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Article

On Joint Approximation of Analytic Functions by Beurling Zeta-Functions

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Abstract

For $j = 1, \ldots, r$, let $\mathscr{P}_j$ be a system of generalized prime numbers, $\mathscr{N}_{\mathscr{P}_j}$ the corresponding system of generalized integers, and $\zeta_{\mathscr{P}_j}(s)$, $s = \sigma + it$, the Beurling zeta-function. In the paper, we consider simultaneous approximation of a collection of analytic functions by shifts $(\zeta_{\mathscr{P}_1}(s_1 + i\tau), \ldots, \zeta_{\mathscr{P}_r}(s_r + i\tau))$. For this, we require that the summatory functions satisfy the bounds $\mathscr{M}_{\mathscr{P}_j}(x) - a_j x \ll_{\mathscr{P}_j} x^{\theta_j}$ with $a_j > 0$ and $0 \leqslant \theta_j < 1$, $j = 1, \ldots, r$. Moreover, we suppose that the mean square of $\zeta_{\mathscr{P}_j}(s)$ is bounded for $\sigma \in (\sigma_{\mathscr{P}_j}, 1)$ with some $\sigma_{\mathscr{P}_j}$ depending on $\mathscr{P}_j$ and θ_j. Then the main result of the paper asserts that there exists a closed non-empty set $\mathscr{F}_{\mathscr{P}_1,\ldots,\mathscr{P}_r}$ of analytic functions such that its functions $(f_1(s_1), \ldots, f_r(s_r))$ are approximated simultaneously by the above shifts of Beurling zeta-functions. It is proved that the set of approximating shifts has a positive lower density (or density with at most countably many exceptions of approximation accuracy). This shows a good joint approximation of Beurling zeta-functions.

Keywords: approximation of analytic functions; Beurling zeta-function; generalized integers; generalized primes; weak convergence of probability measures

MSC: 11M41

1. Introduction

Approximation of analytic functions is an old, important problem of function theory and has deep theoretical and practical applications. Many efforts were devoted to approximation by rational functions and polynomials; for results and references, see [1]. A significant result in the field has been obtained by S.N. Mergelyan [2,3]. He obtained a final result on the approximation of analytic functions by polynomials. The Mergelyan theorem says that every function continuous on a compact set $K \subset \mathbb{C}$ with connected complement and analytic in the interior of K can be approximated with desired accuracy by a polynomial. Examples show that the hypotheses on the set K cannot be reduced.

In 1975, new analytic objects for the approximation of analytic functions were found. It turned out that good approximation properties belong to so-called zeta-functions widely cultivated in analytic number theory. Roughly speaking, zeta-functions are functions of complex variables $s = \sigma + it$, in some half-plane $\sigma > \sigma_0$ defined by a Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{or, more generally,} \quad \sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s}$$

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with coefficients $a_m \in \mathbb{C}$ having a certain arithmetical sense. Here $\{\lambda_m\} \subset \mathbb{R}$ is an increasing sequence to $+\infty$. The name "zeta-functions" comes back to the Riemann zeta-function $\zeta(s)$, which, for $\sigma > 1$, is given by the series

$$\zeta(s) = \sum_{m=}^{\infty} \frac{1}{m^s},$$

and has the meromorphic continuation to the whole complex plane. Namely, the function $\zeta(s)$ was the first example of a zeta-function for which the universality in terms of approximation of analytic functions has been described. More precisely, S.M. Voronin fifty years ago proved [4], see also [5–8], that, for any non-vanishing analytic function $f(s)$ in the disc $|s| < r$, $r \in (0, 1/4)$, and continuous in $|s| \leqslant r$, and every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Voronin called the latter property of $\zeta(s)$ universality, because the shifts $\zeta(s + i\tau)$ approximate an entire class of analytic functions.

Now, the Voronin universality theorem has a more general form. Let $\mathscr{D} = \{s \in \mathbb{C} : \sigma \in (1/2, 1)\}$. Denote by $\mathscr{K}$ the class of compact sets of $\mathscr{D}$ with connected complements, and by $H^0(K)$, with $K \in \mathscr{K}$, the class of continuous functions on K that are analytic inside of K. Moreover, let $\text{meas}A$ stand for the Lebesgue measure of a measurable set $A \in \mathbb{R}$, and

$$\mathscr{U}_T(\dots) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : \dots\},$$

where in place of dots, a condition satisfied by τ is to be written. Then the following statement is known.

Proposition 1. *Suppose that $K \in \mathscr{K}$ and $f(s) \in H^0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathscr{U}_T\left(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \mathscr{U}_T\left(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right)$$

exists and is positive for all $\varepsilon > 0$, possibly except for at most countably many values of ε.

The first part of the proposition can be found in [9–11], while the second has been obtained in [12,13].

There exists a Linnik–Ibragimov conjecture that all functions defined in some half-plane by a Dirichlet series that admit analytic continuation to the left of this half-plane and satisfy certain growth hypotheses are universal in the above sense; see [10]. On the other hand, there are Dirichlet series that obviously are not universal. For example, if

$$a_m = \begin{cases} 2^l & \text{for } l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise}, \end{cases}$$

then

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s} = \sum_{l=0}^{\infty} \frac{1}{2^{ls}} = \frac{1}{1 - 2^{-s}}$$

is not a universal function [10]. Therefore, it is important to find new universal functions in order to use them in resulting applications.

In [14], the universality of Beurling zeta-functions has begun to be studied. These zeta-functions were introduced in [15]; we recall their definition. Arbitrary infinity system $\mathscr{P} = \{p_n\}$,

$$1 < p_1 \leqslant \cdots \leqslant p_n \leqslant \cdots, \qquad \lim p_n = +\infty,$$

is called generalized prime numbers. The system $\mathscr{P}$ generates generalized integers

$$p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \quad p_j \in \mathscr{P},\ \alpha_j \in \mathbb{N}_0,\ j = 1, \dots, r,\ r \in \mathbb{N}.$$

Denote their system by $\mathscr{N}_{\mathscr{P}}$. Notice that the numbers in the systems $\mathscr{P}$ and $\mathscr{N}_{\mathscr{P}}$ are not necessarily different.

For the investigation of the distribution of generalized primes

$$\pi_{\mathscr{P}}(x) = \sum_{\substack{p \leqslant x \\ p \in \mathscr{P}}} 1, \quad x > 1,$$

A. Beurling introduced the zeta-function $\zeta_{\mathscr{P}}(s)$, defined, for $\sigma > \sigma_{\mathscr{P}}$, by the series

$$\zeta_{\mathscr{P}}(s) = \sum_{m \in \mathscr{N}_{\mathscr{P}}} \frac{1}{m^s},$$

or by the product

$$\zeta_{\mathscr{P}}(s) = \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For a precise definition of $\zeta_{\mathscr{P}}(s)$, the summatory function of generalized integers

$$\mathscr{M}_{\mathscr{P}}(x) = \sum_{\substack{m \leqslant x \\ m \in \mathscr{N}_{\mathscr{P}}}} 1$$

can be applied. Recall that $a \ll_\varepsilon b$, $a \in \mathbb{C}$, $b > 0$, indicates that there exists a constant $c = c(\varepsilon) > 0$ such that $|a| \leqslant cb$.

There are several works that connect estimates for $\pi_{\mathscr{P}}(x)$ and $\mathscr{M}_{\mathscr{P}}(x)$; see [16] for references. Among the results for $\mathscr{M}_{\mathscr{P}}(x)$, the bound

$$\mathscr{M}_{\mathscr{P}}(x) - ax \ll x^\theta, \quad a > 0,\ 0 \leqslant \theta < 1, \tag{1}$$

is popular. It is known [16] that, under (1), the Dirichlet series defining $\zeta_{\mathscr{P}}(s)$ is absolutely convergent in the half-plane $\sigma > 1$, and

$$\zeta_{\mathscr{P}}(s) = \sum_{m \in \mathscr{N}_{\mathscr{P}}} \frac{1}{m^s} = \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Furthermore, the function $\zeta_{\mathscr{P}}(s)$ can be continued analytically to the half-plane $\sigma > \theta$, except for a simple pole at the point $s = 1$ with $\text{Res}_{s=1}\zeta_{\mathscr{P}}(s) = a$.

In [14], under (1), a theorem on the approximation of analytic functions by shifts $\zeta_{\mathscr{P}}(s + i\tau)$, $\tau \in \mathbb{R}$, has been obtained. Define

$$M_{\mathscr{P}}(\sigma, T) = \frac{1}{T} \int_0^T |\zeta_{\mathscr{P}}(\sigma + it)|^2 \, \mathrm{d}t, \quad T > 0,$$

and denote by $\widehat{\sigma}$ the infimum of such σ such that $M_{\mathscr{P}}(\sigma, T) \ll_\sigma 1$. Set $\sigma_{\mathscr{P}} = \max(\theta, \widehat{\sigma})$, and $\mathscr{D}_{\mathscr{P}} = \{s \in \mathbb{C} : \sigma \in (\sigma_{\mathscr{P}}, 1)\}$. Denote by $\mathscr{H}(\mathscr{D}_{\mathscr{P}})$ the space of analytic functions on $\mathscr{D}_{\mathscr{P}}$ endowed with the topology of uniform convergence on compact sets. The main result of [14] is the following theorem.

Theorem 1. *Suppose that the estimate* (1) *holds. Then there is a closed non-empty set $\mathscr{F}_{\mathscr{P}} \subset \mathscr{H}(\mathscr{D}_{\mathscr{P}})$ such that, for all compact sets $K \subset \mathscr{D}_{\mathscr{P}}$, $f(s) \in \mathscr{F}_{\mathscr{P}}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathscr{U}_T\left(\sup_{s\in K}|\zeta_{\mathscr{P}}(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \mathscr{U}_T\left(\sup_{s\in K}|\zeta_{\mathscr{P}}(s+i\tau) - f(s)| < \varepsilon\right)$$

exists and is positive for all $\varepsilon > 0$, possibly except for at most countably many values of ε.

In the universality theory of zeta-functions, the joint universality occupies a significant place. In this case, a collection of analytic functions is simultaneously approximated by a collection of shifts of zeta-functions. A characterizing example of joint universality is that of Dirichlet L-functions. We recall that every arithmetic function χ is a Dirichlet character modulo $q \in \mathbb{N}$ if χ is periodic with period q, completely multiplicative ($\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all $m_1, m_2 \in \mathbb{N}$), $\chi(m) = 0$ for $(m, q) > 1$ ((m, q) is the largest common divisor of m and q), and $\chi(m) \neq 0$ for $(m, q) = 1$. A Dirichlet L-function $L(s, \chi)$ with character χ is defined by the series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

and has a meromorphic continuation to the whole complex plane.

The character χ modulo q is called generated by a character $\widehat{\chi}(m)$ modulo $\widehat{q} \mid q$ if

$$\chi(m) = \begin{cases} \widehat{\chi}(m) & \text{if } \widehat{q} \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

The character χ is primitive if it is not generated by a character modulo $\widehat{q} \mid q$. The characters χ and $\widehat{\chi}$ are called non-equivalent if they are not generated by the same primitive character.

The joint universality for Dirichlet L-functions is the following statement.

Proposition 2 (see [17,18])**.** *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathscr{K}$ and $f_j(s) \in H^0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathscr{U}_T\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|L(s+i\tau,\chi_j) - f_j(s)\right| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \mathscr{U}_T\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|L(s+i\tau,\chi_j) - f_j(s)\right| < \varepsilon\right)$$

exists and is positive for all $\varepsilon > 0$, possibly except for at most countably many values of ε.

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The latter proposition shows that, in the case of joint universality, the approximating functions must be in a certain sense independent.

We observe that Dirichlet L-functions are well-studied, comparatively simple, and similar to each other. In the case of Beurling zeta-functions, we have another picture: every function depends on its system of generalized primes and can have absolutely different properties. Therefore, investigation of the simultaneous behavior of Beurling zeta-functions is a deep and important problem of pure and applied mathematics.

The aim of this paper is a joint extension of Theorem 1. Thus, for $j = 1, \dots, r$, $r \in \mathbb{N}$, let $\mathscr{P}_j$ be a system of generalized primes, $\mathscr{N}_{\mathscr{P}_j}$ the corresponding system of generalized integers, and suppose that the estimate

$$\mathscr{M}_{\mathscr{P}_j}(x) - a_j x \ll_{\mathscr{P}_j} x^{\theta_j}, \quad a_j > 0,\ 0 \leqslant \theta_j < 1, \tag{2}$$

holds. We will consider the approximation of a collection of analytic functions by shifts $(\zeta_{\mathscr{P}_1}(s+i\tau), \dots, \zeta_{\mathscr{P}_r}(s+i\tau))$, where $\zeta_{\mathscr{P}_j}(s)$ is given by

$$\zeta_{\mathscr{P}_j}(s) = \sum_{m \in \mathscr{N}_{\mathscr{P}_j}} \frac{1}{m^s} = \prod_{p \in \mathscr{P}_j} \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 1,\ j = 1, \dots, r,$$

and by analytic continuation for $\theta_j < \sigma < 1$. Denote by $\widehat{\sigma}_j$ the infimum of σ satisfying $M_{\mathscr{P}_j}(\sigma, T) \ll_\sigma 1$. Let $\sigma_{\mathscr{P}_j} = \max(\theta_j, \widehat{\sigma}_j)$ and $\mathscr{D}_{\mathscr{P}_j} = \{s \in \mathbb{C} : \sigma \in (\sigma_{\mathscr{P}}, 1)\}$, and introduce the space $\mathscr{H}(\mathscr{D}_{\mathscr{P}_j})$ of analytic functions on $\mathscr{D}_{\mathscr{P}_j}$. Moreover, let

$$\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r}) = \mathscr{H}(\mathscr{D}_{\mathscr{P}_1}) \times \dots \times \mathscr{H}(\mathscr{D}_{\mathscr{P}_r}).$$

Now, we state the main result of the paper.

Theorem 2. *Suppose that the estimates (2) are valid. Then there exists a closed non-empty set $\mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r} \subset \mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$ such that, for compact sets $K_1 \subset \mathscr{D}_{\mathscr{P}_1}, \dots, K_r \subset \mathscr{D}_{\mathscr{P}_r}$, $(f_1(s_1), \dots, f_r(s_r)) \in \mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r}$ and every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathscr{U}_T \left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| \zeta_{\mathscr{P}_j}(s+i\tau) - f_j(s) \right| < \varepsilon \right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \mathscr{U}_T \left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| \zeta_{\mathscr{P}_j}(s+i\tau) - f_j(s) \right| < \varepsilon \right)$$

exists and is positive for all $\varepsilon > 0$, possibly except for at most countably many values of ε.

Mean square estimates for Beurling zeta-functions occupy an important place in the statements of Theorems 1 and 2. Mean squares of $\zeta_{\mathscr{P}}(s)$ have been considered in [19,20]. Write the system of generalized integers $\mathscr{N}_{\mathscr{P}}$ as a strictly increasing unbounded sequence $\{n_j : j \in \mathbb{N}\}$ with multiplicity h_j of n_j. Then

$$\zeta_{\mathscr{P}}(s) = \sum_{j=1}^{\infty} \frac{h_j}{n_j^s},$$

and (1) can be written in the form

$$\sum_{n_j \leqslant x} h_j - ax \ll x^{\theta}. \tag{3}$$

In [20], it was obtained under estimate (3), that, for $\sigma > (1+\theta)/2$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_{\mathscr{P}}(\sigma+it)|^2 \, \mathrm{d}t = \sum_{j=1}^{\infty} \frac{h_j^2}{n_j^{2\sigma}}.$$

The latter series is convergent for $\sigma > (1+\theta)/2$ [20], thus,

$$\frac{1}{T} \int_0^T |\zeta_{\mathscr{P}}(\sigma+it)|^2 \, \mathrm{d}t \ll_{\sigma} 1$$

in that half-plane. In [19], the same result has been obtained under an additional requirement that with every $\delta > 0$

$$n_{j+1} - n_j \gg \exp\{-n_j^{\delta}\}.$$

For the proof of Theorem 2, we will apply a limit theorem on weakly convergent probability measures in the space $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$. We recall that the principles of weak convergence theory have been given by A.N. Kolmogorov [21], P. Erdös and M. Kac [22], J.L. Doob [23], and M. Donsker [24] and were developed by Yu.V. Prokhorov [25], A.V. Skorokhod [26], and V.S. Varadarajan [27]. The modern theory of weak convergence of probability measures is fixed in the monograph [28].

We will introduce a certain $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$-valued random element and will show that a probability measure defined by means of $(\zeta_{\mathscr{P}_1}(s+i\tau), \dots, \zeta_{\mathscr{P}_r}(s+i\tau))$ converges weakly to the distribution of that random element. For this, classical elements of weak convergence of probability measures will be applied.

2. Case of *r*-Dimensional Torus

Denote by $\mathscr{B}(\mathscr{X})$ the Borel σ-field of the space $\mathscr{X}$. For brevity, let

$$\underline{\zeta}_{\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1, \dots, s_r) = \big(\zeta_{\mathscr{P}_1}(s_1), \dots, \zeta_{\mathscr{P}_r}(s_r)\big).$$

We will derive Theorem 2 from the weak convergence of the probability measure

$$P^r_{T,\underline{\zeta}}(A) = \mathscr{U}_T\Big(\underline{\zeta}_{\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1+i\tau, \dots, s_r+i\tau) \in A\Big), \quad \mathscr{B}(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})),$$

as $T \to \infty$. We recall that $P^r_{T,\underline{\zeta}}$ converges weakly to a certain probability measure $P^r_{\underline{\zeta}}$ ($P^r_{T,\underline{\zeta}} \xrightarrow[T\to\infty]{} P^r_{\underline{\zeta}}$) if, for every real continuous bounded function h on $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$,

$$\lim_{T\to\infty} \int\limits_{\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1},\dots,\mathscr{D}_{\mathscr{P}_r})} h \, \mathrm{d}P^r_{T,\underline{\zeta}} = \int\limits_{\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1},\dots,\mathscr{D}_{\mathscr{P}_r})} h \, \mathrm{d}P^r_{\underline{\zeta}}. \tag{4}$$

Also, there are several equivalents of weak convergence in terms of some classes of sets [28]. It is difficult to prove equality (4) directly. Therefore, we will divide the proof into parts and use some transforms and other classical methods of weak convergence theory.

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In this section, we start with the weak convergence of probability measures on a certain group. Let Δ be the unit circle on the complex plane, i.e., $\Delta = \{s \in \mathbb{C} : |s| = 1\}$. For $j = 1, \dots, r$, by the Cartesian product, define the set

$$\mathbb{T}_j = \prod_{p \in \mathscr{P}_j} \Delta.$$

($\mathbb{T}_j$ consists of all functions $\mathrm{t}_j = (t_j(p) : |t_j(p)| = 1, p \in \mathscr{P}_j)$). With the operation of pointwise multiplication and product topology, the torus $\mathbb{T}_j$ is a compact topological group. Put

$$\mathbb{T} = \mathbb{T}_1 \times \cdots \times \mathbb{T}_r.$$

Then again, by the Tikhonov theorem [29], $\mathbb{T}$ is a compact topological group. From this, it follows that, on $(\mathbb{T}, \mathscr{B}(\mathbb{T}))$, the probability Haar measure μ_H can be defined, and we have the probability space $(\mathbb{T}, \mathscr{B}(\mathbb{T}), \mu_H)$.

For $A \in \mathscr{B}(\mathbb{T})$, define

$$P_{T,r}^{\mathbb{T}}(A) = \mathscr{U}_T\Big(\Big(\big(p^{-i\tau} : p \in \mathscr{P}_1\big), \dots, \big(p^{-i\tau} : p \in \mathscr{P}_r\big)\Big) \in A\Big).$$

Lemma 1. *On $(\mathbb{T}, \mathscr{B}(\mathbb{T}))$, there exists a probability measure $P_r^{\mathbb{T}}$ such that $P_{T,r}^{\mathbb{T}} \underset{T\to\infty}{\Longrightarrow} P_r^{\mathbb{T}}$.*

Proof. Since the group $\mathbb{T}$ is compact, for the investigation of weak convergence of $P_{T,r}^{\mathbb{T}}$, it is convenient [30] to apply the Fourier transform method. For the definition of the Fourier transform in $\mathbb{T}$, the characters of $\mathbb{T}$ are involved. For $j = 1, \dots, r$, denote by $\mathrm{t}_j = (t_j(p) : p \in \mathscr{P}_j)$ elements of $\mathbb{T}_j$, and by $\mathrm{t} = (\mathrm{t}_1, \dots, \mathrm{t}_r)$ elements of $\mathbb{T}$. Then the characters of $\mathbb{T}$ are of the form

$$\prod_{j=1}^{r} \prod_{p \in \mathscr{P}_j} t_j^{k_{jp}}(p),$$

where $k_{jp} \in \mathbb{Z}$ and only a finite number of them are not zero. Hence, the Fourier transform $F_{T,r}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathscr{P}_j)$, $j = 1, \dots, r$, is given by

$$F_{T,r}(\underline{k}_1, \dots, \underline{k}_r) = \int_{\mathbb{T}} \left(\prod_{j=1}^{r} \prod_{p \in \mathscr{P}_j} t_j^{k_{jp}}(p) \right) \mathrm{d}P_{T,r}^{\mathbb{T}} = \frac{1}{T} \int_0^T \left(\prod_{j=1}^{r} \prod_{p \in \mathscr{P}_j}{}^{*} p^{-i\tau k_{jp}} \right) \mathrm{d}\tau,$$

where the star $*$ indicates that only a finite number of k_{jp} are not zeros. Thus, we have

$$F_{T,r}(\underline{k}_1, \dots, \underline{k}_r) = \frac{1}{T} \int_0^T \exp\left\{ -i\tau \sum_{j=1}^{r} \sum_{p \in \mathscr{P}_j}{}^{*} k_{jp} \log p \right\} \mathrm{d}\tau. \tag{5}$$

Divide the set of all k_{jp} into two parts

$$\mathcal{K}_1 \overset{\mathrm{def}}{=} \left\{ (\underline{k}_1, \dots, \underline{k}_r) : \sum_{j=1}^{r} \sum_{p \in \mathscr{P}_j}{}^{*} k_{jp} \log p = 0 \right\}$$

and

$$\mathcal{K}_2 \overset{\mathrm{def}}{=} \left\{ (\underline{k}_1, \dots, \underline{k}_r) : \sum_{j=1}^{r} \sum_{p \in \mathscr{P}_j}{}^{*} k_{jp} \log p \neq 0 \right\}.$$

Then equality (5) implies

$$F_{T,r}(\underline{k}_1,\dots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in\mathcal{K}_1,\\ \frac{1-\exp\{-iT\sum_{j=1}^{r}\sum^{*}_{p\in\mathscr{P}_j}k_{jp}\log p\}}{iT\exp\{-i\sum_{j=1}^{r}\sum^{*}_{p\in\mathscr{P}_j}k_{jp}\log p\}} & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in\mathcal{K}_2.\end{cases}$$

Thus,

$$\lim_{T\to\infty}F_{T,r}(\underline{k}_1,\dots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in\mathcal{K}_1,\\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in\mathcal{K}_2.\end{cases}$$

Since $\mathbb{T}$ is a compact group, by Theorem 1.4.2 of [30], it is a Lévy group. Hence, the convergence of Fourier transforms implies weak convergence of the corresponding probability measures. Therefore, $P^{\mathbb{T}}_{T,r}$ converges weakly to the probability measure $P^{\mathbb{T}}_{r}$ on $(\mathbb{T},\mathscr{B}(\mathbb{T}))$ defined by the Fourier transform

$$F_{T,r}(\underline{k}_1,\dots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in\mathcal{K}_1,\\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in\mathcal{K}_2.\end{cases}$$

□

For $j=1,\dots,r$, denote $\underline{0}_j=(k_{jp}=0,p\in\mathscr{P}_j)$. Then

$$F_{\mu_H}(\underline{k}_1,\dots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)=(\underline{0}_1,\dots,\underline{0}_r),\\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\neq(\underline{0}_1,\dots,\underline{0}_r),\end{cases} \tag{6}$$

is the Fourier transform of the Haar measure μ_H on $(\mathbb{T},\mathscr{B}(\mathbb{T}))$.

Define the set

$$\mathscr{L}(\mathscr{P}_1,\dots,\mathscr{P}_r)=\big\{(\log p:p\in\mathscr{P}_j),j=1,\dots,r\big\}.$$

Actually, $\mathscr{L}(\mathscr{P}_1,\dots,\mathscr{P}_r)$ is a multiset; its elements are not necessarily different. Lemma 1 and the Fourier transform (6) give the following corollary.

Corollary 1. *Suppose that the set $\mathscr{L}(\mathscr{P}_1,\dots,\mathscr{P}_r)$ is linearly independent over the field of rational numbers. Then $P^{\mathbb{T}}_{T,r}\xrightarrow[T\to\infty]{}\mu_H$.*

We notice that the probability space $(\mathbb{T},\mathscr{B}(\mathscr{P}),\mu_H)$ and Corollary 1 can be applied to construct on $(\mathbb{T},\mathscr{B}(\mathbb{T}),\mu_H)$ a certain $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1},\dots,\mathscr{D}_{\mathscr{P}_r})$-valued random element and to consider weak convergence for $P^{r}_{T,\underline{\zeta}}$ to the distribution of that element. We are planning to realize this idea for the identification of the set $\mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r}$ of approximating functions in forthcoming papers.

3. Absolute Convergence

We notice that the application of the probabilistic limit theorem for the characterization of behavior of zeta-functions comes back to H. Bohr and B. Jessen [31,32] and was continued by B. Jessen and A. Wintner [33], V. Borchsenius and B. Jessen [34], A. Selberg [35], and D. Joyner [36]; for results, see the survey paper [37] and [9–11].

In this section, we introduce absolutely convergent Dirichlet series connected to the zeta-functions $\zeta_{\mathscr{P}_j}(s)$, $j=1,\dots,r$. We fix a number $\alpha>1-\min_{1\leqslant j\leqslant r}\sigma_{\mathscr{P}_j}$ and define

$$b_{n,j}(m)=\exp\Big\{-\Big(\frac{m}{n}\Big)^{\alpha}\Big\},\quad n\in\mathbb{N},\ m\in\mathscr{P}_j,\ j=1,\dots,r.$$

Suppose that the estimates (2) are satisfied. Then the series

$$\zeta_{n;\mathscr{P}_j}(s) = \sum_{m \in \mathscr{N}_j} \frac{b_{n,j}(m)}{m^s}, \quad j = 1, \dots, r,$$

are absolutely convergent in any half-plane $\sigma > \sigma_0$ with arbitrary σ_0. Let

$$\underline{\zeta}_{n;\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1, \dots, s_r) = \big(\zeta_{n;\mathscr{P}_1}(s_1), \dots, \zeta_{n;\mathscr{P}_r}(s_r)\big).$$

In this section, we will consider weak convergence of probability measure

$$P_{T,\underline{\zeta}_n}(A) = \mathscr{U}_T\Big(\underline{\zeta}_{n;\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1 + i\tau, \dots, s_r + i\tau) \in A\Big), \quad A \in \mathscr{B}(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})).$$

For this, we will apply Lemma 1 and the continuous mapping theorem. For convenience of the reader, we briefly recall a situation. Let $(\mathscr{X}_1, \mathscr{B}(\mathscr{X}_1))$ and $(\mathscr{X}_2, \mathscr{B}(\mathscr{X}_2))$ be two measurable spaces, and $\psi : \mathscr{X}_1 \to \mathscr{X}_2$ a $(\mathscr{B}(\mathscr{X}_1), \mathscr{B}(\mathscr{X}_2))$-measurable mapping, i.e., $\psi^{-1}\mathscr{B}(\mathscr{X}_2) \subset \mathscr{B}(\mathscr{X}_1)$. Then every probability measure Q on $(\mathscr{X}_1, \mathscr{B}(\mathscr{X}_1))$ defines the unique probability measure $Q\psi^{-1}$ on $(\mathscr{X}_2, \mathscr{B}(\mathscr{X}_2))$ by the equality

$$Q\psi^1(A) = Q(\psi^{-1}A), \quad A \in \mathscr{B}(\mathscr{X}_2).$$

Moreover, the following convenient statement holds.

Lemma 2. *Suppose that Q_n, $n \in \mathbb{N}$, and Q are probability measures on $(\mathscr{X}_1, \mathscr{B}(\mathscr{X}_1))$ such that $Q_n \underset{n\to\infty}{\Longrightarrow} Q$, and ψ is a continuous mapping. Then $Q_n\psi^{-1} \underset{n\to\infty}{\Longrightarrow} Q\psi^{-1}$.*

The lemma is discussed at the beginning of Section 2.3 of [28]. Also, it is a particular case of a more general Theorem 2.7 from [28].

Extend the functions $t_j(p)$, $p \in \mathscr{P}_j$, to the set $\mathscr{N}_{\mathscr{P}_j}$ of generalized integers by using the formula

$$t_j(m) = t_j^{\alpha_{1j}}(p_{1j}) \cdots t_j^{\alpha_{lj}}(p_{lj}) \tag{7}$$

if $m = p_{1j}^{\alpha_{1j}} \cdots p_{lj}^{\alpha_{lj}} \in \mathscr{N}_{\mathscr{P}_j}$, $j = 1, \dots r$.

Now, we are ready to state a limit lemma for the measure $P_{T,\underline{\zeta}_n}$.

Lemma 3. *Suppose that the systems $\mathscr{P}_j$ satisfy axioms (2), $j = 1, \dots, r$. Then, on $(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r}), \mathscr{B}(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})))$, there exists a probability measure$P_{\underline{\zeta}_n}$ such that $P_{T,\underline{\zeta}_n} \underset{T\to\infty}{\Longrightarrow} P_{\underline{\zeta}_n}$.*

Proof. Introduce the mapping $\psi_{n,r} : \mathbb{T} \to \mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$ defined by

$$\psi_{n,r}(\mathrm{t}) = \left(\sum_{m \in \mathscr{N}_{\mathscr{P}_1}} \frac{t_1(m) b_{n,1}(m)}{m^{s_1}}, \dots, \sum_{m \in \mathscr{N}_{\mathscr{P}_r}} \frac{t_r(m) b_{n,r}(m)}{m^{s_r}} \right).$$

Hence, we have

$$\begin{aligned} \psi_{n,r}\Big(\Big(p^{-i\tau} : p \in \mathscr{P}_1\Big), \dots, \Big(p^{-i\tau} : p \in \mathscr{P}_r\Big)\Big) &= \left(\sum_{m \in \mathscr{N}_{\mathscr{P}_1}} \frac{b_{n,1}(m)}{m^{s_1 + i\tau}}, \dots, \sum_{m \in \mathscr{N}_{\mathscr{P}_r}} \frac{b_{n,r}(m)}{m^{s_r + i\tau}} \right) \\ &= (\zeta_{n;\mathscr{P}_1}(s_1 + i\tau), \dots, \zeta_{n;\mathscr{P}_r}(s_r + i\tau)) \end{aligned} \tag{8}$$

because, in view of (7), in this case

$$t_j(m) = p_{1j}^{-i\alpha_{1j}\tau} \cdots p_{lj}^{-i\alpha_{lj}\tau} = m^{-i\tau}, \quad j = 1, \ldots, r.$$

Since $|t_j(m)| = 1$, the series

$$\sum_{m \in \mathcal{N}_{\mathcal{P}_j}} \frac{t_j(m) b_{n,j}(m)}{m^{s_j}}$$

are absolutely convergent for $\sigma > \sigma_0$ with every σ_0, and the mapping $\psi_{n,m}$ is continuous. Moreover, the definitions of the measures $P_{T,\underline{\zeta}_n}$ and $P^{\mathbb{T}}_{T,r}$, and (8) imply that, for all $A \in \mathcal{B}(\mathcal{H}^r(\mathcal{D}_{\mathcal{P}_1}, \ldots, \mathcal{D}_{\mathcal{P}_r}))$,

$$P_{T,\underline{\zeta}_n}(A) = \mathcal{U}_T\Big(\Big(\Big(p^{-i\tau} : p \in \mathcal{P}_1\Big), \ldots, \Big(p^{-i\tau} : p \in \mathcal{P}_r\Big)\Big) \in \psi^{-1}_{n,r} A\Big) = P^{\mathbb{T}}_{T,r}\Big(\psi^{-1}_{n,r} A\Big).$$

Therefore, the equality $P_{T,\underline{\zeta}_n} = P^{\mathbb{T}}_{T,r}\psi^{-1}_{n,r}$ holds. This and continuity of $\psi_{n,r}$ allow an application of Lemmas 1 and 2, and we obtain that $P_{T,\underline{\zeta}_n} \underset{T\to\infty}{\Longrightarrow} P^r_{\underline{\zeta}_m}$ with $P^r_{\underline{\zeta}_n} = P^{\mathbb{T}}_{T,r}\psi^{-1}_{n,r}$. □

For the proof the weak convergence of the measure $P^r_{T,\underline{\zeta}}$, we will need good convergence properties for $P^r_{\underline{\zeta}_n}$ as $n \to \infty$. More precisely, it suffices to show that the measure $P_{\underline{\zeta}_n}$ is tight. Recall that the family of probability measures $\{Q\}$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is tight if, for every $\varepsilon > 0$, there is a compact set $K = K_\varepsilon$ such that

$$Q(K) > 1 - \varepsilon$$

for all Q.

Lemma 4. *Suppose that the systems $\mathcal{P}_j$ satisfy axioms (2). Then the measure $P^r_{\underline{\zeta}_n}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K_\varepsilon \subset \mathcal{H}^r(\mathcal{D}_{\mathcal{P}_1}, \ldots, \mathcal{D}_{\mathcal{P}_r})$ such that*

$$P^r_{\underline{\zeta}_n}(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

Proof. For $j = 1, \ldots, r$, denote by $P^r_{\underline{\zeta}_n,j}$ the marginal measure of $P^r_{\underline{\zeta}_n}$, i.e.,

$$P^r_{\underline{\zeta}_n,j}(A) = P^r_{\underline{\zeta}_n}(\underbrace{\mathcal{H}(\mathcal{D}_{\mathcal{P}_1}), \ldots, \mathcal{H}(\mathcal{D}_{\mathcal{P}_{j-1}})}_{j-1}, A, \mathcal{H}(\mathcal{D}_{\mathcal{P}_{j+1}}), \ldots, \mathcal{H}(\mathcal{D}_{\mathcal{P}_r})), \quad A \in \mathcal{H}(\mathcal{D}_{\mathcal{P}_j}).$$

Then it is known by Lemma 6 of [14] that $P^r_{\underline{\zeta}_n,j}$ is tight. Hence, for every $\varepsilon > 0$, there is a compact subset $K_j = K_j(\varepsilon) \subset \mathcal{H}(\mathcal{D}_{\mathcal{P}_j})$ satisfying

$$P^r_{\underline{\zeta}_n,j}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \ldots, r, \tag{9}$$

for all $n \in \mathbb{N}$. Denote by K the Cartesian product $K_1 \times \cdots \times K_r$. Then K is a compact subset in the space $\mathcal{H}^r(\mathcal{D}_{\mathcal{P}_1}, \ldots, \mathcal{D}_{\mathcal{P}_r})$. Moreover, in virtue of (9),

$$\begin{aligned} P^r_{\underline{\zeta}_n}(\mathcal{H}^r(\mathcal{D}_{\mathcal{P}_1}, \ldots, \mathcal{D}_{PP_r}) \setminus K) &= P_{\underline{\zeta}^r_n}\Big(\bigcup_{j=1}^{r}\Big(\mathcal{H}(\mathcal{D}_{\mathcal{P}_j}) \setminus K_j\Big)\Big) \\ &\leqslant \sum_{j=1}^{r} P_{\underline{\zeta}_n,j}\Big(\mathcal{H}(\mathcal{D}_{\mathcal{P}_j}) \setminus K_j\Big) < \sum_{j=1}^{r} \frac{\varepsilon}{r} = \varepsilon \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $P^r_{\underline{\zeta}_n} > 1 - \varepsilon$ for all $n \in \mathbb{N}$. The lemma is proved. □

To pass from the measure $P^r_{\underline{\zeta}_n}$ to $P^r_{T,\underline{\zeta}}$, we need the estimate in the mean between $\underline{\zeta}(s_1, \dots, s_r)$ and $\underline{\zeta}_n(s_1, \dots, s_r)$. This will be realized in the next section.

4. Some Estimates

First, we recall the metric in $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$ inducing its topology. We start with metrics in $\mathscr{H}(\mathscr{D}_{\mathscr{P}_j})$, $j = 1, \dots, r$. For $j = 1, \dots, r$, there exists a sequence $\{K_{k,j}\} \subset \mathscr{D}_{\mathscr{P}_j}$ of compact embedded subsets satisfying

$$\mathscr{D}_{\mathscr{P}_j} = \bigcup_{k=1}^{\infty} K_{k,j},$$

and $K_j \subset K_{k,j}$ with some k for every compact set $K_j \in \mathscr{D}_{\mathscr{P}_j}$ [38]. For $h_1, h_2 \in \mathscr{H}(\mathscr{D}_{\mathscr{P}_j})$, define

$$d_j(h_1, h_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{\sup_{s \in K_{k,j}} |h_1(s) - h_2(s)|}{1 + \sup_{s \in K_{k,j}} |h_1(s) - h_2(s)|}.$$

Then d_j is a metric in $\mathscr{H}(\mathscr{D}_{\mathscr{P}_j})$ inducing its topology of uniform convergence on compacta. Let $\underline{h}_l = (h_{l1}, \dots, h_{lr}) \in \mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$, $l = 1, 2$. Then setting

$$d(\underline{h}_1, \underline{h}_2) = \max_{1 \leqslant j \leqslant r} d_j(h_{1j}, h_{2j})$$

is a metric in $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$ which induces the product topology.

For approximation $\underline{\zeta}(s_1, \dots, s_r)$ by $\underline{\zeta}_n(s_1, \dots, s_r)$ in the mean, we will state some one-dimensional results from [14]. Let the number α be the same as in the definition of the coefficients $b_{n,j}(m)$. Define

$$\kappa_n(s) = \frac{1}{\alpha} \Gamma\left(\frac{s}{\alpha}\right) n^s, \quad n \in \mathbb{N},$$

where, as usual, $\Gamma(s)$ is the Euler gamma function.

Lemma 5. *Suppose that the systems $\mathscr{P}_j$ satisfy axioms (2). Then, for $s \in \mathscr{D}_{\mathscr{P}_j}$, the representation*

$$\zeta_{n;\mathscr{P}_j}(s) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \zeta_{\mathscr{P}_j}(s + z) \kappa_n(z) \, \mathrm{d}z, \quad j = 1, \dots, r,$$

is valid.

Proof. By the definition of α, it follows that, for all $j = 1, \dots, r$, the inequality $\operatorname{Re}(s + z) > 1$. Therefore, repeating the arguments used in the proof of Lemma 3 of [14] with application of the Mellin formula and the series

$$\zeta_{\mathscr{P}_j}(s + z) = \sum_{m \in \mathscr{N}_{\mathscr{P}_j}} \frac{1}{m^{s+z}},$$

we obtain the statement of the lemma. □

Now, we state the main result of the section.

Proposition 3. *Suppose that the systems $\mathscr{P}_j$ satisfy axioms (2). Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T d\Big(\underline{\zeta}_{\mathscr{P}_j}(s_1+i\tau,\ldots,s_r+i\tau),\underline{\zeta}_{n;\mathscr{P}_j}(s_1+i\tau,\ldots,s_r+i\tau)\Big)\,\mathrm{d}\tau=0.$$

Proof. The definitions of the metrics d and d_j show that it suffices to obtain that, for every compact set $K_j \subset \mathscr{D}_{\mathscr{P}_j}$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K_j}\Big|\zeta_{\mathscr{P}_j}(s+i\tau)-\zeta_{n;\mathscr{P}_j}(s+i\tau)\Big|\,\mathrm{d}\tau=0,\quad j=1,\ldots,r. \tag{10}$$

To prove (10), we fix $K_j \subset \mathscr{D}_{\mathscr{P}_j}$ and apply Lemma 5. We fix $\varepsilon=\varepsilon(K_j)$ such that the set K_j lies in the strip $\sigma \in [\sigma_{\mathscr{P}_j}+2\varepsilon, 1-\varepsilon]$. Take $\alpha = 1-\varepsilon$ and $\alpha_1=\sigma_{\mathscr{P}_j}+\varepsilon-\sigma$. Obviously, $\alpha_1<0$. It is well known that the function $\Gamma(s)$ has a simple pole at the point $s=0$, and $\zeta_{\mathscr{P}_j}(s)$ has a simple pole at the point $s=1$. Therefore, an application of Lemma 5 and the residue theorem, for all $s=\sigma+it\in K_j$, yields

$$\zeta_{n;\mathscr{P}_j}(s)-\zeta_{\mathscr{P}_j}(s)=\frac{1}{2\pi i}\int_{\alpha_1-i\infty}^{\alpha_1+i\infty}\zeta_{\mathscr{P}_j}(s+z)\kappa_n(z)\,\mathrm{d}z+a_j\kappa_n(1-s).$$

From this, it follows that, for $s\in K_j$,

$$\begin{aligned}\zeta_{n;\mathscr{P}_j}(s+i\tau)-\zeta_{\mathscr{P}_j}(s+i\tau) \ll & \int_{-\infty}^{\infty}\Big|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+iu+i\tau)\Big|\Big|\kappa_n(\sigma_{\mathscr{P}_j}+\varepsilon-s+iu)\Big|\,\mathrm{d}u\\ &+|a_j||\kappa_n(1-s-i\tau)|.\end{aligned}$$

Hence, we find

$$\begin{aligned}&\frac{1}{T}\int_0^T\sup_{s\in K_j}\Big|\zeta_{n;\mathscr{P}_j}(s+i\tau)-\zeta_{\mathscr{P}_j}(s+i\tau)\Big|\,\mathrm{d}\tau\\ &\qquad\ll\int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\Big|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+iu+i\tau)\Big|\,\mathrm{d}\tau\right)\sup_{s\in K_j}\Big|\kappa_n(\sigma_{\mathscr{P}_j}+\varepsilon-s+iu)\Big|\,\mathrm{d}u\\ &\qquad\quad+\frac{1}{T}\int_0^T\sup_{s\in K_j}|\kappa_n(1-s-i\tau)|\,\mathrm{d}\tau.\end{aligned} \tag{11}$$

The definition of the strip $\mathscr{D}_{\mathscr{P}_j}$ shows that the function $\zeta_{\mathscr{P}_j}(s)$, for $\sigma=\sigma_{\mathscr{P}_j}+\varepsilon$, has a bounded mean square; therefore,

$$\int_0^T\Big|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+it)\Big|^2\,\mathrm{d}t\ll_{\varepsilon,j}T.$$

This implies the bound

$$\int_0^T \left|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+iu+i\tau)\right| \mathrm{d}\tau \ll \left(T \int_0^T \left|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+iu+i\tau)\right|^2 \mathrm{d}\tau \right)^{1/2}$$
$$\ll \left(T \int_{-|u|}^{T+|u|} \left|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+i\tau)\right|^2 \right)^{1/2}$$
$$\ll_{\varepsilon,j} (T(T+|u|))^{1/2} \ll_{\varepsilon,j} (T+u^{1/2}). \tag{12}$$

The Stirling formula for $\Gamma(s)$ implies the estimate

$$\Gamma(\sigma+it) \ll |t|^{\sigma-1/2} \exp\left\{-\frac{\pi}{2}|t|\right\}, \quad |t| \geqslant t_0, \tag{13}$$

which is uniform in any interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Therefore, by the definition of $\kappa_n(s)$, for $s \in K_j$, we obtain

$$\kappa_n(\sigma_{\mathscr{P}_j}+\varepsilon-s+iu) \ll n^{\sigma_{\mathscr{P}_j}+\varepsilon-\sigma} |u-t|^{\sigma_{\mathscr{P}_j}+\varepsilon-\sigma-1/2} \exp\left\{\frac{\pi}{2}|u-t|\right\} \ll_{K_j} n^{-\varepsilon} \exp\{-c|u|\}$$

with $c > 0$. This, together with estimate (12), shows that

$$\int_{-\infty}^{\infty} \left(\frac{1}{T}\int_0^T \left|\zeta_{\mathscr{P}_j}(\sigma_{\mathscr{P}_j}+\varepsilon+iu+i\tau)\right| \mathrm{d}\tau \right) \sup_{s\in K_j} \left|\kappa_n(\sigma_{\mathscr{P}_j}+\varepsilon-s+iu)\right| \mathrm{d}u$$
$$\ll_{K_j} n^{-\varepsilon} \int_{-\infty}^{\infty} (1+u^{1/2}) \exp\{-c|u|\} \,\mathrm{d}u \ll_{K_j} n^{-\varepsilon}. \tag{14}$$

Again, in view of (13), for $s \in K_j$, we find

$$\kappa_n(1-s-i\tau) \ll n^{1-\sigma} |t+\tau|^{1-\sigma-1/2} \exp\left\{-\frac{\pi}{2}|t+\tau|\right\} \ll_{K-j} n^{1-\sigma_{\mathscr{P}_j}-\varepsilon} \exp\{-c_1|\tau|\}$$

with $c_1 > 0$. Therefore,

$$\frac{1}{T}\int_0^T \sup_{s\in K_j} |\kappa_n(1-s-i\tau)| \,\mathrm{d}\tau \ll_{K_j} n^{1-\sigma_{\mathscr{P}_j}-\varepsilon} \frac{1}{T}\int_0^T \exp\{-c|\tau|\} \,\mathrm{d}\tau \ll_K T^{-1} n^{1-\sigma_{\mathscr{P}_j}-\varepsilon}.$$

The latter estimate, (14) and (11) give

$$\frac{1}{T}\int_0^T \sup_{s\in K_j} \left|\zeta_{n;\mathscr{P}_j}(s+i\tau) - \zeta_{\mathscr{P}_j}(s+i\tau)\right| \mathrm{d}\tau \ll_{K_j} n^{-\varepsilon} + T^{-1} n^{1-\sigma_{\mathscr{P}_j}-\varepsilon}.$$

Thus, taking $T \to \infty$ and then $n \to \infty$, we obtain equality (10), and, as it was noted above, this proves the proposition. □

5. Limit Theorem

In this section, we will prove a limit theorem on weak convergence of probability measure $P^r_{T,\underline{\zeta}}$ as $T \to \infty$. This will be derived from Lemmas 3 and 4, Proposition 1, and a statement on convergence in distribution ($\xrightarrow{\mathcal{D}}$). Let ξ_n and ξ be $\mathscr{X}$-valued random elements, and let Q_n and Q denote their distributions, respectively. Then $\xi_n \xrightarrow[n\to\infty]{\mathcal{D}} \xi$ if $Q_n \underset{n\to\infty}{\Longrightarrow} Q$.

Lemma 6. *Suppose that* ξ_{nm} *and* η_n, $n, m \in \mathbb{N}$, *are* $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$*-valued random elements defined on a certain probability space* $(\Omega, \mathcal{B}, \mu)$ *and*

$$\xi_{nm} \xrightarrow[n\to\infty]{\mathcal{D}} \xi_m, \quad \text{and} \quad \xi_m \xrightarrow[m\to\infty]{\mathcal{D}} \xi.$$

If, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mu(d(\eta_n, \xi_{nm}) \geqslant \varepsilon) = 0,$$

then $\eta_n \xrightarrow[n\to\infty]{\mathcal{D}} \xi$.

Proof. Since the space $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$ is separable, the lemma is a partial case of Theorem 3.2 from [28]. □

The main result of this section is the following proposition.

Proposition 4. *Suppose that the systems* $\mathscr{P}_j$ *satisfy axioms* (2). *On* $(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r}), \mathscr{B}(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})))$, *there exists a probability measure* $P^r_{\underline{\zeta}}$ *such that* $P^r_{T,\underline{\zeta}} \xrightarrow[T\to\infty]{} P^r_{\underline{\zeta}}$.

Proof. Denote by γ_T the random variable defined on $(\Omega, \mathcal{B}, \mu)$, which is uniformly distributed in the interval $[0, T]$. Now, introduce the $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$-valued random element

$$\underline{\xi}^r_{T,n} = \underline{\zeta}_{n;\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1 + i\gamma_T, \dots, s_r + i\gamma_T) = \big(\zeta_{n;\mathscr{P}_1}(s_1 + i\gamma_T), \dots \zeta_{n;\mathscr{P}_r}(s_r + i\gamma_T)\big).$$

Moreover, let $\underline{\xi}^r_n = \big(\zeta_{n;\mathscr{P}_1}(s_1), \dots \zeta_{n;\mathscr{P}_r}(s_r)\big)$ be the $\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})$-valued random element with distribution $P^r_{\underline{\zeta}_n}$. By Lemma 4, the measure $P^r_{\underline{\zeta}_n}$ is tight. The classical Prokhorov theorem (Theorem 5.1 of [28]) asserts that if the measure is tight, then it is relatively compact. This means that every sequence $P^r_{\underline{\zeta}_{n_l}}$ contains a subsequence $P^r_{\underline{\zeta}_{n_v}}$ weakly convergent to a certain probability measure $P^r_{\underline{\zeta}}$ on $(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r}), \mathscr{B}(\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r})))$ as $v \to \infty$. This can be written in the form

$$\underline{\xi}_{n_v} \xrightarrow[v\to\infty]{\mathcal{D}} P^r_{\underline{\zeta}} \tag{15}$$

what is understood that $\underline{\xi}_{n_v}$ converges in distribution to a random element having the distribution $P^r_{\underline{\zeta}}$. Now, we apply Lemma 3 and have

$$\underline{\xi}_{T,n_v} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{\xi}_{n_v}. \tag{16}$$

Let

$$\underline{\xi}^r_T = \underline{\zeta}^r_{\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1 + i\gamma_T, \dots, s_r + i\gamma_T) = \big(\zeta_{\mathscr{P}_1}(s_1 + i\gamma_T), \dots, \zeta_{\mathscr{P}_r}(s_r + i\gamma_T)\big).$$

Then, in view of Proposition 3, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{v\to\infty} \limsup_{T\to\infty} \mu\Big\{d\Big(\underline{\xi}^r_T, \underline{\xi}^r_{T,n_v}\Big) \geqslant \varepsilon\Big\} \\
&= \lim_{v\to\infty} \limsup_{T\to\infty} \mathscr{U}_T\Big(d\Big(\underline{\zeta}_{\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1 + i\tau, \dots, s_r + i\tau), \underline{\zeta}_{n_v;\mathscr{P}_1,\dots\mathscr{P}_r}(s_1 + i\tau, \dots, s_r + i\tau)\Big) \geqslant \varepsilon\Big) \\
&\leqslant \lim_{v\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T d\Big(\underline{\zeta}_{\mathscr{P}_1,\dots,\mathscr{P}_r}(s_1 + i\tau, \dots, s_r + i\tau), \underline{\zeta}_{n_v;\mathscr{P}_1,\dots\mathscr{P}_r}(s_1 + i\tau, \dots, s_r + i\tau)\Big)\, d\tau \\
&= 0.
\end{aligned}$$

Now, this and relations (15) and (16) show that Lemma 6 can be applied for the random elements $\underline{\xi}^r_T$, $\underline{\xi}^r_{T,n_v}$, $\underline{\xi}^r_{n_v}$, and the measure $P^r_{\underline{\xi}}$. Thus, we obtain that

$$\underline{\xi}^r_T \xrightarrow[T\to\infty]{\mathcal{D}} P^r_{\underline{\zeta}}, \tag{17}$$

and this shows that the measure $P^r_{T,\underline{\zeta}}$ converges weakly to $P^r_{\underline{\zeta}}$ as $T \to \infty$. The proposition is proved. □

Observe that the relation (17) implies that the measure $P^r_{\underline{\zeta}}$ is independent of the sequence $\underline{\xi}_{n_v}$. Thus, we have that

$$\underline{\xi}_n \xrightarrow[n\to\infty]{\mathcal{D}} P^r_{\underline{\zeta}}.$$

6. Proof of Approximation

In this section, we will prove Theorem 2. It is a simple consequence of Proposition 4. Before that, we recall some equivalents of weak convergence of probability measures. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$; ∂A denotes the boundary of $A \in \mathscr{B}(\mathscr{X})$. The set $A \in \mathscr{B}(\mathscr{X})$ is called a continuity set of the measure P if $P(\partial A) = 0$.

Lemma 7. *The following statements are equivalent:*

1°

$$P_n \underset{n\to\infty}{\Longrightarrow} P;$$

2° *For every open set $G \subset \mathscr{X}$,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

3° *For every continuity set A of the measure P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 from [28], where its proof is given.

Suppose that the space $\mathscr{X}$ is separable. The support of the measure P is a minimal closed set $S_P \subset \mathscr{X}$ such that $P(S_P) = 1$. If $x \in S_P$, then, for every neighborhood G of x, the inequality $P(G) > 0$ is valid.

Proof of Theorem 2. By Proposition 4, we have $P^r_{T,\underline{\zeta}} \underset{T\to\infty}{\Longrightarrow} P^r_{\underline{\zeta}}$. Denote by $\mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r}$ the support of the measure $P^r_{\underline{\zeta}}$. By the definition of the support, we have $P^r_{\underline{\zeta}}(\mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r}) = 1$; hence, $\mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r} \neq \varnothing$. Moreover, $\mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r}$ is a closed set.

For $(f_1, \dots, f_r) \in \mathscr{F}_{\mathscr{P}_1,\dots,\mathscr{P}_r}$, define

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in \mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1}, \dots, \mathscr{D}_{\mathscr{P}_r}) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

Then G_ε is an open neighbourhood of the element $(f_1, \dots, f_r)$ of the support of the measure $P^r_{\underline{\zeta}}$. Therefore,

$$P^r_{\underline{\zeta}}(G_\varepsilon) > 0 \tag{18}$$

by a support property. Proposition 4 and 1° and 2° of Lemma 7 imply

$$\liminf_{T\to\infty} P^r_{T,\underline{\zeta}}(G_\varepsilon) \geqslant P^r_{\underline{\zeta}}(G_\varepsilon).$$

This, (18), and the definitions of G_ε and $P^r_{T,\underline{\zeta}}$ yield the inequality

$$\liminf_{T\to\infty} \mathscr{U}_T\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|\zeta_{\mathscr{P}_j}(s+i\tau)-f_j(s)\right|<\varepsilon\right)>0.$$

The first statement of Theorem 2 is proved.

The set G_ε can be written in the form

$$G_\varepsilon = \left\{(g_1,\dots,g_r)\in\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1},\dots,\mathscr{D}_{\mathscr{P}_r}):\right.$$
$$\left.\sup_{s\in K_1}|g_1(s)-f_1(s)|<\varepsilon,\dots\sup_{s\in K_r}|g_r(s)-f_r(s)|<\varepsilon\right\}.$$

Therefore, see Chapter 1 from [28],

$$\partial G_\varepsilon \subset \left\{(g_1,\dots,g_r)\in\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1},\dots,\mathscr{D}_{\mathscr{P}_r}):\right.$$
$$\left.\sup_{s\in K_1}|g_1(s)-f_1(s)|=\varepsilon,(g_2,\dots,g_r)\in\mathscr{H}^{r-1}(\mathscr{D}_{\mathscr{P}_2},\dots,\mathscr{D}_{\mathscr{P}_r})\right\}$$
$$\cup\cdots\cup\left\{(g_1,\dots,g_r)\in\mathscr{H}^r(\mathscr{D}_{\mathscr{P}_1},\dots\mathscr{D}_{\mathscr{P}_r}):\right.$$
$$\left.\sup_{s\in K_r}|g_r(s)-f_r(s)|=\varepsilon,(g_1,\dots,g_{r-1})\in\mathscr{H}^{r-1}(\mathscr{D}_{\mathscr{P}_1},\dots,\mathscr{D}_{\mathscr{P}_{r-1}})\right\}.$$

This shows that the boundaries ∂G_ε do not intersect with different ε. Hence, it follows that $P^r_{\underline{\zeta}}(\partial G_\varepsilon)>0$ for at most countably many $\varepsilon>0$. This means that the G_ε is a continuity set of the measure $P^r_{\underline{\zeta}}$ for all but at most countably many values of ε. Hence, Proposition 4 and 1° and 3° of Lemma 7 give

$$\lim_{T\to\infty} P^r_{T,\underline{\zeta}}(G_\varepsilon)=P^r_{\underline{\zeta}}(G_\varepsilon)$$

for all but at most countably many $\varepsilon>0$. This, (18), and the definitions $P^r_{\underline{\zeta}}$ and G_ε show that

$$\lim_{T\to\infty} \mathscr{U}_T\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|\zeta_{\mathscr{P}_j}(s+i\tau)-f_j(s)\right|<\varepsilon\right)$$

exists and is positive for all but at most countably many values of $\varepsilon>0$. The theorem is proved. □

7. Conclusions

Beurling zeta-functions are interesting analytic objects introduced by A. Beurling almost 90 years ago for the study of generalized prime numbers and generalized integers. In recent years, Beurling zeta-functions have been used for the description of vibration for fractal membranes and for the structure study of quasicrystals; thus, they even have practical applications. This stimulates their studies.

In the paper, we considered joint approximation of analytic functions by shifts of Beurling zeta-functions. Let $\zeta_{\mathscr{P}_1}(s), \dots, \zeta_{\mathscr{P}_r}(s)$ be a collection of Beurling zeta-functions corresponding to the systems $\mathscr{P}_1, \dots, \mathscr{P}_r$ of generalized primes. Using the relations

$$\sum_{\substack{m \leqslant x \\ m \in \mathscr{N}_{\mathscr{P}_j}}} 1 - a_j x \ll x^{\theta_j},$$

where $\mathscr{N}_{\mathscr{P}_j}$ is a system of generalized integers corresponding to $\mathscr{P}_j$, $a_j > 0$ and $0 \leqslant \theta_j < 1$, as well as

$$\frac{1}{T} \int_0^T \left| \zeta_{\mathscr{P}_j}(\sigma + it) \right|^2 \mathrm{d}t \ll_\sigma 1,$$

with $\sigma_{\mathscr{P}_j} \leqslant \sigma < 1$, $j = 1, \dots, r$, we obtained that the shifts $(\zeta_{\mathscr{P}_1}(s_1 + i\tau), \dots, \zeta_{\mathscr{P}_r}(s_r + i\tau))$, $\tau \in \mathbb{R}$, approximate a certain set of analytic functions $(f_1(s_1), \dots, f_r(s_r))$. More precisely, the set of these shifts is sufficiently rich; it has a positive lower density (or density with at most countably many exceptions for accuracy of approximation). Unfortunately, the mentioned set of analytic functions is not effectively defined, and this problem remains for future studies of Beurling zeta-functions. Also, we are planning to investigate the approximation of analytic functions by discrete shifts of Beurling zeta-functions when τ runs a certain discrete set. Moreover, the results of the paper confirm that the probabilistic method based on a limit theorem in spaces of analytic functions works well for the approximation of analytic functions by shifts of Beurling zeta-functions.

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