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A STOCHASTIC PROCESS DEFINED VIA THE RANDOM PERMUTATION DIVISORS

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Abstract

The normalised partial sums of values of a nonnegative multiplicative function over divisors with appropriately restricted lengths of a random permutation from the symmetric group define trajectories of a stochastic process. We prove a functional limit theorem in the Skorokhod space when the permutations are drawn uniformly at random. Furthermore, we show that the paths of the limit process almost surely belong to the space of continuous functions on the unit interval and, exploiting results from number-theoretic papers, we obtain rather complex formulas for the limits of joint power moments of the process.

Keywords: Symmetric group; Ewens probability; beta distribution; functional limit theorem; Skorokhod space

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1. Introduction and result

Stochastic processes appear in various constructions based upon permutations σ drawn at random from the symmetric group $\mathbb{S}_n$. Sometimes their distribution limit as $n \to \infty$ is Brownian motion (see [2, 13]); in other cases, it is some other process with independent or dependent increments (see, e.g., [3, 4]). The Poisson–Dirichlet process has received greater attention [1, Sections 5.5, 5.7, and 8.2]. In the present paper, we propose a new type of process construction based upon permutation divisors and prove a functional limit theorem in the Skorokhod space $(\mathbb{D}, \mathcal{D})$. Here, $\mathbb{D} = \mathbb{D}[0, 1]$ denotes the space of right-continuous functions on $[0, 1]$ that have left-hand limits, endowed with the Skorokhod topology, and $\mathcal{D}$ represents the Borel sigma-algebra over $\mathbb{D}$ [11, Chapter 3]. Let $\mathbb{C} = \mathbb{C}[0, 1]$ be the subset of $\mathbb{D}$ comprising continuous functions. The present paper has been highly influenced by progress in probabilistic number theory; in particular, the problem statement goes back to [30]. Therefore, it is worth taking a wider look at the prehistory.

In 1979 [14] established the following law for the natural divisors d of a uniformly drawn random natural number $m \in [n] := \{1, \dots, n\}$. Namely, it was proved that

$$\frac{1}{n}\sum_{m\le n}\Big(\sum_{d|m}1\Big)^{-1}\Big(\sum_{\substack{d|m\\ \log d\le t\log n}}1\Big) = \frac{2}{\pi}\arcsin\sqrt{t} + O\big((\log n)^{-1/2}\big) \tag{1}$$

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uniformly in $0 \le t \le 1$ and $n \ge 2$. Many generalisations of the last relation, nowadays called the DDT theorem, have appeared since then. In particular, [9] dealt with the ratio

$$T_n(m, t; g) := \left(\sum_{d|m} g(d) \right)^{-1} \sum_{\substack{d|m \\ d \le n^t}} g(d), \quad 0 \le t \le 1,$$

where g is a nonnegative multiplicative arithmetic function taking a fixed value $g(p) = \vartheta > 0$ for all primes p and arbitrary values $g(p^s)$ if $s \ge 2$. It was shown that

$$\frac{1}{n} \sum_{m \le n} T_n(m, t; g) = B(t; \theta, 1-\theta) + O\big((\log n)^{-\min\{\theta, 1-\theta\}}\big), \quad \theta := \vartheta/(1+\vartheta). \tag{2}$$

Here,

$$B(t; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^t \frac{\mathrm{d}v}{v^a (1-v)^b}, \quad 0 < a, \ b < 1,$$

and $\Gamma(\cdot)$ denotes Euler's gamma function. For a concise bibliography of more recent results, we refer to [8].

Long ago, in a personal discussion with the author, G. Tenenbaum asked if (1) relates to the so-called invariance principle where the arcsine law appears as the distribution of some functionals on the Wiener process. Actually, in the earlier article [15] announcing (1), an attempt to relate it to a random walk problem can be seen. However, in [30], a new stochastic process lying behind the formulas (1) and (2) was revealed by proving a functional limit theorem. More specifically, the sequence $T_n := T_n(m, t; g)$ was treated as a random process in the Skorokhod space $(\mathbb{D}, \mathcal{D})$ under the uniform probability measure on the set $[n]$. It was proved that T_n converges in distribution as $n \to \infty$ to a limit process depending on ϑ. In [34], Tenenbaum relaxed the condition $g(p) = \vartheta$ for primes p and, in addition, showed that the limit trajectories belong to the space $\mathbb{C}$. In the following, we will return in more detail to the process that appears.

Recent decades have been marked by numerous works intertwining probabilistic number theory and random permutations (see, e.g., [1]). Reflecting this, and presenting new results, we will use the following notation and definitions. A permutation $\sigma \in \mathbb{S}_n$ is a one-to-one (bijective) mapping $\sigma : [n] \to [n]$. It can be represented by the table

$$\sigma = \begin{pmatrix} 1 \, 2 & \cdots & n \\ i_1 \, i_2 & \cdots & i_n \end{pmatrix},$$

where $\sigma(r) = i_r$ for $1 \le r \le n$, or by the labelled directed graph $G(\sigma)$ with vertex set $V(\sigma) = [n]$ and edges $r \to i_r$, where $1 \le r \le n$. Its components are oriented cycles. A typical cycle has a vertex set $V(\varkappa) = \{k_1, \ldots, k_j\} \subset [n]$ defined by

$$k_1 \xrightarrow{\sigma} k_2 \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} k_j \xrightarrow{\sigma} k_1.$$

Using mapping multiplication, we obtain the unique (up to the order of factors) decomposition of σ into cycles $\varkappa_i$ on pairwise disjoint subsets $V(\varkappa_i)$, namely,

$$\sigma = \varkappa_1 \cdots \varkappa_w. \tag{3}$$

Here and in what follows, $w = w(\sigma)$ denotes the number of cycles and $[n] = V(\varkappa_1) \cup \cdots \cup V(\varkappa_w)$. Let us also introduce the empty permutation $\varnothing$ and $\mathbb{S}_0 = \{\varnothing\}$.

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A one-to-one mapping $\delta\colon V(\delta)\to V(\delta)$ with $V(\delta)\subset[n]$ is a *divisor* of σ if $\sigma|_{V(\delta)}=\delta$. The domain $V(\delta)$ (the vertex set in the subgraph $G(\delta)\subset G(\sigma)$) is a *fixed set* of σ, that is, it satisfies the relation $\sigma(V(\delta))=V(\delta)$. The empty permutation $\varnothing$ will be considered a divisor of each $\sigma\in\mathbb{S}_n$. A divisor is uniquely determined by an appropriate partial product of cycles in (3), thus each $\sigma\in\mathbb{S}_n$ has $2^{w(\sigma)}$ divisors. The cycles are analogues of primes in number theory. Following the latter, for a divisor δ of σ, we will write $\delta|\sigma$ and $\sigma=\delta\tau$, where τ is another divisor acting on $[n]\setminus V(\delta)$ and determined by the cycles from (3) not included in δ. The notation $\tau=\sigma/\delta$ will also be used where convenient.

Let us define some numerical parameters. If $k_j(\delta)$ denotes the number of cycles in δ of length j and $1\le j\le n$, then $\bar{k}(\delta):=(k_1(\delta),\dots,k_n(\delta))$, $\delta|\sigma$, will be called the *cycle vector* of δ. The zero vector corresponds to the empty permutation $\delta=\varnothing$. We introduce the notation $\ell(\bar{s}):=1s_1+\cdots+ns_n$ for a vector $\bar{s}=(s_1,\dots,s_n)\in\mathbb{Z}_+^n$. Throughout the paper, the cardinality $\#V(\delta)=\ell(\bar{k}(\delta))=:L(\delta)$ will be called the *length* of δ. Let ν_n denote the uniform (Haar) probability measure on $\mathbb{S}_n$. The facts about uniformly sampled permutations that we use can be found in [1].

Construction of the processes we are interested in is based on the real-valued *multiplicative functions* q defined on $\delta|\sigma$ where $\sigma\in\mathbb{S}_n$. Adopting the definition of a multiplicative function on partitions [35, Section 2], we begin with a family of functions $q_j\colon\mathbb{Z}_+\to\mathbb{R}$, $j\in\mathbb{N}$, such that $q_j(0)=1$ for every $j\in\mathbb{N}$, and put

$$q(\delta):=\prod_{j\le n}q_j(k_j(\delta)),\qquad \delta|\sigma,\ \sigma\in\mathbb{S}_n,\tag{4}$$

agreeing on $q(\varnothing)=1$. We say that $q_j(s)$, $0\le s\le n$, is the q function value prescribed to an s-subset of cycles of length j in a permutation divisor δ. If $q_j(k)=q_j(1)^k$ for each $j\le n$ and $k\ge 0$, where $0^0:=1$, the function q will be called *completely multiplicative*; its definition dates back to [25]. Note that the multiplicative functions are *structure dependent*, that is, their values $q(\delta)$ depend only on the vector $\bar{k}(\delta)$, irrespective of the numbers in $V(\delta)$. In the following, the multiplicative functions f, h, g, and q will have expressions as in (4) with $f_j(\cdot)$, $h_j(\cdot)$, $g_j(\cdot)$, and $q_j(\cdot)$, respectively. Let $\mathcal{M}$ and $\mathcal{M}_{\mathrm{c}}$ denote the classes of multiplicative and completely multiplicative functions. Given $g,h\in\mathcal{M}$, we introduce the convolution

$$q(\sigma):=\sum_{\sigma=\delta\tau}g(\delta)h(\tau)=\sum_{\delta|\sigma}g(\delta)h(\sigma/\delta).$$

Here, the first summation is over the ordered decompositions of σ into the product of divisors. Observe that the latter implies

$$q_j(k)=\sum_{s=0}^{k}\binom{k}{s}g_j(s)h_j(k-s),\qquad j\le n.\tag{5}$$

Indeed, for each $j\le n$, any $\delta|\sigma$ has an s-subset, $0\le s\le k_j(\sigma)=:k$, of the cycles of length j from σ. The subset can be chosen in $\binom{k}{s}$ ways and, moreover, the values of g on these subsets coincide, as do the values of h on the $(k-s)$ subsets of the remaining cycles of the same length. Note also that the classes $\mathcal{M}$ and $\mathcal{M}_{\mathrm{c}}$ are closed under convolution. The recalled toolkit will be used in what follows.

Given a nonnegative $g\in\mathcal{M}$, we define the multiplicative function

$$f(\sigma):=\sum_{\delta|\sigma}g(\delta),\tag{6}$$

and the family of cumulative distribution functions supported by [0, 1],

$$X_n(\sigma, t) := X_n(\sigma, t; g) = \frac{1}{f(\sigma)} \sum_{\substack{\delta|\sigma \\ L(\delta)\le tn}} g(\delta), \quad \sigma \in \mathbb{S}_n,\ 0 \le t \le 1. \tag{7}$$

In particular,

$$X_n(\sigma, t; 1) = 2^{-w(\sigma)} \#\{M \subset \mathbb{N}_n : \#M \le tn,\ \sigma(M) = M\}, \quad \sigma \in \mathbb{S}_n,\ 0 \le t \le 1,$$

is the relative density of the σ-invariant subsets $M \subset \mathbb{N}_n$ with cardinality not exceeding tn. The recent result in [24] gives the following analogue of the DDT theorem:

$$\frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} X_n(\sigma, t; 1) = \frac{2}{\pi} \arcsin\sqrt{t} + O(n^{-1/2}), \quad 0 \le t \le 1.$$

The latter concerns the asymptotic behaviour as $n \to \infty$ of the averaged distribution of the random variable $L(\delta)/n$ when δ is drawn uniformly from the set of divisors of σ. A more general problem was explored in [10] in which the divisor $\delta|\sigma$ was taken according to the Ewens probability

$$\mathrm{Prob}_\vartheta(\{\delta\}) = \vartheta^{w(\delta)} \Bigg(\sum_{\delta|\sigma} \vartheta^{w(\delta)} \Bigg)^{-1} = \vartheta^{w(\delta)} (1 + \vartheta)^{-w(\sigma)}.$$

Here, $\vartheta > 0$ is a fixed constant and the last equality stems from (6) and (5). It was proved that

$$\frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} \mathrm{Prob}_\vartheta(\delta|\sigma : L(\delta) \le nt) = B(t; \theta, 1 - \theta) + O\big(n^{-\min\{\theta,1-\theta\}}\big), \tag{8}$$

where $\theta := \vartheta/(1 + \vartheta)$ and the constant in $O(\cdot)$ can depend on ϑ. Evidently, (8) corresponds to the number-theoretic result (2).

The exposed parallels between phenomena in number theory and combinatorics lead to the thought that an appropriate functional limit theorem is available for permutations. Let us denote by ν_n the uniform (Haar) probability measure on $\mathbb{S}_n$ and focus on the distribution $\mathbb{P}_n := \nu_n \cdot X_n^{-1}$ defined on $\mathcal{D}$ by

$$\mathbb{P}_n(A) = \nu_n \cdot X_n^{-1}(A) = \nu_n(\sigma \in \mathbb{S}_n : X_n(\sigma, \cdot) \in A), \quad A \in \mathcal{D}.$$

Theorem 1. *Let $\vartheta > 0$ be a constant and $g \in \mathcal{M}$. Assume that $g_j(1) = \vartheta > 0$ for all $j \ge 1$, and $g_j(k) \ge 0$ are arbitrary if $j \ge 1$ and $k \ge 2$. The sequence of distributions $\mathbb{P}_n$ converges weakly as $n \to \infty$ to a measure $\mathbb{P}$ supported by $\mathbb{C}$.*

In the proof of Proposition 2 below, we will see that the limit process X with distribution $\mathbb{P}$ on the sigma-algebra $\mathcal{D}$ coincides with that found in [30]. Starting from [34], it has subsequently been explored in many aspects. Let φ be the variable in $\mathbb{D}$ that is the space of trajectories, and let

$$E(l, \bar{t}) := \mathbb{E}\Bigg(\prod_{i\le l} X(t_i) \Bigg) = \int_{\mathbb{D}} \prod_{i\le l} \varphi(t_i)\, \mathrm{d}P(\varphi) \tag{9}$$

be the mixed moment of the process values $X(t_i)$, where $\bar{t} := (t_1, \ldots, t_l) \in [0, 1]^l$ and $l \in \mathbb{N}$. Set $\mathbb{E}_n$ to be the mean value with respect to the probability ν_n.

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Corollary 1. *Assume that the conditions of Theorem 1 are satisfied, and* $l \in \mathbb{N}$ *is fixed. Then*

$$\lim_{n\to\infty} \mathbb{E}_n\left(\prod_{i\le l} X_n(\sigma, t_i)\right) = E(l, \bar{t})$$

uniformly in $\bar{t} \in [0, 1]^l$.

Proof. Since $0 < X_n(t) \le 1$, based on the dominated convergence theorem, from Theorem 1 we obtain convergence of the mixed moments to the expression defined in (9). □

In the proof of Proposition 1, we will observe that X depends only on $g_j(1) = \vartheta$, regardless of the values $g_j(k)$ if $j \ge 1$ and $k \ge 2$. In other words, the limit process is the same as if X_n were defined via $g(\sigma) = \vartheta^{w(\sigma)}$ and $f(\sigma) = (1+\vartheta)^{w(\sigma)}$.

If $l = 1$, Corollary 1 recovers (8) without the remainder term estimate. Let us exploit the known expressions for the mixed moments, starting with the simple case.

Corollary 2. *Let* $E(l, \bar{t})$ *be as defined in (9). Assume that the conditions of Theorem 1 are satisfied,* $l = 2$, $\bar{t} = (t, t)$ *where* $0 \le t \le 1$, *and* $\theta = \vartheta/(1+\vartheta)$. *Then Corollary 1 holds with*

$$E(2, \bar{t}) = \begin{cases} I(t, 1-\theta) & \text{if } 0 \le t \le \frac{1}{2}, \\ I(1-t, \theta) + 2B(t; \theta, 1-\theta) - 1 & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

Here,

$$I(t, a) = \frac{2}{\Gamma(a^2)\Gamma(b^2)\Gamma^2(ab)} \int_0^t \frac{\mathrm{d}w}{w^{1-ab}} \int_0^{t-w} \frac{\mathrm{d}v}{v^{1-b^2}} \int_0^w \frac{u^{ab-1}\,\mathrm{d}u}{(1-w-v-u)^{1-a^2}}$$

if $0 \le t \le \frac{1}{2}$, $0 < a < 1$, *and* $b = 1 - a$.

Proof. Apply Corollary 1 and [7, Theorem 2.2]. That result states that the second moment of the number-theoretic process $T_n(m, t; 1)$ defined above equals

$$\frac{1}{n}\sum_{m\le n} T_n^2(m, t; 1) = E(2, \bar{t}) + \text{remainder}, \quad \bar{t} = (t, t),\ 0 \le t \le 1.$$

Since the limits of $T_n(m, t; 1)$ and $X_n(\sigma, t; 1)$, as well as the limits of their moments, coincide, we can exploit this. □

A few computer-drawn illustrations of $I_t'(t, a)$ are presented in [7]. Generalising [5] and [7], [12] succeeded in writing rather complex formulas for all $E(l, \bar{t})$ in (9). For the reader's convenience, we include them here.

Let $l \ge 1$ be fixed, $r = 2^l - 1$, $\bar{v} = (v_1, \ldots, v_r) \in [0, 1]^r$, and $\bar{t} = (t_1, \ldots, t_l) \in [0, 1]^l$. For a nonnegative integer m, introduce the base-2 digits $d_j(m) \in \{0, 1\}$ of m and the sum of digits $d(m) = d_0(m) + d_1(m) + \cdots$. Given $\bar{v} \in \mathbb{R}^r$, define the vector $\bar{u}(\bar{v}) = (u_1(v), \ldots, u_l(v)) \in \mathbb{R}^l$ by $u_j(v) := \sum_{1\le m\le r} d_j(m) v_m$, $j \le l$. For an arbitrary vector $\bar{u} = (u_1, \ldots, u_l) \in \mathbb{R}^l$, we will write $\bar{u} \le \bar{t}$ if $u_j \le t_j$ for each $j \le l$. Set $s(\bar{v}) = v_1 + \cdots + v_r$ and $\mathrm{d}\bar{v} = \mathrm{d}v_1 \cdots \mathrm{d}v_r$. Define the region $\Omega(\bar{t}) = \{\bar{v} \in [0, 1]^r : \bar{u}(\bar{v}) \le \bar{t},\ s(\bar{v}) \le 1\}$.

Corollary 3. *Assume that the conditions of Theorem 1 are satisfied,* $l \ge 1$, $\bar{t} = (t_1, \ldots, t_l) \in [0, 1]^l$, *and* $E(l, \bar{t})$ *is as defined in (9). Then Corollary 1 holds with*

$$E(l, \bar{t}) = \prod_{0\le m\le r} \Gamma\left(\vartheta^{d(m)}(1+\vartheta)^{-l}\right)^{-1} \int_{\Omega(\bar{t})} \prod_{1\le m\le r} v_m^{\vartheta^{d(m)}(1+\vartheta)^{-l}-1} (1 - s(\bar{v}))^{(1+\vartheta)^{-l}-1}\,\mathrm{d}\bar{v}.$$

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Proof. Apply Corollary 1 and [12, (1.7), p. 4]. □

In a similar manner, to the result in [24] holding in the case $\vartheta = 1$, we can add more facts concerning the limit process. Here are two examples.

Corollary 4. *Assume that the process X_n is defined in (7) with $\vartheta = 1$. Then*:

(i) *For all $0 < s < t < 1$, the increments $X_n(t) - X_n(s)$ converge in distribution to a discrete random variable whose values are dyadic rational numbers.*

(ii) *If $\varphi(t)$ is a trajectory of the limit process, then, P-almost surely, $\varphi'(t) = 0$ for every $0 < t < 1$.*

Proof. Claim (i) is just [20, Theorem 20]. Claim (ii) is formulated in [34, p. 16]. □

Note that the latter corollary does not exhaust all the possible applications of item (i) of the theorem from [32].

The main lemmata required are collected in the next section. The proof of Theorem 1, split into three parts, is presented in Section 3. Namely, Proposition 1 shows that the influence of short cycles is negligible and provides the possibility of reducing the process to a simpler case. Proposition 2 establishes the convergence of marginal laws of finite order. Proposition 3 verifies the tightness criteria for the sequence of measures $\{\mathbb{P}_n\}_{n=1}^{\infty}$, in fact in a stronger form than needed for the application of [11, Theorem 15.5].

2. Lemmata

Let us recall the classical formulas due to Cauchy and Goncharov concerning the distribution of the cycle vector and its components.

Lemma 1 *Let $\bar{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ and $n \geq 1$. Then*

$$\nu_n(\bar{k}(\sigma) = \bar{k}) = \mathbf{1}\{\ell(\bar{k}) = n\} \prod_{j \leq n} \frac{1}{j^{k_j} k_j!}.$$

If $j \leq n$, then

$$\nu_n(k_j(\sigma) = k) = \frac{1}{j^k k!} \sum_{0 \leq s \leq (n/j)-k} \frac{(-1)^s}{j^s s!} \leq \frac{1}{j^k k!}.$$

Furthermore, $\mathbb{E}_n k_j(\sigma) = 1/j$ and $\mathbb{E}_n k_j(\sigma)(k_j(\sigma) - 1) = \mathbf{1}\{j \leq n/2\}/j^2$ for each $j \leq n$.

Proof. See [1, (1.3), (1.7), and (1.5), pp. 11–12]. □

The next lemma gives an analogue of the Turán–Kubilius inequality, well known for additive number-theoretic functions. The result below corresponds to that obtained in [22] and refined in [23].

Lemma 2. *Let a_j, $1 \leq j \leq n$, be arbitrary real numbers. Then*

$$\frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \left(\sum_{j \leq n} a_j k_j(\sigma) - \sum_{j \leq n} \frac{a_j}{j} \right)^2 = \sum_{j \leq n} \frac{a_j^2}{j} - \sum_{\substack{i,j \leq n \\ i+j>n}} \frac{a_i a_j}{ij} \leq \frac{3}{2} \sum_{j \leq n} \frac{a_j^2}{j}.$$

Proof. See [21]. □

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Note that $\frac{3}{2}$ is the optimal constant if $n \geq 2$, and it can be substituted by 1 if $a_j \geq 0$.

The next lemma is an analogue of the number-theoretic result in [19].

Lemma 3. *If $q \in \mathcal{M}$ satisfies $0 \leq q_j(k) \leq 1$ for $jk \leq n$, then*

$$\mathbb{E}_n q(\sigma) \ll \exp\left\{\sum_{j \leq n} \frac{q_j(1)-1}{j}\right\}.$$

Proof. If $q \in \mathcal{M}_c$, the required estimate is known [28, Proposition 5] even with the exact asymptotic constant.

According to the definition of a multiplicative function, q takes the same value for all permutations having the cycle structure vector $\bar{k}$; therefore, by Lemma 1,

$$\mathbb{E}_m q(\sigma) = \sum_{\ell(\bar{k})=m} \prod_{j \leq m} \frac{q_j(k_j)}{j^{k_j} k_j!}, \quad 1 \leq m \leq n,$$

and $\mathbb{E}_0 q(\sigma) = 1$. This leads to the formal product expression for the generating series

$$Z(x; q) := \sum_{m=0}^{\infty} \mathbb{E}_m q(\sigma) x^m = \prod_{j=1}^{\infty} \left(1 + \sum_{k \geq 1} \frac{q_j(k) x^{jk}}{j^k k!}\right).$$

If $jk > n$, the values $q_j(k)$ are not present in the expressions for $\mathbb{E}_m q(\sigma)$ for $m \leq n$. We may assume them to be equal to zero. Thus, $Z(x,q)$ may be considered as a finite product of polynomials. Consequently,

$$\begin{aligned}\sum_{0 \leq m \leq n} \mathbb{E}_m q(\sigma) \leq \prod_{j \leq n} \left(1 + \sum_{k \leq n/j} \frac{q_j(k)}{j^k k!}\right) &= \exp\left\{\sum_{j \leq n} \frac{q_j(1)}{j}\right\} \prod_{j \leq n} \mathrm{e}^{-q_j(1)/j} \left(1 + \sum_{k \leq n/j} \frac{q_j(k)}{j^k k!}\right) \\ &\ll \exp\left\{\sum_{j \leq n} \frac{q_j(1)}{j}\right\}. \qquad (10)\end{aligned}$$

The proof that the product over $j \leq n$ is bounded by an absolute constant is straightforward; we omit it here.

Next, differentiating the generating series, we obtain the following equality:

$$xZ'(x, q) = \sum_{j \geq 1} \sum_{k \geq 1} \frac{q_j(k) x^{jk}}{j^{k-1}(k-1)!} Z(x, q^{(j)}) = \sum_{j \geq 1} \sum_{k \geq 1} \frac{q_j(k) x^{jk}}{j^{k-1}(k-1)!} \sum_{m=0}^{\infty} \mathbb{E}_m q^{(j)}(\sigma) x^m.$$

Here, $q^{(j)} = q^{(j)}(\sigma) \in \mathcal{M}$ are obtained from q by setting $q_j(k) = 0$ for all $k \geq 1$. Equalising the coefficients of x^n and applying the condition of the lemma, we have

$$n\mathbb{E}_n q(\sigma) = \sum_{jk \leq n} \frac{q_j(k)}{j^{k-1}(k-1)!} \mathbb{E}_{n-jk} q^{(j)}(\sigma) \leq \sum_{m \leq n} \mathbb{E}_{n-m} q(\sigma) \sum_{jk=m} \frac{1}{j^{k-1}(k-1)!}.$$

Since the inner sum is bounded by an absolute constant, (10) yields

$$\mathbb{E}_n q(\sigma) \ll \frac{1}{n} \sum_{0 \leq m < n} \mathbb{E}_m q(\sigma) \ll \exp\left\{\sum_{j \leq n} \frac{q_j(1)-1}{j}\right\}.$$

The lemma is proved. □

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If $1 \le r \le m$ and $m \ge 2$ are arbitrary integers, then we can uniquely decompose $\sigma \in \mathbb{S}_m$ into the product $\sigma = \sigma'\sigma''$, where σ' is the so-called *r-friable* (smooth) divisor and σ'' the *r-free* divisor. Namely, σ' comprises all cycles in σ whose lengths do not exceed r, while σ'' contains the remaining ones. Counting such divisors, we will reduce the task to enumerating the r-friable and the r-free permutations in $\mathbb{S}_m$ where $r \le m \le n$. Now we present a few known results.

As usual, let $\rho\colon [0, \infty[\to]0, 1]$ denote the Dickman–de Bruijn function defined as $\rho(u) = 1$ for $[0, 1]$ and, for the rest of its range, by the delay differential equation $u\rho'(u) + \rho(u-1) = 0$. Recall (see, e.g., [33, Section 5.4]) that $\rho(u) = u^{-u+o(u)}$ as $u \to \infty$.

Lemma 4. *If* $1 \le r \le m$ *and* $u := m/r$, *then*

$$\nu_m(\sigma \in \mathbb{S}_m\colon \sigma \text{ is } r\text{-friable}) = \rho(u)\left(1 + O\left(\frac{u\log(u+1)}{r}\right)\right).$$

Proof. This is [18, Proposition 1.8]. Actually, the lemma is a quick consequence of [17, Theorem 1.17]. If $\sqrt{m\log(m+1)} \le r \le m$, the result dates back to [29]. □

Let $\omega\colon [0, \infty[\to]0, 1]$ denote Buchstab's function defined as $\omega(u) = 1/u$ for $u \in [1, 2]$ and by the delay differential equation $(u\omega(u))' = \omega(u-1)$ for $u \ge 2$. [33, Theorem 4, p. 402] gives us that $\omega(u) - \mathrm{e}^{-\gamma} \ll \rho(u)\log^{-1}(u+1)$ if $u \ge 1$.

Lemma 5. *If* $1 \le r \le m$ *and* $u := m/r \ge 1$, *then*

$$\nu_m(\sigma \in \mathbb{S}_m\colon \sigma \text{ is } r\text{-free}) = \exp\left\{-\sum_{j\le r}\frac{1}{j}\right\}\big(\mathrm{e}^{\gamma}\omega(u) + O(r^{-1})\big).$$

Proof. This is [26, Theorem 3]. See [17, 31] for state-of-the-art surveys on enumeration of the r-free permutations.

3. Proof of Theorem 1

We split the proof into three parts.

3.1. Long cycles are essential

We will discover that the divisors having only long cycles $\varkappa$, that is, large $L(\varkappa)$, determine the asymptotic behaviour of the process $X_n(\sigma, t)$ defined in (7) via the function $g \in \mathcal{M}$. If $0 < \varepsilon < 1$, we let

$$\sigma'(\varepsilon) = \prod_{\substack{\varkappa|\sigma\\ L(\varkappa)\le\varepsilon n}} \varkappa, \qquad \sigma(\varepsilon) = \prod_{\substack{\varkappa|\sigma\\ \varepsilon n<L(\varkappa)\le n}} \varkappa$$

denote the (εn)-friable and (εn)-free divisors, respectively. We introduce the process

$$X_n(\sigma(\varepsilon), t) = \frac{1}{f(\sigma(\varepsilon))}\sum_{\substack{\delta|\sigma(\varepsilon)\\ L(\delta)\le tn}} g(\delta), \qquad 0 \le t \le 1. \tag{11}$$

For brevity, set $\alpha = (\log(1/\varepsilon))^{1/2}$.

Proposition 1. *Let* $X_n(\sigma, t)$ *and* $X_n(\sigma(\varepsilon), t)$ *be the processes defined in (7) and (11). There exist absolute positive constants* c_0, ε_0, *and* C *such that* $\nu_n(X_n(\sigma, t) \ne X_n(\sigma(\varepsilon), t)) \le C\alpha^{-1}$ *uniformly in* $\varepsilon^c \le t \le 1 - \varepsilon^c$, *provided that* $0 < c \le c_0$, $0 < \varepsilon \le \varepsilon_0$, *and* $n \ge n_0(\varepsilon)$.

Furthermore, the estimate continues to hold if, defining $X_n(\sigma(\varepsilon), t)$ by (11), we replace $g(\delta)$ by $\vartheta^{w(\delta)}$ and $f(\sigma(\varepsilon))$ by $(1+\vartheta)^{w(\sigma(\varepsilon))}$.

Proof. We adopt the original arguments used in [30]. We may start with $\varepsilon_0 \le \mathrm{e}^{-1}$ and $n_0(\varepsilon) > \varepsilon^{-1} \ge \mathrm{e}$ and refine the choice in the proof process.

Each $\delta|\sigma'(\varepsilon)\sigma(\varepsilon)$ splits into a product of two divisors, $\delta = \delta_1\delta_2$, such that $\delta_1|\sigma'(\varepsilon)$ and $\delta_2|\sigma(\varepsilon)$. Hence,

$$\begin{aligned} X_n(\sigma, t) &= \frac{1}{f(\sigma'(\varepsilon))} \sum_{\delta_1|\sigma'(\varepsilon)} g(\delta_1) \cdot \frac{1}{f(\sigma(\varepsilon))} \sum_{\substack{\delta_2|\sigma(\varepsilon) \\ L(\delta_2)\le nt}} g(\delta_2) \\ &\quad - \frac{1}{f(\sigma)} \sum_{\delta_1|\sigma'(\varepsilon)} g(\delta_1) \sum_{\substack{\delta_2|\sigma(\varepsilon) \\ tn - L(\delta_1) < L(\delta_2) \le nt}} g(\delta_2) \\ &=: X_n(\sigma(\varepsilon), t) - Y_n(\sigma, t). \end{aligned}$$

The largest (εn)-friable divisor of σ is $\sigma'(\varepsilon)$. Observe that the subset of σ having comparatively large $L(\sigma'(\varepsilon))$ is sparse. Indeed,

$$\begin{aligned} \nu_n(\sigma \in \mathbb{S}_n \colon L(\sigma'(\varepsilon)) > \varepsilon\alpha n) &= \nu_n\Bigg(\sum_{j\le\varepsilon n} jk_j(\sigma) > \varepsilon\alpha n\Bigg) \\ &\le \frac{1}{\varepsilon\alpha n} \sum_{j\le\varepsilon n} j\mathbb{E}_n k_j(\sigma) = \frac{1}{\varepsilon\alpha n}\lfloor \varepsilon n\rfloor \le \frac{1}{\alpha} \end{aligned}$$

by Lemma 1. Hence,

$$\begin{aligned} &\nu_n(Y_n(\sigma, t) \ne 0) \\ &\quad \le \alpha^{-1} + \nu_n\big(\sigma \in \mathbb{S}_n \colon L(\sigma'(\varepsilon)) \le \varepsilon\alpha n, \\ &\qquad\qquad \text{there exists } \delta_2|\sigma(\varepsilon),\ tn - L(\sigma'(\varepsilon)) < L(\delta_2) \le tn\big) \\ &\quad =: \alpha^{-1} + \mu_n(t). \end{aligned} \tag{12}$$

Let us focus on the σs counted in $\mu_n(t)$. If $\sigma(\varepsilon) = \delta_2\tilde{\delta}_2$, then $\sigma = \delta_2\tilde{\delta}_2\sigma'(\varepsilon)$. Solving the inequalities between the parentheses in $\mu_n(t)$, by virtue of $L(\delta_2) + L(\tilde{\delta}_2) = n - L(\sigma'(\varepsilon))$ we have

$$(t - \varepsilon\alpha)n \le L(\delta_2) \le tn, \qquad (1 - t - \varepsilon\alpha)n \le L(\tilde{\delta}_2) \le (1-t)n \tag{13}$$

provided that $\varepsilon\alpha \le t \le 1 - \varepsilon\alpha$. Now, it is essential to verify that at least one of δ_2 and $\tilde{\delta}_2$ has a comparatively small number of cycles. By the definition of the number-of-cycles function,

$$w(\sigma(\varepsilon)) = \sum_{\varepsilon n < j \le n} k_j(\sigma).$$

According to Lemma 2, we have

$$\frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} (w(\sigma(\varepsilon)) - h(\varepsilon n, n))^2 \le h(\varepsilon n, n), \qquad n \ge 2, \tag{14}$$

where $h(y, x) = \sum_{y<j\le x} 1/j$, which satisfies $|h(y, x) - \log(x/y)| \le 1/y$. This and the Chebyshev inequality yield

$$\nu_n(w(\sigma(\varepsilon)) > (3/2)\alpha^2) \le \nu_n(w(\sigma(\varepsilon)) - h(\varepsilon n, n) > \alpha^2/4) \le 32\alpha^{-2} \tag{15}$$

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if $n \geq 4\varepsilon^{-1}\alpha^{-2}$. Thus, for all but $O(n!\alpha^{-2})$ permutations $\sigma \in \mathbb{S}_n$, one of the above δ_2 and $\tilde{\delta}_2$ has no more than $\frac{3}{4}\alpha^2$ cycles. In either case, apart from $O(n!\alpha^{-2})$ permutations σ counted in the frequency $\mu_n(t)$, we obtain a decomposition $\sigma = \delta\tau(\varepsilon)$, where $\tau(\varepsilon)$ is (εn)-free and δ belongs to the set $\Delta_t := \big\{\delta|\sigma : w(\delta(\varepsilon)) \leq \frac{3}{4}\alpha^2,\ (t-\varepsilon\alpha)n \leq L(\delta) \leq (t+\varepsilon\alpha)n\big\}$ for t (or $(1-t)$) from $[\varepsilon\alpha, 1-\varepsilon\alpha]$, as indicated in (13).

Indeed, if $w(\delta_2) \leq \frac{3}{4}\alpha^2$, then we set $\tau(\varepsilon) = \tilde{\delta}_2$ and $\delta = \delta_2\sigma'(\varepsilon)$, where $\delta_2 = \delta(\varepsilon)$ satisfies the left-hand-side inequalities in (13). In the second case, if $w(\tilde{\delta}_2) \leq \frac{3}{4}\alpha^2$, then we take $\tau(\varepsilon) = \delta_2$ and $\delta = \tilde{\delta}_2\sigma'(\varepsilon)$, where $\tilde{\delta}_2 = \delta(\varepsilon)$ satisfies the right-hand-side inequalities in (13). Consequently, excluding the permutations counted in (15), for the shorter interval $[\varepsilon^c, 1-\varepsilon^c] \subset [\varepsilon\alpha, 1-\varepsilon\alpha]$, where $0 < c \leq \frac{1}{4}$, we have

$$\begin{aligned}\mu_n &:= \max_{\varepsilon^c \leq t \leq 1-\varepsilon^c} \mu_n(t)\\ &\ll \alpha^{-2} + \nu_n(\sigma = \delta\tau(\varepsilon) \in \mathbb{S}_n : \text{there exists } t \in [\varepsilon^c, 1-\varepsilon^c] \text{ such that } \delta \in \Delta_t)\\ &\ll \alpha^{-2} + \max_{\varepsilon^c \leq t \leq 1-\varepsilon^c} \frac{1}{n!} \sum_{\substack{\sigma = \delta\tau(\varepsilon) \in \mathbb{S}_n\\ \delta \in \Delta_t}} 1. \end{aligned} \tag{16}$$

As described in [10] or [16, Chapter II], the sum over decompositions in (16) can be reduced to a summation over permutations belonging to respective symmetric groups of lower order. For that, the vertex labels in $V(\tau(\varepsilon))$ and, simultaneously, the labels in $V(\delta)$ belonging to $[n]$, can be substituted by the numbers from $[L(\tau(\varepsilon))]$ and $[L(\delta)]$ so that the former orders of the labels for either of the divisors are preserved. If $L(\delta) = k$, then exactly $\binom{n}{k}$ of the pairs $(\delta, \tau(\varepsilon))$ are reduced to one pair $(\delta, \tau(\varepsilon))$, with $\delta \in \mathbb{S}_k$ and $\tau(\varepsilon) \in \mathbb{S}_{n-k}$. Here, we are leaving the same notation after the relabelling since the reduction does not change the cycle structure of the divisors; in particular, neither the lengths nor $w(\delta(\varepsilon))$. Consequently, (16) attains the form

$$\mu_n \ll \alpha^{-2} + \max_{\varepsilon^c \leq t \leq 1-\varepsilon^c} \frac{1}{n!} \sum_{(t-\varepsilon\alpha)n \leq k \leq (t+\varepsilon\alpha)n} \binom{n}{k} \sum_{\substack{\delta \in \mathbb{S}_k\\ w(\delta(\varepsilon)) \leq (3/4)\alpha^2}} \sum_{\tau(\varepsilon) \in \mathbb{S}_{n-k}} 1. \tag{17}$$

The innermost sum counts the (εn)-free permutations in the symmetric group $\mathbb{S}_{n-k}$. Since

$$\frac{(n-k)}{(\varepsilon n)} \geq \frac{(1-t-\varepsilon\alpha)}{\varepsilon} \geq \frac{\varepsilon^{c-1}}{2} \geq 1$$

for $0 < c \leq \frac{1}{4}$, ensuring $\varepsilon\alpha \leq \frac{1}{2}\varepsilon^c$ if $\varepsilon \leq \mathrm{e}^{-2}$, we can apply Lemma 5 to get

$$\frac{1}{(n-k)!} \sum_{\tau(\varepsilon) \in \mathbb{S}_{n-k}} 1 \ll \frac{1}{\varepsilon n}.$$

Inserting this into the estimate (17), we obtain

$$\mu_n \ll \alpha^{-2} + \max_{\varepsilon^c \leq t \leq 1-\varepsilon^c} \frac{1}{\varepsilon n} \sum_{(t-\varepsilon\alpha)n \leq k \leq (t+\varepsilon\alpha)n} \nu_k\left(\delta \in \mathbb{S}_k : w(\delta(\varepsilon)) \leq \frac{3}{4}\alpha^2\right). \tag{18}$$

The summation is over large k, namely $k \geq (t-\varepsilon\alpha)n \geq \frac{1}{2}\varepsilon^c n$. Therefore, we can again use the inequality (14) with k instead of n, centralising $w(\delta(\varepsilon))$ by $h(\varepsilon n, k)$. The frequency under the sum does not exceed $\nu_k := \nu_k\big(\delta \in \mathbb{S}_k : w(\delta(\varepsilon)) - h(\varepsilon n, k) \leq -\frac{1}{4}\alpha^2 + R\big)$, where

$$R = \alpha^2 - \log\left(\frac{k}{\varepsilon n}\right) + \frac{1}{\varepsilon n} = \log\left(\frac{n}{k}\right) + \frac{1}{\varepsilon n} \leq c\alpha^2 + \log 2 + \frac{1}{\varepsilon n}.$$

If $c \le 1/24 =: c_0$, $\varepsilon < 2^{-24} =: \varepsilon_0$, and $n \ge n_0(\varepsilon) := 24\varepsilon^{-1}\alpha^{-2}$, then $R \le \frac{1}{8}\alpha^2$. Hence, by Chebyshev's inequality and Lemma 2,

$$\nu_k \le \nu_k\Big(\delta \in \mathbb{S}_k : w(\delta(\varepsilon)) - h(\varepsilon n, k) \le -\frac{1}{8}\alpha^2\Big) \ll \alpha^{-4} h(\varepsilon n, k) \ll \alpha^{-2}.$$

The last estimate and (18) yield $\mu_n \ll \alpha^{-2} + (\varepsilon n)^{-1} \cdot \alpha^{-2} \cdot \varepsilon\alpha n \ll \alpha^{-1}$. Combining the inequalities (16) and (12), we complete the proof of the main part of Proposition 1.

Next, recall that $g_j(1) = \vartheta$ for every j, and that all the cycle lengths of $\sigma(\varepsilon)$ belong to the interval $(\varepsilon n, n]$. Consequently, $X_n(\sigma(\varepsilon), t)$ with $g(\delta)$ as defined in (11) differs from the process

$$(1+\vartheta)^{-w(\sigma(\varepsilon))} \sum_{\substack{\delta|\sigma(\varepsilon) \\ L(\delta) \le tn}} \vartheta^{w(\delta)}$$

at most for $\sigma \in \mathbb{S}_n$ having a cycle $\varkappa$ of length $j \in (\varepsilon n, n]$ such that $k_j(\sigma) \ge 2$. Using Lemma 1, we can estimate the frequency of such permutations:

$$\begin{aligned} &\nu_n(\sigma \in \mathbb{S}_n : \text{ there exists } j \in (\varepsilon n, n] \text{ such that } k_j(\sigma) \ge 2) \\ &\quad \le \sum_{\varepsilon n < j \le n} \sum_{2 \le m \le n/j} \nu_n(k_j(\sigma) = m) \le \sum_{\varepsilon n < j \le n} \sum_{2 \le m \le n/j} \frac{1}{j^m m!} \le \sum_{\varepsilon n < j \le n} \frac{1}{j^2} \ll \frac{1}{\varepsilon n} \ll \alpha^2. \end{aligned}$$

This furnishes the proof of Proposition 1. □

3.2. Convergence of finite-dimensional distributions

This step is devoted to the finite-dimensional distributions of the process $X_n(t)$. Let $l \in \mathbb{N}$ and $0 \le t_1 < \cdots < t_l \le 1$ be arbitrary fixed numbers. Define the vectors $T = (t_1, \ldots, t_l)$, $U = (u_1, \ldots, u_l) \in [0, 1]^l$, and the distribution function

$$F_n(U, T) = \nu_n\big(X_n(t_1) \le u_1, \ldots, X_n(t_l) \le u_l\big).$$

Proposition 2. *For all vectors T and U, the distribution function $F_n(U, T)$ converges as $n \to \infty$ to an l-dimensional distribution function.*

Proof. Recall that the function $f \in \mathcal{M}$ is defined by $f_j(k)$, where, according to (5), $f_j(0) = 1$, $f_j(1) = 1 + \vartheta$, and

$$f_j(k) = \sum_{s=0}^{k} \binom{k}{s} g_j(k) \ge 1 + \vartheta, \quad k \ge 2.$$

Since $0 < 1/f(\sigma) = X_n(\sigma, 0) \le X_n(\sigma, t) \le X_n(\sigma, 1) = 1$, $\sigma \in \mathbb{S}_n$ and $0 \le t \le 1$, without loss of generality we can assume that $t_l < 1$ and $u_1 > 0$.

To settle the case $t_1 = 0$, we observe that, by virtue of $1/f \in \mathcal{M}$ and $0 < 1/f(\sigma) \le 1$, Lemma 3 yields

$$\mathbb{E}_n\Big(\frac{1}{f}\Big) \ll \exp\Bigg\{\sum_{j \le n} \frac{(1+\vartheta)^{-1} - 1}{j}\Bigg\} \ll \exp\{-\vartheta/(1+\vartheta)\log n\} \ll n^{-\vartheta/(1+\vartheta)}.$$

Using this estimate, we can evaluate the difference

$$\begin{aligned} &0 \le \nu_n(X_n(t_2) \le u_2, \ldots, X_n(t_l) \le u_l) - \nu_n(X_n(0) \le u_1, X_n(t_2) \le u_2, \ldots, X_n(t_l) \le u_l) \\ &\quad \le \nu_n(X_n(0) > u_1) \le u_1^{-1}\mathbb{E}_n(1/f) \ll n^{-\vartheta/(1+\vartheta)}. \end{aligned}$$

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Thus, if $t_1 = 0$, our task reduces to the $(l-1)$-dimensional problem.

Henceforth, let $0 < \varepsilon < \min\{t_1^{1/c_0}, (1-t_l)^{1/c_0}\}$, where c_0 was found in Proposition 1. We introduce the distribution functions

$$G_n(U, T) := \nu_n(\sigma : X_n(\sigma(\varepsilon), t_1) \le u_1, \ldots, X_n(\sigma(\varepsilon), t_l) \le u_l) =: \nu_n\big(\sigma : \overline{X}_n(\sigma(\varepsilon), T) \in A\big),$$

where the process $X_n(\sigma(\varepsilon), t)$ was defined in the second assertion of Proposition 1 via $g(\delta) = \vartheta^{w(\delta)}$, and

$$\overline{X}_n(\sigma(\varepsilon), T) := (X_n(\sigma(\varepsilon), t_1), \ldots, X_n(\sigma(\varepsilon), t_l)), \qquad A := \prod_{i \le l}]0, u_i[\subset]0, 1[^l.$$

By Proposition 1,

$$F_n(U, T) = G_n(U, T) + O(\alpha^{-1}) = \frac{1}{n!} \sum_{\substack{\sigma = \sigma(\varepsilon)\delta \in \mathbb{S}_n, \\ \delta \text{ is } (\varepsilon n)\text{-friable}}} \mathbf{1}\big\{\overline{X}_n(\sigma(\varepsilon), T) \in A\big\} + O(\alpha^{-1}).$$

As in the derivation of (17), we can apply the reduction of labels (then $\binom{n}{k}$ of the (εn)-free permutations $\sigma(\varepsilon)$ reduce to one (εn)-free permutation $\tau \in \mathbb{S}_k$) and rewrite

$$\begin{aligned} F_n(U, T) &= \sum_{\varepsilon n \le k \le n} \frac{1}{k!} \sum_{\substack{\tau \in \mathbb{S}_k \\ \tau \text{ is } (\varepsilon n)-\text{free}}} \mathbf{1}\big\{\overline{X}_n(\tau, T) \in A\big\} \\ &\qquad \times \nu_{n-k}(\delta \in \mathbb{S}_{n-k} : \delta \text{ is } (\varepsilon n)\text{-friable}) + O(\alpha^{-1}) \\ &= \sum_{\varepsilon n \le k \le n} \frac{1}{k!} \sum_{\substack{\tau \in \mathbb{S}_k \\ \tau \text{ is } (\varepsilon n)-\text{free}}} \mathbf{1}\big\{\overline{X}_n(\tau, T) \in A\big\} \rho\left(\frac{n-k}{\varepsilon n}\right) + O(\alpha^{-1}) \end{aligned}$$

by Lemma 4 with $m = n - k \ge n(1-\varepsilon)$ and $r = \lfloor \varepsilon n \rfloor$, taking into account that $\rho(u) \ll u^{-u/2}$ for $u = (n-k)/(\varepsilon n) \ge \varepsilon^{-1}/2$. Similarly, here we can get rid of τ having cycles with repeated lengths. For that, we can apply the next estimate obtained using Lemma 1. Namely,

$$\begin{aligned} &\sum_{\varepsilon n \le k \le n} \nu_k(\tau \in \mathbb{S}_k : \tau \text{ is } (\varepsilon n)\text{-free, there exists } j \text{ such that } k_j(\tau) \ge 2) \\ &\qquad \le \frac{1}{2} \sum_{\varepsilon n \le k \le n} \sum_{\varepsilon n \le j \le k} \mathbb{E}_k(k_j(\tau)(k_j(\tau) - 1)) \le \frac{1}{2} \sum_{\varepsilon n \le k \le n} \sum_{\varepsilon n \le j \le k} \frac{1}{j^2} \ll \frac{1}{\varepsilon}. \end{aligned}$$

Multiplied by $\rho(\varepsilon^{-1}/2)$, this quantity gives also the remainder $O(\alpha^{-1})$. So. after simplifications, we arrive at

$$\begin{aligned} &F_n(U, T) \\ &= \sum_{\varepsilon n \le k \le n} \rho\left(\frac{n-k}{\varepsilon n}\right) \nu_k\big(\tau \in \mathbb{S}_k : \tau \text{ is } (\varepsilon n)\text{-free, } k_j(\tau) \le 1,\ \varepsilon n \le j \le k,\ \overline{X}_n(\tau, T) \in A\big) \\ &\quad + O(\alpha^{-1}) \\ &= \sum_{m \le \varepsilon^{-1}} \sum_{\varepsilon n \le k \le n} \rho\left(\frac{n-k}{\varepsilon n}\right) \nu_k\big(\tau \in \mathbb{S}_k(m) : \overline{X}_n(\tau, T) \in A\big) + O(\alpha^{-1}). \end{aligned} \tag{19}$$

Here, $\mathbb{S}_k(m)$ denotes the subset of (εn)-free permutations τ in $\mathbb{S}_k$ that have exactly m, $1 \le m \le \varepsilon^{-1}$, cycles of different lengths. We further intend to change the summation over k by a summation over m-tuples of the cycle lengths appearing in all accounted τ.

Let $\tau \in \mathbb{S}_k(m)$ and its cycle lengths in the fixed order be $(j_1, \ldots, j_m) \in]\varepsilon n, n]^m$ provided that $j_1 + \cdots + j_m = k$. The number of such τs equals $k!/(j_1 \cdots j_m)$. What is the meaning of the condition $\overline{X}_n(\tau, T) \in A$ in terms of the cycle lengths $(j_1, \ldots, j_m)$ for the given τ?

The following observations hold for any such τ. Firstly, $f(\tau) = (1 + \vartheta)^m$. Secondly, the divisors $\delta|\tau$ are enumerated by the vector of indicators $(i_1, \ldots, i_m)$, where $i_r = 1$ if the cycle of length j_r appears in δ. Hence, $L(\delta) = i_1 j_1 + \cdots + i_m j_m$. Thirdly, the vector $(j_1/n, \ldots, j_m/n)$ necessarily belongs to the intersection, denoted by $D_m(\varepsilon)$, of the following two sets of the vectors $\bar{x} = (x_1, \ldots, x_m)$: $\{\bar{x} \in]\varepsilon, 1]^m : x_1 + \cdots + x_m \le 1\}$ and

$$\bigcap_{j \le l} \left\{ \bar{x} : \sum_{(i_1, \ldots, i_m) \in \{0,1\}^m} \vartheta^{i_1 + \cdots + i_m} \mathbf{1}\{i_1 x_1 + \cdots + i_m x_m \le t_j\} \le u_j (1 + \vartheta)^m \right\}.$$

Having all this in mind, we obtain, from (19),

$$\begin{aligned} F_n(U, T) &= \sum_{m \le \varepsilon^{-1}} \sum_{(j_1/n, \ldots, j_m/n) \in D_m(\varepsilon)} \rho\left(\frac{1}{\varepsilon}\left(1 - \sum_{r \le m} \frac{j_r}{n}\right)\right) \frac{1}{j_1 \cdots j_m} + O(\alpha^{-1}) \\ &= \sum_{m \le \varepsilon^{-1}} \int_{D_m(\varepsilon)} \rho\left(\frac{1}{\varepsilon}\left(1 - \sum_{r \le m} x_r\right)\right) \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_m}{x_1 \cdots x_m} + O(\alpha^{-1}) + o_\varepsilon(1) \end{aligned}$$

as $n \to \infty$, by the definition of the m-dimensional Riemann integral. We have arrived at the situation described in [30, p. 6]. Successively letting $n \to \infty$ and $\varepsilon \to 0$, by virtue of the notation $\alpha = (\log(1/\varepsilon))^{-1/2}$, we verify that

$$\limsup_{\varepsilon \to 0} I(\varepsilon) \le \liminf_{n \to \infty} F_n(U, T) \le \limsup_{n \to \infty} F_n(U, T) \le \liminf_{\varepsilon \to 0} I(\varepsilon),$$

where $I(\varepsilon)$ denotes the sum of integrals in the above relation. This shows that, as claimed in Proposition 2, the limits $\lim_{\varepsilon \to 0} I(\varepsilon) = \lim_{n \to \infty} F_n(U, T)$ exist for all vectors T and U. □

We stress once more that the limiting finite-dimensional distribution agrees with that found in [30] for the corresponding number-theoretic process $T_n(m, t)$.

3.3. Tightness

This subsection is devoted to showing the tightness of the sequence of distributions $\{\mathbb{P}_n\}_{n=1}^{\infty}$. We take advantage of the idea [34] of verifying a stronger tightness criterion than that used in the Skorokhod space $\mathbb{D}[0, 1]$. According to [11, Theorem 15.5], the following assertion also ensures that the weak limit $\mathbb{P}$ of a subsequence $\{\mathbb{P}_{n'}\}$ as $n' \to \infty$ is supported by a subset of the space $\mathbb{C}[0, 1]$.

Proposition 3. *For every* $0 < a, \epsilon < 1$, $\nu_n\big(\sup_{|s-t| \le a} |X_n(t) - X_n(s)| \ge \epsilon\big) \ll \epsilon^{-1} a^{\vartheta/(1+\vartheta)^2}$.

Proof. As noted in Proposition 1, in the definition in (7) of $X_n(t) = X_n(\sigma, t)$, we may take $g(\delta) = \vartheta^{w(\delta)}$ and $f(\sigma) = (1 + \vartheta)^{w(\sigma)}$. By virtue of the monotonicity of $X_n(t)$, $0 \le t \le 1$, the modulus of continuity is

$$\sup_{|s-t| \le a} |X_n(t) - X_n(s)| = \sup_{0 \le t \le 1-a} (1 + \vartheta)^{-w(\sigma)} \sum_{\substack{\delta|\sigma \\ tn < L(\delta) \le (t+a)n}} \vartheta^{w(\delta)} =: Q_n(\sigma, a),$$

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where $0 \le s \le t \le 1$ and $0 < a < 1$. Thus, the estimate of Proposition 3 will follow from

$$\mathbb{E}_n Q_n(\sigma, a) \ll a^{\vartheta/(1+\vartheta)^2}. \tag{20}$$

For a given $\sigma \in \mathbb{S}_n$, the introduced $Q_n(\sigma, a)$ is just the concentration function of the random variable taking values $L(\delta)/n$ with probability $\vartheta^{w(\delta)}(1+\vartheta)^{-w(\sigma)}$ for each $\delta|\sigma$. By [33, Chapter III, Lemma 2.6.1], $Q_n(\sigma, a) \le 3an \int_0^{1/(an)} |H(\sigma, v)|\, \mathrm{d}v$, where, according to the toolkit presented in the introduction,

$$\begin{aligned} H(\sigma, v) &= (1+\vartheta)^{-w(\sigma)} \sum_{\delta|\sigma} \vartheta^{w(\delta)} \mathrm{e}^{ivL(\delta)} \\ &= (1+\vartheta)^{-w(\sigma)} \sum_{\delta|\sigma} \prod_{j\le n} (\vartheta \mathrm{e}^{ivj})^{k_j(\delta)} = \prod_{j \le n} \left(\frac{1+\vartheta \mathrm{e}^{ivj}}{1+\vartheta} \right)^{k_j(\sigma)} \in \mathcal{M}_{\mathrm{c}}. \end{aligned}$$

Consequently,

$$\mathbb{E}_n Q_n(\sigma, a) \ll an \int_0^{1/(an)} \mathbb{E}_n |H(\sigma, v)|\, \mathrm{d}v. \tag{21}$$

Since $|H(\sigma, v)| \le 1$, by Lemma 3 we obtain

$$\mathbb{E}_n |H(\sigma, v)| \ll \exp\left\{ \sum_{j\le n} \left(\frac{|1+\vartheta \mathrm{e}^{ivj}|}{1+\vartheta} - 1 \right) \frac{1}{j} \right\} \le \exp\left\{ -\frac{\vartheta}{(1+\vartheta)^2} \sum_{j \le n} \frac{1-\cos vj}{j} \right\}. \tag{22}$$

Here we have applied the inequality

$$|1+\vartheta \mathrm{e}^{ix}| \le 1 + \vartheta - \frac{\vartheta}{1+\vartheta}(1-\cos x), \quad x \in \mathbb{R},$$

whose proof is straightforward. To estimate the sum in (22), we apply the relation

$$\begin{aligned} &\sum_{j\le n} \frac{1-\cos vj}{j} - \log \frac{|1-\mathrm{e}^{-1/n+iv}|}{1-\mathrm{e}^{-1/n}} \\ &\quad = \sum_{j \le n} \frac{1-\cos vj}{j}(1-\mathrm{e}^{-j/n}) - \sum_{j>n} \frac{1-\cos vj}{j} \mathrm{e}^{-j/n} \le 2 + 2\int_1^\infty \mathrm{e}^{-v} \frac{\mathrm{d}v}{v} \ll 1. \end{aligned}$$

Thus, (22) yields

$$\mathbb{E}_n |H(\sigma, v)| \ll \left(\frac{1-\mathrm{e}^{-1/n}}{|1-\mathrm{e}^{-1/n+iv}|} \right)^{\vartheta/(1+\vartheta)^2} \le \left(\frac{1}{n|1-\mathrm{e}^{-1/n+iv}|} \right)^{\vartheta/(1+\vartheta)^2}$$

with an absolute constant in the symbol $\ll$. Returning to (21), we proceed as follows:

$$\begin{aligned} \mathbb{E}_n Q_n(\sigma, a) &\ll an^{1-\vartheta/(1+\vartheta)^2} \int_0^{1/(an)} \frac{\mathrm{d}v}{|1-\mathrm{e}^{-1/n+iv}|^{\vartheta/(1+\vartheta)^2}} \\ &\ll a + an^{1-\vartheta/(1+\vartheta)^2} \int_{1/n}^{1/(an)} v^{-\vartheta/(1+\vartheta)^2}\, \mathrm{d}v \ll a^{\vartheta/(1+\vartheta)^2}. \end{aligned}$$

This is the desired estimate (20). □

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4. Concluding remarks

Doing more cumbersome work, we can extend Theorem 1 in two ways. First, the function $g \in \mathcal{M}$ might be more general. Adopting the argument from [34], we can replace the condition on $g_j(1) = \vartheta > 0$ so that their weighted sums over $\varepsilon n \le j \le n$ behave in the required manner. The analytic technique, analogous to that developed in [7], would work here as well. Second, as seen from [10, Theorem 3], similar results could be established for permutations drawn according to the Ewens probability on $\mathbb{S}_n$ with another parameter $\vartheta_1 > 0$. Then, a stochastic process depending on ϑ and ϑ_1 should appear in the limit. Use of the generalised Ewens probabilities defined by multiplicative weights (see, e.g., [27]) should also be possible.

Permutations comprise just one class of decomposable combinatorial structures. The distribution of divisors of a random element in an arithmetical semigroup has already attracted attention. Announcing [24, Theorem 1.2] for polynomials over a finite field, Leung rediscovered the result in [6] from 2013. We are convinced that the methodology in proving the functional limit theorem elaborated in the present paper will have further combinatorial extensions.

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