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Article

Application of Stochastic Elements in the Universality of the Periodic Zeta-Function: The Case of Short Intervals

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Abstract

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a multiplicative periodic sequence of complex numbers. In this paper, we consider the approximation of analytic functions defined in the strip $\{s = \sigma + it : 1/2 < \sigma < 1\}$ by shifts $\zeta(s + i\tau; \mathfrak{a})$ of the zeta-function defined, for $\sigma > 1$, by $\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} a_m m^{-s}$ and by analytic continuation elsewhere. Using stochastic techniques, we obtain that the set of the above shifts approximating a given analytic function has a positive lower density (or density with at most countably many exceptions) in the interval $[T, T + V]$ with $T^{23/70} \leqslant V \leqslant T^{1/2}$ as $T \to \infty$. The proofs are based on a limit theorem with an explicitly given limit probability measure in the space of analytic functions.

Keywords: approximation of analytic functions; limit theorem; periodic zeta-function; universality; weak convergence of probability measures

MSC: 11M41

1. Introduction and Results

Let $s = \sigma + it$ be a complex variable and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\} \subset \mathbb{C}$ a periodic sequence with minimal period $q \in \mathbb{N}$; i.e., $a_{m+q} = a_m$ for all $m \in \mathbb{N}$. The periodic zeta-function $\zeta(s; \mathfrak{a})$, for $\sigma > 1$, is defined by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

For analytic continuation of $\zeta(s; \mathfrak{a})$ to the left of the half-plane $\sigma > 1$, the Hurwitz zeta-function is used. Let $0 < \alpha \leqslant 1$ be a fixed number. The classical Hurwitz zeta-function $\zeta(s, \alpha)$ in the half-plane $\sigma > 1$ is given by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and is analytically continued to the whole complex plane, except for the unique point $s = 1$, which is a simple pole with $\operatorname{Res}_{s=1} \zeta(s, \alpha) = 1$; see, for example, Ref. [1]. Since the periodicity of the sequence $\mathfrak{a}$ leads, for $\sigma > 1$, to the equality

$$\zeta(s; \mathfrak{a}) = q^{-s} \sum_{k=1}^{q} a_k \zeta\left(s, \frac{k}{q}\right), \tag{1}$$

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the above properties of $\zeta(s,\alpha)$ imply that $\zeta(s;\mathfrak{a})$ is an analytic function in the entire complex plane, except for a possible simple pole at the point $s=1$ with residue

$$r \stackrel{\text{def}}{=} q^{-1} \sum_{k=1}^{q} a_k.$$

If $r=0$, then the periodic zeta-function $\zeta(s;\mathfrak{a})$ is entire.

If $a_1 = 1$ and $q = 1$, then $\zeta(s;\mathfrak{a})$ becomes the Riemann zeta-function $\zeta(s)$. Denote by (m_1, m_2) the largest common divisor of $m_1, m_2 \in \mathbb{N}$, and recall that every arithmetic periodic function $\chi(m)$ with a period q, which is completely multiplicative ($\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all $m_1, m_2 \in \mathbb{N}$), and where $\chi(m) = 0$ for $(m,q) > 1$ and $\chi(m) \neq 0$ for $(m,q) = 1$, is called a Dirichlet character modulo q. If $\chi(m)$ is a Dirichlet character modulo q, and $a_m = \chi(m)$, then $\zeta(s;\mathfrak{a})$ is a Dirichlet L-function $L(s,\chi)$,

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1.$$

These examples demonstrate that the periodic zeta-function is a generalization of the classical functions $\zeta(s)$ and $L(s,\chi)$. Notice that the functions $\zeta(s)$ and $L(s,\chi)$ are the main analytic tools for investigation of the distribution of prime numbers in $\mathbb{N}$ and arithmetical progressions, respectively.

After Voronin's works [2–6], it is known that some zeta-functions $Z(s)$ are universal in the sense that their shifts $Z(s+i\tau)$ approximate with a desired accuracy a wide class of analytic functions. Voronin's investigations have been improved and extended by various authors; see [7–13]. We recall some results involving the function $\zeta(s;\mathfrak{a})$. Let $\mathfrak{D} = \{s \in \mathbb{C} : \sigma \in (1/2, 1)\}$ and $\mathfrak{K}$ be the class of compact subsets of $\mathfrak{D}$ with connected complements, and let $H(K)$, $K \in \mathfrak{K}$, be the set of functions continuous on K and analytic inside of K. Finally, let $H_0(K)$, $K \in \mathfrak{K}$, be a subset of $H(K)$ of non-vanishing functions on K. Moreover, denote by $\mu_{\mathfrak{L}}$ the Lebesgue measure on $\mathbb{R}$.

The first universality result connected to periodic sequence $\{a_m\}$ was proved by B. Bagchi in [9].

Proposition 1 (see Corollary 5.3.5 of [9])**.** *Let*

$$\zeta_1(s;\mathfrak{a}) = \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{a_m}{m^s}, \quad s \in \mathfrak{D}.$$

Then the following alternatives hold:

1° *There is a constant $\beta \in \mathbb{C}$ and a Dirichlet character χ modulo q such that $\zeta_1(s;\mathfrak{a}) = \beta L(s,\chi)$;*

2° *If $K \in \mathfrak{K}$ and $f(s) \in H(K)$, and $\varepsilon > 0$, then*

$$\liminf_{T\to\infty} \frac{1}{2T} \mu_{\mathfrak{L}} \left\{ \tau \in [-T,T] : \sup_{s\in K} |\zeta_1(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

In [11] (Theorem 11.8), another proof of Proposition 1 was proposed.

Recall that the sequence $\{a_m\}$ is multiplicative if, for all $m_1, m_2 \in \mathbb{N}$, $(m_1, m_2) = 1$, the equality $a_{m_1 m_2} = a_{m_1} a_{m_2}$ holds. Observe that, in view of [14], the sequence $\{a_m\}$ in Proposition 1 is not multiplicative.

In [15], the function $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ was discussed.

Proposition 2 (see [15], Theorem 2)**.** *Assume that the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathfrak{K}$ and $f(s) \in H_0(K)$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mu_{\mathfrak{L}}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

In the latter inequality, the shifts $\zeta(s+i\tau,\mathfrak{a})$ are not effectively defined; we only know that their set has a positive lower density in the interval $[0,T]$. This shortcoming suggests the idea of decreasing the length of intervals for τ. It is obvious that shorter intervals contain more information on approximating shifts $\zeta(s+i\tau;\mathfrak{a})$. The method of short intervals is widely used in analytic number theory for investigation of the distribution of prime numbers and zeros of zeta-functions. Thus, we arrive at the approximation of the analytic function by shifts $\zeta(s+i\tau;\mathfrak{a})$ in the so-called short intervals, i.e., with length $o(T)$ as $T\to\infty$. Of course, it is desirable to obtain universality of zeta-functions in intervals of length as short as possible. The first attempt in this direction for the function $\zeta(s;\mathfrak{a})$ was made in [16].

Proposition 3 (see [16], Theorem 1)**.** *Suppose that $T^{23/70} \leqslant V \leqslant T^{1/2}$. Then there is a closed non-empty set $F_{\mathfrak{a}} \subset \mathfrak{H}(\mathfrak{D})$ such that, for every compact set $K \subset \mathfrak{D}$, function $f(s) \in F_{\mathfrak{a}}$, and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{V}\mu_{\mathfrak{L}}\left\{\tau \in [T,T+V] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0. \tag{2}$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{V}\mu_{\mathfrak{L}}\left\{\tau \in [T,T+V] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Unfortunately, a structure of the set $F_{\mathfrak{a}}$ is not known. On the other hand, there is not any restriction on the sequence $\mathfrak{a}$. The aim of this paper is the identification of the set $F_{\mathfrak{a}}$ for a multiplicative sequence $\mathfrak{a}$. We will prove the following theorem.

Theorem 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and $T^{23/70} \leqslant V \leqslant T^{1/2}$. Let $K \in \mathfrak{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$, inequality (2) holds, and the second assertion of Proposition 3 is valid.*

We observe that the constants 23/70 and 1/2 come from the estimate for the mean square

$$\int_T^{T+V} |\zeta(\sigma+it,\alpha)|^2 \,\mathrm{d}t$$

of the Hurwitz zeta-function with $\sigma > 1/2$ and rational α. The lower bound for V is the essential and complicated problem. The complication is illustrated by a fact that, even in the case of the Riemann zeta-function, it was observed [17] that $V \geqslant T^{1273/4053}$. The upper bound $V \leqslant T^{1/2}$ is technical, and we expect that it can be extended until $V \leqslant T$.

Since Dirichlet characters are periodic completely multiplicative functions, Theorem 1 is valid for all Dirichlet L-functions. The full description of periodic multiplicative arithmetic functions is given in [14], Satz 2, and allows us to construct these functions.

The proof of Theorem 1 is based on identification of the limit measure in a limit theorem for the function $\zeta(s;\mathfrak{a})$ in the space of analytic functions. We will study weak convergence of the probability measure as $T \to \infty$, defined by using shifts $\zeta(s + i\tau; \mathfrak{a})$ with $\tau \in [T, T + V]$. First we will prove the existence of the limit measure. Then, using elements of ergodic theory, we will identify this limit measure. The last part of the proof of a limit theorem is devoted to identification of the support of the limit measure. For this, the properties of exponential functions as well as periodicity and multiplicativity of the coefficients a_m will be applied. The multiplicativity of the sequence $\mathfrak{a}$ ensures the representation of $\zeta(s; \mathfrak{a})$ by the Euler product over primes, and this plays a crucial role in characterization of the limit measure. Having a full description of the limit measure and using the Mergelyan theorem on the approximation of analytic functions by polynomials, we will obtain Theorem 1.

We note that probabilistic methods for investigating the chaotic behavior of zeta-functions were proposed by H. Bohr and B. Jessen [18–20] and later developed by various authors; see a survey paper [21]. B. Bagchi cleverly observed [9] that a probabilistic approach also works well in the universality theory of zeta-functions. This observation is quite natural. Actually, probabilistic limit theorems for zeta-functions are stated in terms of weak convergence of probability measures defined by the density of shifts of zeta-functions, and universality theorems also are stated in terms of a positive limit density of those shifts. Thus, a relation between probability and universality theories is obvious. It remains, in every concrete case, to realize this connection using additional requirements.

2. Limit Theorems

Let $\mathfrak{B}(\mathfrak{X})$ be the Borel σ-field of the topological space $\mathfrak{X}$ and $\mathfrak{H}(\mathfrak{D})$ stand for the space of analytic functions on $\mathfrak{D}$ equipped with the topology of uniform convergence on compacta. We will consider parallelly the probability measures

$$P_{T,V,\mathfrak{a}}(A) = \frac{1}{V}\mu_{\mathfrak{L}}\{\tau \in [T, T+V] : \zeta(s + i\tau; \mathfrak{a}) \in A\}, \quad A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D})),$$

and

$$P_{T,\mathfrak{a}}(A) = \frac{1}{T}\mu_{\mathfrak{L}}\{\tau \in [0, T] : \zeta(s + i\tau; \mathfrak{a})\}, \quad A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D})),$$

as $T \to \infty$.

Denote by $\mathbb{P}$ the set of all prime numbers, and define the set

$$\mathbb{T} = \prod_{p \in \mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

Then $\mathbb{T}$ with pointwise multiplication and the product topology is a compact topological group. Therefore, on $(\mathbb{T}, \mathfrak{B}(\mathbb{T}))$, the probability Haar measure μ_H exists, and we obtain the probability space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), \mu_H)$. Denote by $\mathfrak{t} = (t(p) : p \in \mathbb{P})$ the elements of $\mathbb{T}$. For $A \in \mathfrak{B}(\mathbb{T})$, define

$$P^{\mathbb{T}}_{T,V}(A) = \frac{1}{V}\mu_{\mathfrak{L}}\Big\{\tau \in [T, T+V] : \Big(p^{-i\tau} : p \in \mathbb{P}\Big) \in A\Big\}.$$

Let Q_n, $n \in \mathbb{N}$, and Q be probability measures on $(\mathfrak{X}, \mathfrak{B}(\mathfrak{X}))$. By the definition, Q_n converges weakly to Q as $n \to \infty$ if, for every real continuous bounded function g on $\mathfrak{X}$,

$$\int_{\mathfrak{X}} g \, \mathrm{d}Q_n \xrightarrow[n \to \infty]{} \int_{\mathfrak{X}} g \, \mathrm{d}Q.$$

Lemma 1 (see Lemma 5 of [16])**.** *Suppose that $V \to \infty$ as $T \to \infty$. Then $P^{\mathbb{T}}_{T,V}$ converges weakly to the Haar measure μ_H as $T \to \infty$.*

Proof. Denote by $f_{T,V}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, the Fourier transform of the measure $P^{\mathbb{T}}_{T,V}$. Then, in [16], it was obtained that

$$\lim_{T\to\infty} f_{T,V}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the limit of $f_{T,V}$ is the Fourier transform of the measure μ_H, this gives the assertion of the lemma. □

Now, introduce an absolutely convergent Dirichlet series connected to $\zeta(s;\mathfrak{a})$. Suppose that $g_n(m) = \exp\{-(m/n)^\delta\}$ with a fixed $\delta > 1/2$. Here and further, we use the notation $\exp\{a\} = \mathrm{e}^a$. Set

$$\zeta_n(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m g_n(m)}{m^s}, \quad n \in \mathbb{N}.$$

By virtue of the exponential decreasing of $g_n(m)$ with respect to m, the latter series is absolutely convergent in every half-plane $\sigma > \sigma_0$ with arbitrary σ_0.

For $A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D}))$, define

$$P_{T,V,n,\mathfrak{a}}(A) = \frac{1}{V}\mu_{\mathfrak{L}}\{\tau \in [T, T+V] : \zeta_n(s+i\tau;\mathfrak{a}) \in A\}.$$

Extend the functions $t(p)$ to the set $\mathbb{N}$ by the formula

$$t(m) = \sum_{\substack{p^l | m \\ p^{l+1} \nmid m}} t^l(p), \quad m \in \mathbb{N},$$

and introduce a mapping $u_{n,\mathfrak{a}} : \mathbb{T} \to \mathfrak{H}(\mathfrak{D})$ given by

$$u_{n,\mathfrak{a}}(\mathfrak{t}) = \sum_{m=1}^{\infty} \frac{a_m g_n(m) t(m)}{m^s}, \quad \mathfrak{t} \in \mathbb{T}.$$

Moreover, let $P_{n,\mathfrak{a}}$ be a probability measure on $(\mathfrak{H}(\mathfrak{D}), \mathfrak{B}(\mathfrak{H}(\mathfrak{D})))$ defined by $P_{n,\mathfrak{a}} = \mu_H u^{-1}_{n,\mathfrak{a}}$. This means that, for $A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D}))$,

$$P_{n,\mathfrak{a}}(A) = \mu_H u^{-1}_{N,\mathfrak{a}}(A) = \mu_H(u^{-1}_{n,\mathfrak{a}} A).$$

Lemma 2 (see Lemma 6 of [16])**.** *Suppose that $V \to \infty$ as $T \to \infty$. Then $P_{T,V,n,\mathfrak{a}}$ converges weakly to $P_{n,\mathfrak{a}}$ as $T \to \infty$.*

Proof. The lemma follows from the continuity of $u_{n,\mathfrak{a}}$, Lemma 1, and the continuous mapping theorem; see Theorem 5.1 of [22]. □

Let d be the metric in $\mathfrak{H}(\mathfrak{D})$ inducing its topology of uniform convergence on compacta; i.e., for $f_1, f_2 \in \mathfrak{H}(\mathfrak{D})$,

$$d(f_1, f_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s\in K_j} |f_1(s) - f_2(s)|}{1 + \sup_{s\in K_j} |f_1(s) - f_2(s)|},$$

where $\{K_j : j \in \mathbb{N}\} \subset \mathfrak{D}$ is a sequence of embedded compact sets such that

$$\mathfrak{D} = \bigcup_{j=1}^{\infty},$$

and every compact set $K \subset \mathfrak{D}$ lies in some K_j. Then the following equality is valid.

Lemma 3 (see Lemma 2 of [16])**.** *Suppose that $T^{23/70} \leqslant V \leqslant T^{1/2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{V} \int_T^{T+V} d(\zeta_n(s+i\tau;\mathfrak{a}), \zeta(s+i\tau;\mathfrak{a}))\, d\tau = 0.$$

Lemmas 2 and 3 together with the tightness of the sequence $\{P_{n,\mathfrak{a}} : n \in \mathbb{N}\}$ and Theorem 4.2 of [22] imply the following statement.

Lemma 4 (see Theorem 3 of [16])**.** *Suppose that $T^{23/70} \leqslant V \leqslant T^{1/2}$. Then, on $(\mathfrak{H}(\mathfrak{D}), \mathfrak{B}(\mathfrak{H}(\mathfrak{D})))$, there exists a probability measure $P_\mathfrak{a}$ such that $P_{T,V,\mathfrak{a}}$ converges weakly to $P_\mathfrak{a}$ as $T \to \infty$. Moreover, $P_{n,\mathfrak{a}}$ also converges weakly to $P_\mathfrak{a}$ as $n \to \infty$.*

Proof. The first part of the lemma is Theorem 3 of [16]. We explain only the second assertion of the lemma. Since $\{P_{n,\mathfrak{a}}\}$ is tight, it is relatively compact. Thus, there exist $P_\mathfrak{a}$ and a subsequence $\{P_{n_l,\mathfrak{a}}\}$ such that $P_{n_l,\mathfrak{a}}$ converges weakly to $P_\mathfrak{a}$ as $l \to \infty$. By the first part of the lemma, $P_{T,V,\mathfrak{a}}$ converges weakly to $P_\mathfrak{a}$ as $T \to \infty$. Thus, $P_\mathfrak{a}$ is independent of the subsequence $\{P_{n_l,\mathfrak{a}}\}$. This implies that $P_{n,\mathfrak{a}}$ converges weakly to $P_\mathfrak{a}$ as $n \to \infty$. □

Our next purpose is the identification of the limit measure $P_\mathfrak{a}$ in Lemma 4. Unfortunately, in the case of short intervals, we cannot use some elements of ergodic theory, namely, the Birkhoff–Khintchine theorem. Therefore, we propose an another way: to perform this for the limit measure of $P_{T,\mathfrak{a}}$ and to expect that it will coincide with $P_\mathfrak{a}$.

For $A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D}))$, define

$$P_{T,n,\mathfrak{a}}(A) = \frac{1}{T}\mu_\mathfrak{L}\{\tau \in [0,T] : \zeta_n(s+i\tau;\mathfrak{a}) \in A\},$$

and, for $\mathfrak{t} \in \mathbb{T}$,

$$\widehat{P}_{T,n,\mathfrak{a}}(A) = \frac{1}{T}\mu_\mathfrak{L}\{\tau \in [0,T] : \zeta_n(s+i\tau,\mathfrak{t};\mathfrak{a}) \in A\},$$

where

$$\zeta_n(s,\mathfrak{t};\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m t(m) g_n(m)}{m^s}.$$

We recall that $\mathfrak{t} = (t(p) : p \in \mathbb{P})$ and $t(m)$ is an extension of $t(p)$.

Lemma 5. *The measures $P_{T,n,\mathfrak{a}}$ and $\widehat{P}_{T,n,\mathfrak{a}}$ both converge weakly to the limit measure $P_{n,\mathfrak{a}}$ of Lemma 2 as $T \to \infty$.*

Proof. We use the same mapping $u_{n,\mathfrak{a}}$ as in Lemma 2. Then we have

$$u_{n,\mathfrak{a}}(p^{-i\tau} : p \in \mathbb{P}) = \sum_{m=1}^{\infty} \frac{a_m g_n(m)}{m^s} \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} p^{-i\tau l} = \sum_{m=1}^{\infty} \frac{a_m g_n(m)}{m^{s+i\tau}} = \zeta_n(s+i\tau;\mathfrak{a}).$$

Hence, $P_{T,n,\mathfrak{a}}(A) = P_T^{\mathbb{T}}(u_{n,\mathfrak{a}}^{-1} A)$; i.e., $P_{T,n,\mathfrak{a}} = P_T^{\mathbb{T}} u_{n,\mathfrak{a}}^{-1}$, where

$$P_T^{\mathbb{T}}(A) = \frac{1}{T}\mu_\mathfrak{L}\{\tau \in [0,T] : (p^{-i\tau} : p \in \mathbb{P}) \in A\}.$$

Along the same lines as in the proof of Lemma 1, we find that $P_T^{\mathbb{T}}$ converges weakly to the Haar measure μ_H as $T \to \infty$. Thus, repeating the proof of Lemma 2, we obtain that $P_{T,n,\mathfrak{a}}$ converges weakly to $P_{n,\mathfrak{a}} = \mu_H u_{n,\mathfrak{a}}^{-1}$ as $T \to \infty$.

In the case of the measure $\widehat{P}_{T,n,\mathfrak{a}}$, we introduce a new mapping $\widehat{u}_{n,\mathfrak{a}}; \mathbb{T} \to \mathfrak{H}(\mathfrak{D})$ given by

$$\widehat{u}_{n,\mathfrak{a}}(\mathfrak{t}_1) = \sum_{m=1}^{\infty} \frac{a_m t(m) t_1(m) g_n(m)}{m^s}, \quad \mathfrak{t}_1 \in \mathbb{T}.$$

Then, analogically to the case of $P_{T,n,\mathfrak{a}}$, we get that $\widehat{P}_{T,n,\mathfrak{a}}$ converges weakly to $\mu_H \widehat{u}_{n,\mathfrak{a}}^{-1}$ as $T \to \infty$. Let $u : \mathbb{T} \to \mathbb{T}$ be given by $u(\mathfrak{t}) = \mathfrak{t}_1 \mathfrak{t}$. Then $\widehat{u}_{n,\mathfrak{a}} = u_{n,\mathfrak{a}}(u)$. Hence, in view of the invariance of the Haar measure with respect to shifts by elements of $\mathbb{T}$, we find that

$$\mu_H \widehat{u}_{n,\mathfrak{a}} = \mu_H (u_{n,\mathfrak{a}} u)^{-1} = (\mu_H u^{-1}) u_{n,\mathfrak{a}}^{-1} = \mu_H u_{n,\mathfrak{a}}^{-1} = P_{n,\mathfrak{a}}.$$

Thus, $\widehat{P}_{n,\mathfrak{a}} = \mu_H \widehat{u}_{n,\mathfrak{a}}^{-1} = P_{n,\mathfrak{a}}$, and the lemma is proved. □

Throughout this paper, we use the useful synonymity $x \ll_\delta y$, $x \in \mathbb{C}$, $y > 0$ of the notation $O(\dots)$, which indicates that there is a constant $c = c(\delta) > 0$ such that $|x| \leqslant cy$.

Obviously, the coefficients a_m are bounded by a constant $C = C(\mathfrak{a})$. Therefore, equality (1) together with the well-known property of the Hurwitz zeta-function, see, for example, Ref. [23], Theorem 3.3.1, yields, for $\sigma > 1/2$,

$$\int_0^T |\zeta(\sigma + it; \mathfrak{a})|^2 \, dt \ll_{\mathfrak{a},\sigma} T. \tag{3}$$

We also need a similar mean square estimate for

$$\zeta(s, \mathfrak{t}; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m t(m)}{m^s}, \quad \mathfrak{t} \in \mathbb{T}.$$

Lemma 6. *$\zeta(s, \mathfrak{t}; \mathfrak{a})$ is an $\mathfrak{H}(\mathfrak{D})$-valued random element given on the probability space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), \mu_H)$.*

Proof. Let $\mathbb{E}[\eta]$ denote the expectation of the random variable η, and set, for a fixed $\sigma_1 > 1/2$,

$$X_m = \frac{a_m t(m)}{m^{\sigma_1}}.$$

Then $\{X_m : m \in \mathbb{N}\}$ is a sequence of complex-valued random variables defined on the probability space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), \mu_H)$. Thus,

$$\mathbb{E}\Big[|X_m|^2\Big] = \frac{|a_m|^2}{m^{2\sigma_1}} \int_{\mathbb{T}} |t(m)|^2 \, d\mu_H = \frac{|a_m|^2}{m^{2\sigma_1}}, \tag{4}$$

and, for $m_1 \neq m_2$,

$$\mathbb{E}[X_{m_1} \overline{X}_{m_2}] = \frac{a_{m_1} \overline{a_{m_2}}}{m_1^{\sigma_1} m_2^{\sigma_1}} \int_{\mathbb{T}} t(m_1) \overline{t(m_2)} \, d\mu_H = 0, \tag{5}$$

where $\overline{a}$ means the complex conjugate of $a \in \mathbb{C}$, because the Haar measure is a product of Haar measures μ_γ on unit circles $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\int_\gamma t(p) \, d\mu_\gamma = 0.$$

Equality (5) shows that $\{X_m\}$ is a sequence of pairwise orthogonal random variables. Moreover, in view of (4),

$$\sum_{m=1}^{\infty} \mathbb{E}\Big[|X_m|^2\Big] \log^2 m = \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma_1}} \log^2 m \leqslant C^2(\mathfrak{a}) \sum_{m=1}^{\infty} \frac{\log^2 m}{m^{2\sigma_1}} < \infty.$$

Therefore, by the Radamacher theorem, see, for example, Ref. [24], the series

$$\sum_{m=1}^{\infty} X_m = \sum_{m=1}^{\infty} \frac{a_m t(m)}{m^{\sigma_1}},$$

for almost all $\mathfrak{t} \in \mathbb{T}$, is convergent. Hence, the series

$$\sum_{m=1}^{\infty} \frac{a_m t(m)}{m^s},$$

for almost all $\mathfrak{t} \in \mathbb{T}$, is uniformly convergent on compact subsets of the half-plane $\sigma > \sigma_1$. Now, let $\sigma_1 = 1/2 + 1/k$, $k \in \mathbb{N}$, and $\mathbb{T}_k$ be a subset of $\mathbb{T}$ such that, for $\mathfrak{t} \in \mathbb{T}$, the series defining $\zeta(s, \mathfrak{t}; \mathfrak{a})$ is uniformly convergent on compact subsets of the half-plane $\sigma > 1/2 + 1/k$. We have $\mu_H(\mathbb{T}_k) = 1$ for all $k \in \mathbb{N}$. Set

$$\widehat{\mathbb{T}} = \bigcap_{k=1}^{\infty} \mathbb{T}_k.$$

Then it follows that $\mu_H(\widehat{\mathbb{T}}) = 1$, and, for $\mathfrak{t} \in \mathbb{T}$, the series for $\zeta(s, \mathfrak{t}; \mathfrak{a})$ is uniformly convergent on compact sets of the strip $\mathfrak{D}$. Thus, $\zeta(s, \mathfrak{t}; \mathfrak{a})$ is analytic on $\mathfrak{D}$ for almost all $\mathfrak{t} \in \mathbb{T}$ with respect to the measure μ_H. The lemma is proved. □

Let, for $\tau \in \mathbb{R}$,

$$b_\tau = \{p^{-i\tau} : p \in \mathbb{P}\}.$$

On $\mathbb{T}$, define the transformation

$$\psi_\tau(\mathfrak{t}) = b_\tau \mathfrak{t}, \quad \tau \in \mathbb{R},\ \mathfrak{t} \in \mathbb{T}.$$

Then $\Psi \stackrel{\text{def}}{=} \{\psi_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure-preserving transformations on $\mathbb{T}$. A set $A \in \mathfrak{B}(\Psi))$ is called invariant with respect to Ψ if, for each τ, the sets A and $A_\tau = \psi_\tau(A)$ differ from one another by a set of μ_H-measure zero. All invariant sets form a sub-σ-field of $\mathfrak{B}(\Psi)$.

Lemma 7 (see Lemma 4.6 of [11])**.** *The one-parameter group Ψ is ergodic; i.e., its σ-field of invariant sets consists only of sets with μ_H-measure equal to* 1 *or* 0*.*

Lemma 8. *Suppose that $\sigma > 1/2$. Then, for almost all $\mathfrak{t} \in \mathbb{T}$,*

$$\int_0^T |\zeta(\sigma + it, \mathfrak{t}; \mathfrak{a})|^2 \, dt \ll_\sigma T.$$

Proof. In the proof of Lemma 6, we have seen that the random variables X_m are pairwise orthogonal. Therefore,

$$\mathbb{E}\left[\left|\sum_{m=1}^{\infty} \frac{a_m t(m)}{m^\sigma}\right|^2\right] = \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} < \infty. \tag{6}$$

The definition of $\psi_\tau(\mathfrak{t})$ gives

$$|\zeta(\sigma, \psi_\tau \mathfrak{t}; \mathfrak{a})|^2 = |\zeta(\sigma + i\tau, \mathfrak{t}; \mathfrak{a})|^2.$$

Moreover, in view of Lemma 7, the random process $|\zeta(\sigma + it, \mathfrak{t}; \mathfrak{a})|^2$ is ergodic. Therefore, by the classical Birkhoff–Khintchine theorem, see, for example, Ref. [25],

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(\sigma + it, \mathfrak{t}; \mathfrak{a})|^2 \, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(\sigma, \psi_\tau \mathfrak{t}; \mathfrak{a})|^2 \, \mathrm{d}t = \mathbb{E}\left[\left| \sum_{m=1}^{\infty} \frac{a_m t(m)}{m^\sigma} \right|^2 \right]$$

for almost all $\mathfrak{t} \in \mathbb{T}$. This together with (6) proves the lemma. □

Now, we are ready to state analogs of Lemma 3. Recall that d is the metric in $\mathfrak{H}(\mathfrak{D})$ inducing its topology of uniform convergence on compacta.

Lemma 9. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T d(\zeta_n(s + i\tau; \mathfrak{a}), \zeta(s + i\tau; \mathfrak{a})) \, \mathrm{d}\tau = 0$$

is valid.

Proof. We use the representation

$$\zeta_n(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \zeta(s + z; \mathfrak{a}) g_n(z) \, \mathrm{d}z$$

which is valid for $\sigma > 1/2$, and follow the proof of Theorem 2 of [16], applying (3). The number δ is from the definition of $g_n(s)$. □

Lemma 10. *For almost all* $\mathfrak{t} \in \mathbb{T}$*, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T d(\zeta_n(s + i\tau, \mathfrak{t}; \mathfrak{a}), \zeta(s + i\tau, \mathfrak{t}; \mathfrak{a})) \, \mathrm{d}\tau = 0$$

is valid.

Proof. Suppose that $\sigma > 1/2$. Then, for almost all $\mathfrak{t} \in \mathbb{T}$, the representation

$$\zeta_n(s, \mathfrak{t}; \mathfrak{a}) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \zeta(s + z, \mathfrak{t}; \mathfrak{a}) g_n(z) \, \mathrm{d}z$$

holds. Thus, the further proof coincides with the proof of Lemma 9 using Lemma 8. Observe that, in the case of $\zeta(s, \mathfrak{t}; \mathfrak{a})$, the pole at the point $s = 1$ does not exist. □

Now, we have sufficient information to consider weak convergence of the measures $P_{T,\mathfrak{a}}$ and

$$\widehat{P}_{T,\mathfrak{a}}(A) \stackrel{\text{def}}{=} \frac{1}{T} \mu_{\mathfrak{L}}\{\tau \in [0, T] : \zeta(s + iz, \mathfrak{t}; \mathfrak{a}) \in A\}, \quad A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D})).$$

Lemma 11. *On $(\mathfrak{H}(\mathfrak{D}), \mathfrak{B}(\mathfrak{H}(\mathfrak{D})))$, there exists a probability measure $P_{\mathfrak{a}}$ such that the measures $P_{T,\mathfrak{a}}$ and $\widehat{P}_{T,\mathfrak{a}}$, for almost all $\mathfrak{t} \in \mathbb{T}$, converge weakly to $P_{\mathfrak{a}}$ as $T \to \infty$.*

Proof. In view of the separability of $\mathfrak{H}(\mathfrak{D})$, we can apply Theorem 4.2 of [22]. We return to the limit measure $P_{n,\mathfrak{a}}$ in Lemmas 2 and 5. By Lemma 7 of [16], the measure $P_{n,\mathfrak{a}}$ is tight. This means that, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathfrak{H}(\mathfrak{D})$ such that $P_{n,\mathfrak{a}}(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. The classical Prokhorov theorem, see Theorem 6.1 of [22], states that every tight probability measure is relatively compact; i.e., every sequence $\{P_{n,\mathfrak{a}}\}$ contains a subsequence weakly convergent to a certain probability measure. Thus, we may suppose that there exists a probability measure $P_{\mathfrak{a}}$ on $(\mathfrak{H}(\mathfrak{D}), \mathfrak{B}(\mathfrak{H}(\mathfrak{D})))$ and $\{P_{n_l,\mathfrak{a}}\}$ such that $P_{n_l,\mathfrak{a}}$ converges weakly to $P_{\mathfrak{a}}$ as $l \to \infty$. Theorem 4.2 of [22] is stated in terms of convergence of random elements in distribution ($\xrightarrow{\mathcal{D}}$), which is equivalent to weak convergence of distributions. Thus, denoting by $X_{n,\mathfrak{a}}$ the $\mathfrak{H}(\mathfrak{D})$-valued random element with the distribution $P_{n,\mathfrak{a}}$, we have

$$X_{n_l,\mathfrak{a}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\mathfrak{a}}. \tag{7}$$

Introduce a random variable η_T defined on a certain probability space $(\Omega, \mathfrak{B}, \nu)$ and uniformly distributed in the interval $[0, T]$. Let

$$X_{T,n,\mathfrak{a}} = X_{T,n,\mathfrak{a}}(s) = \zeta_n(s + i\eta_T; \mathfrak{a}),$$

and

$$\widehat{X}_{T,n,\mathfrak{a}} = \widehat{X}_{T,n,\mathfrak{a}}(s) = \zeta_n(s + i\eta_T, \mathfrak{t}; \mathfrak{a}).$$

Thus, in view of Lemma 5,

$$X_{T,n,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\mathfrak{a}}, \tag{8}$$

and

$$\widehat{X}_{T,n,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\mathfrak{a}}. \tag{9}$$

Additionally, define two $\mathfrak{H}(\mathfrak{D})$-valued random elements

$$Y_{T,\mathfrak{a}} = Y_{T,\mathfrak{a}}(s) = \zeta(s + i\eta_T; \mathfrak{a}),$$

and

$$\widehat{Y}_{T,\mathfrak{a}} = \widehat{Y}_{T,\mathfrak{a}}(s) = \zeta(s + i\eta_T, \mathfrak{t}; \mathfrak{a}).$$

Then, by Lemmas 9 and 10, for every $\varepsilon > 0$, we get

$$\begin{aligned}\lim_{l\to\infty}\lim_{T\to\infty} \nu\{d(Y_{T,\mathfrak{a}}, X_{T,n_l,\mathfrak{a}}) \geqslant \varepsilon\} &= \frac{1}{T}\mu_{\mathfrak{L}}\{\tau \in [0,T] : d(\zeta(s+i\tau;\mathfrak{a}), \zeta_{n_l}(s+i\tau;\mathfrak{a})) \geqslant \varepsilon\} \\ &\leqslant \frac{1}{T\varepsilon}\int_0^T d(\zeta(s+i\tau;\mathfrak{a}), \zeta_{n_l}(s+i\tau;\mathfrak{a}))\,\mathrm{d}\tau = 0\end{aligned}$$

and

$$\lim_{l\to\infty}\lim_{T\to\infty} \nu\left\{d(\widehat{Y}_{T,\mathfrak{a}}, \widehat{X}_{T,n_l,\mathfrak{a}}) \geqslant \varepsilon\right\} = 0,$$

respectively. The latter equalities (7)–(9), together with Theorem 4.2 of [22], lead to

$$Y_{T,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\mathfrak{a}}$$

and
$$\widehat{Y}_{T,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\mathfrak{a}}.$$
In other words, this means that the probability measures $P_{T,\mathfrak{a}}$ and $\widehat{P}_{T,\mathfrak{a}}$ both converge weakly to $P_{\mathfrak{a}}$ as $T \to \infty$. Moreover, the latter weak convergence shows that the measure $P_{\mathfrak{a}}$ is independent of the sequence $\{P_{n_l,\mathfrak{a}}\}$. From this, it follows, by Theorem 2.3 of [22], that $P_{n,\mathfrak{a}}$ converges weakly to $P_{\mathfrak{a}}$ as $n \to \infty$. □

Denote by $P_{\zeta,\mathfrak{a}}$ the distribution of the random element $\zeta(s,\mathfrak{t};\mathfrak{a})$; i.e.,
$$P_{\zeta,\mathfrak{a}}(A) = \mu_H\{\mathfrak{t}\in\mathbb{T} : \zeta(s,\mathfrak{t};\mathfrak{a}) \in A\}, \quad A \in \mathfrak{B}(\mathfrak{H}(\mathfrak{D})).$$

Theorem 2. *Suppose that $T^{23/70} \leqslant V \leqslant T^{1/2}$. Then $P_{T,V,\mathfrak{a}}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $T \to \infty$.*

Proof. By Lemma 4, the limit measure of $P_{T,V,\mathfrak{a}}$ is the same as the limit measure of $P_{n,\mathfrak{a}}$. Thus, it suffices to show that $P_{\mathfrak{a}} = P_{\zeta,\mathfrak{a}}$.

We will use the equivalent of weak convergence of probability measures in terms of continuity sets. Recall that $A \in \mathfrak{B}(\mathfrak{X})$ is a continuity set of the probability measure P on $(\mathfrak{X},\mathfrak{B}(\mathfrak{X}))$ if $P(\partial A) = 0$, where ∂A is the boundary of the set A. The equivalent of weak convergence says that P_n, as $n \to \infty$, converges to P if and only if, for every continuity set A of P, the relation
$$\lim_{n\to\infty} P_n(A) = P(A)$$
holds [22].

Thus, let A be a fixed continuity set of the measure $P_{\mathfrak{a}}$. On $(\mathbb{T},\mathfrak{B}(\mathbb{T}))$, define the random variable
$$\eta(\mathfrak{t}) = \begin{cases} 1 & \text{if} \quad \zeta(s,\mathfrak{t};\mathfrak{a}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$
Then we have
$$\mathbb{E}[\eta] = \int\limits_{\mathbb{T}} \eta \, \mathrm{d}\mu_H = \mu_H\{\mathfrak{t}\in\mathbb{T} : \zeta(s,\mathfrak{t};\mathfrak{a}) \in A\} = P_{\zeta,\mathfrak{a}}(A). \tag{10}$$

From Lemma 7, it follows that the random process $\eta(\psi_\tau(\mathfrak{t}))$ is ergodic. Therefore, an application of the Birkhoff–Khintchine ergodic theorem gives
$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \eta(\psi_\tau(\mathfrak{t}))\, \mathrm{d}\tau = \mathbb{E}[\eta] \tag{11}$$
for almost all $\mathfrak{t} \in \mathbb{T}$. Moreover, the definitions of η and ψ_τ show that
$$\frac{1}{T} \int\limits_0^T \eta(\psi_\tau(\mathfrak{t}))\, \mathrm{d}\tau = \frac{1}{T}\mu_{\mathfrak{L}}\{\tau \in [0,T] : \zeta(s,\psi(\mathfrak{t});\mathfrak{a}) \in A\}$$
$$= \frac{1}{T}\mu_{\mathfrak{L}}\{\tau \in [0,T] : \zeta(s+i\tau,\mathfrak{t};\mathfrak{a}) \in A\}.$$
This equality together with (10) and (11) implies that
$$\lim_{T\to\infty} \frac{1}{T}\mu_{\mathfrak{L}}\{\tau \in [0,T] : \zeta(s+i\tau,\mathfrak{t};\mathfrak{a})\} = P_{\zeta,\mathfrak{a}}. \tag{12}$$

However, by Lemma 11,

$$\lim_{T\to\infty}\frac{1}{T}\mu_{\mathfrak{L}}\{\tau\in[0,T]:\zeta(s+i\tau,\mathfrak{t};\mathfrak{a})\in A\}=P_{\mathfrak{a}}(A).$$

Thus, by (12), we obtain $P_n(A)=P_{\zeta,\mathfrak{a}}(A)$. Since A is an arbitrary continuity set of $P_{\mathfrak{a}}$, the equality

$$P_{\mathfrak{a}}(A)=P_{\zeta,\mathfrak{a}}(A) \tag{13}$$

is true for all continuity sets A of the measure $P_{\mathfrak{a}}$. It is well known, see [22], that all continuity sets of the probability measure form a determining class. Therefore, (13) shows that $P_{\mathfrak{a}}=P_{\zeta,\mathfrak{a}}$. The theorem is proved. □

We observe that the auxiliary measures $P_{T,\mathfrak{a}}$ and $\widehat{P}_{T,\mathfrak{a}}$ were involved because the Birkhoff–Khintchine theorem in short intervals is not known.

3. Support

Let the space $\mathfrak{X}$ be separable and P be a probability measure on $(\mathfrak{X},\mathfrak{B}(\mathfrak{X}))$. The support of the measure P is a minimal closed set $S_P\subset\mathfrak{X}$ such that $P(S_P)=1$. The set S_P consists of all $x\in\mathfrak{X}$ such that, for every neighborhood $\mathfrak{G}$ of x, the inequality $P(\mathfrak{G})>0$ holds. The support of a random element is the support of its distribution.

Throughout this section, we suppose that the sequence $\{a_m:m\in\mathbb{N}\}$ is multiplicative. We will consider the support of the $\mathfrak{H}(\mathfrak{D})$-valued random element. We start with the following lemma.

Lemma 12. *For almost all $\mathfrak{t}\in\mathbb{T}$, the equality*

$$\zeta(s,\mathfrak{t};\mathfrak{a})=\prod_{p\in\mathbb{P}}\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}t^l(p)}{p^{ls}}\right)$$

is valid.

Proof. By virtue of Lemma 6 and analytic continuation, it suffices to show that the product, for almost all $\mathfrak{t}$, converges uniformly on compact subsets of the strip $\mathfrak{D}$ because, by $|a_m|\leqslant C$, the Dirichlet series for $\zeta(s,\mathfrak{t};\mathfrak{a})$ converges absolutely for $\sigma>1$, and, by the Euler identity,

$$\sum_{m=1}^{\infty}\frac{a_m\mathfrak{t}(m)}{m^s}=\prod_{p\in\mathbb{P}}\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}t^l(p)}{p^{ls}}\right).$$

It is well known that the product

$$\prod_{m=1}^{\infty}(1+a(m))$$

is convergent if the series

$$\sum_{m=1}^{\infty}|a(m)|^2 \quad\text{and}\quad \sum_{m=1}^{\infty}a(m)$$

are convergent. In our case, the series

$$\sum_{p\in\mathbb{P}}\left|\sum_{l=1}^{\infty}\frac{a_{p^l}t^l(p)}{p^{ls}}\right|^2\leqslant C^2\sum_{p\in\mathbb{P}}\frac{1}{(p^\sigma-1)^2}$$

is uniformly convergent on compact subsets of the half-plane $\sigma > 1/2$. Obviously,

$$\left|\sum_{l=2}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right| \leqslant C \frac{1}{p^{2\sigma} - p^{\sigma}}.$$

Thus, the series

$$\sum_{p \in \mathbb{P}} \sum_{l=2}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}} \tag{14}$$

is uniformly convergent on compact subsets of the half-plane $\sigma > 1/2$. Since

$$\mathbb{E}\left[\left|\frac{a_p t(p)}{p^{\sigma}}\right|^2\right] = \frac{|a_p|^2}{p^{2\sigma}},$$

repeating the proof of Lemma 6 shows that the series

$$\sum_{p \in \mathbb{P}} \frac{a_p t(p)}{p^s},$$

for almost all $\mathfrak{t}$, converges uniformly on compact subsets of the half-plane $\sigma > 1/2$. This remark, together with the convergence of series (14), proves that the series

$$\sum_{p \in \mathbb{P}} \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}},$$

for almost all $\mathfrak{t} \in \mathbb{T}$, converges uniformly on compact subsets of the half-plane $\sigma > 1/2$. From this, we obtain that, for almost all $\mathfrak{t} \in \mathbb{T}$, the product

$$\prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right)$$

converges uniformly on compact subsets of the region $\sigma > 1/2$. The lemma is proved. □

Write

$$\zeta(s, \mathfrak{t}; \mathfrak{a}) = X_1(s) X_2(s),$$

where

$$X_1(s) = \prod_{\substack{p \leqslant p_0 \\ p \in \mathbb{P}}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right), \quad X_2(s) = \prod_{\substack{p > p_0 \\ p \in \mathbb{P}}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right),$$

and p_0 is such that

$$\left|\sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right| < 1$$

for $p > p_0$ and $\sigma \geqslant 1/2$. Before investigation of $X_2(s)$, we state some results from analytic function theory.

Lemma 13 (see Theorem 6.3.10 of [10])**.** *Suppose that* $\{h_m(s) : m \in \mathbb{N}\} \subset \mathfrak{H}(\mathfrak{D})$ *such that the following hypotheses hold:*

1° *If* $\mu_{\mathfrak{B}}$ *is a complex Borel measure on* $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ *with compact support contained in* $\mathfrak{D}$ *such that*

$$\sum_{m=1}^{\infty} \left|\int_{\mathbb{C}} h_m(s) \, \mathrm{d}\mu_{\mathfrak{B}}\right| < \infty,$$

then

$$\int_{\mathbb{C}} s^k \, \mathrm{d}\mu_{\mathfrak{B}} = 0 \quad \text{for all } k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};$$

2° *The series*

$$\sum_{m=1}^{\infty} h_m(s)$$

is convergent in $\mathfrak{H}(\mathfrak{D})$;

3° *For any compact set* $K \subset \mathfrak{D}$,

$$\sum_{m=1}^{\infty} \sup_{s \in K} |h_m(s)|^2 < \infty.$$

Then the set of all convergent series

$$\sum_{m=1}^{\infty} b_m h_m(s)$$

with $|b_m| = 1$ *is dense in* $\mathfrak{H}(\mathfrak{D})$.

Let $0 < \alpha_0 \leqslant \pi$. A function $h(s)$ analytic in the region $|\arg s| \leqslant \alpha_0$ is said to be of exponential type if

$$\limsup_{r \to \infty} \frac{\log |h(re^{i\alpha})|}{r} < \infty$$

uniformly in α, $|\alpha| \leqslant \alpha_0$.

Lemma 14 (see Theorem 6.4.14 of [10])**.** *Suppose that* $h(s)$ *is an entire function of exponential type and*

$$\limsup_{r \to \infty} \frac{\log |h(r)|}{r} > -1.$$

Then

$$\sum_{p \in \mathbb{P}} |h(\log p)| = +\infty.$$

Now, we return to $X_2(s)$. For $p > p_0$, define

$$h_p(s) = \log\left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right), \quad |t(p)| = 1,$$

where, for $|u| < 1$,

$$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \dots.$$

Lemma 15. *The set of all convergent series*

$$\sum_{p > p_0} h_p(s)$$

is dense in $\mathfrak{H}(\mathfrak{D})$.

Proof. Clearly, we have

$$\log\left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l}}{p^{ls}}\right) = \frac{a_p}{p^s} + R_p(s), \tag{15}$$

where

$$R_p(s) = \sum_{l=2}^{\infty} \frac{a_{p^l}}{p^{ls}} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{1}{k} \left(\sum_{l=1}^{\infty} \frac{a_{p^l}}{p^{ls}} \right)^k \ll \sum_{l=2}^{\infty} \frac{1}{p^{l\sigma}} + \sum_{k=2}^{\infty} \frac{1}{k} \left(\sum_{l=1}^{\infty} \frac{1}{p^{2\sigma}} \right)^k \ll \frac{1}{p^{2\sigma}}.$$

Therefore, the series

$$\sum_{p>p_0} R_p(s) \tag{16}$$

converges uniformly on compact subsets of $\mathfrak{D}$.

The series

$$\sum_p \left| \frac{a_p}{p^s} \right|^2$$

converges for $\sigma > 1/2$. Therefore, in view of Lemma 6.5.3 from [10], there exists a sequence $\{\delta_p = \pm 1\}$ of independent random variables such that the series

$$\sum_p \frac{a_p \delta_p}{p^{\sigma_0}}$$

converges almost surely for $\sigma_0 > 1/2$. Hence, there exists $|\widetilde{t}(p)| = 1$ such that the series

$$\sum_p \frac{a_p \widetilde{t}(p)}{p^s}$$

converges on compact subsets of $\mathfrak{D}$. This and the convergence of the series (16) imply that the series

$$\sum_{p>p_0} \widetilde{t}(p) \log \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l}}{p^{ls}} \right)$$

converges uniformly on compact subsets of $\mathfrak{D}$.

For brevity, let

$$\widehat{h}_p(s) = \widetilde{t}(p) \log \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l}}{p^{ls}} \right).$$

We will prove that the set of all convergent series

$$\sum_{p>p_0} \widehat{t}(p) \widehat{h}_p(s) \quad \text{with } |\widehat{t}(p)| = 1$$

is dense in $\mathfrak{H}(\mathfrak{D})$. For this, we will apply Lemma 13. We will check the hypotheses of the latter lemma.

1° Let $\mu_{\mathfrak{B}}$ be a complex Borel measure with compact support in $\mathfrak{D}$ such that

$$\sum_{p>p_0} \left| \int_{\mathbb{C}} \widehat{h}_p(s) \, \mathrm{d}\mu_{\mathfrak{B}}(s) \right| < \infty. \tag{17}$$

Setting

$$\widetilde{h}_p(s) = \frac{a_p \widetilde{t}(p)}{p^s},$$

we find by (15) that

$$\sum_{p>p_0} \left| \widehat{h}_p(s) - \widetilde{h}_p(s) \right| = \sum_{p>p_0} |R_p(s)| < \infty$$

uniformly on compact subsets of $\mathfrak{D}$. Hence, by (17),

$$\sum_{p>p_0} |a_p| \left| \int_{\mathbb{C}} \frac{1}{p^s} \, \mathrm{d}\mu_{\mathfrak{B}}(s) \right| < \infty. \tag{18}$$

If among the values $a_1, \ldots, a_q$ there are no zeros, then (18) implies that

$$\sum_{p>p_0} |I(\log p)| < \infty,$$

where

$$I(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz} \, \mathrm{d}\mu_{\mathfrak{B}}(s).$$

In the opposite case, the problem is more complicated.

We have $|a_p| \leqslant C = C(\mathfrak{a})$. We may write

$$|a_p| = C \cos \alpha_p, \quad \alpha_p \in \left[0, \frac{\pi}{2}\right].$$

Let

$$B = B(\mathfrak{a}) = \frac{1}{C\varphi(q)} \sum_{k=1}^{q} |a_k|,$$

where $\varphi(q)$ is the Euler totient function. Fix $0 < \beta < \min(1, B)$ and denote

$$\mathbb{P}_\beta = \{p \in \mathbb{P} : p > p_0, \cos \alpha_p > \beta\}.$$

Then, in view of (18),

$$\sum_{p \in \mathbb{P}_\beta} |I(\log p)| < \infty. \tag{19}$$

Since the support of the measure $\mu_{\mathfrak{B}}$ is compact, there exists $\mathcal{V} > 0$ such that the support of $\mu_{\mathfrak{B}}$ lies in the rectangle

$$\left\{s \in \mathbb{C} : \tfrac{1}{2} < \sigma < 1, |t| \leqslant \mathcal{V}\right\}.$$

Then, for $y > 0$, we have

$$|I(\pm y)| \leqslant \exp\{\mathcal{V}y\} \int_{\mathbb{C}} |\mathrm{d}\mu_{\mathfrak{B}}(s)|,$$

and

$$\limsup_{y \to \infty} \frac{\log |I(\pm y)|}{y} \leqslant \mathcal{V}. \tag{20}$$

Suppose that the number $0 < \gamma < \pi\mathcal{V}^{-1}$ is fixed, and consider the set

$$A = \left\{m \in \mathbb{N} : \exists r \in \left(\left(m - \tfrac{1}{4}\right)\gamma, \left(m + \tfrac{1}{4}\right)\gamma\right) \text{ with } |I(r)| \leqslant \mathrm{e}^{-r}\right\}.$$

For brevity, let

$$a = \exp\left\{\left(m - \tfrac{1}{4}\right)\gamma\right\} \quad \text{and} \quad b = \exp\left\{\left(m + \tfrac{1}{4}\right)\gamma\right\}.$$

By the definitions of A and I, we find that, for $m \notin A$,

$$|I(\log p)| > \mathrm{e}^{-\log p} = \frac{1}{p}, \quad p \in (a, b].$$

Hence,

$$\sum_{\substack{p\in\mathbb{P}\\ p\in(a,b]}} |I(\log p)| \geqslant \sum_{m\notin A}\sum_{\substack{p\in\mathbb{P}_\beta\\ p\in(a,b]}} |I(\log p)| \geqslant \sum_{m\notin A}\sum_{\substack{p\in\mathbb{P}_\beta\\ p\in(a,b]}} \frac{1}{p}.$$

Thus, by (19),

$$\sum_{m\notin A}\sum_{\substack{p\in\mathbb{P}_\beta\\ p\in(a,b]}} \frac{1}{p} < \infty. \tag{21}$$

Setting

$$\pi_\beta(x) = \sum_{\substack{p\leqslant x\\ p\in\mathbb{P}_\beta}} 1, \qquad \pi(x) = \sum_{p\leqslant x} 1,$$

for $v \in (a,b]$ we find

$$\sum_{p\in(a,v]} \cos\alpha_p \leqslant \sum_{\substack{p\in\mathbb{P}_\beta\\ p\in(a,v]}} 1 + \beta \sum_{\substack{p\notin\mathbb{P}_\beta\\ p\in(a,v]}} 1 = (1-\beta)(\pi_\beta(v)-\pi_\beta(a)) + \beta(\pi(v)-\pi(a)). \tag{22}$$

Define

$$\pi(x;c,q) = \sum_{\substack{p\leqslant x\\ p\equiv c(\operatorname{mod} q)}} 1.$$

It is well known that, for $(c,q)=1$,

$$\pi(x;c,q) = \frac{1}{\varphi(q)}\int_2^x \frac{\mathrm{d}u}{\log u}(1+o(1)), \qquad \pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u}(1+o(1)), \quad x\to\infty.$$

Therefore, as $x\to\infty$,

$$\begin{aligned}\sum_{p\leqslant x}\cos\alpha_p &= \frac{1}{C}\sum_{p\leqslant x}|a_p| = \frac{1}{C}\sum_{k=1}^{q}|a_k|\sum_{\substack{p\leqslant x\\ p\equiv k(\operatorname{mod} q)}} 1 = \frac{1}{C\varphi(q)}\sum_{k=1}^{q}|a_k|\int_2^x\frac{\mathrm{d}u}{\log u}(1+o(1))\\ &= B(\mathfrak{a})\int_2^x\frac{\mathrm{d}u}{\log u}(1+o(1)) = B(\mathfrak{a})\pi(x)(1+o(1))\end{aligned} \tag{23}$$

with the same $B(\mathfrak{a})>0$ as above. Suppose that $v \geqslant a(1+d)$ with small $d>0$. Then (22) and (23) yield

$$\pi(v)-\pi(a) \leqslant (1+o(1))\frac{1}{B(\mathfrak{a})}\sum_{p\in(a,v]}\cos\alpha_p$$

and

$$\pi_\beta(v)-\pi_\beta(a) \geqslant \frac{B(\mathfrak{a})-\beta}{1-\beta}(1+o(1))(\pi(v)-\pi(a))$$

as $m\to\infty$. Hence, using the Styltjes integral, we get

$$\begin{aligned}\sum_{\substack{p\in\mathbb{P}_\beta\\ p\in(a,b]}}\frac{1}{p} &= \int_a^b\frac{\mathrm{d}\pi_\beta(u)}{u} \geqslant \left(\frac{B(\mathfrak{a})-\beta}{1-\beta}+o(1)\right)\int_a^b\frac{\mathrm{d}\pi(u)}{u}\\ &\geqslant \left(\frac{B(\mathfrak{a})-\beta}{1-\beta}+o(1)\right)\sum_{p\in(a(1+\delta),b]}\frac{1}{p}\end{aligned} \tag{24}$$

as $m \to \infty$. Using the formula

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c_1 + O(\exp\{-c_2\sqrt{\log x}\}), \quad c_1, c_2 > 0,$$

we obtain

$$\sum_{p \in (a(1+\delta), b]} \frac{1}{p} = \log \frac{\log b}{\log a(1+\delta)} + O(\exp\{-c_3\sqrt{m}\})$$
$$= \left(\frac{1}{2} - \frac{\log(1+\delta)}{\delta}\right)\frac{1}{m} + O\left(\frac{1}{m^2}\right), \quad c_3 > 0.$$

Therefore, by (24), we have

$$\sum_{\substack{p \in \mathbb{P}_\beta \\ p \in (a,b]}} \frac{1}{p} \geqslant \left(\frac{b-\beta}{1-\beta} + o(1)\right)\left(\frac{1}{2} - \frac{\log(1+\delta)}{\gamma}\right)\frac{1}{m} + O\left(\frac{1}{m^2}\right)$$

as $m \to \infty$. This together with (21) shows that

$$\sum_{m \notin A} \frac{1}{m} < \infty. \tag{25}$$

Denote $A = \{b_m : b_m < b_{m+1}, m \in \mathbb{N}\}$. Then (25) implies that

$$\lim_{m \to \infty} \frac{b_m}{m} = 1. \tag{26}$$

The definition of the set A shows that there is a sequence $\{\eta_m\}$ such that

$$\left(b_m - \tfrac{1}{4}\right)\gamma < \eta_m \leqslant \left(b_m + \tfrac{1}{4}\right)\gamma \quad \text{and} \quad |I(\eta_m)| \leqslant \mathrm{e}^{-\eta_m}.$$

Therefore, by (26),

$$\lim_{m \to \infty} \frac{\eta_m}{m} = \gamma \quad \text{and} \quad \limsup_{m \to \infty} \frac{\log|I(\eta_m)|}{\eta_m} \leqslant -1. \tag{27}$$

Now, an application of the Bernstein theorem, see, for example, Lemma 5.9 of [11], leads to

$$\limsup_{r \to \infty} \frac{\log I(r)}{r} \leqslant -1.$$

From this, by a standard way, it follows that, for all $r \in \mathbb{N}_0$,

$$\int_{\mathbb{C}} s^r \, \mathrm{d}\mu_{\mathfrak{B}}(s) = 0,$$

and 1° of Lemma 13 is proved.

The series

$$\sum_{p > p_0} \widehat{h}_p(s)$$

converges uniformly on compact subsets of $\mathfrak{D}$. Thus, it converges in $\mathfrak{H}(\mathfrak{D})$; i.e., hypothesis 2° of Lemma 13 is satisfied. Clearly, for any compact set $K \subset \mathfrak{D}$,

$$\sum_{p > p_0} |\widehat{h}_p(s)|^2 < \infty,$$

i.e., hypothesis 3° of Lemma 13 is satisfied as well. Now, by Lemma 13, the set of all convergent series

$$\sum_{p>p_0} \widehat{t}(p)\widehat{h}_p(s), \quad |\widehat{t}(p)| = 1, \tag{28}$$

is dense in $\mathfrak{H}(\mathfrak{D})$.

Fix a function $f(s) \in \mathfrak{H}(\mathfrak{D})$, $\varepsilon > 0$, and a compact set $K \subset \mathfrak{D}$. The denseness of the series (28) implies the existence of $|\widehat{t}(p)| = 1$ such that

$$\sup_{s\in K}\left| f(s) - \sum_{p_0<p\leqslant n} \log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}}{p^{ls}}\right) - \sum_{p>n}\widehat{t}(p)\log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}}{p^{ls}}\right)\right| < \frac{\varepsilon}{2}. \tag{29}$$

Let

$$t(p) = \begin{cases} \widehat{t}(p) & \text{if} \quad p > n, \\ 1 & \text{otherwise.} \end{cases}$$

Then, by (29),

$$\begin{aligned}
&\sup_{s\in K}\left| f(s) - \sum_{p>p_0}\log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}t^l(p)}{p^{ls}}\right)\right| \\
&\quad\leqslant \sup_{s\in K}\left| f(s) - \sum_{p_0<p\leqslant n}\log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}}{p^{ls}}\right) - \sum_{p>n}\log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}t^l(p)}{p^{ls}}\right)\right| \\
&\qquad + \sup_{s\in K}\left|\sum_{p>n}\widehat{t}(p)\log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}}{p^{ls}}\right) - \sum_{p>n}\log\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}t^l(p)}{p^{ls}}\right)\right| \\
&\quad< \frac{\varepsilon}{2} + \widehat{C}\sup_{s\in K}\sum_{p>n}\frac{1}{p^{2\sigma}} < \varepsilon, \quad \widehat{C} > 0,
\end{aligned}$$

provided $n \in \mathbb{N}$ is such that

$$\widehat{C}\sup_{s\in K}\sum_{p>n}\frac{1}{p^{2\sigma}} < \frac{\varepsilon}{2}.$$

The lemma is proved. □

For identification of the support of the random element $\zeta(s, \mathfrak{t}; \mathfrak{a})$, we recall the Hurwitz theorem; see, for example, Ref. [26], Section 3.4.5.

Lemma 16. *Let $\{g_n(s)\}$ be a sequence of functions analytic in the region G that converges uniformly on G to the function $g(s) \not\equiv 0$. Then an interior point s_0 of G is a zero of $g(s)$ if and only if there exists a sequence $\{s_n\} \subset G$ that converges to s_0 as $n \to \infty$ and $g_n(s_n) = 0$ for all sufficiently large n.*

Define the set

$$\mathfrak{S} = \{g \in \mathfrak{H}(\mathfrak{D}) : g(s) \neq 0 \text{ for } s \in \mathfrak{D}, \text{ or } g(s) \equiv 0\}.$$

Theorem 3. *The support of the measure $P_{\zeta,\mathfrak{a}}$ is the set $\mathfrak{S}$.*

Proof. We will show that the support of the random element $\zeta(s, \mathfrak{t}; \mathfrak{a})$ is the set $\mathfrak{S}$. Recall that

$$\zeta(s, \mathfrak{t}; \mathfrak{a}) = X_1(s)X_2(s),$$

where

$$X_1(s) = \prod_{p \leqslant p_0} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right), \qquad X_2(s) = \prod_{p > p_0} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right).$$

By the definition, $\{t(p) : p \in \mathbb{P}\}$ is a square of independent random variables defined on the probability space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), \mu_H)$. Then

$$\{x_p(s)\} \stackrel{\text{def}}{=} \left\{ \log\left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} t^l(p)}{p^{ls}}\right) : p > p_0 \right\}$$

is a sequence of independent $\mathfrak{H}(\mathfrak{D})$-valued random elements on the space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), \mu_H)$. The support of each $t(p)$ is the unit circle. Hence, the support of $x_p(s)$ is the set

$$\left\{ g \in \mathfrak{H}(\mathfrak{D}) : g(s) = \log\left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} a^l}{p^{ls}}\right), |a| = 1 \right\}.$$

Therefore, by Lemma 3.16 of [11], the support of

$$\sum_{p > p_0} x_p(s)$$

is the closure of all convergent series

$$\sum_{p > p_0} \log\left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} a^l(p)}{p^{ls}}\right), \quad |a(p)| > 1.$$

By Lemma 15, the latter set of series is dense in $\mathfrak{H}(\mathfrak{D})$. The map $M : \mathfrak{H}(\mathfrak{D}) \to \mathfrak{H}(\mathfrak{D})$ given by $M(g) = \mathrm{e}^g$, $g \in \mathfrak{H}(\mathfrak{D})$, is continuous, sending $\log X_2(s)$ to $X_2(s)$ and $\mathfrak{H}(\mathfrak{D})$ to $\mathfrak{S} \setminus \{0\}$. Hence, the support of $X_2(s)$ contains $\mathfrak{S} \setminus \{0\}$. Moreover, the support of the $\mathfrak{H}(\mathfrak{D})$-valued random element is a closed set. By virtue of Lemma 16, the closure of $\mathfrak{S} \setminus \{0\}$ is $\mathfrak{S}$. This shows that the support of $X_2(s)$ contains the set $\mathfrak{S}$.

By the definition, $X_2(s)$ is almost surely convergent for non-vanishing multipliers. Therefore, by Lemma 16, $X_2(s)$ lies in $\mathfrak{S}$ almost surely. Thus, the support of $X_2(s)$ lies in $\mathfrak{S}$. These remarks show that the support of $X_2(s)$ is the set $\mathfrak{S}$.

The element $\zeta(s, \mathfrak{t}; \mathfrak{a})$ is an almost surely convergent product. Therefore, the product $X_1(s)$ is not degenerate at zero. This and the independence of the variables $t(p)$ together with the support of $X_2(s)$ prove the theorem. □

4. Proof of Theorem 1

The proof of Theorem 1 is based on Theorems 2 and 3 and the Mergelyan theorem on the approximation of analytic functions by polynomials [27,28]. We state the Mergelyan theorem in a convenient form.

Lemma 17. *Suppose that $K \subset \mathbb{C}$ is a compact set with a connected complement and $g(s)$ is a function continuous on K and analytic inside of K. Then, for every $\varepsilon > 0$, there is a polynomial $p(s) = p_{\varepsilon,K,g}(s)$ such that*

$$\sup_{s \in K} \left| g(s) - p_{\varepsilon,K,g}(s) \right| < \varepsilon.$$

Proof of Theorem 1. By Lemma 17, for $\log f(s)$, there is a polynomial $p(s)$ such that

$$\sup_{s \in K} \left| f(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2}. \tag{30}$$

In view of Theorem 3, the function $e^{p(s)}$ belongs to the support of the measure $P_{\zeta,\mathfrak{a}}$. Hence,

$$P_{\zeta,\mathfrak{a}}(\mathfrak{G}_\varepsilon) > 0, \tag{31}$$

where

$$\mathfrak{G}_\varepsilon = \left\{ g \in \mathfrak{H}(\mathfrak{D}) : \sup_{s \in K} \left| g(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Define one more set

$$\widehat{\mathfrak{G}}_\varepsilon = \left\{ g \in \mathfrak{H}(\mathfrak{D}) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, taking into account (30), we see that $\mathfrak{G}_\varepsilon \subset \widehat{\mathfrak{G}}_\varepsilon$. This inclusion together with (31) implies that

$$P_{\zeta,\mathfrak{a}}(\widehat{\mathfrak{G}}_\varepsilon) > 0. \tag{32}$$

Now, we apply Theorem 2 using the equivalent of weak convergence of probability measures in terms of open sets; see Theorem 2.1 of [22]. This and (32) give

$$\liminf_{T \to \infty} P_{T,V,\mathfrak{a}}(\widehat{\mathfrak{G}}_\varepsilon) \geqslant P_{\zeta,\mathfrak{a}}(\widehat{\mathfrak{G}}_\varepsilon) > 0,$$

and the definitions of $P_{T,V,\mathfrak{a}}$ and $\widehat{\mathfrak{G}}_\varepsilon$ prove the first statement of Theorem 1.

For the proof of the second statement of Theorem 1, we observe that the boundaries $\partial\widehat{\mathfrak{G}}_\varepsilon$ of the sets $\widehat{\mathfrak{G}}_\varepsilon$ lie in the sets

$$\left\{ g \in \mathfrak{H}(\mathfrak{D}) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\},$$

hence, they do not intersect for different ε. From this, we have $P_{\zeta,\mathfrak{a}}(\partial\widehat{\mathfrak{G}}_\varepsilon) > 0$ for at most countably many values of ε. Hence, $P_{\zeta,\mathfrak{a}}(\partial\widehat{\mathfrak{G}}_\varepsilon) = 0$ for all but at most countably many $\varepsilon > 0$. In other words, the set $\widehat{\mathfrak{G}}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 2 and the equivalent of weak convergence of probability measures in terms of continuity sets, see Theorem 2.1 of [22], yield

$$\lim_{T \to \infty} P_{T,V,\mathfrak{a}}(\widehat{\mathfrak{G}}_\varepsilon) = P_{\zeta,\mathfrak{a}}(\widehat{\mathfrak{G}}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. This, and the definitions of $P_{T,V,\mathfrak{a}}$ and $\widehat{\mathfrak{G}}_\varepsilon$, give the second statement of the theorem. The theorem is proved. □

5. Conclusions

Approximation of analytic functions by simpler ones is an important problem in mathematics and applied natural sciences. In this paper, for the approximation of analytic functions, we apply zeta-functions

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \mathrm{Re}s > 1,$$

with periodic multiplicative coefficients a_m. For the reason of more effective detection of approximating shifts $\zeta(s+i\tau;\mathfrak{a})$, $\tau \in \mathbb{R}$, we consider the approximation in the so-called short intervals $\tau \in [T, T+V]$ with $T^{23/70} \leqslant V \leqslant T^{1/2}$. Using a method involving mean square estimates for $\zeta(s;\mathfrak{a})$ in short intervals and probabilistic limit theorems in the space of analytic functions, we proved that the set of the above shifts has a positive lower density (or density) as $T \to \infty$ for every non-vanishing analytic function defined in the strip $1/2 < \text{Re}s < 1$. The result obtained is also valid for all Dirichlet L-functions. The problems remain to decrease the lower bound for V, as well as to refuse the multiplicativity for coefficients a_m. Also, we are planning to consider the approximation of analytic functions by discrete shifts of the function $\zeta(s; \mathfrak{a})$ and to obtain a joint version of Theorem 1. Moreover, it is known [29] that the Bagchi theorem on the equivalent of the Riemann hypothesis (all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$) in terms of self-approximation can be proved by using topological dynamics. Therefore, we support the suggestions of the anonymous referee to extend probabilistic approaches in the study of dynamical systems [30] and fractional integrals [31] associated with zeta-functions.

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