related objects are isomorphic to one of $n = 2$ algebras $Cl_{0,2}$, $Cl_{1,1}$, $Cl_{2,0}$, and therefore, the quaternionic square root analysis can be easily rewritten in terms of CA (see Appendix B). In this paper, we shall mainly be interested in higher, namely, $n = 3$ Clifford algebras (CAs), where the main object is the eight-component multivector (abbreviated as MV through the article).

For CAs of dimension $n \geq 3$, the investigation and understanding of square root properties is still in infancy. The most akin to the present paper are the investigation of conditions for the existence of square root of -1 [8–10]. The existence of such roots allows to extend the Fourier transform to MVs, where they are used in formulating Clifford–Fourier transform and CA-based wavelet theories [11].

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Our preliminary investigation [12] on this subject was concerned with square roots of individual multivector grades such as scalar, vector, bivector, pseudoscalar, or their simple combinations. For this purpose, we have applied the Gröbner basis algorithm to analyze the system of nonlinear polynomial equations that ensue from the MV equation $\mathsf{A}^2 = \mathsf{B}$, where A and B are the MVs. The Gröbner basis is accessible in many symbolic mathematical packages. Specifically, *Mathematica* commands such as **Reduce[ ]**, **Solve[ ]**, **Eliminate[ ]** and others also employ the Gröbner basis to solve nonlinear problems. With their help, we were able to find new properties of roots for the $n = 3$ case, namely, that the MVs may have no roots, a single or multiple isolated roots, or even an infinite number (continuum) of roots in 4D parameter spaces or spaces of smaller dimension. A short overview of current Clifford algebra software is described in the introduction of [13].

In this paper, we continue our [12] investigations of the square root problem in real CAs for the $n = 3$ case. In particular, we examine and provide explicit conditions for an MV to have discrete and continuum of roots, and how to express real root coefficients in radicals. For this purpose, a symbolic package was used [14] that appeared to be invaluable both for detecting specific solutions of the nonlinear equation $\mathsf{A}^2 = \mathsf{B}$ and for numerical checks in general.

In Section 2, the notation is introduced. The algorithm to calculate the square root of a generic MV and special cases that follow are given in Sections 3–5 for $Cl_{3,0} \simeq Cl_{1,2}$, $Cl_{0,3}$, and $Cl_{2,1}$ algebras, respectively. The computation is illustrated by a number of examples. The conclusions are drawn in Section 6. For completeness, in Appendixes A and B, the MV square roots are presented for lower-dimensional CAs. Appendix C provides summary for the two most important algebras, $Cl_{3,0}$ and $Cl_{0,3}$ algebras, which will be useful for implementation. Appendix D is devoted to MV determinants.

2. Notation

For $n = 3$, a general MV can be expanded in the orthonormal basis that consists of $2^n = 8$ elements listed in inverse degree lexicographic ordering (note an increasing order of digits in indices). Therefore, we write $\mathbf{e}_{13}$ instead of $\mathbf{e}_{31} = -\mathbf{e}_{13}$. This convention is reflected in opposite signs of some terms in formulas.

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}, \tag{1}$$

where $\mathbf{e}_i$ are basis vectors and $\mathbf{e}_{ij}$ are the bivectors (oriented planes). The last term is the pseudoscalar. The number of subscripts indicates the grade of basis element. The scalar is a grade-0 element, the vectors $\mathbf{e}_i$ are the grade-1 elements, etc. In the orthonormalized basis, the geometric (Clifford) products of basis vectors satisfy the anticommutation relation,

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}. \tag{2}$$

For $Cl_{3,0}$ and $Cl_{0,3}$ algebras, the squares of basis vectors, correspondingly, are $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_i^2 = -1$, where $i = 1, 2, 3$. For mixed signature algebras such as $Cl_{2,1}$ and $Cl_{1,2}$, we have $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3^2 = -1$ and $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = -1$, respectively. The sign of squares of higher grade elements is determined by squares of vectors and property (2). For example, in $Cl_{3,0}$, we have $\mathbf{e}_{12}^2 = \mathbf{e}_{12}\mathbf{e}_{12} = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1(+1)\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_1 = -1$. However, in $Cl_{1,2}$, a similar computation gives $\mathbf{e}_{12}^2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1(-1)\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_1 = +1$.

When $n = 3$, an MV A in real CA can be expanded in the basis (1),

$$\begin{aligned}\mathsf{A} =& a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13} + a_{123}I \\ \equiv& a_0 + \mathbf{a} + \mathcal{A} + a_{123}I,\end{aligned} \tag{3}$$

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where a_i, a_{ij}, and a_{123} are the real coefficients, and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13}$ are, respectively, the vector and the bivector. We will seek a real MV A (with real coefficients), the square of which satisfies

$$\mathsf{AA} \equiv \mathsf{A}^2 = \mathsf{B} = b_0 + \mathbf{b} + \mathcal{B} + b_{123}I. \tag{4}$$

The MV A is called a square root of B. In Equation (4), the square A^2 has been expanded in the orthonormal basis where $b_0, \mathbf{b}, \mathcal{B}$ and $I \equiv I_3$ denote, respectively, a scalar, a vector ($\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$), a bivector ($\mathcal{B} = b_{12}\mathbf{e}_{12} + b_{23}\mathbf{e}_{23} + b_{13}\mathbf{e}_{13}$), and a pseudoscalar. The representation (3) is not convenient for our problem; therefore, for all 3D algebras $Cl_{3,0}$, $Cl_{0,3}$, $Cl_{1,2}$, and $Cl_{2,1}$ a more symmetric representation is introduced,

$$\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I, \tag{5}$$

where now both s and S are the real scalars and both $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{V} = V_1\mathbf{e}_1 + V_2\mathbf{e}_2 + V_3\mathbf{e}_3$ are the vectors with real coefficients v_i and V_i. The MV representation (5) allows to disentangle the coupled nonlinear equations in a regular manner for all listed algebras. To select the scalar s in (5), the grade selector $\langle \mathsf{A} \rangle \equiv \langle \mathsf{A} \rangle_0 = s$ is used. The pseudoscalar part can be extracted by $\langle \mathsf{A} \rangle \equiv \langle -\mathsf{A}I \rangle_0 = S$, and similarly for other grades. More about CAs and MV properties can be found, for example, in books [15,16].

When $n = 1, 2$, the MV square root algorithm simplifies substantially. All needed formulas are presented in Appendix A and Appendix B, respectively.

3. Square Roots in $Cl_{3,0}$ and $Cl_{1,2}$ Algebras

This section describes the method of substitution of variables in CA which paves a direct way to square root algorithm. The Euclidean $Cl_{3,0}$ algebra is the most simple one among $n = 3$ algebras. The algebra $Cl_{1,2}$ is isomorphic to $Cl_{3,0}$; therefore, the computations for this algebra follows the same route except that, there, some notational differences appear.

The goal is to solve nonlinear MV equation $\mathsf{A}^2 = \mathsf{B}$, where $\mathsf{B} = b_0 + b_1\mathbf{e}_2 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{23} + b_{13}\mathbf{e}_{13} + b_{23}\mathbf{e}_{23} + b_{123}I$ and A is unknown. The latter may have the general form (5). Expanding A^2 in components and equating (real) coefficients at basis elements to respective coefficients in B, one obtains a system of eight nonlinear equations:

$$b_0 = s^2 - S^2 + \mathbf{v}^2 - \mathbf{V}^2, \qquad b_{123} = 2(sS + \mathbf{v}\cdot\mathbf{V}), \tag{6}$$

$$b_1 = 2(sv_1 - SV_1), \qquad b_{23} = 2(sV_1 + Sv_1), \tag{7}$$

$$b_2 = 2(sv_2 - SV_2), \qquad b_{13} = 2(sV_2 + Sv_2), \tag{8}$$

$$b_3 = 2(sv_3 - SV_3), \qquad b_{12} = 2(sV_3 + Sv_3). \tag{9}$$

These equations were written in [12], but they were not solved properly, so the results presented there did not cover all possible cases. Here, we provide a complete solution of square root problem for $n = 3$ Clifford algebras. The square root computations split into two cases: the *generic* case where either $s^2 + S^2 \neq 0$ (in $Cl_{3,0}$ and $Cl_{1,2}$) or $s^2 - S^2 \neq 0$ (in $Cl_{0,3}$ and $Cl_{2,1}$), and the *special* case where $s^2 + S^2 = 0$ (in $Cl_{3,0}$ and $Cl_{1,2}$) or $s^2 - S^2 = 0$ (in $Cl_{0,3}$ and $Cl_{2,1}$).

3.1. The Generic Case $s^2 + S^2 \neq 0$

The system of six Equations (7)–(9) is linear in new variables v_i and V_i in (5). It has a very simple solution which is a key to the analysis that follows. As a note, the symmetry of Equations (10) and (11) with respect to pairs (v_2, V_2), (v_1, V_1), and (v_3, V_3) differ as it was explained in the beginning of Section 2.

It can be restored if b_{13} is replaced by $-b_{31}$.

$$v_1 = \frac{b_1 s + b_{23} S}{2(s^2 + S^2)}, \qquad v_2 = \frac{b_2 s - b_{13} S}{2(s^2 + S^2)}, \qquad v_3 = \frac{b_3 s + b_{12} S}{2(s^2 + S^2)}, \tag{10}$$

$$V_1 = \frac{b_{23} s - b_1 S}{2(s^2 + S^2)}, \qquad V_2 = -\frac{b_{13} s + b_2 S}{2(s^2 + S^2)}, \qquad V_3 = \frac{b_{12} s - b_3 S}{2(s^2 + S^2)}. \tag{11}$$

The Equations (10) and (11) express the components of vectors $\mathbf{v}$ and $\mathbf{V}$ in terms of scalars s and S, which are to be determined from a pair of Equation (6). The solution is valid when $s^2 + S^2 \neq 0$, i.e., when either $s \neq 0$ or $S \neq 0$, or both s and S are non-zero scalars. If these conditions are not satisfied, we have the subcase $s = S = 0$. After substitution of (10) and (11), i.e., of (v_1, v_2, v_3) and (V_1, V_2, V_3), into (6), we get a system of two coupled algebraic equations for two unknowns s and S,

$$\begin{aligned} 4(b_0 - s^2 + S^2)(s^2 + S^2)^2 &= + (b_1 s + b_{23} S)^2 + (b_2 s - b_{13} S)^2 + (b_3 s + b_{12} S)^2 \\ &\quad - (b_{23} s - b_1 S)^2 - (b_{13} s + b_2 S)^2 - (b_{12} s - b_3 S)^2, \\ 2(b_{123} - 2sS)(s^2 + S^2)^2 &= + (b_1 s + b_{23} S)(b_{23} s - b_1 S) - (b_2 s - b_{13} S) \\ &\quad \times (b_{13} s + b_2 S) + (b_3 s + b_{12} S)(b_{12} s - b_3 S). \end{aligned} \tag{12}$$

The system (12) has exactly four solutions that can be expressed in radicals. If new variables t and T are introduced and substitution

$$s\,S = t, \qquad \tfrac{1}{2}(-s^2 + S^2) = T \tag{13}$$

is used, the system (12) reduces to

$$\begin{aligned} &(b_0 + 4T)(4t - b_{123}) - b_I/2 = 0, \\ &b_S - (b_0 - b_{123} + 4T + 4t)(b_0 + b_{123} + 4T - 4t) = 0. \end{aligned} \tag{14}$$

In (14), coordinate-free abbreviations b_S and b_I have been introduced,

$$\begin{aligned} b_S &= \langle \mathsf{B}\tilde{\hat{\mathsf{B}}} \rangle_0 = b_0^2 - b_1^2 - b_2^2 - b_3^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 - b_{123}^2, \\ b_I &= \langle \mathsf{B}\tilde{\hat{\mathsf{B}}} I \rangle_0 = 2b_3 b_{12} - 2b_2 b_{13} + 2b_1 b_{23} - 2b_0 b_{123}. \end{aligned} \tag{15}$$

In (15), the MV $\tilde{\hat{B}}$ denotes the Clifford conjugate of B, where tilde is the grade reversion and cap is the grade inversion. Note that for remaining algebras $Cl_{0,3}$, $Cl_{2,1}$, and $Cl_{2,1}$, the signs of individual terms inside b_S and b_I all are different. As we shall see below, the square roots for all $n = 3$ algebras are predetermined by four real coefficients only, namely, b_0, b_{123}, b_S, and b_I.

After substitution of (13), the resulting system of Equation (14) is of degree ≤ 4. Thus, we conclude that the initial system (12) can be explicitly solved in radicals. In particular, two real solutions of (13) have the form

$$\left(s_{1,2} = \pm\sqrt{-T + \sqrt{T^2 + t^2}}, \qquad S_{1,2} = \pm\frac{t}{\sqrt{-T + \sqrt{T^2 + t^2}}} \right), \tag{16}$$

where the signs in pairs (s_i, S_i) must be identical, plus or minus. The denominator of $S_{1,2}$ becomes zero if $s = S = 0$. The remaining two solutions of (13), which can be obtained from (16) by the substitution $\sqrt{T^2 + t^2} \to -\sqrt{T^2 + t^2}$, are complex-valued due to the inequality $\sqrt{T^2 + t^2} \geq T$ and therefore must be rejected.

The two real-valued solutions of Equation (14) are

$$\begin{cases} \left(t_{1,2} = \frac{1}{4}\left(b_{123} \pm \frac{1}{\sqrt{2}}\sqrt{-b_S + \sqrt{D}}\right),\ T_{1,2} = \frac{1}{4}\left(\frac{\pm b_I}{\sqrt{2}\sqrt{-b_S+\sqrt{D}}} - b_0\right)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad -b_S + \sqrt{D} > 0, \\ \left(t_{1,2} = b_{123}/4,\ T_{1,2} = \frac{1}{4}(\pm\sqrt{b_S} - b_0)\right),\ \text{if } -b_S + \sqrt{D} = 0 \wedge b_S > 0. \end{cases} \tag{17}$$

No additional conditions are required for the determinant $D = b_S^2 + b_I^2 \geq 0$ of the MV B, since for $Cl_{3,0}$ algebra, it is always positive definite $D \geq b_S$ (see in Appendix D how to compute the MV determinant). Again, we should take the same signs for t_i and T_i. The two complex-valued solutions of (14), which can be obtained from (17) by substitution $\sqrt{D} \to -\sqrt{D}$, must be rejected. The denominator of $T_{1,2}$ in (17) turns into zero when $b_S = \sqrt{D}$, i.e., when $b_I = 0$.

To summarize, starting from (17) and then going to (16), and finally to Formulas (10) and (11), one obtains four explicit real solutions which completely determine the square root $\mathsf{A} = \sqrt{\mathsf{B}}$ of generic MV B in terms of radicals $\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I$ of real Clifford algebra $Cl_{3,0}$.

3.2. The Special Case $s^2 + S^2 = 0$

The only special case in $Cl_{3,0}$ corresponds to $s = S = 0$. In the subcases $s = S \neq 0$ and $s = -S \neq 0$, one can rewrite expressions (16) in a simpler form. In particular, when $s = S \neq 0$, we have

$$s_{1,2} = \begin{cases} \pm\frac{1}{2}\sqrt{b_{123} + \dfrac{b_I}{2b_0}} & \text{if} \quad b_0 \neq 0, \\ \pm\frac{1}{2}\sqrt{b_{123} \pm \sqrt{-b_S}} & \text{if} \quad b_0 = 0, \end{cases} \tag{18}$$

and when $s = -S \neq 0$,

$$s_{1,2} = \begin{cases} \pm\frac{1}{2}\sqrt{-b_{123} - \dfrac{b_I}{2b_0}} & \text{if} \quad b_0 \neq 0, \\ \pm\frac{1}{2}\sqrt{-b_{123} \pm \sqrt{-b_S}} & \text{if} \quad b_0 = 0, \end{cases} \tag{19}$$

where all expressions inside square roots are assumed to be positive.

The case $s = S = 0$ is special, because the condition implies that the number of square roots of B may be infinite (the case of simple MV roots is given in [12]). Indeed, in this case, Equations (7)–(9) are compatible only if the vector (b_1, b_2, b_3) and bivector (b_{12}, b_{13}, b_{23}) coefficients are zeros. Then, Equation (6) reduces to

$$b_0 = \mathbf{v}^2 - \mathbf{V}^2, \quad b_{123} = 2(\mathbf{v} \cdot \mathbf{V}), \tag{20}$$

where $\mathbf{v}^2 = v_1^2 + v_2^2 + v_3^2$ and $\mathbf{V}^2 = V_1^2 + V_2^2 + V_3^2$ for $Cl_{3,0}$. Since, in general, we have $3 + 3 = 6$ unknowns which must satisfy Equation (20), we are left with four real arbitrary parameters as will be explicitly demonstrated in Example 1. The solution therefore makes a four-dimensional (or smaller) set of real-valued MV coefficients. It is interesting that both expressions in (20) have a very clear geometric interpretation. Indeed, if the ends of vectors $\mathbf{v}$ and $\mathbf{V}$ represent two concentric spheres, then the coefficient b_0 controls the lengths of radii $|\mathbf{v}|$ and $|\mathbf{V}|$, while the pseudoscalar coefficient b_{123} controls the angle between the vectors $\mathbf{v}$ and $\mathbf{V}$. From this follows that, due to periodicity of the angle, one can introduce principal value for coefficient b_{123}. Similar properties, i.e., the multiplicity of roots and the existence of principal angle in a complex plane, are well-known in case complex numbers [17].

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3.3. $Cl_{1,2} \simeq Cl_{3,0}$ *Algebra*

In paper [18], it is shown that "...for odd $n \geq 3$, there are three classes of isomorphic Clifford algebras what is consistent with Cartan's classification of real Clifford algebras." In particular, two algebras, $Cl_{3,0}$ and $Cl_{1,2}$, are represented by 2×2 complex matrices $\mathbb{C}(2)$. The similarity between square root expressions obtained below also confirms that these two algebras fall into the same isomorphism class. On the other hand, the algebras $Cl_{0,3}$ and $Cl_{2,1}$ are represented by blocked 2×2 and 1×1 matrices, respectively, ${}^2\mathbb{R}(2)$ and ${}^2\mathbb{H}(1)$. Therefore, they belong to different classes. Indeed, as we shall show later, the analysis of roots in $Cl_{2,1}$ is only roughly similar to that in $Cl_{0,3}$. However, between $Cl_{2,1}$ and $Cl_{0,3}$, there are distinctions: they are isomorphic to different, real, and quaternionic matrices.

As far as $Cl_{1,2}$ algebra is concerned, its difference from $Cl_{3,0}$ is contained in the explicit expression for b_S,

$$b_S = \langle \mathsf{B}\tilde{\widehat{\mathsf{B}}} \rangle_0 = b_0^2 - b_1^2 + b_2^2 + b_3^2 - b_{12}^2 - b_{13}^2 + b_{23}^2 - b_{123}^2, \tag{21}$$

$$b_I = \langle \mathsf{B}\tilde{\widehat{\mathsf{B}}} I \rangle_0 = 2b_3b_{12} - 2b_2b_{13} + 2b_1b_{23} - 2b_0b_{123}, \tag{22}$$

$$D = b_S^2 + b_I^2, \qquad Cl_{1,2} \tag{23}$$

and expressions for v_i and V_i

$$v_1 = \frac{b_1 s + b_{23} S}{2(s^2 + S^2)}, \qquad v_2 = \frac{b_2 s + b_{13} S}{2(s^2 + S^2)}, \qquad v_3 = \frac{b_3 s - b_{12} S}{2(s^2 + S^2)}, \tag{24}$$

$$V_1 = \frac{b_{23} s - b_1 S}{2(s^2 + S^2)}, \qquad V_2 = \frac{b_{13} S - b_2 S}{2(s^2 + S^2)}, \qquad V_3 = -\frac{b_{12} s + b_3 S}{2(s^2 + S^2)}. \tag{25}$$

The expressions for b_I and D (the determinant of B) remain the same. Note that in (20), the scalar product in $Cl_{1,2}$ has both plus/minus signs, in particular $\mathbf{v}^2 = v_1^2 - v_2^2 - v_3^2$. Before considering other algebras, it is helpful to analyze a few examples.

3.4. *Examples for* $Cl_{3,0}$ *and* $Cl_{1,2}$

Example 1. *The Case* $s \neq S$.

The square root of $\mathsf{B} = \mathbf{e}_1 - 2\mathbf{e}_{23}$ in $Cl_{3,0}$. The coefficients in this case are $b_1 = 1$ and $b_{23} = -2$, and all remaining ones are equal to zero. Then, from (15) follows that $b_I = -4$ and $b_S = 3$. The expression (17) gives $t_{1,2} = (\frac{1}{4}, -\frac{1}{4})$ and $T_{1,2} = (-\frac{1}{2}, \frac{1}{2})$. Finally, using (16), we find the real solutions for s and S,

$$\left(s_{1,2} = \mp\tfrac{1}{2}c_1,\ S_{1,2} = \pm\tfrac{1}{2}c_2\right) \quad \text{and} \quad \left(s_{3,4} = \pm\tfrac{1}{2}c_2,\ S_{3,4} = \pm\tfrac{1}{2}c_1\right), \tag{26}$$

where $c_1 = \sqrt{-2 + \sqrt{5}}$ and $c_2 = \sqrt{2 + \sqrt{5}}$. Thus, the MV is regular. Using (10) and (11), we have the following four sets of non-zero coefficients:

$$\begin{array}{llll} \left(s_1 = -\tfrac{1}{2}c_1,\right. & S_1 = \tfrac{1}{2}c_2, & v_1 = -\tfrac{1}{2}c_2, & \left.V_1 = -\tfrac{1}{2}c_1\right), \\ \left(s_2 = \tfrac{1}{2}c_1,\right. & S_2 = -\tfrac{1}{2}c_2, & v_1 = \tfrac{1}{2}c_2, & \left.V_1 = \tfrac{1}{2}c_1\right), \\ \left(s_3 = \tfrac{1}{2}c_2,\right. & S_3 = \tfrac{1}{2}c_1, & v_1 = \tfrac{1}{2}c_1, & \left.V_1 = -\tfrac{1}{2}c_2\right), \\ \left(s_4 = -\tfrac{1}{2}c_2,\right. & S_4 = -\tfrac{1}{2}c_1, & v_1 = -\tfrac{1}{2}c_1, & \left.V_1 = \tfrac{1}{2}c_2\right). \end{array} \tag{27}$$

The remaining coefficients are equal to zero, $v_2 = v_3 = V_2 = V_3 = 0$. Finally, inserting the coefficients (27) into (5), one can find four different roots,

$$\begin{aligned} \mathsf{A}_{1,2} &= \mp\tfrac{1}{2}c_2\big(-2+\sqrt{5}+\mathbf{e}_1+(-2+\sqrt{5})\mathbf{e}_{23}-\mathbf{e}_{123}\big), \\ \mathsf{A}_{3,4} &= \pm\tfrac{1}{2}c_1\big(2+\sqrt{5}+\mathbf{e}_1-(2+\sqrt{5})\mathbf{e}_{23}+\mathbf{e}_{123}\big), \end{aligned} \tag{28}$$

squares of which give the initial MV $\mathsf{B} = \mathbf{e}_1 - 2\mathbf{e}_{23}$.

Example 2. *The Case $s = S$.*

The square root of $\mathsf{B} = -1 + \mathbf{e}_3 - \mathbf{e}_{12} + \frac{1}{2}\mathbf{e}_{123}$ in $Cl_{3,0}$. Now, $b_0 = -1$, $b_{123} = \frac{1}{2}$, $b_I = -1$, $b_S = \frac{3}{4}$. Then, from (16) and (17) follows real solutions for s_i and S_i,

$$\big(s_{1,2} = \pm\tfrac{1}{2},\quad S_{1,2} = \pm\tfrac{1}{2}\big) \quad \text{and} \quad \big(s_{3,4} = 0,\quad S_{3,4} = \pm 1\big). \tag{29}$$

Then, for case $(s_{1,2}, S_{1,2})$, Equations (10) and (11) yield

$$\begin{aligned} &\big(s_1 = -\tfrac{1}{2},\quad v_1 = v_2 = v_3 = 0,\quad V_1 = V_2 = 0,\quad V_3 = 1\big), \\ &\big(s_2 = \tfrac{1}{2},\quad\ \ v_1 = v_2 = v_3 = 0,\quad V_1 = V_2 = 0,\quad V_3 = -1\big). \end{aligned} \tag{30}$$

The case $(s_{3,4}, S_{3,4})$ is treated exactly as in Example 1. The final answer consists of four roots too:

$$\begin{aligned} \mathsf{A}_{1,2} &= \pm\tfrac{1}{2}\big(-1+2\mathbf{e}_{12}-\mathbf{e}_{123}\big), \\ \mathsf{A}_{3,4} &= \pm\tfrac{1}{2}\big(\mathbf{e}_3+\mathbf{e}_{12}-2\mathbf{e}_{123}\big). \end{aligned} \tag{31}$$

Example 3. *The Case $s = S = 0$.*

The square root of $\mathsf{B} = -1 + \mathbf{e}_{123}$, which is the center of $Cl_{3,0}$. The coefficients $b_0 = -1$, $b_{123} = 1$ give $b_I = 2$, $b_S = 0$. Then, from expressions (17) and (16) follows

$$\big(s_{1,2} = \pm c_1,\ S_{1,2} = \pm c_2\big) \quad \text{and} \quad \big(s_3 = 0,\ S_3 = 0\big), \tag{32}$$

where $c_1 = \sqrt{-1/2 + 1/\sqrt{2}}$ and $c_2 = \sqrt{1/2 + 1/\sqrt{2}}$.

The case $(s_{1,2}, S_{1,2})$ in (32) can be computed similarly as in Example 1. The two square roots, which are obtained from case $\big(s_{1,2} = \pm c_1,\ S_{1,2} = \pm c_2\big)$, therefore are

$$\mathsf{A}_{1,2} = \pm(c_1 + c_2\mathbf{e}_{123}). \tag{33}$$

The set of two roots above should be extended by adding a set of roots provided by the case $(s_3 = 0,\ S_3 = 0)$ in (32), which is special. Indeed, some of coefficients in this case remain unspecified and therefore may be treated as free parameters that yield an uncountable number (continuum) of roots. The coefficients (b_1, b_2, b_3) and (b_{12}, b_{13}, b_{23}) in this case are zeroes; however, the compatibility of (7)–(9) is satisfied and the solution set is not empty. Indeed, as seen from (20), the system can be solved for an arbitrary pair of coefficients $(v_1, v_2, v_3, V_1, V_2, V_3)$, for example with (v_1, V_1). If (v_1, V_1) is inserted into (5), one gets an MV with four free parameters,

$$\mathsf{A} = f_1(v_2, v_3, V_2, V_3)\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + f_2(v_2, v_3, V_2, V_3)\mathbf{e}_{23} - V_2\mathbf{e}_{13} + V_3\mathbf{e}_{12}, \tag{34}$$

where $v_1 = f_1(v_2, v_3, V_2, V_3)$ and $V_1 = f_2(v_2, v_3, V_2, V_3)$ denote explicit solutions of (20),

$$\begin{aligned} v_1 &= \mp \frac{c_1}{\sqrt{2}}, \qquad V_1 = \pm \frac{1}{c_1} \frac{-b_{123} + 2(v_2 V_2 + v_3 V_3)}{\sqrt{2}}, \qquad \text{with} \\ c_1 &= \Big(\pm \sqrt{\big(b_0 - v_2^2 - v_3^2 + V_2^2 + V_3^2\big)^2 + (b_{123} - 2(v_2 V_2 + v_3 V_3))^2} \\ &\qquad + \Big(b_0 - v_2^2 - v_3^2 + V_2^2 + V_3^2\Big)\Big)^{\frac{1}{2}}. \end{aligned} \tag{35}$$

For example, by setting all free parameters to zero, $v_2 = V_2 = v_3 = V_3 = 0$, we select from a continuum two roots, which we denote

$$\mathsf{A}_{3,4} = \pm(c_1 \mathbf{e}_1 + c_2 \mathbf{e}_{23}). \tag{36}$$

It is important to realize, however, that the number of roots provided by case $(s_3 = 0,\ S_3 = 0)$ in general is infinite and the two roots in (36) represent the simplest choice of free parameters. All roots A_j satisfy $\mathsf{A}_j^2 = \mathsf{B} = -1 + \mathbf{e}_{123}$.

If, instead of $\mathsf{B} = -1 + \mathbf{e}_{123}$, we had tried to find the square root of MV that does not belong to the center, for example, if we had worked with $\mathsf{B} = \mathbf{e}_1 + \mathbf{e}_{12}$, which is directly related to polarized electromagnetic wave in $Cl_{3,0}$, we would have ended up with an empty solution set. Indeed, in the latter case, $s_1 = 0$, $S_1 = 0$ and $b_0 = b_{123} = b_I = b_S = 0$. Then, after substitution of $s \to s_1 = 0$ and $S \to S_1 = 0$ into Equations (7)–(9), one obtains the contradiction $1 = 0$.

Example 4. *The case of Quaternion.*

Quaternions are isomorphic to even subalgebra $Cl_{3,0}^+$ with elements $\{1, \mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{13}\}$; therefore, the provided formulas allow to find quaternionic square root too. Taking into account that quaternion imaginary units are $\mathbf{i} = \mathbf{e}_{12}$, $\mathbf{j} = -\mathbf{e}_{13}$ and $\mathbf{k} = \mathbf{e}_{23}$, let us compute the square root of $\mathsf{B} = 1 + \mathbf{e}_{12} - \mathbf{e}_{13} + \mathbf{e}_{23} = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. In this example, we have $b_0 = 1$, $b_{123} = 0$, and $b_I = 0$, $b_S = 4$. Starting from (17) and then using Equation (16), it is easy to find that the MV represents a regular case with four different coefficients

$$\big(s_{1,2} = 0, \quad S_{1,2} = \pm 1/\sqrt{2}\big) \quad \text{and} \quad \Big(s_{3,4} = \pm\sqrt{3/2}, \quad S_{3,4} = 0\Big). \tag{37}$$

Using (10)–(11) and (5), we can write the answer:

$$\begin{aligned} \mathsf{A}_{1,2} &= \pm(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_{123})/\sqrt{2}\,, \\ \mathsf{A}_{3,4} &= \pm(3 + \mathbf{e}_{12} - \mathbf{e}_{13} + \mathbf{e}_{23})/\sqrt{6} \equiv \pm(3 + \mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{6}\,. \end{aligned} \tag{38}$$

The squares of all roots yield the initial MV. It should be noticed that in $\mathsf{A}_{3,4}$, the quaternion imaginary units have remained in even algebra only. The source of this 'strange' difference is related to the algebra where the square root problem is solved. In particular, here, the roots are computed in $Cl_{3,0}$ algebra rather than in $Cl_{0,2}$, i.e., algebra of quaternions. However, the formulas for $Cl_{0,2}$ (see resp. equation in Appendix B) give two roots only.

Example 5. *The regular case of $Cl_{1,2}$ algebra.*

Using the same initial MV, $\mathsf{B} = \mathbf{e}_1 - 2\mathbf{e}_{23}$, as in Example 1, one obtains the same values for (b_S, b_I) and (s, S). After substitution into (24), (25), and then into (5), the square roots are found to be

$$\begin{aligned} \mathsf{A}_{1,2} &= \pm\tfrac{1}{2}(c_2(-\mathbf{e}_1 + \mathbf{e}_{123}) - c_1(1 + \mathbf{e}_{23})), \\ \mathsf{A}_{3,4} &= \pm\tfrac{1}{2}(-c_1(\mathbf{e}_1 + \mathbf{e}_{123}) + c_2(-1 + \mathbf{e}_{23})), \end{aligned} \tag{39}$$

where $c_1 = \sqrt{-2 + \sqrt{5}}$ and $c_2 = \sqrt{2 + \sqrt{5}}$.

4. Square Roots in $Cl_{0,3}$ Algebra

The similar approach to the root problem allows to write down explicit square root formulas for $Cl_{0,3}$ algebra as well. Using the same notation (5) for A and B and equating coefficients at same basis elements in $\mathsf{A}^2 = \mathsf{B}$, now we obtain the following system of equations. The formulas have the same structure and differ in signs of some constituent terms only. Below, for easier reading and application, all formulas, including those for mixed algebras, are written explicitly without introducing a large number of sign epsilons $\varepsilon_{\pm} = \pm 1$. In fact, appearance of different signs in structurally similar expressions brings in different conditions for real root existence in distinct algebras.

$$b_0 = s^2 + S^2 + \mathbf{v}^2 + \mathbf{V}^2, \qquad b_{123} = 2(sS + \mathbf{v}\cdot\mathbf{V}), \tag{40}$$

$$b_1 = 2(sv_1 + SV_1), \qquad b_{23} = -2(sV_1 + Sv_1), \tag{41}$$

$$b_2 = 2(sv_2 + SV_2), \qquad b_{13} = 2(sV_2 + Sv_2), \tag{42}$$

$$b_3 = 2(sv_3 + SV_3), \qquad b_{12} = -2(sV_3 + Sv_3), \tag{43}$$

where, now, $\mathbf{v}^2 = -v_1^2 - v_2^2 - v_3^2$ and $\mathbf{v}\cdot\mathbf{V} = -v_1V_1 - v_2V_2 - v_3V_3$.

4.1. The Generic Case $s^2 - S^2 \neq 0$

The solution of Equations (41)–(43) is

$$v_1 = \frac{b_1 s + b_{23} S}{2(s^2 - S^2)}, \qquad v_2 = -\frac{b_2 s - b_{13} S}{2(s^2 - S^2)}, \qquad v_3 = \frac{b_3 s + b_{12} S}{2(s^2 - S^2)}, \tag{44}$$

$$V_1 = -\frac{b_{23} s + b_1 S}{2(s^2 - S^2)}, \qquad V_2 = \frac{b_{13} s - b_2 S}{2(s^2 - S^2)}, \qquad V_3 = -\frac{b_{12} s + b_3 S}{2(s^2 - S^2)}, \tag{45}$$

which is valid when $s^2 - S^2 \neq 0$, and corresponds to the generic case. After substitution of (44) and (45) into (40), one obtains two coupled nonlinear algebraic equations for two unknowns s and S,

$$\begin{aligned} & b_S + 4s^2(-6S^2 + b_0) + 8sSb_{123} = 4s^4 + (-2S^2 + b_0)^2 + b_{123}^2, \\ & b_I = 2(2(s^2 + S^2) - b_0)(4sS - b_{123}), \end{aligned} \tag{46}$$

where, again, the coordinate-free notation is introduced,

$$\begin{aligned} b_S &= \langle \mathsf{B}\widetilde{\widehat{\mathsf{B}}}\rangle_0 = b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 + b_{123}^2, \\ b_I &= \langle \mathsf{B}\widetilde{\widehat{\mathsf{B}}} I\rangle_0 = -2b_3b_{12} + 2b_2b_{13} - 2b_1b_{23} + 2b_0b_{123}. \end{aligned} \tag{47}$$

Note change of signs as compared to the $Cl_{3,0}$ case. The determinant D in $Cl_{0,3}$ is expressed as a difference, $D = b_S^2 - b_I^2$, which is always positive, $D > 0$.

To reduce the degree of the above equations, the substitution

$$s\,S = t; \qquad \tfrac{1}{2}(s^2 + S^2) = T, \tag{48}$$

is used that transforms the system (46) into a simpler one. To eliminate s and S, *Mathematica* commands **Eliminate[ ]**, **GroebnerBasis[ ]** have been used. They allow to rewrite the initial Equation (48) in a number of equivalent forms.

$$b_S = (4T - b_0)^2 + (4t - b_{123})^2, \qquad b_I = 2(4T - b_0)(4t - b_{123}). \tag{49}$$

The solution of (49) when $(b_S \pm \sqrt{D}) > 0$ is

$$\begin{cases} \Big(t_{1,2} = \tfrac{1}{4}\big(b_{123} \pm \tfrac{1}{\sqrt{2}}\sqrt{b_S - \sqrt{D}}\big), \quad T_{1,2} = \tfrac{1}{4}\big(b_0 \pm \tfrac{b_I}{\sqrt{2}\sqrt{b_S - \sqrt{D}}}\big)\Big), \\ \hfill \text{if } b_S - \sqrt{D} > 0, \\ \big(t_{1,2} = \tfrac{1}{4}b_{123}, \quad T_{1,2} = \tfrac{1}{4}(\pm\sqrt{b_S} + b_0)\big), \text{ if } \quad b_S - \sqrt{D} = 0 \text{ and } b_S > 0. \end{cases} \tag{50}$$

The $\pm$ signs in the above formulas are mutually related; thus, there are only two possibilities that correspond to either plus or minus signs inside t_i and T_i formulas. The remaining two solutions of (49), which were obtained from (50) after replacement $-\sqrt{D} \to +\sqrt{D}$, yield a complex-valued expression for $T \pm \sqrt{T^2 - t^2}$ (see Equation (51) below); therefore, they were dismissed in advance.

Once the equations in (50) are computed, they can be substituted back into solutions of (48),

$$\begin{cases} \Big(s_{1,2,3,4} = \pm\sqrt{T \pm \sqrt{T^2 - t^2}},\ S_{1,2,3,4} = \frac{\pm t}{\sqrt{T \pm \sqrt{T^2 - t^2}}}\Big) \text{if } T \geq 0,\ t \neq 0, \\ (s_{1,2} = S_{3,4} = \pm\sqrt{2T}, \qquad S_{1,2} = s_{3,4} = 0) \qquad \text{if} \quad T \geq 0,\ t = 0. \end{cases} \tag{51}$$

In the obtained equations, the same signs must be chosen in the same index positions in $s_{1,2,3,4}$ and $S_{1,2,3,4}$ (four possibilities).

Thus, starting from pairs (t_1, T_1) and (t_2, T_2) in Equation (50) and then going to (51), and finally to Formulas (44), (45), and (5), one obtains explicit real solutions that completely determine the square root of equation $\mathsf{B} = \mathsf{A}^2$ (with $\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I$) of the generic MV B of real $Cl_{0,3}$ algebra in radicals. It appears that, at most, only four isolated real solutions (because the condition $T \geq 0$ in (51) selects only single sign from (50)) are possible in this algebra too, since other choices of signs in (50) and (51) yield negative expressions inside square roots.

4.2. *The Special Case* $s^2 - S^2 = 0$

There are three subcases: (1) $s = S \neq 0$, (2) $s = -S \neq 0$, and (3) $s = S = 0$.

4.2.1. The Subcase $s = S \neq 0$

Here, the system of Equations (41)–(43) has a special solution,

$$v_1 = \frac{b_1}{2s} - V_1, \quad v_2 = \frac{b_2}{2s} - V_2, \quad v_3 = \frac{b_3}{2s} - V_3, \tag{52}$$

if and only if the MV B coefficients satisfy $b_1 = -b_{23}$, $b_2 = b_{13}$, $b_3 = -b_{12}$. In (52), v_i is expressed in terms of V_i. Appearance of s in the denominators implies that the case $s = S = 0$ must be investigated separately. After substituting the solution (52) into (40) and taking into account the mentioned conditions ($b_1 = -b_{23}, b_2 = b_{13}, b_3 = -b_{12}$) one gets two equations,

$$\begin{aligned} &-\frac{b_1^2+b_2^2+b_3^2}{4s^2}+\frac{b_1V_1+b_2V_2+b_3V_3}{s}-b_0+2s^2+2(\mathbf{V}\cdot\mathbf{V})=0,\\ &-\frac{b_1V_1+b_2V_2+b_3V_3}{s}-b_{123}+2s^2-2(\mathbf{V}\cdot\mathbf{V})=0, \end{aligned} \tag{53}$$

that should be kept mutually compatible. To this end, we subtract and add the above equations to get

$$\begin{aligned} &-\frac{4s^2(b_0-b_{123}-4(\mathbf{V}\cdot\mathbf{V}))-8s(b_1V_1+b_2V_2+b_3V_3)+b_1^2+b_2^2+b_3^2}{4s^2}=0,\\ &-\frac{4s^2\big(b_0+b_{123}-4s^2\big)+b_1^2+b_2^2+b_3^2}{4s^2}=0. \end{aligned} \tag{54}$$

Then, making use of expanded form of $b_S = \langle \mathrm{B}\widetilde{\widehat{\mathrm{B}}}\rangle_0 = b_0^2 + 2(b_1^2+b_2^2+b_3^2) + b_{123}^2$, where the conditions $b_1 = -b_{23}, b_2 = b_{13}, b_3 = -b_{12}$ have been taken into account, one can express the sum $b_1^2+b_2^2+b_3^2$ from the second equation in (54), $b_1^2+b_2^2+b_3^2 = \frac{1}{2}(b_S - b_0^2 - b_{123}^2)$, and substitute the latter into the first of the equations. The result is the quadratic equations for V_i's. After solving, for example, with respect to V_1, one can express V_1 in terms of, now, arbitrary free parameters V_2 and V_3,

$$\begin{aligned} V_1 =& \frac{\sqrt{2}}{8s}\Big(\sqrt{2}b_1 \pm \Big(-8s^2\Big(b_0-b_{123}+4\Big(V_2^2+V_3^2\Big)\Big)+\\ &16s(b_2V_2+b_3V_3)+b_0^2+2b_1^2+b_{123}^2-b_S\Big)^{1/2}\Big), \end{aligned} \tag{55}$$

that warrants compatibility of the system (53). Thus, further analysis may be restricted to the simplest single equation: the second equation in (54) that after introduction of shortcut b_S can be cast to form

$$b_S = -8b_0s^2 - 8b_{123}s^2 + b_0^2 + b_{123}^2 + 32s^4. \tag{56}$$

The solution of (56) with respect to s can be expressed in radicals,

$$s_{1,2} = \pm\tfrac{1}{2\sqrt{2}}\sqrt{\sqrt{2b_S-(b_0-b_{123})^2}+b_0+b_{123}}\,, \tag{57}$$

where all expressions inside square roots are assumed to be positive. The expressions (57), (55), and (52) after substitution into (5) yield the final answer for this special case under conditions for MV B coefficients: $b_1 = -b_{23}, b_2 = b_{13}, b_3 = -b_{12}$ that in an abridged version reduce to $b_S - (b_0 - b_{123})^2 = b_I$. In conclusion, the solution set contains two free parameters, V_2 and V_3, and therefore represents continuum of roots on a two-dimensional manifold in the parameter space.

4.2.2. The Subcase $s = -S \neq 0$

Performing exactly the same analysis as in Section 4.2.1, one obtains the conditions for the existence of a solution: $b_1 = b_{23}$, $b_2 = -b_{13}$, and $b_3 = b_{12}$, or in short, $-b_S + (b_0 + b_{123})^2 = b_I$. Similarly, expressing v_i in terms of V_i, one gets

$$v_1 = \frac{b_1}{2s} + V_1, \quad v_2 = \frac{b_2}{2s} + V_2, \quad v_3 = \frac{b_3}{2s} + V_3. \tag{58}$$

If V_1 is expressed in terms of V_2 and V_3,

$$V_1 = \frac{\sqrt{2}}{8s}\Big(-\sqrt{2}b_1 \pm \big(-8s^2\big(b_0 + b_{123} + 4(V_2^2 + V_3^2)\big) - 16s(b_2V_2 + b_3V_3) + b_0^2 + 2b_1^2 + b_{123}^2 - b_S\big)^{1/2}\Big), \tag{59}$$

we find two real solutions for s,

$$s_{1,2} = \pm\frac{1}{2\sqrt{2}}\sqrt{\sqrt{2b_S - (b_0 + b_{123})^2} + b_0 - b_{123}}\,, \tag{60}$$

After substitution into (5), the above expressions again yield the final MV, provided the conditions $b_1 = b_{23}, b_2 = -b_{13}, b_3 = b_{12}$ are satisfied.

4.2.3. The Subcase $s = S = 0$

The analysis of this special subcase is very similar to that in $Cl_{3,0}$. Equations (41)–(43) satisfy the compatibility condition if vector (b_1, b_2, b_3) and bivector (b_{12}, b_{13}, b_{23}) coefficients are equated to zero. Then, Equation (40) assumes the following form

$$b_0 = \mathbf{v}^2 + \mathbf{V}^2, \quad b_{123} = 2(\mathbf{v}\cdot\mathbf{V}) \tag{61}$$

from which follows that four parameters remain unspecified. For example, if Equation (61) is solved with respect to pair (v_1, V_1), one gets

$$\begin{aligned} &v_1 = \mp\frac{c_1}{\sqrt{2}}, \qquad V_1 = \pm\frac{1}{c_1}\frac{b_{123} + 2(v_2V_2 + v_3V_3)}{\sqrt{2}}, \qquad \text{where}\\ &c_1 = \Big(\pm\sqrt{\big(b_0 + v_2^2 + v_3^2 + V_2^2 + V_3^2\big)^2 - (b_{123} + 2(v_2V_2 + v_3V_3))^2} - b_0 - v_2^2 - v_3^2 - V_2^2 - V_3^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{62}$$

The pairs (v_2, V_2) and (v_3, V_3) may be interpreted as free parameters that generate a continuum of roots in a four parameter space. The geometric interpretation of Equation (61) is similar to those in (20) for $Cl_{3,0}$.

4.3. Examples for $Cl_{0,3}$

Example 6. *The regular case.*

As in Example 1, let the initial MV be $\mathsf{B} = \mathbf{e}_1 - 2\mathbf{e}_{23}$, the coefficients of which are $b_1 = 1$, $b_{12} = -2$. The shortcuts b_I and b_S in (47) have the values $b_I = 4$ and $b_S = 5$. The formulas in (50) give $(T_1, t_1) = (\frac{1}{2}, \frac{1}{4})$ and $(T_2, t_2) = (-\frac{1}{2}, -\frac{1}{4})$. Since T_1 is positive, the pair (T_1, t_1) is used in the following. Then, from the system (51), we find four values of (s_i, S_i) and then from (44), (45), the coefficients v_i and V_i,

$$\begin{aligned} &\big(s_1 = V_1 = \tfrac{1}{4}d_3, && S_1 = v_1 = -\tfrac{1}{2}d_2, && v_2 = v_3 = V_2 = V_3 = 0\big),\\ &\big(s_2 = V_1 = \tfrac{1}{2}d_1, && S_2 = v_1 = \tfrac{1}{2}d_2, && v_2 = v_3 = V_2 = V_3 = 0\big),\\ &\big(s_3 = V_1 = -\tfrac{1}{2}d_2, && S_3 = -\tfrac{1}{2d_2},\ v_1 = \tfrac{1}{4}d_3, && v_2 = v_3 = V_2 = V_3 = 0\big),\\ &\big(s_4 = V_1 = \tfrac{1}{2}d_2, && S_4 = \tfrac{1}{2d_2},\ v_1 = \tfrac{1}{2}d_1, && v_2 = v_3 = V_2 = V_3 = 0\big). \end{aligned} \tag{63}$$

where $d_1 = \sqrt{2-\sqrt{3}}$, $d_2 = \sqrt{2+\sqrt{3}}$, and $d_3 = \sqrt{2} - \sqrt{6}$. Finally, in the same way as in *Example* 1, we find four different square roots,

$$\begin{aligned}
\mathsf{A}_1 &= \tfrac{1}{2}\big(d_1 + d_2\mathbf{e}_1 - d_1\mathbf{e}_{23} + d_2\mathbf{e}_{123}\big),\\
\mathsf{A}_2 &= \tfrac{1}{2}\big(\tfrac{1}{2}d_3 - d_2\mathbf{e}_1 - \tfrac{1}{2}d_3\mathbf{e}_{23} - d_2\mathbf{e}_{123}\big).\\
\mathsf{A}_3 &= \tfrac{1}{2}\big(d_2 + d_1\mathbf{e}_1 - d_2\mathbf{e}_{23} + \frac{\mathbf{e}_{123}}{d_2}\big),\\
\mathsf{A}_4 &= \tfrac{1}{2}\big(-d_2 + \tfrac{1}{2}d_3\mathbf{e}_1 + d_2\mathbf{e}_{23} - \frac{\mathbf{e}_{123}}{d_2}\big),
\end{aligned} \tag{64}$$

Noting that $d_3 = -2d_1$ and $d_2^{-1} = d_1$, the roots may be rewritten in the standard form

$$\begin{aligned}
\mathsf{A}_{1,2} &= \pm\tfrac{1}{2}\big(d_1 + d_2\mathbf{e}_1 - d_1\mathbf{e}_{23} + d_2\mathbf{e}_{123}\big),\\
\mathsf{A}_{3,4} &= \pm\tfrac{1}{2}\big(d_2 + d_1\mathbf{e}_1 - d_2\mathbf{e}_{23} + d_1\mathbf{e}_{123}\big).
\end{aligned} \tag{65}$$

Example 7. *The case $s = S \neq 0$.*

The square root of $\mathsf{B} = -\mathbf{e}_3 + \mathbf{e}_{12} + 4\mathbf{e}_{123}$. The shortcuts in (b_I, b_S) have values $b_I = 2$ and $b_S = 18$, and afterwards, the expression (50) gives $(T_1, t_1) = (\frac{c_1}{4}, \frac{c_1}{4})$, where $c_1 = (2+\sqrt{5})$. The negative T solution has been omitted. All this gives $(s_1, S_1) = (-\frac{\sqrt{c_1}}{2}, -\frac{\sqrt{c_1}}{2})$ and $(s_2, S_2) = (\frac{\sqrt{c_1}}{2}, \frac{\sqrt{c_1}}{2})$. The coefficients satisfy the relations $b_1 = -b_{23}$, $b_2 = b_{13}$, $b_3 = -b_{12}$; therefore, a special solution consisting of four MVs exists:

$$\begin{aligned}
\mathsf{A}_1 &= -\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_{23})\sqrt{-4V_2^2 + 4(c_1 - V_3)V_3 + c_3} - \mathbf{e}_{12}V_3 + (\mathbf{e}_{13} - \mathbf{e}_2)V_2\\
&\qquad + \mathbf{e}_3(c_1 - V_3) - \frac{1}{2}c_2(\mathbf{e}_{123} + 1),\\
\mathsf{A}_2 &= \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_{23})\sqrt{-4V_2^2 + 4(c_1 - V_3)V_3 + c_3} - \mathbf{e}_{12}V_3 + (\mathbf{e}_{13} - \mathbf{e}_2)V_2\\
&\qquad + \mathbf{e}_3(c_1 - V_3) - \frac{1}{2}c_2(\mathbf{e}_{123} + 1),\\
\mathsf{A}_3 &= \frac{1}{2}(-(\mathbf{e}_1 + \mathbf{e}_{23})\sqrt{-4V_2^2 - 4V_3(V_3 + c_1) + c_3} - 2\mathbf{e}_{12}V_3\\
&\qquad + 2(\mathbf{e}_{13} - \mathbf{e}_2)V_2 - 2\mathbf{e}_3(V_3 + c_1) + c_2(\mathbf{e}_{123} + 1)),\\
\mathsf{A}_4 &= \frac{1}{2}((\mathbf{e}_1 + \mathbf{e}_{23})\sqrt{-4V_2^2 - 4V_3(V_3 + c_1) + c_3} - 2\mathbf{e}_{12}V_3\\
&\qquad + 2(\mathbf{e}_{13} - \mathbf{e}_2)V_2 - 2\mathbf{e}_3(V_3 + c_1) + c_2(\mathbf{e}_{123} + 1)),
\end{aligned} \tag{66}$$

where $c_1 = \sqrt{\sqrt{5}-2}$, $c_2 = \sqrt{\sqrt{5}+2}$, $c_3 = -\sqrt{5}+6$. Assuming concrete values of parameters V_2 and V_3, one can check that the root formulas give real MVs.

It should be noted, however, that symbolical expressions do not guarantee that we will always be able to find *real* parameters V_2 and V_3, what would ensure real square roots. For example, if instead of the above MV $\mathsf{B} = -\mathbf{e}_3 + \mathbf{e}_{12} + 4\mathbf{e}_{123}$, we tried to find the square root of MV $\mathsf{B} = -\mathbf{e}_3 + \mathbf{e}_{12}$ in $Cl_{0,3}$ (the MV was used earlier in the Example 2), we would find $s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$. The first value then yields $v_1 = -V_1 = \frac{1}{2}\sqrt{-4V_2^2 - (1+2V_3)^2}$, $v_2 = -V_2$, $v_3 = -1 - V_3$, and the second one yields $V_1 = -v_1 = \frac{1}{2}\sqrt{-4V_2^2 - (1-2V_3)^2}$, $v_2 = -V_2$, and $v_3 = 1 - V_3$. Taking the square of symbolical expressions, one can easily check that, formally, we indeed obtain the MV $\mathsf{B} = -\mathbf{e}_3 + \mathbf{e}_{12}$. It is obvious, however, that in both cases ($s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$), the expression under square root can be made non-negative (i.e., only zero in this case) for a single choice of parameters. In particular, in the case $s_1 = \frac{1}{2}$, the requirement $-4V_2^2 - (1+2V_3)^2 \geq 0$ yields $V_2 = 0, V_3 = -1/2$. Alternatively, in the case $s_2 = -\frac{1}{2}$, from equation $-4V_2^2 - (1-2V_3) \geq 0$ follows $V_2 =$

$0, V_3 = 1/2$. Both cases yield an isolated root $\pm\frac{1}{2}(1 - \mathbf{e}_3 + \mathbf{e}_{12} + \mathbf{e}_{123})$. Therefore, in this algebra, there exists only the isolated real square root of $\mathsf{B} = -\mathbf{e}_3 + \mathbf{e}_{12}$.

5. Square Roots in $Cl_{2,1}$ Algebra

5.1. The Generic Case $s^2 - S^2 \neq 0$

The system of nonlinear equations is

$$b_0 = s^2 + S^2 + \mathbf{v}^2 + \mathbf{V}^2, \qquad b_{123} = 2(sS + \mathbf{v}\cdot\mathbf{V}), \tag{67}$$
$$b_1 = 2(sv_1 + SV_1), \qquad b_{23} = 2(sV_1 + Sv_1), \tag{68}$$
$$b_2 = 2(sv_2 + SV_2), \qquad b_{13} = -2(sV_2 + Sv_2), \tag{69}$$
$$b_3 = 2(sv_3 + SV_3), \qquad b_{12} = -2(sV_3 + Sv_3), \tag{70}$$

where, now, $\mathbf{v}^2 = v_1^2 + v_2^2 - v_3^2$ and $\mathbf{v}\cdot\mathbf{V} = v_1V_1 + v_2V_2 - v_3V_3$. When $s^2 - S^2 \neq 0$, the solutions of systems (68)–(70) are

$$v_1 = \frac{b_1 s - b_{23}S}{2(s^2 - S^2)}, \qquad v_2 = \frac{b_2 s + b_{13}S}{2(s^2 - S^2)}, \qquad v_3 = \frac{b_3 s + b_{12}S}{2(s^2 - S^2)}, \tag{71}$$
$$V_1 = \frac{b_{23}s - b_1 S}{2(s^2 - S^2)}, \qquad V_2 = -\frac{b_{13}s + b_2 S}{2(s^2 - S^2)}, \qquad V_3 = -\frac{b_{12}s + b_3 S}{2(s^2 - S^2)}. \tag{72}$$

Insertion of v_i and V_i into (67) gives two coupled equations for unknowns s, S

$$\begin{aligned} &b_S + 4s^2(-6S^2 + b_0) + 8sSb_{123} = 4s^4 + (-2S^2 + b_0)^2 + b_{123}^2, \\ &b_I = 2(2(s^2 + S^2) - b_0)(4sS - b_{123}), \end{aligned} \tag{73}$$

where b_S and b_I are functions of coefficients in B,

$$\begin{aligned} b_S &= \langle \mathsf{B}\widehat{\tilde{\mathsf{B}}}\rangle_0 = b_0^2 - b_1^2 - b_2^2 + b_3^2 + b_{12}^2 - b_{13}^2 - b_{23}^2 + b_{123}^2, \\ b_I &= \langle \mathsf{B}\widehat{\tilde{\mathsf{B}}}I\rangle_0 = -2b_3b_{12} + 2b_2b_{13} - 2b_1b_{23} + 2b_0b_{123}. \end{aligned} \tag{74}$$

Because Equations (73) and (47) have the same shape (the concrete equations for b_S and b_I, of course, are different) we can make use of (48) with the purpose of lowering the order of the system. However, there arises an important difference: the determinant, $D = b_S^2 - b_I^2$, in $Cl_{2,1}$ is not always positive. It may happen that for some B, the MV determinant may become negative, $D < 0$. In such a case, the solution set becomes empty. The other particularity is that in the solution (50), instead of single sign ($-\sqrt{D}$), we have to take into account both signs, i.e., $\pm\sqrt{D}$, what doubles the number of possible solutions in the case $D > 0$,

$$\begin{cases} \left(t_{1,2,3,4} = \frac{1}{4}\left(b_{123} \pm \frac{1}{\sqrt{2}}\sqrt{b_S \pm \sqrt{D}}\right),\ T_{1,2,3,4} = \frac{1}{4}\left(\frac{\pm b_I}{\sqrt{2}\sqrt{b_S \pm \sqrt{D}}} + b_0\right)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad b_S \pm \sqrt{D} > 0, \\ \left(t_{1,2} = \frac{1}{4}b_{123},\ T_{1,2} = \frac{1}{4}(\pm\sqrt{b_S} + b_0)\right), \quad \text{if} \quad b_S \pm \sqrt{D} = 0 \text{ and } b_S > 0. \end{cases} \tag{75}$$

Here, again, the sign of t_i must be taken in all possible combinations, and the sign of T must follow the same upper–lower sign position as in t_i. The condition $b_S \pm \sqrt{D} = 0$ implies that $b_I = 0$. Since we already have four sign combinations in the solution for s, S (as in (51)), we end up with 16 different square roots of MV in a generic case of $Cl_{2,1}$.

5.2. *The Special Case* $s^2 - S^2 = 0$

The analysis again closely follows the $Cl_{0,3}$ case, except that now, different signs appear in expressions.

5.2.1. The Subcase $s = S \neq 0$

Now, the coefficients satisfy the conditions $b_1 = b_{23}$, $b_2 = -b_{13}$, $b_3 = -b_{12}$, which allows to eliminate the singularity at $s = S$. As a result, the system of Equations (68)–(70) has a special solution,

$$v_1 = \frac{b_1}{2s} - V_1, \quad v_2 = \frac{b_2}{2s} - V_2, \quad v_3 = \frac{b_3}{2s} - V_3, \tag{76}$$

which coincides with the same solution for $Cl_{0,3}$ (see Equation (52)). Thus, after similar calculations, one finds that Equation (55) becomes

$$\begin{aligned} V_1 =& \frac{\sqrt{2}}{8s}\Big(\sqrt{2}b_1 \pm \big(8s^2\big(b_0 - b_{123} + 4(-V_2^2 + V_3^2)\big) + 16s(b_2V_2 - b_3V_3) \\ & - b_0^2 + 2b_1^2 - b_{123}^2 + b_S\big)^{1/2}\Big). \end{aligned} \tag{77}$$

The coefficients s_1 and s_2 are similar to (57), except that now, we have to take into account all sign combinations in inner square root,

$$s_{1,2} = \pm \tfrac{1}{2\sqrt{2}} \sqrt{\pm\sqrt{2b_S - (b_0 - b_{123})^2} + b_0 + b_{123}}\,. \tag{78}$$

The above listed formulas solve the square root problem in the case $s = S \neq 0$.

5.2.2. The Subcase $s = -S \neq 0$

The only formulas which differ from $Cl_{0,3}$ algebra are connected with the coefficient compatibility condition $b_1 = -b_{23}$, $b_2 = b_{13}$, $b_3 = b_{12}$. Now, the coefficients must be replaced by

$$\begin{aligned} V_1 =& \frac{\sqrt{2}}{8s}\Big(-\sqrt{2}b_1 \pm \big(8s^2\big(b_0 + b_{123} + 4(-V_2^2 + V_3^2)\big) \\ & - 16s(b_2V_2 - b_3V_3) - b_0^2 + 2b_1^2 - b_{123}^2 + b_S\big)^{1/2}\Big), \end{aligned} \tag{79}$$

$$s_{1,2} = \pm \tfrac{1}{2\sqrt{2}} \sqrt{\pm\sqrt{2b_S - (b_0 + b_{123})^2} + b_0 - b_{123}}\,. \tag{80}$$

The remaining formulas which are needed for final answer exactly match the formulas in the corresponding subcase of $Cl_{0,3}$ algebra.

5.2.3. The Subcase $s = S = 0$

The only distinct formulas from $Cl_{0,3}$ are listed below:

$$\begin{aligned} &v_1 = \pm\frac{c_1}{\sqrt{2}}, \qquad V_1 = \pm\frac{1}{c_1}\frac{b_{123} + 2(-v_2V_2 + v_3V_3)}{\sqrt{2}}, \qquad \text{where} \\ &c_1 = \Big(\pm\sqrt{\big(b_0 - v_2^2 + v_3^2 - V_2^2 + V_3^2\big)^2 - (b_{123} + 2(-v_2V_2 + v_3V_3))^2} \\ &\qquad + b_0 - v_2^2 + v_3^2 - V_2^2 + V_3^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{81}$$

This ends the investigation of the square root formulas for all real 3D CAs.

5.3. Examples for $Cl_{2,1}$

Example 8. *The regular case.*

First, we shall show that MV $\mathsf{B} = \mathbf{e}_1 - 2\mathbf{e}_{23}$ has no real square roots. Indeed, we have $b_S = -5$, $b_I = 4$ and $D = b_S^2 - b_I^2 = (3)^2$. As a result, the expression under square root in (75), namely $b_S \pm \sqrt{D} = -5 \pm 3$, is always negative and therefore, there are no real-valued solutions.

Next, we shall calculate the roots of $\mathsf{B} = 2 + \mathbf{e}_1 + \mathbf{e}_{13}$. The values of b_I and b_S are 0 and 2, respectively. The determinant of the MV is positive, $D = 4 > 0$. The Equation (75) give four real values for pairs: $(T_1, t_1) = \left(\frac{1}{4}(2-\sqrt{2}), 0\right)$, $(T_2, t_2) = \left(\frac{1}{4}(2+\sqrt{2}), 0\right)$, $(T_3, t_3) = \left(\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$, and $(T_4, t_4) = \left(\frac{1}{2}, \frac{1}{2\sqrt{2}}\right)$. After insertion into (51), 16 pairs of scalars (s_i, S_i) are found:

$$
\begin{array}{lll}
(s_1 = 0, S_1 = -\frac{c_2}{\sqrt{2}}), & (s_2 = 0, S_2 = \frac{c_2}{\sqrt{2}}), & (s_3 = -\frac{c_2}{\sqrt{2}}, S_3 = 0), \\
(s_4 = \frac{c_2}{\sqrt{2}}, S_4 = 0), & (s_5 = 0, S_5 = -\frac{c_1}{\sqrt{2}}), & (s_6 = 0, S_6 = \frac{c_1}{\sqrt{2}}), \\
(s_7 = -\frac{c_1}{\sqrt{2}}, S_7 = 0), & (s_8 = \frac{c_1}{\sqrt{2}}, S_8 = 0), & (s_9 = -\frac{c_2}{2}, S_9 = \frac{c_1}{2}), \\
(s_{10} = \frac{c_2}{2}, S_{10} = -\frac{c_1}{2}), & (s_{11} = -\frac{c_1}{2}, S_{11} = \frac{1}{\sqrt{2}c_1}), & (s_{12} = \frac{c_1}{2}, S_{12} = -\frac{1}{\sqrt{2}c_1}), \\
(s_{13} = -\frac{c_2}{2}, S_{13} = -\frac{c_1}{2}), & (s_{14} = \frac{c_2}{2}, S_{14} = \frac{c_1}{2}), & (s_{15} = -\frac{c_1}{2}, S_{15} = -\frac{1}{\sqrt{2}c_1}), \\
(s_{16} = \frac{c_1}{2}, S_{16} = \frac{1}{\sqrt{2}c_1}), & &
\end{array}
$$

where $c_1 = \sqrt{2+\sqrt{2}}$ and $c_2 = \sqrt{2-\sqrt{2}}$. After substitution of (s_i, S_i) into Equation (71) and then into Equation (5), we obtain 16 roots $\mathsf{A}_{i,j} = \pm\sqrt{2 + \mathbf{e}_1 + \mathbf{e}_{13}}$:

$$
\begin{aligned}
\mathsf{A}_{1,2} &= \pm\tfrac{1}{2}\big(c_1\mathbf{e}_2 - c_1\mathbf{e}_{23} - \sqrt{2}c_2\mathbf{e}_{123}\big), \\
\mathsf{A}_{3,4} &= \pm\tfrac{1}{\sqrt{2}}\big(-c_1^{-1}\mathbf{e}_2 + c_1^{-1}\mathbf{e}_{23} + c_1\mathbf{e}_{123}\big), \\
\mathsf{A}_{5,6} &= \pm\tfrac{1}{2}\big(\sqrt{2}c_2 + c_1\mathbf{e}_1 + c_1\mathbf{e}_{13}\big), \\
\mathsf{A}_{7,8} &= \pm\tfrac{1}{\sqrt{2}}\big(c_1 + c_1^{-1}\mathbf{e}_1 + c_1^{-1}\mathbf{e}_{13}\big), \\
\mathsf{A}_{9,10} &= \pm\tfrac{1}{2\sqrt{2}}\big(\sqrt{2}c_2 - c_2\mathbf{e}_1 - c_1\mathbf{e}_2 - c_2\mathbf{e}_{13} + c_1\mathbf{e}_{23} + \sqrt{2}c_1\mathbf{e}_{123}\big), \\
A_{11,12} &= \pm\tfrac{1}{2\sqrt{2}}\big(\sqrt{2}c_1 + c_1\mathbf{e}_1 - c_2\mathbf{e}_2 + c_1\mathbf{e}_{13} + c_2\mathbf{e}_{23} - 2c_1^{-1}\mathbf{e}_{123}\big), \\
\mathsf{A}_{13,14} &= \pm\tfrac{1}{2\sqrt{2}}\big(\sqrt{2}c_1 + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_1\mathbf{e}_{13} - c_2\mathbf{e}_{23} + 2c_1^{-1}\mathbf{e}_{123}\big), \\
\mathsf{A}_{15,16} &= \pm\tfrac{1}{2\sqrt{2}}\big(-\sqrt{2}c_2 + c_2\mathbf{e}_1 - c_1\mathbf{e}_2 + c_2\mathbf{e}_{13} + c_1\mathbf{e}_{23} + \sqrt{2}c_1\mathbf{e}_{123}\big).
\end{aligned}
\tag{82}
$$

In the end, it is worth noting that the necessary (but not sufficient) condition for a square root of MV B to exist in real Clifford algebras $Cl_{p,q}$ requires the positivity of the multivector determinant $\det(\mathsf{B})$ [19,20]. Indeed, if the MV A exists and $\mathsf{AA} = \mathsf{B}$, then the determinant of both sides gives $\det(\mathsf{A})\det(\mathsf{A}) = \det(\mathsf{B})$, where we have used the multiplicative property of the determinant [21]. Since the determinant of A in real CAs is a real quantity, the condition can be satisfied if and only if $\det(\mathsf{B}) \geq 0$. This is in agreement with explicit formulas for $n \leq 3$.

6. Conclusions

First, we have shown analytically that the square root of general MV in $n = p + q \leq 3$ Clifford algebras (CAs) can be expressed in radicals and have provided a detailed analysis and formulas to accomplish the task. For a general MV, the computation algorithm is rather complicated, where many conditions are controlled by plus/minus signs. Our first

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paper [12] was limited to the roots of individual grades, where it is possible to write down explicit formulas in a coordinate-free form.

Second, the paper shows that MV roots may be isolated (up to 16 roots in case of $Cl_{2,1}$) and/or continuous, or conversely, there may be no roots at all. Thus, the MV algebras may also accommodate number-free parameters that bring in a continuum of roots on respective parameter hypersurface.

Third, the described algorithm was implemented in the *Mathematica* system [14] and applied in checking up algorithms with purely numerical root search. For this purpose, the *Mathematica* universal root search algorithm was realized and used in the system function **FindInstance[ ]** to check whether there are cases when the isolated root algorithm fails. No such cases were found. The only complication we encountered in the algorithm programming was that *Mathematica* symbolic zero detection algorithm **PossibleZeroQ[ ]**, in the more complicated cases, often switched over to numerical procedure to detect that the involved symbolic expression with nested radicals indeed represents zero. This is quite understandable, since it is well-known that a two-expression equivalence problem is, in general, undecidable.

Fourth, we found that for algebras $Cl_{3,0}$ and $Cl_{1,2}$, the square root solution in general is a union of the following sets: (1) when $s^2 \neq S^2$, the set consists of (up to) four different isolated roots; (2) when $s^2 = S^2 \neq 0$, the set consists of two isolated roots; and (3) when $s^2 = S^2 = 0$, there appears a continuum of roots that belong to four-or-smaller-dimensional parameter manifolds. Similar sets with minor modifications exist for remaining algebras $Cl_{2,1}$ and $Cl_{0,3}$ as well.

The proposed method is a step forward in solving general quadratic equations in CAs (examples are given in [12]), and may find new applications in the control and systems theory [22], partly because presented solutions uncover totally new properties of square roots of MVs; for example, the root multiplicity and appearance of free parameters in the roots. Due to intricacies of square root algorithms, it is recommended to perform all calculations with prepared-in-advance numerical/symbolic subroutines.

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Appendix A. Square Roots in $Cl_{1,0}$ and $Cl_{0,1}$ Algebras

In $Cl_{1,0}$ and $Cl_{0,1}$, the square root of general MV $\mathsf{B} = b_0 + b_1\mathbf{e}_1$ has the solution $\mathsf{A} = \sqrt{\mathsf{B}} = s + v_1\mathbf{e}_1$, where the real coefficients s and v_1 are

$$v_1 = \frac{b_1}{2s}; \quad s = \begin{cases} \pm\frac{1}{\sqrt{2}}\sqrt{b_0 - \sqrt{D}} & \text{if} \quad b_0 - \sqrt{D} > 0 \quad \text{and} \quad D \geq 0, \\ \pm\frac{1}{\sqrt{2}}\sqrt{b_0 + \sqrt{D}} & \text{if} \quad b_0 + \sqrt{D} > 0 \quad \text{and} \quad D \geq 0, \end{cases}$$

where

$$D = \begin{cases} b_0^2 - b_1^2, & \text{for } Cl_{1,0}, \\ b_0^2 + b_1^2, & \text{for } Cl_{0,1}. \end{cases}$$

When $s = 0$ (i.e., when $b_0 \pm \sqrt{D} = 0$) and $b_1 = 0$, the square roots are

$$\mathsf{A} = \begin{cases} \pm\sqrt{b_0}, & \text{if } \ b_1 = 0, \quad \text{for } Cl_{1,0}, \\ \pm\sqrt{-b_0}, & \text{if } \ b_1 = 0, \quad \text{for } Cl_{0,1}. \end{cases}$$

Note that the $Cl_{0,1}$ algebra is isomorphic to the algebra of complex numbers, so we know in advance that any MV in this algebra has two roots. The MV determinant for this algebra is positive definite $D = b_0^2 + b_1^2 \geq 0$ and represents the square of module of a complex number. We shall always assume that expressions under square roots are non-negative. For example, in this case, the square root can only exist when $D \geq 0$, and either $(b_0 - \sqrt{D}) \geq 0$ or $(b_0 + \sqrt{D}) \geq 0$. If these conditions cannot be satisfied, then square roots are absent.

Appendix B. Square Roots in $Cl_{2,0}$, $Cl_{1,1}$, and $Cl_{0,2}$ Algebras

Square root A of general MV $\mathsf{B} = b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_{12}$ in all three algebras is $\mathsf{A} = s + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + S\mathbf{e}_{12}$. The coefficients (s, S) are

$$\begin{cases} \left(s = \pm\frac{1}{\sqrt{2}}\sqrt{b_0 - \sqrt{D}}, S = \pm\frac{1}{\sqrt{2}}\frac{b_3}{\sqrt{b_0 - \sqrt{D}}}\right), & \text{if} \quad b_0 - \sqrt{D} > 0 \text{ and } D \geq 0, \\ \left(s = \pm\frac{1}{\sqrt{2}}\sqrt{b_0 + \sqrt{D}}, S = \pm\frac{1}{\sqrt{2}}\frac{b_3}{\sqrt{b_0 + \sqrt{D}}}\right), & \text{if} \quad b_0 + \sqrt{D} > 0 \text{ and } D \geq 0, \end{cases}$$

where the determinant of MV B is [19,20],

$$D = \begin{cases} b_0^2 - b_1^2 - b_2^2 + b_3^2, & \text{for } Cl_{2,0}, \\ b_0^2 - b_1^2 + b_2^2 - b_3^2, & \text{for } Cl_{1,1}, \\ b_0^2 + b_1^2 + b_2^2 + b_3^2, & \text{for } Cl_{0,2}. \end{cases}$$

The case $s \neq 0$. The coefficients $v_1, v_2 \in \mathsf{A}$ are then given by formulas

$$v_1 = \frac{b_1}{2s}, \quad v_2 = \frac{b_2}{2s}.$$

The case $s = 0$. When $b_0 - \sqrt{D} = 0$, or $b_0 - \sqrt{D} = 0$ and $b_1 = b_2 = b_3 = 0$, the coefficients v_1, v_2, and S are connected by the single equation $\pm v_1^2 \pm v_2^2 \pm b_0 \pm S^2 = 0$. Therefore, one can search the solution with respect to any of coefficients v_1, v_2, or S, and assume that the remaining two coefficients are the free parameters. For example, if we solve with respect to S, then the square root for each algebra is

$$\mathsf{A} = \begin{cases} v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \pm \sqrt{-b_0 + v_1^2 + v_2^2}\,\mathbf{e}_{12}, & \text{for } Cl_{2,0}, \quad \text{if} \quad b_1 = b_2 = b_3 = 0, \\ v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \pm \sqrt{b_0 - v_1^2 + v_2^2}\,\mathbf{e}_{12}, & \text{for } Cl_{1,1}, \quad \text{if} \quad b_1 = b_2 = b_3 = 0, \\ v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \pm \sqrt{-b_0 - v_1^2 - v_2^2}\,\mathbf{e}_{12}, & \text{for } Cl_{0,2}, \quad \text{if} \quad b_1 = b_2 = b_3 = 0. \end{cases}$$

Since the coefficient S is real, the roots exist only when the expressions under square root are positive. The algebra $Cl_{2,0}$ is isomorphic to $Cl_{1,1}$.

Example.
The square root of $\mathsf{B} = 6 + 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_{12}$ in various 2D algebras:

$$\mathsf{A} = \begin{cases} \pm \frac{1}{\sqrt{2(6+\sqrt{39})}}(6 + \sqrt{39} + 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_{12}) & Cl_{2,0}, \\ \pm \frac{1}{\sqrt{2}}(1 + 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_{12}) \text{ and } \pm \frac{1}{\sqrt{22}}(11 + 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_{12}) & Cl_{1,1}, \\ \pm \frac{1}{\sqrt{2(6+\sqrt{65})}}(6 + \sqrt{65} + 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_{12}) & Cl_{0,2}. \end{cases}$$

Note that in $Cl_{1,1}$, there are four roots.

Appendix C. Summary for $n = 3$ Algebras

In this section, we provide a summary of results for the two most often used CAs, $Cl_{3,0}$ and $Cl_{0,3}$. The summary is primarily intended for implementation purposes. By putting in some more efforts, they can also be formulated as theorems, constructive proof of which is described in the corresponding sections of the article. The notations common for both algebras:

$$\mathsf{B} = b_0 + b_1\mathbf{e}_2 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{23} + b_{13}\mathbf{e}_{13} + b_{23}\mathbf{e}_{23} + b_{123}I, \tag{A1}$$

$$\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I, \tag{A2}$$

$$b_S = \langle \mathsf{B}\widetilde{\widehat{\mathsf{B}}}\rangle_0, \qquad b_I = \langle \mathsf{B}\widetilde{\widehat{\mathsf{B}}}\mathbf{e}_{123}\rangle_0, \tag{A3}$$

Appendix C.1. Computation Flow in $Cl_{3,0}$

$Cl_{3,0}$ specific expressions:

$$\sqrt{D} = \sqrt{b_S^2 + b_I^2},$$
$$\mathrm{bSD} = -b_S + \sqrt{D}.$$

Computation-ready s and S expressions:

$$\begin{aligned} &F_1 : s = -\sqrt{-\mathrm{s2S2} + \mathrm{sSsq}}, && S = -(\mathrm{stS}/\sqrt{-\mathrm{s2S2} + \mathrm{sSsq}}); \\ &F_2 : s = \sqrt{-\mathrm{s2S2} + \mathrm{sSsq}}, && S = (\mathrm{stS}/\sqrt{-\mathrm{s2S2} + \mathrm{sSsq}}); \\ &F_3 : s = 0, && S = -\sqrt{2}\sqrt{\mathrm{s2S2}}; \\ &F_4 : s = 0, && S = \sqrt{2}\sqrt{\mathrm{s2S2}}; \\ &F_5 : s = -\sqrt{2}\sqrt{-\mathrm{s2S2}}, && S = 0; \\ &F_6 : s = \sqrt{2}\sqrt{-\mathrm{s2S2}}, && S = 0; \\ &F_7 : s = 0, && S = 0, \end{aligned} \tag{A4}$$

where $\mathrm{sSsq} = \sqrt{\mathrm{s2S2}^2 + \mathrm{stS}^2}$. Substitutions that are used for condition checks in (A5) and for computation of final s, S values in Formula (A4):

$$\begin{aligned} &T_1 : \mathrm{stS} = (\sqrt{\mathrm{bSD}}/\sqrt{2} + b_{123})/4, && \mathrm{s2S2} = ((\sqrt{2}b_I)/\sqrt{\mathrm{bSD}} - 2b_0)/8, \\ &T_2 : \mathrm{stS} = (-\sqrt{\mathrm{bSD}}/\sqrt{2} + b_{123})/4, && \mathrm{s2S2} = ((-\sqrt{2}b_I)/\sqrt{\mathrm{bSD}} - 2b_0)/8, \\ &T_3 : \mathrm{stS} = b_{123}/4, && \mathrm{s2S2} = (-\sqrt{b_S} - b_0)/4, \\ &T_4 : \mathrm{stS} = b_{123}/4, && \mathrm{s2S2} = (\sqrt{b_S} - b_0)/4. \end{aligned}$$

After T_i quantities are determined, we start checking all conditions on the right-hand side of the piecewise function:

$$s,S=\begin{cases} F_1|_{T_{1,2}}, & \text{bSD}>0\wedge(-\text{s2S2}+\text{sSsq})|_{T_{1,2}}>0\wedge\text{stS}|_{T_{1,2}}\neq 0,\\ F_2|_{T_{1,2}}, & \text{bSD}>0\wedge(-\text{s2S2}+\text{sSsq})|_{T_{1,2}}>0\wedge\text{stS}|_{T_{1,2}}\neq 0,\\ F_3|_{T_{1,2}}, & \text{bSD}>0\wedge\text{s2S2}|_{T_{1,2}}>0\wedge\text{stS}|_{T_{1,2}}=0,\\ F_4|_{T_{1,2}}, & \text{bSD}>0\wedge\text{s2S2}|_{T_{1,2}}>0\wedge\text{stS}|_{T_{1,2}}=0,\\ F_5|_{T_{1,2}}, & \text{bSD}>0\wedge-\text{s2S2}|_{T_{1,2}}>0\wedge\text{stS}|_{T_{1,2}}=0,\\ F_6|_{T_{1,2}}, & \text{bSD}>0\wedge-\text{s2S2}|_{T_{1,2}}>0\wedge\text{stS}|_{T_{1,2}}=0,\\ F_1|_{T_{3,4}}, & \text{bSD}=0\wedge(-\text{s2S2}+\text{sSsq})|_{T_{3,4}}>0\wedge b_S\geq 0\wedge\text{stS}|_{T_{3,4}}\neq 0,\\ F_2|_{T_{3,4}}, & \text{bSD}=0\wedge(-\text{s2S2}+\text{sSsq})|_{T_{3,4}}>0\wedge b_S\geq 0\wedge\text{stS}|_{T_{3,4}}\neq 0,\\ F_3|_{T_{3,4}}, & \text{bSD}=0\wedge\text{s2S2}|_{T_{3,4}}>0\wedge b_S\geq 0\wedge\text{stS}|_{T_{3,4}}=0,\\ F_4|_{T_{3,4}}, & \text{bSD}=0\wedge\text{s2S2}|_{T_{3,4}}>0\wedge b_S\geq 0\wedge\text{stS}|_{\text{T}_{3,4}}=0,\\ F_5|_{T_{3,4}}, & \text{bSD}=0\wedge-\text{s2S2}|_{T_{3,4}}>0\wedge b_S\geq 0\wedge\text{stS}|_{T_{3,4}}=0,\\ F_6|_{T_{3,4}}, & \text{bSD}=0\wedge-\text{s2S2}|_{T_{3,4}}>0\wedge b_S\geq 0\wedge\text{stS}|_{T_{3,4}}=0,\\ F_7, & b_1=b_2=b_3=b_{12}=b_{13}=b_{23}=b_S-b_0^2+b_{123}^2=b_I+2b_0b_{123}=0 \end{cases} \tag{A5}$$

The notation $|_{T_{1,2}}$ means that we test substitutions T_1 and T_2 independently, i.e., first, substitute T_1 values into (A5) conditions and if they are satisfied, select F_i expression for s,S and substitute T_1 values into the formula. Then, repeat the same procedure with T_2 values (a single entry represents two separate entries in the piecewise function (A5)). By checking combinations of all 25 conditions encoded in (A5), we establish all valid values of s,S coefficients. If $s\neq 0$ or $S\neq 0$ (cases F_1-F_6), then using (10) and (11), we compute $\mathbf{v}$ and $\mathbf{V}$. When $s=S=0$, we use (20) and solve any two coefficients, for example v_1 and V_1. The remaining four coefficients v_2,v_3,V_2,V_3 then become free (real) parameters, provided that the solved coefficients v_1 and V_1 can acquire real values. The determination of whether these real values exist is known to be a hard problem (especially for B with symbolic coefficients). It can be solved by using the famous quantifier elimination algorithm, the explanation of which is out of the scope of this article (our implementation uses *Mathematica* command **Resolve[ ]** for the task).

Lastly, substitute $s,S,\mathbf{v},\mathbf{V}$ into (A2) to obtain A, which then represents all possible square roots of B. In $Cl_{3,0}$ algebra, we can have 0, 2, or 4 isolated roots and 4D continuum of roots (in an exceptional case, continuum can turn into an isolated real root).

Appendix C.2. Computation Flow in $Cl_{0,3}$

$Cl_{0,3}$ specific notations:

$$\sqrt{D}=\sqrt{b_S^2-b_I^2},$$
$$\text{bSD}=b_S-\sqrt{D},$$

Formulas for s and S are as follows:

$$\begin{aligned}
&F1\ : s = -\sqrt{\mathrm{s2S2} - \mathrm{sSsq}}, && S = -(\mathrm{stS}/\sqrt{\mathrm{s2S2} - \mathrm{sSsq}}),\\
&F2\ : s = \sqrt{\mathrm{s2S2} - \mathrm{sSsq}}, && S = (\mathrm{stS}/\sqrt{\mathrm{s2S2} - \mathrm{sSsq}}),\\
&F3\ : s = -\sqrt{\mathrm{s2S2} + \mathrm{sSsq}}, && S = -(\mathrm{stS}/\sqrt{\mathrm{s2S2} + \mathrm{sSsq}}),\\
&F4\ : s = \sqrt{\mathrm{s2S2} + \mathrm{sSsq}}, && S = (\mathrm{stS}/\sqrt{\mathrm{s2S2} + \mathrm{sSsq}}),\\
&F5\ : s = 0, && S = -\sqrt{2}\sqrt{\mathrm{s2S2}},\\
&F6\ : s = 0, && S = \sqrt{2}\sqrt{\mathrm{s2S2}},\\
&F7\ : s = -\sqrt{2}\sqrt{\mathrm{s2S2}}, && S = 0,\\
&F8\ : s = \sqrt{2}\sqrt{\mathrm{s2S2}}, && S = 0,\\
&F9\ : s = -\frac{1}{2\sqrt{2}}\sqrt{b_0 + b_{123} - \sqrt{2b_S - (b_0 - b_{123})^2}} && S = s,\\
&F10 : s = \frac{1}{2\sqrt{2}}\sqrt{b_0 + b_{123} - \sqrt{2b_S - (b_0 - b_{123})^2}} && S = s,\\
&F11 : s = -\frac{1}{2\sqrt{2}}\sqrt{b_0 + b_{123} + \sqrt{2b_S - (b_0 - b_{123})^2}} && S = s,\\
&F12 : s = \frac{1}{2\sqrt{2}}\sqrt{b_0 + b_{123} + \sqrt{2b_S - (b_0 - b_{123})^2}} && S = s,\\
&F13 : s = -\frac{1}{2\sqrt{2}}\sqrt{b_0 - b_{123} - \sqrt{2b_S - (b_0 + b_{123})^2}} && S = -s,\\
&F14 : s = \frac{1}{2\sqrt{2}}\sqrt{b_0 - b_{123} - \sqrt{2b_S - (b_0 + b_{123})^2}} && S = -s,\\
&F15 : s = -\frac{1}{2\sqrt{2}}\sqrt{b_0 - b_{123} + \sqrt{2b_S - (b_0 + b_{123})^2}} && S = -s,\\
&F16 : s = \frac{1}{2\sqrt{2}}\sqrt{b_0 - b_{123} + \sqrt{2b_S - (b_0 + b_{123})^2}} && S = -s,\\
&F17 : s = 0, && S = 0,
\end{aligned} \tag{A6}$$

where $\mathrm{sSsq} = \sqrt{\mathrm{s2S2}^2 - \mathrm{stS}^2}$. Substitutions that are used in condition checks (A7) and s, S Formula (A6) are

$$\begin{aligned}
&T_1 : \mathrm{stS} = (\sqrt{\mathrm{bSD}}/\sqrt{2} + b_{123})/4, && \mathrm{s2S2} = ((\sqrt{2}b_I)/\sqrt{\mathrm{bSD}} + 2b_0)/8,\\
&T_2 : \mathrm{stS} = (-\sqrt{\mathrm{bSD}}/\sqrt{2} + b_{123})/4, && \mathrm{s2S2} = ((-\sqrt{2}b_I)/\sqrt{\mathrm{bSD}} + 2b_0)/8,\\
&T_3 : \mathrm{stS} = b_{123}/4, && \mathrm{s2S2} = (-\sqrt{b_S} + b_0)/4,\\
&T_4 : \mathrm{stS} = b_{123}/4, && \mathrm{s2S2} = (\sqrt{b_S} + b_0)/4.
\end{aligned}$$

Once the above quantities are computed, we start checking all conditions on the right-hand side of the piecewise function:

$$s, S = \begin{cases} F_{1,2}|_{T_{1,2}}, & \text{bSD} > 0 \wedge (\text{s2S2} - \text{sSsq})|_{T_{1,2}} > 0 \wedge \text{s2S2}|_{T_{1,2}} \geq 0 \wedge \text{stS}|_{T_{1,2}} \neq 0 \wedge s^2 - S^2 \neq 0, \\ F_{3,4}|_{T_{1,2}}, & \text{bSD} > 0 \wedge (\text{s2S2} + \text{sSsq})|_{T_{1,2}} > 0 \wedge \text{s2S2}|_{T_{1,2}} \geq 0 \wedge \text{stS}|_{T_{1,2}} \neq 0 \wedge s^2 - S^2 \neq 0, \\ F_{1,2}|_{T_{3,4}}, & \text{bSD} = 0 \wedge (\text{s2S2} - \text{sSsq})|_{T_{3,4}} > 0 \wedge \text{s2S2}|_{T_{3,4}} \geq 0 \wedge \text{stS}|_{T_{3,4}} \neq 0 \wedge s^2 - S^2 \neq 0, \\ F_{3,4}|_{T_{3,4}}, & \text{bSD} = 0 \wedge (\text{s2S2} + \text{sSsq})|_{T_{3,4}} > 0 \wedge \text{s2S2}|_{T_{3,4}} \geq 0 \wedge \text{stS}|_{T_{3,4}} \neq 0 \wedge s^2 - S^2 \neq 0, \\ F_{5,6}|_{T_{1,2}}, & \text{bSD} > 0 \wedge (\text{s2S2} - \text{sSsq})|_{T_{1,2}} > 0 \wedge \text{s2S2}|_{T_{1,2}} > 0 \wedge \text{stS}|_{T_{1,2}} = 0 \wedge s^2 - S^2 \neq 0, \\ F_{7,8}|_{T_{1,2}}, & \text{bSD} > 0 \wedge (\text{s2S2} + \text{sSsq})|_{T_{1,2}} > 0 \wedge \text{s2S2}|_{T_{1,2}} > 0 \wedge \text{stS}|_{T_{1,2}} = 0 \wedge s^2 - S^2 \neq 0, \\ F_{5,6,7,8}|_{T_{3,4}}, & \text{bSD} = 0 \wedge b_S > 0 \wedge \text{s2S2}|_{T_{3,4}} > 0 \wedge \text{stS}|_{T_{3,4}} = 0 \wedge s^2 - S^2 \neq 0, \\ F_{9,10,11,12}, & \text{cond}_1 \wedge 2b_S - (b_0 - b_{123})^2 > 0 \wedge b_0 + b_{123} - \sqrt{2b_S - (b_0 - b_{123})^2} > 0, \\ F_{9,10,11,12}, & \text{cond}_1 \wedge 2b_S - (b_0 - b_{123})^2 > 0 \wedge b_0 + b_{123} + \sqrt{2b_S - (b_0 - b_{123})^2} > 0, \\ F_{13,14,15,16}, & \text{cond}_2 \wedge 2b_S - (b_0 + b_{123})^2 > 0 \wedge b_0 - b_{123} - \sqrt{2b_S - (b_0 + b_{123})^2} > 0, \\ F_{13,14,15,16}, & \text{cond}_2 \wedge 2b_S - (b_0 + b_{123})^2 > 0 \wedge b_0 - b_{123} + \sqrt{2b_S - (b_0 + b_{123})^2} > 0, \\ F_{17}, & b_1 = b_2 = b_3 = b_{12} = b_{13} = b_{23} = b_S - b_0^2 - b_{123}^2 = b_I - 2b_0 b_{123} = 0, \end{cases} \quad \text{(A7)}$$

where cond_1 denotes triple logical conjunction $((b_1 + b_{23} = 0(\wedge(b_2 - b_{13} = 0) \wedge (b_3 + b_{12} = 0))$ and cond_2 logical conjunction $((b_1 - b_{23} = 0) \wedge (b_2 + b_{13} = 0) \wedge (b_3 - b_{12} = 0))$. The $F_{i,k,\ldots}|_{T_{1,2}}$ is the abridgment of lines $F_i|_{T_{1,2}}$ and $F_k|_{T_{1,2}}$, etc. The notation $|_{T_{1,2}}$ means two independent lines with substitutions $|_{T_1}$ and $|_{T_2}$. Therefore, for example, $F_{5,6,7,8}|_{T_{3,4}}$ is equivalent to $4 \times 2 = 8$ piecewise function (represented as cases) entries. Note that in (A7), the condition list includes the post-selection rule $(s^2 - S^2) \neq 0$, which should be checked after values s and S were (successfully) computed.

By checking combinations of all 49 entries present in (A7), we establish all valid values of s, S coefficients. If $s^2 - S^2 \neq 0$ (first 7 entries in (A7), i.e., formulas $F_1 - F_8$), then use (45) for $\mathbf{v}$ and $\mathbf{V}$.

For s, S given by formulas $F_9 - F_{12}$, use (52) and the solutions

$$V_1 = (2b_1 + \sqrt{2}\sqrt{-b_S + b_0^2 + 2b_1^2 + b_{123}^2 + 16s(b_2V_2 + b_3V_3) - 8s^2(b_0 - b_{123} + 4(V_2^2 + V_3^2))})/(8s),$$
$$V_1 = (2b_1 - \sqrt{2}\sqrt{-b_S + b_0^2 + 2b_1^2 + b_{123}^2 + 16s(b_2V_2 + b_3V_3) - 8s^2(b_0 - b_{123} + 4(V_2^2 + V_3^2))})/(8s)$$

to find $\mathbf{v}, \mathbf{V}$ (coefficients V_2 and V_3 are free by our choice). The solution represents 2D parameter manifold which consist of two pieces.

In a similar way, for s, S given by formulas $F_{13} - F_{16}$, use (58) and the solutions

$$V_1 = (-2b_1 - \sqrt{2}\sqrt{-b_S + b_0^2 + 2b_1^2 + b_{123}^2 - 16s(b_2V_2 + b_3V_3) - 8s^2(b_0 + b_{123} + 4(V_2^2 + V_3^2))})/(8s),$$
$$V_1 = (-2b_1 + \sqrt{2}\sqrt{-b_S + b_0^2 + 2b_1^2 + b_{123}^2 - 16s(b_2V_2 + b_3V_3) - 8s^2(b_0 + b_{123} + 4(V_2^2 + V_3^2))})/(8s)$$

The solution represents 2D parameter manifold which consist of two pieces, where parameters V_2 and V_3 are made free by our choice.

For F_{17}, case $s = S = 0$, we again have 4D parameter solution manifold, which is obtained by solving any two parameters from (61), for example, the solution (62).

Lastly, substitute $s, S, \mathbf{v}, \mathbf{V}$ into (A2) to obtain A, which then represents all possible square roots of B. In $Cl_{0,3}$ algebra, we can have 0, 2, or 4 isolated roots and 2D or 4D continuum of roots and/or their mixture.

Since Clifford algebra $Cl_{2,1}$ is rarely used (for this algebra, a piecewise function has 81 entries) and the algebra $Cl_{1,2}$ is isomorphic to $Cl_{3,0}$, we refer a reader to our implementation [14], from which all summary formulas were extracted.

Appendix D. Determinant of Multivector

The characteristic polynomial of MV in arbitrary Clifford algebra $Cl_{p,q}$ can be computed using the recursive Faddeev–LeVerrier–Souriau algorithm [19], where each recursion step produces one of the coefficients $C_{(k)}(\mathsf{A})$ of the polynomial $\chi_{\mathsf{A}}(\lambda) = \sum_{k=0}^{d} C_{(d-k)}(\mathsf{A})\,\lambda^k$. The first recursion gives $C_{(1)}(\mathsf{A})$. Each subsequent step produces the coefficient $C_{(k)}(\mathsf{A}) = \frac{d}{k}\langle \mathsf{A}_{(k)}\rangle_0$ and a new MV $\mathsf{A}_{(k+1)} = \mathsf{A}\big(\mathsf{A}_{(k)} - C_{(k)}(\mathsf{A})\big)$ according to the rules

$$
\begin{aligned}
\mathsf{A}_{(1)} = \mathsf{A} \quad &\rightarrow \quad C_{(1)}(\mathsf{A}) = \tfrac{d}{1}\langle \mathsf{A}_{(1)}\rangle_0,\\
\mathsf{A}_{(2)} = \mathsf{A}\big(\mathsf{A}_{(1)} - C_{(1)}(\mathsf{A})\big) \quad &\rightarrow \quad C_{(2)}(\mathsf{A}) = \tfrac{d}{2}\langle \mathsf{A}_{(2)}\rangle_0,\\
&\ \vdots\\
\mathsf{A}_{(d)} = \mathsf{A}\big(\mathsf{A}_{(d-1)} - C_{(d-1)}(\mathsf{A})\big) \quad &\rightarrow \quad C_{(d)}(\mathsf{A}) = \tfrac{d}{d}\langle \mathsf{A}_{(d)}\rangle_0.
\end{aligned}
\tag{A8}
$$

The last step of this procedure returns the determinant of MV with opposite sign: $-\det(\mathsf{A}) = \mathsf{A}_{(d)} = C_{(d)}(\mathsf{A}) = \mathsf{A}\big(\mathsf{A}_{(d-1)} - C_{(d-1)}(\mathsf{A})\big)$. For low-dimensional $n \leq 7$ algebras, an MV determinant can be computed using explicit involutions. In particular, for $n = 1,2$, the determinant of MV can be computed [20] as $\det(\mathsf{A}) = \mathsf{A}\bar{\mathsf{A}}$ and for $n = 3,4$, as $\det(\mathsf{A}) = \frac{1}{3}\big(\mathsf{A}\mathsf{A}\overline{\mathsf{A}\mathsf{A}} + 2\mathsf{A}\overline{\bar{\mathsf{A}}\bar{\bar{\mathsf{A}}}\bar{\mathsf{A}}}\big)$, where the overbar denotes a negation of all grades except of the scalar, $\overline{\mathsf{A}} := 2\langle \mathsf{A}\rangle_0 - \mathsf{A}$. The quantity $\sqrt{D}$, which enters square root formulas, can be identified as a square of the determinant norm $|\mathsf{A}| = \big(\mathrm{abs}(\det(\mathsf{A}))\big)^{1/k} \geq 0$, with $k = 2^{\lceil n/2\rceil}$. In 3D algebras $n = 3$, we have $k = 2^{\lceil 3/2\rceil} = 2^2 = 4$, which is the degree of characteristic polynomial $\det(\mathsf{A})$. Since, in the 3D case, the determinant is $D = b_S^2 \pm b_I^2$, we have $\sqrt{D} = |\mathsf{A}|^2$. It generalizes well-known norms $\sqrt{\mathsf{A}\widetilde{\mathsf{A}}}$ (or $\sqrt{\mathsf{A}\widetilde{\widehat{\mathsf{A}}}}$) for blades. For example, the determinant norm of MV of $Cl_{3,0}$ $\mathsf{A} = 1 + \mathbf{e}_{12}$ is $|\mathsf{A}| = \sqrt[4]{\mathrm{abs}(\det(\mathsf{A}))} = \sqrt{2}$, which may be identified with a module of a complex number. This is a generalization of $\sqrt{\mathsf{A}\widetilde{\mathsf{A}}} = \sqrt{(1+\mathbf{e}_{12})(1-\mathbf{e}_{12})} = \sqrt{2}$.

References

1. Grant, H.; Kleiner, I. *Turning Points in the History of Mathematics*; Springer: New York, NY, USA, 2015.
2. Cayley, A. On the extraction of square root of matrix of the third order. *Proc. R. Soc. Edinburgh* **1872**, *7*, 675–682. [CrossRef]
3. Higham, N.J. *Functions of Matrices (Theory and Computation)*; SIAM: Philadelphia, PA, USA, 2008.
4. Niven, I. The roots of quaternion. *Am. Mon.* **1942**, *49*, 386–388. [CrossRef]
5. Opfer, G. Niven's algorithm applied to the roots of the companion polynomial over $\mathbb{R}^4$ algebras. *Adv. Appl. Clifford Algebr.* **2017**, *27*, 2659–2675. [CrossRef]
6. Falcão, M.I.; Miranda, F.; Severino, R.; Soares, M.J. On the roots of coquaternions. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 97. [CrossRef]
7. Özdemir, M. The roots of a split quaternion. *Appl. Math. Lett.* **2009**, *22*, 258–263. [CrossRef]
8. Sangwine, S.J. Biquaternion (complex quaternion) roots of -1. *Adv. Appl. Clifford Algebr.* **2006**, *16*, 63–68. [CrossRef]
9. Hitzer, E.; Abłamowicz, R. Geometric roots of -1 in Clifford algebras $Cl_{p,q}$ with $p+q \leq 4$. *Adv. Appl. Clifford Algebr.* **2011**, *21*, 121–144. [CrossRef]
10. Hitzer, E.; Helmstetter, J.; Abłamowicz, R. Square roots of -1 in real Clifford algebras. In *Quaternion and Clifford-Fourier Transforms and Wavelets*; Hitzer, E., Sangwine, S.J., Eds.; Springer: Basel, Siwtzerland, 2013; pp. 123–153. [CrossRef]
11. Hitzer, E.; Sangwine, S.J. (Eds.) *Quaternion and Clifford-Fourier Transforms and Wavelets*; Springer: Basel, Siwtzerland, 2013.
12. Dargys, A.; Acus, A. Square root of a multivector in 3d Clifford algebras. *Nonlinear Anal. Model. Control* **2020**, *25*, 301–320. [CrossRef]
13. Prodanov, D. Computation of Minimal Polynomials and Multivector Inverses in Non-Degenerate Clifford Algebras. *Mathematics* **2025**, *13*, 1106. [CrossRef]

14. Acus, A.; Dargys, A. Geometric algebra Mathematica Package. Available online: https://github.com/ArturasAcus/GeometricAlgebra (accessed on 31 December 2025).
15. Doran, C.; Lasenby, A. *Geometric Algebra for Physicists*; Cambridge University Press: Cambridge, UK, 2003.
16. Lounesto, P. *Clifford Algebra and Spinors*; Cambridge University Press: Cambridge, UK, 1997.
17. Korn, G.A.; Korn, T.M. *Mathematical Handbook for Scientists and Engineers*; McGraw-Hill Book Company: New York, NY, USA, 1961.
18. Marchuk, N.G. Demonstration representation and tensor products of Clifford algebras, Trudy Matematicheskogo Instituta Imeni V.A. *Steklova* **2015**, *290*, 154–165.
19. Shirokov, D.S. On computing the determinant, other characteristic polynomial coefficients, and inverse in Clifford algebras of arbitrary dimension. *Comput. Appl. Math.* **2021**, *40*, 29. [CrossRef]
20. Acus, A.; Dargys, A. *Calculation of the Exponential in Arbitrary* $Cl_{p,q}$ *Clifford Algebra*; Hitzer, E., Papagiannakis, C., Vasik, P., Eds.; Empowering Novel Geometric Algebra for Graphics and Engineering (ENGAGE 2022, Proceedings); Springer Nature: Cham, Switzerland, 2023; pp. 16–27. [CrossRef]
21. Lundholm, D. Geometric (Clifford) algebra and its applications. *arXiv* **2006**, arXiv:math/0605280. [CrossRef]
22. Abou-Kandil, H.; Freiling, G.; Ionescu, V.; Jank, G. Matrix Riccati Equations. In *Control and Systems Theory*; Birkhäuser Verlag: Basel, Switzerland, 2003. [CrossRef]