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Finite Difference Scheme for Two-Dimensional Poisson Equation with the Multiple Integral Boundary Condition

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Abstract

This article investigates the numerical solution of the two-dimensional Poisson equation defined over a rectangular domain subject to a double integral nonlocal boundary condition. We propose a finite difference scheme by discretizing the integral term using the two-dimensional trapezoidal rule. The main difficulty of this problem is that, in the non-classical case, we cannot use the method of separation of variables and decompose the problem into one-dimensional problems. Our approach involves reducing the integral boundary condition from the complete domain to the interior points and strategically partitioning the computational domain into the boundary and interior points. We propose a method that allows us to find a solution by solving the Poisson equation with classical boundary conditions, and using the solutions found to construct a solution to a problem with a nonlocal integral condition. This method requires solving a linear system whose dimension is much smaller than the original. Under certain conditions on the kernel, the proposed method is correct.

Keywords: Poisson equation; integral condition; finite difference method

MSC: 65N06; 65N22

1. Introduction

Problems with various types of Boundary Conditions (BC) for partial differential equations (elliptic, parabolic, and hyperbolic) are one of the intensively studied branches in the theory of differential equations and numerical methods.

Nonlocal Boundary Value Problems (BVP) arise naturally in the mathematical modelling of many processes of physics, thermoelasticity, heat conduction, chemical diffusion, population dynamics, inverse problems, and control theory. A short review on applications of problems on Nonlocal Boundary Conditions (NBC) can be found in many articles (see [1,2]). The method of fundamental solutions and several applications of this method are presented in [3].

In the article [4], a nonlinear elliptic equation with the integral condition

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = \gamma(u), \quad x \in \Omega \subset \mathbb{R}^n,$$
$$u|_{\partial\Omega} = \int_{\Omega} k(x,y)u(y)dy + g. \tag{1}$$

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is considered. Under the assumption

$$k(x,y) \geq 0, \quad \int_{\Omega} k(x,y)dy \leq 1,$$

the existence and dynamic behaviour of solutions to linear and semilinear equations are proved. In the case $k(x,y) = \text{const}$, the corresponding Sturm–Liouville Problem (SLP) is considered.

At the same time, it is possible to interpret nonlocal BVPs as a very interesting generalisation of classical BVPs [5]. In the first article about the solution of elliptic equations with NBCs instead of condition (1), a simpler nonlocal condition was analysed. Usually two-point NBC, multi-point NBC or 1D integral conditions were investigated. In [6], the two-dimensional elliptical equation was solved using the finite difference method with NBC of a type

$$u(x,0) = \gamma \int_{\xi_1}^{\xi_2} u(x,y)dy + \mu_1(x).$$

The elliptic equation with another type of integral conditions (or one condition)

$$\int_0^{\xi_1} u(x,y)dx = 0, \quad \int_{\xi_2}^{1} u(x,y)dx = 0$$

was analysed in [7–9].

In the articles mentioned below, the main goal is the investigation of the existence and uniqueness of the solution of the difference problem, as well as the estimation of the error in a certain norm. However, depending on the type of NBCs, solving a system of difference equations with NBCs is often a rather difficult task (complicated problem), especially if the differential equation is nonlinear. To find solutions to such systems, special iterative methods were studied based on the investigation of structure of the spectrum of the difference problem [10–14]. The conditions for the convergence of the Alternating Direction Implicit (ADI) method were investigated in [15,16]. Having supplemented this technique (approach) with the theory of M-matrices, other problems, namely, convergence of the finite difference method and stability of the Finite-Difference Scheme (FDS), were also studied [17–19]. Note that differential and difference eigenvalue problems with NBCs are much more difficult than similar problems with classical BCs. The main reason is that the linear operators are no longer symmetric (self-adjoint).

The finite difference method for the linear two-dimensional parabolic equation with NBC (1) in the square domain $\Omega = [0,1]^2$ was studied in [20]. It is proven that the semi-implicit and fully-implicit backward Euler schemes are unconditionally stable under the assumption

$$\int_{\Omega} |k(x,y,\xi,\eta)| d\xi d\eta \leq \varrho < 1. \tag{2}$$

Some cases of FDS for such a problem was investigated using properties of the spectrum of one-dimensional problems in [21,22]. Parabolic equations with nonlocal integral conditions are described as mathematical models of real processes in the work [23], and the literature cited therein.

In the paper [24] the initial-boundary problem for equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x,y) \in \Omega = \{0 < x < 1, \quad 0 < y < 1\}, \tag{3}$$

with two conditions

$$u(x,0,t) = \mu(t)h_0(x), \quad \int_0^1 \int_0^{s(x)} u(x,y,t)dydx = m(t). \tag{4}$$

Dirichlet BCs are considered on the other three sides of the rectangle. Here, $s(x)$, $h_0(x)$, and $m(t)$ are known functions, while the function $\mu(t)$ is unknown. The existence and uniqueness of the solution are also demonstrated, and the numerical procedure is discussed. To solve the problem (3) and (4), various FDS were investigated and used, including locally one-dimensional method [25].

In [26] FDSs for two-dimensional elliptic problems with an integral condition

$$\int_\Omega k(x)udx = V$$

were investigated.

In this article, we consider a finite difference approximation [27] to the solution of the following boundary value problem with nonlocal conditions involving multiple integral

$$L(u) := -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2, \tag{5}$$

$$u(x,y) = \int_\Omega k(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta + g(x,y), \quad (x,y) \in \partial\Omega, \tag{6}$$

where Ω is a rectangular domain. The study of this problem was initiated by Prof. M. Sapagovas [28]. The kernel k satisfies the condition (2). In the case $k \equiv 0$, we have classical Dirichlet BC

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega. \tag{7}$$

The main idea of this study is to approximate a nonlocal double integral boundary condition and develop an efficient method for solving the presented finite difference scheme.

NBC (1) for elliptic problems has previously been studied only in the differential case [4], and the FDS method has been used to solve parabolic equations [20,24,25]. In [24,26] the additional conditions are not NBC. These are the only papers related to the study of this type of NBC (with double integral). The main difficulty of this problem is that, in the non-classical case, we cannot use the method of separation of variables and decompose the problem into one-dimensional problems. We consider a second-order FDS where the Poisson equation is approximated using a five-point stencil that does not use the vertex nodes of the rectangle, and the integral BC is approximated using a two-dimensional trapezoidal rule that uses those nodes. Our approach involves reducing the integral boundary condition from the complete domain to the interior points. We propose a method that uses solutions of the Poisson equation with classical Dirichlet BC, and does not use vertex nodes to find them. Using these classical solutions, we construct a solution to the nonlocal problem at the inner nodes. This method requires solving a linear system whose dimension is much smaller than the original. An explicit inverse method for block matrices is presented, which can be used for theoretical studies in the two-dimensional case. Finally, we find the solution values at all boundary nodes, including the vertices of the rectangle.

In Section 2, we introduce the notation for matrices, grids, grid functions, and describe linear systems of equations. We consider FDS for the two-dimensional Poisson equation with the multiple integral boundary condition in Section 3. Additionally, we consider the FDS′ in which BC is rewritten so that the boundary nodes of the vertices of the rectangle are not used. Next, methods for solving such problems with classical BC are reviewed, and explicit formula for inverse matrix is derived. We are looking for a solution of the problem with NBC that would be a combination of classical fundamental solutions of discrete Laplace equation. Section 4 presents the results of the numerical experiments, which demonstrate the precision and effectiveness of FDS.

2. Notation

2.1. Differential Problem

We consider two-dimensional Poisson Equation (5) with a multiply integral BC (6), when the kernel k satisfies the condition (2) and $k \in C^2(\partial\Omega \times \overline{\Omega})$, where $\Omega := \Omega_x \times \Omega_y$, $\Omega_x := \{x\colon 0 < x < l_x\}$, $\Omega_y := \{y\colon 0 < y < l_y\}$, $|\Omega| = l_x l_y$. Note that $|k| \leq \varkappa < +\infty$. The classical solution of this equation $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\overline{\Omega} = \Omega \cup \partial\Omega$ [29,30].

We will briefly review the compatibility conditions for Poisson's Equation (5) and the integral condition (6) (see more in ([31] Ch. Nonlocal Elliptic Boundary Value Problems), ([32] Ch. 2–6), [33,34]). Define the nonlocal boundary operator $\mathcal{K}\colon L^2(\Omega) \to L^2(\partial\Omega)$ and its adjoint operator $\mathcal{K}^*\colon L^2(\partial\Omega) \to L^2(\Omega)$:

$$(\mathcal{K}u)(\boldsymbol{x}) := \int_\Omega k(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{\xi})d\boldsymbol{\xi}, \quad \boldsymbol{x} := (x,y) \in \partial\Omega,\ \boldsymbol{\xi} := (\xi,\eta) \in \Omega,$$
$$(\mathcal{K}^*u)(\boldsymbol{\xi}) := \int_{\partial\Omega} k(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{x})d\sigma_x, \quad \boldsymbol{x} \in \partial\Omega,\ \boldsymbol{\xi} \in \Omega.$$

If v solves the inhomogeneous Dirichlet problem

$$-\Delta w_f = f \ \text{ in } \Omega, \quad w_f|_{\partial\Omega} = 0,$$

then

$$w_f(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x},\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi},$$

where $G(\boldsymbol{x},\boldsymbol{\xi})$ is Dirichlet Green's function for $-\Delta$ on Ω ($G = 0$ on $\partial\Omega$).

If $\mathcal{H}\colon L^2(\partial\Omega) \to H^1(\Omega)$ is the harmonic extension (Poisson) operator [35,36]:

$$\Delta(\mathcal{H}\varphi) = 0 \ \text{ in } \Omega, \quad (\mathcal{H}\varphi)|_{\partial\Omega} = \varphi,$$

then, by Green's identity

$$\int_\Omega (\mathcal{H}\varphi)f d\boldsymbol{x} = \int_{\partial\Omega} \varphi\partial_n w_f d\sigma_x.$$

So, the adjoint operator $\mathcal{H}^*\colon H^1(\Omega) \to L^2(\partial\Omega)$ is

$$(\mathcal{H}^*f)(\boldsymbol{x}) = \partial_n w_f(\boldsymbol{x}) = \int_\Omega \frac{\partial G(\boldsymbol{x},\boldsymbol{\xi})}{\partial n_x} f(\boldsymbol{\xi})d\boldsymbol{\xi}$$

and $(\mathcal{KH})^* = \mathcal{H}^*\mathcal{K}^*$. Operator $\mathcal{KH}$ is compact on $L^2(\partial\Omega)$. The integral BC (6) becomes

$$\varphi = \mathcal{K}(w_f + \mathcal{H}\varphi) + g \Leftrightarrow \mathcal{T}\varphi = \mathcal{K}w_f + g, \quad \mathcal{T} = \mathcal{I} - \mathcal{KH}. \tag{8}$$

We consider equation $\psi = \mathcal{H}^*\mathcal{K}^*\psi$. Then, $\psi \in \ker\mathcal{T}^*$ and ψ is the operator $\mathcal{H}^*\mathcal{K}^*$ eigenfunction corresponding to the eigenvalue 1. According to the Fredholm alternative, the Equation (8) is solvable if and only if $\mathcal{K}w_f + g$ is orthogonal to each $\psi \in \mathcal{T}^*$.

Let $\psi \in \ker\mathcal{T}^*$ be arbitrary and define w_ψ by

$$-\Delta w_\psi = \mathcal{K}^*\psi = \int_{\partial\Omega} k(\boldsymbol{x},\boldsymbol{\xi})\partial_n\psi(\boldsymbol{x})d\sigma_x, \quad \boldsymbol{\xi} \in \Omega, \quad w_\psi \text{ on } \partial\Omega,$$

then compatibility condition is

$$\int_\Omega f(\boldsymbol{x})w_\psi(\boldsymbol{x})d\boldsymbol{x} + \int_{\partial\Omega} g(\boldsymbol{x})\psi(\boldsymbol{x})d\sigma_{\boldsymbol{x}} = 0, \quad \forall\psi \in \ker\mathcal{T}^*.$$

Finally, we consider operators $\mathcal{H}\colon C(\partial\Omega) \to C(\overline{\Omega})$ and $\mathcal{K}\colon C(\overline{\Omega}) \to C(\partial\Omega)$. Then, $||\mathcal{H}||_{C\to C} = 1$ and

$$||\mathcal{K}||_{C\to C} \le \sup_{\partial\Omega} \int_\Omega |k(\boldsymbol{x},\boldsymbol{\xi})| d\boldsymbol{\xi}.$$

So, if condition (2) holds, then spectral radius $r(\mathcal{KH}) \le ||\mathcal{KH}||_{C\to C} \le \varrho < 1$, i.e., $1 \notin \sigma(\mathcal{KH})$, then no compatibility restriction is required and the solution is unique.

Example 1. *If $k(\boldsymbol{x},\boldsymbol{\xi}) \equiv \gamma$, then $(Ku)(\boldsymbol{x}) = \gamma \int_\Omega u(\boldsymbol{\xi}) d\boldsymbol{\xi} =: C$ is constant on $\partial\Omega$ and from Maximum Principle for Laplace equation $\mathcal{H}C = C$, i.e., $u = w_f + \mathcal{H}C + \mathcal{H}g = w_f + \mathcal{H}g + C$. From*

$$C = \gamma \int_\Omega u(\boldsymbol{x}) d\boldsymbol{x} = \gamma \int_\Omega w_f(\boldsymbol{x}) d\boldsymbol{x} + \gamma \int_\Omega \mathcal{H}g(\boldsymbol{x}) d\boldsymbol{x} + \gamma C|\Omega|$$

we get

$$C(1-\gamma|\Omega|) = \gamma\Big(\int_\Omega w_f(\boldsymbol{x}) d\boldsymbol{x} + \int_\Omega \mathcal{H}g(\boldsymbol{x}) d\boldsymbol{x}\Big).$$

If $\gamma \ne |\Omega|^{-1}$, then no compatibility needed and we have unique solution

$$u = w_f + \mathcal{H}g + \frac{\gamma}{1-\gamma|\Omega|}\Big(\int_\Omega w_f(\boldsymbol{x}) d\boldsymbol{x} + \int_\Omega \mathcal{H}g(\boldsymbol{x}) d\boldsymbol{x}\Big).$$

If $\gamma = |\Omega|^{-1}$, the compatibility condition is $\int_\Omega w_f d\boldsymbol{x} + \int_\Omega \mathcal{H}g d\boldsymbol{x} = 0$. Uniqueness can be fixed by adding additional condition, for example $\int_\Omega u d\boldsymbol{x} = 0$.

Additional smoothness requirements are required to obtain a second-order accurate FDS [20,27]: $u \in C^{4,\delta}(\overline{\Omega})$, $f \in C^{2,\delta}(\overline{\Omega})$, $g \in C^{4,\delta}(\partial\Omega)$, $0 < \delta < 1$, and $k(\cdot,\boldsymbol{\xi}) \in C^{2,\delta}(\partial\Omega)$ in the boundary variable $\boldsymbol{x} \in \partial\Omega$ (uniformly in $\boldsymbol{\xi}$). For a rectangular domain BC (6) can be written as four equalities:

$$u(x_\alpha, y) = \int_\Omega k_\alpha(y,\boldsymbol{\xi}) u(\boldsymbol{\xi}) d\boldsymbol{\xi} + g_\alpha(y), \quad y \in \overline{\Omega}_y, \quad \alpha = L, R, \quad x_L = 0,\ x_R = l_x,$$
$$u(x, y_\beta) = \int_\Omega k_\beta(x,\boldsymbol{\xi}) u(\boldsymbol{\xi}) d\boldsymbol{\xi} + g_\beta(x), \quad x \in \overline{\Omega}_x, \quad \beta = B, T, \quad y_B = 0,\ x_T = l_y,$$

where $k_\alpha(y,\boldsymbol{\xi}) := k(x_\alpha, y, \boldsymbol{\xi}) \in C^{2,\delta}(\overline{\Omega}_y \times \overline{\Omega})$, $k_\beta(x,\boldsymbol{\xi}) := k(x, y_\beta, \boldsymbol{\xi}) \in C^{2,\delta}(\overline{\Omega}_x \times \overline{\Omega})$, $g_\alpha(y) := g(x_\alpha, y) \in C^{2,\delta}(\overline{\Omega}_y)$, $g_\beta(x) := g(x, y_\alpha) \in C^{2,\delta}(\overline{\Omega}_x)$. We will assume that the compatibility conditions are valid at the vertices of the rectangular Ω.

At the vertex $v = (0,0)$, we write these compatibility conditions (they are analogous at other vertices of the rectangle):

$$g_L(0) = g_B(0), \quad \text{0-order vertex compatibility},$$
$$\int_\Omega \partial_{yy} k_L(0,\boldsymbol{\xi}) u(\boldsymbol{\xi})\, d\boldsymbol{\xi} + \int_\Omega \partial_{xx} k_B(0,\boldsymbol{\xi}) u(\boldsymbol{\xi})\, d\boldsymbol{\xi} + g_B''(0) + g_L''(0) = -f(0,0).$$

Under these conditions $u \in C^{4,\delta}(\overline{\Omega})$ ([37] Ch. 6. Results in Spaces of Holder Functions).

2.2. Matrices and Vectors

The set of all k-by-l real matrices $\mathbf{A} = (a_{ij})_{k\times l} = (a_{ij})$, $a_{ij} \in \mathbb{R}$, is denoted $\mathbb{R}^{k\times l}$. We use the first index (by position) for the row of the matrix: $\mathbf{A} = (a_{ij})$, where $(a^i_j) := (a^{i}_{\cdot j})$. We consider matrices $\mathbf{v} = (v_1, \ \ldots, \ v_k)^\top \in \mathbb{R}^{k\times 1} = \mathbb{R}^k$ (or $\mathbf{v} = (v^1, \ \ldots, \ v^k)^\top$) as vectors. A matrix (or vector) consisting only of zeros is denoted by $\mathbf{O}$ (or $\mathbf{0}$). We denote the identity matrix as $\mathbf{I}$. If clarification is required, the dimensions of these vectors or matrices will be indicated by subscripts ($\mathbf{0}_k \in \mathbb{R}^{k\times 1}$, $\mathbf{O}_{k_1\times k_2} \in \mathbb{R}^{k_1\times k_2}$, $\mathbf{I}_k \in \mathbb{R}^{k\times k}$). A block matrix is a matrix that is subdivided into smaller matrices, called blocks or submatrices. A partitioning of $\mathbf{A} \in \mathbb{R}^{k\times l}$ is a representation of $\mathbf{A}$ in the form $\mathbf{A} = (\mathbf{B}_{ij})_{p\times q}$, where $\mathbf{B}_{ij} \in \mathbb{R}^{k_i\times l_j}$ are contiguous submatrices, $\sum_{i=1}^{p} k_i = k$, $\sum_{j=1}^{q} l_j = l$. In addition to regular matrix multiplication $\mathbf{AB} = (\sum_{q=1}^{p} a_{iq}b_{qj}) \in \mathbb{R}^{k\times l}$, where $\mathbf{A} \in \mathbb{R}^{k\times p}$, $\mathbf{B} \in \mathbb{R}^{p\times l}$, we will use Kronecker product as block matrix $\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})_{k_1\times l_1} \in \mathbb{R}^{k_1k_2\times l_1l_2}$, where $\mathbf{A} \in \mathbb{R}^{k_1\times l_1}$, $\mathbf{B} \in \mathbb{R}^{k_2\times l_2}$. $[\mathbf{A},\mathbf{B}] := \mathbf{AB} - \mathbf{BA}$ is commutator of matrices $\mathbf{A}$ and $\mathbf{B}$ and elements of the commutator $[\mathbf{A},\mathbf{B}]$ we denote $[\mathbf{A},\mathbf{B}]_{ij}$, i.e., $[\mathbf{A},\mathbf{B}] = ([\mathbf{A},\mathbf{B}]_{ij})$. The vectorisation of a matrix is a linear transformation that converts the matrix into a vector: $\mathbf{V} = (v_1, \ldots, v_p)^\top = \mathrm{vec}(\mathbf{A}) := (a_{11}, \ldots, a_{k1}, a_{12}, \ldots, a_{k2}, \ldots, a_{1l}, \ldots, a_{kl})^\top$, $p = kl$.

In this article, we will use the following vector and matrix norms:

$$||\mathbf{v}||_\infty := \max_i |v_i|, \quad ||\mathbf{v}||_2 := \sqrt{\sum_i |v_i|^2},$$

$$||\mathbf{A}||_\infty := \max_{i=1,\ldots,k} \sum_{j=1}^{l} |a_{ij}|, \quad ||\mathbf{A}||_{\max} := \max_{i,j} |a_{ij}|.$$

The matrix norm $||\cdot||_\infty$ is induced by vectors norms $||\cdot||_\infty$ on $\mathbb{R}^k$ and $\mathbb{R}^l$ and consistent with the vectors norms $||\mathbf{Av}||_\infty \leq ||\mathbf{A}||_\infty\, ||\mathbf{v}||_\infty$. This matrix norm is also sub-multiplicative $||\mathbf{AB}||_\infty \leq ||\mathbf{A}||_\infty ||\mathbf{B}||_\infty$. Note that matrix norm $||\mathbf{A}||_{\max}$ is not sub-multiplicative.

2.3. Grids and Grid Functions

First, we introduce one-dimensional grids with uniform steps h_x, h_y, where $h_x = l_x/N$, $h_y = l_y/M$, $0 < N, M \in \mathbb{N}$ [17,18]:

$$\begin{aligned}
\overline{\omega}_x^h &:= \{x_i = i\cdot h_x, i \in \overline{\mathcal{I}}_x\}, \quad \overline{\mathcal{I}}_x := \{i, i = 0, \ldots, N\},\\
\omega_x^h &:= \{x_i = i\cdot h_x, i \in \mathcal{I}_x\}, \quad \mathcal{I}_x := \{i, i = 1, \ldots, N-1\},\\
\overline{\omega}_y^h &:= \{y_j = j\cdot h_y, j \in \overline{\mathcal{I}}_y\}, \quad \overline{\mathcal{I}}_y := \{j, j = 0, \ldots, M\},\\
\omega_y^h &:= \{y_j = j\cdot h_y, i \in \mathcal{I}_y\}, \quad \mathcal{I}_y := \{j, j = 1, \ldots, M-1\},
\end{aligned}$$

and then define two-dimensional grids for $\overline{\Omega}$:

$$\begin{aligned}
\overline{\omega}^h &:= \overline{\omega}_x^h \times \overline{\omega}_y^h = \{x_{ij} = (x_i, y_j), (i,j) \in \overline{\mathcal{I}}\}, \quad \overline{\mathcal{I}} := \overline{\mathcal{I}}_x \times \overline{\mathcal{I}}_y,\\
\omega^h &:= \omega_x^h \times \omega_y^h = \{x_{ij} = (x_i, y_j), (i,j) \in \mathcal{I}\} = \overline{\omega}^h \cap \Omega, \quad \mathcal{I} := \mathcal{I}_x \times \mathcal{I}_y,\\
\partial^2\omega^h &:= \{x_{00}, x_{N0}, x_{0M}, x_{NM}\}, \quad \widetilde{\omega}^h := \overline{\omega}^h \setminus \partial^2\omega^h,\\
\partial\overline{\omega}^h &:= \overline{\omega}^h \setminus \omega^h = \overline{\omega}^h \cap \partial\Omega, \quad \partial\omega^h := \partial\overline{\omega}^h \setminus \partial^2\omega^h.
\end{aligned}$$

We denote $h^2 := h_xh_y$, $|h|^2 := h_x^2 + h_y^2$, $|\partial\Omega^h| := l_xh_y + l_yh_x - h_xh_y$. For small h_x, h_y, the number of the grid $\partial\overline{\omega}^h$ nodes is much smaller than the number of the grid ω^h nodes.

Note that $\widetilde{\omega}^h = \omega^h \cup \partial\omega^h$, $\partial\overline{\omega}^h = \partial\omega^h \cup \partial^2\omega^h$, $\overline{\omega}^h = \omega^h \cup \partial\overline{\omega}^h$. The grids ω^h, $\partial\omega^h$, $\partial^2\omega^h$, $\widetilde{\omega}^h$, $\partial\overline{\omega}^h$, $\overline{\omega}^h$ are related only to the domain $\overline{\Omega}$ and are independent of the problems considered in them. We use index h in notation of these grids.

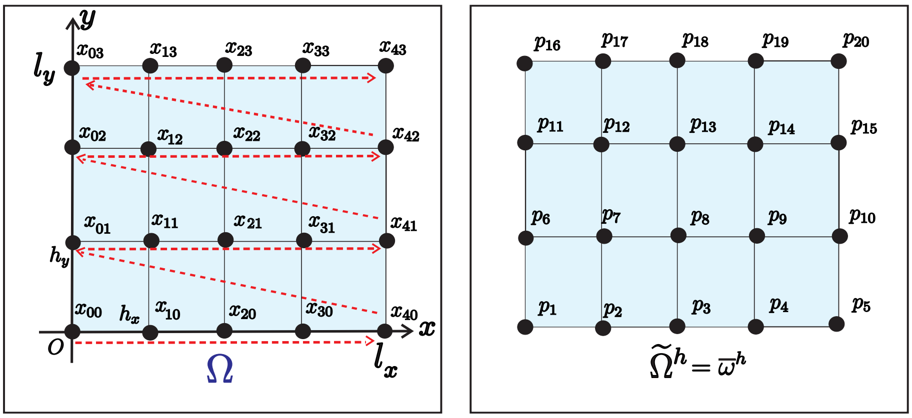
We enumerate the grid $\overline{\omega}^h$ nodes in the lexicographic order ([38] Sec. 9.1.2) (see Figure 1a)

$$x_{00} \prec \ldots \prec x_{N0} \prec x_{01} \prec \ldots \prec x_{N1} \prec \ldots \prec x_{NM},$$

i.e., we can use a row vector of nodes

$$(p_1, \ldots, p_{|\overline{\omega}^h|}) = \mathrm{vec}(x_{ij} \in \overline{\omega}^h), \quad p_1 \prec p_2 \prec \ldots \prec p_k \prec \ldots \prec p_{|\overline{\omega}^h|},$$

numbered with a single index $k \in \widetilde{\mathcal{J}} := \{k, k = 1, \ldots, |\overline{\omega}^h|\}$ (see Figure 1b). We will use notation $U_k := U(p_k) = U(x_{ij}) =: U_{ij}$ for grid functions $U \in \mathbb{R}^{\overline{\omega}^h}$ (the set of all real functions $f\colon A \to \mathbb{R}$, A is a given set, we denote as $\mathbb{R}^A$). The grids ω^h, $\partial\omega^h$, $\partial^2\omega^h \subset \overline{\omega}^h$ are presented in Figure 2b,c for $N = 6$, $M = 5$, and the induced order from $\overline{\omega}^h$ is used for nodes.



(**a**) the nodes $x_{ij} \in \overline{\omega}^h$ (**b**) the nodes $p_k \in \overline{\omega}^h$, $|\overline{\omega}^h| = 20$

Figure 1. The domain Ω, the universal grid $\widetilde{\Omega}^h = \overline{\omega}^h$ ($N = 4$, $M = 3$) with lexicographic order.

If we have functions $\widehat{U} \in \mathbb{R}^{\widehat{\Omega}^h}$, $\widehat{K} \in \mathbb{R}^{\widehat{\Omega}_1^h \times \widehat{\Omega}_2^h}$, $\widehat{\Omega}^h$, $\widehat{\Omega}_1^h$, $\widehat{\Omega}_2^h \subset \overline{\omega}^h$, $|\widehat{\Omega}^h| = k$, $|\widehat{\Omega}_1^h| = k_1$, $|\widehat{\Omega}_2^h| = k_2$, then we define a vector $\widehat{\mathbf{U}} = (\widehat{U}_k)_{p_k \in \widehat{\Omega}^h} \in \mathbb{R}^{k\times 1}$ and a matrix $\widehat{\mathbf{K}} = (\widehat{K}_{k\cdot}^{\cdot l})_{p_l \in \widehat{\Omega}_2^h}^{p_k \in \widehat{\Omega}_1^h} \in \mathbb{R}^{k_1 \times k_2}$ (note that k is the first (row) index, l is the second (column) index). For grid functions $\widehat{U}$ and $\widehat{K}$ we define norms $||\widehat{U}||_\infty := ||\widehat{\mathbf{U}}||_\infty$, $||\widehat{K}||_\infty := ||\widehat{\mathbf{K}}||_\infty$, $||\widehat{K}||_{\max} := ||\widehat{\mathbf{K}}||_{\max}$. On the other hand, a vector $\widehat{\mathbf{U}} \in \mathbb{R}^k$ and a matrix $\widehat{\mathbf{K}} \in \mathbb{R}^{k_1 \times k_2}$ define grid functions $\widehat{U} \in \mathbb{R}^{\widehat{\Omega}^h}$, $\widehat{K} \in \mathbb{R}^{\widehat{\Omega}_1^h \times \widehat{\Omega}_2^h}$.

We denote ω as a grid for inner nodes, $\partial\overline{\omega}$ as a grid for boundary nodes, $\overline{\omega} = \omega + \partial\overline{\omega}$. We approximate the differential Equation (2) at the inner nodes. For functions $U = U_{ij} \in \mathbb{R}^{\widetilde{\omega}^h}$, we use the notation for approximating the second order partial derivatives in Equation (5):

$$\delta_x^2 U_{ij} := \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h_x^2}, \quad \delta_y^2 U_{ij} := \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h_y^2}, \quad x_{ij} \in \omega,$$

and get a discrete equation

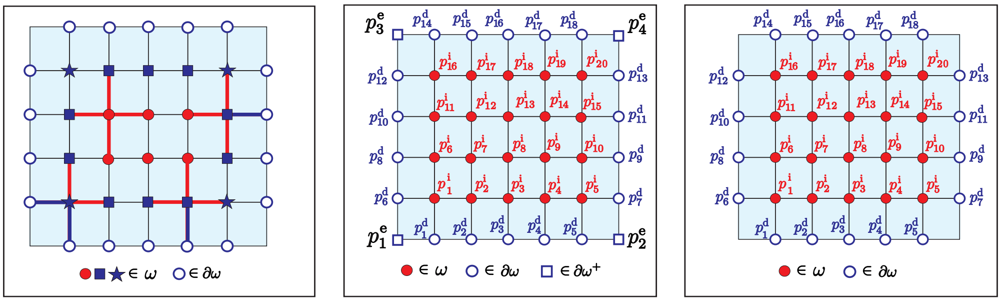
$$L^h U := -\delta_x^2 U - \delta_y^2 U = F, \quad x_{ij} \in \omega, \tag{9}$$

where $F = F_{ij} = f(x_{ij}) \in \mathbb{R}^\omega$. This finite-difference equation uses the five-point stencil

$$\operatorname{supp}_{ij}^h L^h = \{x_{i,j-1}, x_{i-1,j}, x_{ij}, x_{i+1,j}, x_{i,j+1}\}, \quad x_{ij} \in \omega. \tag{10}$$

For this stencil, we define a support $\operatorname{supp}_\omega^h L^h := \cup_{x_{ij}\in\omega} \operatorname{supp}_{ij}^h L^h$ of the operator L^h and boundary nodes for this support $\partial\omega := \operatorname{supp}_\omega^h L^h \setminus \omega$ (see Figure 2a). Additionally, we denote $\widetilde{\omega} := \omega + \partial\omega = \operatorname{supp}_\omega^h L^h$, $n := |\omega|$, $m_1 := |\partial\omega|$. For stencil (10) and $\omega = \omega^h$, we have $\widetilde{\omega} = \widetilde{\omega}^h$, $\partial\omega = \partial\omega^h$, $n = |\omega^h| = (N-1)(M-1)$, $m_1 = |\partial\omega^h| = 2(N+M) - 4$, $|\widetilde{\omega}| = n + m_1$.

The structure of boundary nodes is $\partial\overline{\omega} = \partial\omega + \partial\omega^+$, $m_2 := |\partial\omega^+|$, $m := |\partial\overline{\omega}| = m_1 + m_2$. The grid $\partial\omega^+$ nodes are not used in the Equation (9), they are used only in BC. The nodes of the grids ω, $\partial\omega$, and $\partial\omega^+$ are linearly ordered using the order in $\widetilde{\Omega}^h$: $(p_1^{\mathrm{i}}, \ldots, p_n^{\mathrm{i}}) = \operatorname{vec}(x_{ij} \in \omega)$, $(p_1^{\mathrm{d}}, \ldots, p_{m_1}^{\mathrm{d}}) = \operatorname{vec}(x_{ij} \in \partial\omega)$, $(p_1^{\mathrm{e}}, \ldots, p_{m_2}^{\mathrm{e}}) = \operatorname{vec}(x_{ij} \in \partial\omega^+)$ (see Figure 2b); the nodes of the grid $\partial\overline{\omega}$ are ordered as $(p_1^{\mathrm{b}}, \ldots, p_m^{\mathrm{b}})$, $p_k^{\mathrm{b}} = p_k^{\mathrm{d}}$, $k = 1, \ldots, m_1$, $p_{m_1+k}^{\mathrm{b}} = p_k^{\mathrm{e}}$, $k = 1, \ldots, m_2$. If $\partial\omega^+ = \varnothing$, then $m_2 = 0$, $m = m_1$ and $\partial\overline{\omega} = \partial\omega$.



(**a**) stencils $\mathrm{supp}_{ij}^h L^h$ (**b**) $\omega = \omega^h, \partial\omega = \partial\omega^h$, $\partial\omega^+ = \partial^2\omega^h, \partial\overline{\omega} = \partial\overline{\omega}^h$ (**c**) $\omega = \omega^h, \partial\overline{\omega} = \partial\omega = \partial\omega^h$, $\partial\omega^+ = \varnothing$

Figure 2. The five-point stencils $\mathrm{supp}_{ij}^h L^h$ and the grids $\omega, \partial\omega, \partial\omega^+, \partial\overline{\omega}$, $N = 6$, $M = 5$.

In summary, the grid ω is used for inner nodes, which are the master grid points, the grid $\partial\omega$ is used for boundary nodes, which are used to approximate the equation, the grid $\partial\omega^+$ is used for boundary nodes, which are used only in boundary conditions. These grids may vary for different FDSs. Note that the grid used for all boundary nodes is $\partial\overline{\omega} = \partial\omega + \partial\omega^+$, and this grid is not equal to the grid $\partial\overline{\omega}^h$ in the general case (see Figure 2c).

For grid functions U, we will use column vectors:

$$\begin{aligned}
\mathbf{U}^{\mathrm{i}} &:= (U_1^{\mathrm{i}}, \ldots, U_n^{\mathrm{i}})^\top,\ U^{\mathrm{i}} = U_k^{\mathrm{i}} = U(p_k^{\mathrm{i}}),\ p_k^{\mathrm{i}} \in \omega,\ U^{\mathrm{i}} \in \mathbb{R}^{\omega};\\
\mathbf{U}^{\mathrm{d}} &:= (U_1^{\mathrm{d}}, \ldots, U_{m_1}^{\mathrm{d}})^\top,\ U^{\mathrm{d}} = U_{l_1}^{\mathrm{d}} = U(p_{l_1}^{\mathrm{d}}),\ p_{l_1}^{\mathrm{d}} \in \partial\omega,\ U^{\mathrm{d}} \in \mathbb{R}^{\partial\omega};\\
\mathbf{U}^{\mathrm{e}} &:= (U_1^{\mathrm{e}}, \ldots, U_{m_2}^{\mathrm{e}})^\top,\ U^{\mathrm{e}} = U_{l_2}^{\mathrm{e}} = U(p_{l_2}^{\mathrm{e}}),\ p_{l_2}^{\mathrm{e}} \in \partial\omega^+,\ U^{\mathrm{e}} \in \mathbb{R}^{\partial\omega^+};\\
\mathbf{U}^{\mathrm{b}} &:= (U_1^{\mathrm{b}}, \ldots, U_m^{\mathrm{b}})^\top := (U_1^{\mathrm{d}}, \ldots, U_{m_1}^{\mathrm{d}}, U_1^{\mathrm{e}}, \ldots, U_{m_2}^{\mathrm{e}})^\top,\ U^{\mathrm{b}} \in \mathbb{R}^{\partial\overline{\omega}};\\
\mathbf{U} &:= (U_1, \ldots, U_{n+m})^\top := (U_1^{\mathrm{i}}, \ldots, U_n^{\mathrm{i}}, U_1^{\mathrm{b}}, \ldots, U_m^{\mathrm{b}})^\top,\ U \in \mathbb{R}^{\overline{\omega}};
\end{aligned}$$

and column vector $\mathbf{F}^{\mathrm{i}} = (F_1^{\mathrm{i}}, \ldots, F_n^{\mathrm{i}})^\top$, $F^{\mathrm{i}} = F_k^{\mathrm{i}} = F(p_k^{\mathrm{i}})$, $p_k^{\mathrm{i}} \in \omega$, for the grid function $F \in \mathbb{R}^{\omega}$.

Let us consider integral BC (6). We approximate the function $g = g(x, y)$ with a grid function G, and use column vectors:

$$\begin{aligned}
\mathbf{G}^{\mathrm{d}} &:= (G_1^{\mathrm{d}}, \ldots, G_{m_1}^{\mathrm{d}})^\top,\ G^{\mathrm{d}} = G_{l_1}^{\mathrm{d}} = G(p_{l_1}^{\mathrm{d}}),\ p_{l_1}^{\mathrm{d}} \in \partial\omega,\ G^{\mathrm{d}} \in \mathbb{R}^{\partial\omega};\\
\mathbf{G}^{\mathrm{e}} &:= (G_1^{\mathrm{e}}, \ldots, G_{m_2}^{\mathrm{e}})^\top,\ G^{\mathrm{e}} = G_{l_2}^{\mathrm{e}} = G(p_{l_2}^{\mathrm{e}}),\ p_{l_2}^{\mathrm{e}} \in \partial\omega^+,\ G^{\mathrm{e}} \in \mathbb{R}^{\partial\omega^+};\\
\mathbf{G}^{\mathrm{b}} &:= (G_1^{\mathrm{b}}, \ldots, G_m^{\mathrm{b}})^\top := (G_1^{\mathrm{d}}, \ldots, G_{m_1}^{\mathrm{d}}, G_1^{\mathrm{e}}, \ldots, G_{m_2}^{\mathrm{e}})^\top,\ G^{\mathrm{b}} \in \mathbb{R}^{\partial\overline{\omega}}.
\end{aligned}$$

Remark 1. *If $\partial\omega^+ = \varnothing$, then $\mathbf{U}^{\mathrm{b}} = \mathbf{U}^{\mathrm{d}}$, $\mathbf{G}^{\mathrm{b}} = \mathbf{G}^{\mathrm{d}}$ (see Figure 2c).*

The kernel $k = k(x, y, \xi, \eta)$ is approximated with a discrete kernel $K = K_{ij\cdot\cdot}^{\cdot\cdot i_1 j_1} = k(x_{ij}, x_{i_1 j_1})$, $x_{ij} \in \partial\overline{\omega}$, $x_{i_1 j_1} \in \overline{\omega}$. For the approximation of the two-dimensional integral in the trapezoidal rule case, we introduce a weight function $r \in \mathbb{R}^{\overline{\omega}}$: $r^{i_1 j_1} = h^2/4$ for $x_{i_1 j_1} \in \partial\omega^+$; $r^{i_1 j_1} = h^2/2$ for $x_{i_1 j_1} \in \partial\omega$; else $r = h^2$. After vectorisation by the upper and the lower indexes, we get $K = K_{k\cdot}^{\cdot l} \in \mathbb{R}^{\partial\overline{\omega} \times \overline{\omega}}$ and matrix $\mathbf{K} = (K_{k\cdot}^{\cdot l})_{p_l \in \overline{\omega}}^{p_k \in \partial\overline{\omega}} \in \mathbb{R}^{m \times (n+m)}$. In this article, we use grid functions $K^{\mathrm{i}} = K|_{\partial\overline{\omega} \times \omega}$, $B = r^l K_{k\cdot}^{\cdot l} \in \mathbb{R}^{\partial\overline{\omega} \times \overline{\omega}}$, $B^{\mathrm{i}} = B|_{\partial\overline{\omega} \times \omega}$, $B^{\mathrm{b}} = B|_{\partial\overline{\omega} \times \partial\overline{\omega}}$, and corresponding matrices $\mathbf{K}^{\mathrm{i}}$, $\mathbf{B}$, $\mathbf{B}^{\mathrm{i}}$, $\mathbf{B}^{\mathrm{b}}$.

We use notation

$$\langle K, U \rangle_k^{\omega_2} := \sum_{p_l \in \omega_2} r^l K_{k\cdot}^{\cdot l} U_l, \quad p_k \in \partial\omega_1 \subset \partial\overline{\omega},\ U \in \mathbb{R}^{\omega_2},\ \omega_2 \subset \overline{\omega},$$

where $K = K|_{\partial\omega_1 \times \omega_2}$, and partial cases are

$$[K,U]_k := \langle K,U\rangle_k^{\overline{\omega}},\ (K,U)_k := \langle K,U\rangle_k^{\omega},\ \langle K,U\rangle_k := \langle K,U\rangle_k^{\partial\overline{\omega}},\ p_k \in \partial\omega_1. \tag{11}$$

Since $\overline{\omega} = \omega + \partial\overline{\omega}$ and $B_{k\cdot}^{\cdot l} = r^l K_{k\cdot}^{\cdot l}$, $p_k \in \partial\overline{\omega}$, $p_l \in \overline{\omega}$, then

$$\sum_{p_l \in \overline{\omega}} r^l K_{k\cdot}^{\cdot l} U_l = \sum_{p_l \in \omega} r^l K_{k\cdot}^{\cdot l} U_l + \sum_{p_l \in \partial\overline{\omega}} r^l K_{k\cdot}^{\cdot l} U_l, \quad \sum_{p_l \in \overline{\omega}} B_{k\cdot}^{\cdot l} U_l = \sum_{p_l \in \omega} B_{k\cdot}^{\cdot l} U_l + \sum_{p_l \in \partial\overline{\omega}} B_{k\cdot}^{\cdot l} U_l,$$

$p_k \in \partial\overline{\omega}$. Using definitions (11) and matrix notation, we get:

$$[K,U]_k = (K,U^{\mathrm{i}})_k + \langle K,U^{\mathrm{b}}\rangle_k, \quad p_k \in \partial\overline{\omega},$$
$$\mathbf{B}\mathbf{U} = \mathbf{B}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{B}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}}.$$

For functions $k \in C^2(\partial\Omega \times \overline{\Omega})$, $u \in C^2(\overline{\Omega})$, the integration formula is valid [20]:

$$\int_\Omega k(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta = \sum_{i_1=0}^{N}\sum_{j_1=0}^{M} r^{i_1 j_1} k(x,y,x_{i_1},y_{j_1})U_{i_1 j_1} + \mathcal{O}(|h|^2).$$

This formula follows from ([39] 5.4 Multidimensional Integration), ([40] 4.8 Multiple Integrals)

$$\int_0^{l_x}\int_0^{l_y} u(\xi,\eta)d\xi d\eta = \sum_{i_1=0}^{N}\sum_{j_1=0}^{M} r^{i_1 j_1} U_{i_1 j_1} + \mathcal{O}(|h|^2), \quad u \in C^2(\overline{\Omega})$$

or from a more general formula for multidimensional case ([41] Multidimensional problems). The proof of the two-dimensional case can be found in [42] and error estimation is equal to

$$|R| \le |\Omega| \max_{\overline{\Omega}} \left|\frac{\partial^2 u(x,y)}{\partial x^2}\right| \frac{h_x^2}{12} + |\Omega| \max_{\overline{\Omega}} \left|\frac{\partial^2 u(x,y)}{\partial y^2}\right| \frac{h_y^2}{12},$$

and this bound holds uniformly with respect to the mesh size.

For any $0 \le \varrho < 1$ (see (2)), there exists h_0 such that for $|h| \le h_0$ we have

$$\sum_{i_1=0}^{N}\sum_{j_1=0}^{M} r^{i_1 j_1} |k(x,y,x_{i_1},y_{j_1})| \le \rho_1 = \frac{1+\varrho}{2} < 1, \quad (x,y) \in \partial\Omega.$$

We will suppose that

$$||\mathbf{B}||_\infty = \max_{k=1,\dots,m} \sum_{l=0}^{m+n} r^l |K_{k\cdot}^{\cdot l}| \le \rho_1 < 1. \tag{12}$$

Lemma 1. *If $||K||_{\max} \le \varkappa < +\infty$, then $||\mathbf{B}^{\mathrm{b}}||_\infty = \mathcal{O}(|h|)$.*

Proof. We estimate

$$||\mathbf{B}^{\mathrm{b}}||_\infty = \max_{k=1,\dots,m} \sum_{l=1}^{m} |r^l K_{k\cdot}^{\cdot l}| \le \max_{k=1,\dots,m} \sum_{l=1}^{m} ||K||_{\max} h^2 = 2||K||_{\max}(N+M)h^2$$
$$= 2\varkappa l_x l_y (N^{-1} + M^{-1}) = 2\varkappa(l_y h_x + l_x h_y) = \mathcal{O}(h_x + h_y) = \mathcal{O}(|h|).$$

□

Remark 2. *From this lemma, it follows that instead of (12), we can use a simpler condition*

$$h^2||K||_\infty = \max_{k=1,\dots,m} \sum_{l=0}^{m+n} h^2 |K_{k\cdot}^{\cdot l}| \le \rho = \frac{1+\rho_1}{2} < 1.$$

If $h^2||K||_\infty \leq \rho_1 < 1$*, then we have* (12).

Let us consider linear system $\mathbf{Y} = \mathbf{AY} + \mathbf{G}$, $\mathbf{A} \in \mathbb{R}^{m\times m}$, $\mathbf{G} \in \mathbb{R}^m$. The next lemma gives well-known result.

Lemma 2. *If* $||\mathbf{A}||_\infty \leq \bar{\rho} < 1$*, then the matrix* $\mathbf{I}_m - \mathbf{A}$ *is nonsingular and* $||(\mathbf{I}_m - \mathbf{A})^{-1}||_\infty \leq (1-\bar{\rho})^{-1}$.

For every $\tilde{\rho} \in (\rho, 1)$, we have $0 < 1 - \rho/\tilde{\rho} < 1$. We choose h_0 so that $||\mathbf{B}^{\mathrm{b}}||_\infty \leq 1 - \rho/\tilde{\rho}$.

Corollary 1. *Suppose* $||K||_{\max} \leq \varkappa < +\infty$, $h^2||K||_\infty \leq \rho < 1$*. If* $|h| < h_0$*, then*

$$||\mathbf{B}^{\mathrm{b}}||_\infty, ||\mathbf{B}^{\mathrm{i}}||_\infty \leq ||\mathbf{B}||_\infty \leq \rho < 1,$$

$\mathbf{I}_m - \mathbf{B}^{\mathrm{b}}$ *is nonsingular and*

$$(1 - ||\mathbf{B}^{\mathrm{b}}||_\infty)^{-1} \leq \tilde{\rho}\rho^{-1} \leq \rho^{-1}, \quad ||(\mathbf{I}_m - \mathbf{B}^{\mathrm{b}})^{-1}\mathbf{B}^{\mathrm{i}}||_\infty \leq \tilde{\rho} < 1.$$

2.4. Linear Systems

The discrete Equation (9) and BCs form a linear system that is used to numerically solve the problems under consideration. We will write down the general form of those systems both when we have classical ones and when we have NBCs. The properties of the system matrix allow the use of certain solution methods (for example, decomposition into one-dimensional problems). In the case of NBC, the system has a more general form, but it can, as in the classical case, be reduced to the system only at inner nodes.

In the lecture notes [43], the following linear system of equations was studied:

$$\mathbf{A}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{d}}\mathbf{U}^{\mathrm{d}} = \mathbf{F}^{\mathrm{i}}, \tag{13}$$

$$\mathbf{U}^{\mathrm{d}} = \mathbf{G}^{\mathrm{d}}. \tag{14}$$

This linear system corresponds to discrete Equation (9) ($\omega = \omega^h$, $n = |\omega|$) with classical Dirichlet BC $U = G$, $p_k \in \partial\overline{\omega} = \partial\omega = \partial\omega^h$ ($m = m_1 = |\partial\omega^h|$), $\partial\omega^+ = \varnothing$ (see Figure 2c), and approximates differential problem (5) and (7). The matrix $\mathbf{A}^{\mathrm{i}}$ is a block tridiagonal matrix

$$\mathbf{A}^{\mathrm{i}} = \frac{1}{h_y^2}\begin{pmatrix} 2\mathbf{Z} & -\mathbf{I} & & \\ -\mathbf{I} & 2\mathbf{Z} & \ddots & \\ & \ddots & \ddots & -\mathbf{I} \\ & & -\mathbf{I} & 2\mathbf{Z} \end{pmatrix}_{(M-1)\times(M-1)} \in \mathbb{R}^{n\times n}, \tag{15}$$

$$\mathbf{Z} = \frac{\alpha^2}{2}\begin{pmatrix} 2(1+\alpha^{-2}) & -1 & & \\ -1 & 2(1+\alpha^{-2}) & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2(1+\alpha^{-2}) \end{pmatrix}, \tag{16}$$

where $\mathbf{I}, \mathbf{Z} \in \mathbb{R}^{(N-1)\times(N-1)}$, $\alpha = h_y/h_x$. The matrix $\mathbf{A}^{\mathrm{d}}$ is a block matrix

$$\mathbf{A}^{\mathrm{d}} = \frac{1}{h_y^2} \begin{pmatrix} -\mathbf{I} & -\alpha^2\tilde{\mathbf{J}} & & & \\ & & -\alpha^2\tilde{\mathbf{J}} & & \\ & & & \ddots & \\ & & & & -\alpha^2\tilde{\mathbf{J}} & -\mathbf{I} \end{pmatrix}_{(M-1)\times(M+1)} \in \mathbb{R}^{n\times m_1},$$

where

$$\tilde{\mathbf{J}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^{\top} \in \mathbb{R}^{(N-1)\times 2}.$$

Using the Kronecker product, we express $\mathbf{A}^{\mathrm{i}} = \mathbf{A}^{\mathrm{i}}_x + \mathbf{A}^{\mathrm{i}}_y$, $\mathbf{A}^{\mathrm{i}}_x := \mathbf{I}_{M-1} \otimes \mathbf{\Lambda}_x$, $\mathbf{A}^{\mathrm{i}}_y := \mathbf{\Lambda}_y \otimes \mathbf{I}_{N-1}$, $\mathbf{\Lambda}_x := h_x^{-2}\mathbf{\Lambda}_{N-1}$, $\mathbf{\Lambda}_y := h_y^{-2}\mathbf{\Lambda}_{M-1}$, where

$$\mathbf{\Lambda}_k = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{k\times k}, \quad k = N-1, M-1.$$

It is known that eigenvalues of matrices $\mathbf{\Lambda}_x$ and $\mathbf{\Lambda}_y$ are simple and the eigenvectors are orthogonal and linearly independent [27]:

$$\lambda_k(\mathbf{\Lambda}_x) = 4\sin^2(\pi k h_x/2)/h_x^2, \quad \mathbf{v}^k = (v_1^k, \dots, v_{N-1}^k)^T, \quad k \in \mathcal{I}_x,$$
$$\lambda_l(\mathbf{\Lambda}_y) = 4\sin^2(\pi l h_y/2)/h_y^2, \quad \mathbf{w}^l = (w_1^l, \dots, w_{M-1}^l)^T, \quad l \in \mathcal{I}_y,$$

where $v_i^k = \sin(\pi k x_i)$, $i \in \mathcal{I}_x$, $w_j^l = \sin(\pi l y_j)$, $j \in \mathcal{I}_y$.

Lemma 3. *Suppose matrices* $\mathbf{A}_1, \mathbf{B}_1 \in \mathbb{R}^{k\times k}$, $\mathbf{A}_2, \mathbf{B}_2 \in \mathbb{R}^{l\times l}$ *and* $\mathbf{A}_1\mathbf{B}_1 = \mathbf{B}_1\mathbf{A}_1$, $\mathbf{A}_2\mathbf{B}_2 = \mathbf{B}_2\mathbf{A}_2$. *If* $\mathbf{C}_1 = \mathbf{B}_2 \otimes \mathbf{A}_1$, $\mathbf{C}_2 = \mathbf{A}_2 \otimes \mathbf{B}_1$, *then matrices* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *commute and* $\mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2\mathbf{C}_1 = (\mathbf{B}_2\mathbf{A}_2) \otimes (\mathbf{A}_1\mathbf{B}_1)$.

Proof. The product of two Kronecker products yields another Kronecker product ([44] [Lemma 4.2.10]):

$$(\mathbf{B}_2 \otimes \mathbf{A}_1)(\mathbf{A}_2 \otimes \mathbf{B}_1) = (\mathbf{B}_2\mathbf{A}_2) \otimes (\mathbf{A}_1\mathbf{B}_1),$$
$$(\mathbf{A}_2 \otimes \mathbf{B}_1)(\mathbf{B}_2 \otimes \mathbf{A}_1) = (\mathbf{A}_2\mathbf{B}_2) \otimes (\mathbf{B}_1\mathbf{A}_1).$$

Thus:

$$\begin{aligned} [\mathbf{B}_2 \otimes \mathbf{A}_1, \mathbf{A}_2 \otimes \mathbf{B}_1] &= (\mathbf{B}_2\mathbf{A}_2) \otimes (\mathbf{A}_1\mathbf{B}_1) - (\mathbf{A}_2\mathbf{B}_2) \otimes (\mathbf{B}_1\mathbf{A}_1) \\ &= (\mathbf{B}_2\mathbf{A}_2) \otimes (\mathbf{A}_1\mathbf{B}_1) - (\mathbf{A}_2\mathbf{B}_2) \otimes (\mathbf{A}_1\mathbf{B}_1) + (\mathbf{A}_2\mathbf{B}_2) \otimes (\mathbf{A}_1\mathbf{B}_1) - (\mathbf{A}_2\mathbf{B}_2) \otimes (\mathbf{B}_1\mathbf{A}_1) \\ &= [\mathbf{B}_2, \mathbf{A}_2] \otimes (\mathbf{A}_1\mathbf{B}_1) + (\mathbf{A}_2\mathbf{B}_2) \otimes [\mathbf{A}_1, \mathbf{B}_1]. \end{aligned} \tag{17}$$

Matrices $\mathbf{A}$ and $\mathbf{B}$ commute if and only if commutator $[\mathbf{A}, \mathbf{B}] = \mathbf{O}$. Applying Formula (17) we get $[\mathbf{C}_1, \mathbf{C}_2] = \mathbf{O}$ and $\mathbf{C}_1\mathbf{C}_2 = (\mathbf{B}_2 \otimes \mathbf{A}_1)(\mathbf{A}_2 \otimes \mathbf{B}_1) = (\mathbf{B}_2\mathbf{A}_2) \otimes (\mathbf{A}_1\mathbf{B}_1)$. □

Corollary 2. *If* $\mathbf{B}_1 = \mathbf{I}_k$, $\mathbf{B}_2 = \mathbf{I}_l$, *then*

$$\mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2\mathbf{C}_1 = \mathbf{A}_2 \otimes \mathbf{A}_1.$$

So, the matrices $\mathbf{A}^{\mathrm{i}}_x$ and $\mathbf{A}^{\mathrm{i}}_y$ commute $\mathbf{A}^{\mathrm{i}}_x\mathbf{A}^{\mathrm{i}}_y = \mathbf{A}^{\mathrm{i}}_y\mathbf{A}^{\mathrm{i}}_x = \mathbf{\Lambda}_y \otimes \mathbf{\Lambda}_x$ (see [16], too).

Vectors

$$\mathbf{U}^{k,l} = \mathbf{w}^l \otimes \mathbf{v}^k \in \mathbb{R}^n, \quad (k,l) \in \mathcal{I}, \tag{18}$$

are linearly independent in the vector space $\mathbb{R}^n$ according to the Kronecker product properties. We have

$$\mathbf{A}^{\mathrm{i}}_x \mathbf{U}^{k,l} = (\mathbf{I}_{M-1} \otimes \mathbf{\Lambda}_x)(\mathbf{w}^l \otimes \mathbf{v}^k) = (\mathbf{I}_{M-1}\mathbf{w}^l) \otimes (\mathbf{\Lambda}_x \mathbf{v}^k) = \lambda_k(\mathbf{\Lambda}_x)\mathbf{U}^{k,l},$$
$$\mathbf{A}^{\mathrm{i}}_y \mathbf{U}^{k,l} = (\mathbf{\Lambda}_y \otimes \mathbf{I}_{N-1})(\mathbf{w}^l \otimes \mathbf{v}^k) = (\mathbf{\Lambda}_y\mathbf{w}^l) \otimes (\mathbf{I}_{N-1} \mathbf{v}^k) = \lambda_l(\mathbf{\Lambda}_y)\mathbf{U}^{k,l}.$$

So, matrices $\mathbf{A}^{\mathrm{i}}_x$ and $\mathbf{A}^{\mathrm{i}}_y$ have the same system of eigenvectors (18). Eigenvalues of a matrix $\mathbf{A}^{\mathrm{i}}_x$ are $\lambda_k(\mathbf{\Lambda}_x)$, $k \in \mathcal{I}_x$, and the geometric multiplicity of each eigenvalue is $M-1$; eigenvalues of matrix $\mathbf{A}^{\mathrm{i}}_y$ are $\lambda_l(\mathbf{\Lambda}_y)$, $l \in \mathcal{I}_y$, and the geometric multiplicity of each eigenvalue is $N-1$. Finally, matrix $\mathbf{A}^{\mathrm{i}}$ has the same system of eigenvectors (18) and

$$\mathbf{A}^{\mathrm{i}}\mathbf{U}^{k,l} = \big(\lambda_k(\mathbf{\Lambda}_x) + \lambda_l(\mathbf{\Lambda}_y)\big)\mathbf{U}^{k,l}, \quad (k,l) \in \mathcal{I}.$$

For problems with NBCs, the following system of linear equations was considered [17,18]:

$$\mathbf{A}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} = \mathbf{F}^{\mathrm{i}}, \tag{19}$$
$$\mathbf{U}^{\mathrm{b}} = \mathbf{B}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{B}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} + \mathbf{G}^{\mathrm{b}}, \tag{20}$$

where $\mathbf{A}^{\mathrm{b}} = (\mathbf{A}^{\mathrm{d}}, \mathbf{O}_{n\times m_2}) \in \mathbb{R}^{n\times m}$, $\mathbf{B}^{\mathrm{i}} \in \mathbb{R}^{m\times n}$, $\mathbf{B}^{\mathrm{b}} \in \mathbb{R}^{m\times m}$, $\mathbf{G}^{\mathrm{b}} \in \mathbb{R}^{m}$.

Remark 3. *If* $\partial\omega^+ = \varnothing$, *then* $m_2 = 0$, $m = m_1$ *and* $\mathbf{A}^{\mathrm{b}} = \mathbf{A}^{\mathrm{d}}$, $\mathbf{U}^{\mathrm{b}} = \mathbf{U}^{\mathrm{d}}$, $\mathbf{G}^{\mathrm{b}} = \mathbf{G}^{\mathrm{d}}$.

If matrix $\mathbf{A}^{\mathrm{bb}} = \mathbf{I}_m - \mathbf{B}^{\mathrm{b}}$ is nonsingular (see Lemma 2), then Equation (20) is rewritten as

$$\mathbf{U}^{\mathrm{b}} = \widetilde{\mathbf{B}}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \widetilde{\mathbf{G}}^{\mathrm{b}}, \quad \widetilde{\mathbf{B}}^{\mathrm{i}} \in \mathbb{R}^{m\times n}, \quad \widetilde{\mathbf{G}}^{\mathrm{b}} \in \mathbb{R}^{m}, \tag{21}$$

where $\widetilde{\mathbf{B}}^{\mathrm{i}} = (\mathbf{A}^{\mathrm{bb}})^{-1}\mathbf{B}^{\mathrm{i}}$, $\widetilde{\mathbf{G}}^{\mathrm{b}} = (\mathbf{A}^{\mathrm{bb}})^{-1}\mathbf{G}^{\mathrm{b}}$. If the conditions of Corollary 1 are valid, then $||\widetilde{\mathbf{B}}^{\mathrm{i}}||_\infty \le \tilde{\rho} < 1$ and

$$||\mathbf{U}^{\mathrm{b}}||_\infty \le ||\widetilde{\mathbf{B}}^{\mathrm{i}}||_\infty \cdot ||\mathbf{U}^{\mathrm{i}}||_\infty + ||\widetilde{\mathbf{G}}^{\mathrm{b}}||_\infty \le ||\mathbf{U}^{\mathrm{i}}||_\infty + ||\widetilde{\mathbf{G}}^{\mathrm{b}}||_\infty. \tag{22}$$

The first m_1 equations in (21) are rewritten in the form

$$\mathbf{U}^{\mathrm{d}} = \widetilde{\mathbf{B}}\mathbf{U}^{\mathrm{i}} + \widetilde{\mathbf{G}}, \quad \widetilde{\mathbf{B}} \in \mathbb{R}^{m_1\times n}, \quad \widetilde{\mathbf{G}} = \widetilde{\mathbf{G}}^{\mathrm{d}} \in \mathbb{R}^{m_1}. \tag{23}$$

By substituting the Expression (23) of $\mathbf{U}^{\mathrm{d}}$ into (19), we obtain an equation for $\mathbf{U}^{\mathrm{i}}$:

$$\mathbf{A}\mathbf{U}^{\mathrm{i}} = \mathbf{F}, \tag{24}$$

where $\mathbf{A} = \mathbf{A}^{\mathrm{i}} - \mathbf{C}$, $\mathbf{C} = -\mathbf{A}^{\mathrm{d}}\widetilde{\mathbf{B}}$, $\mathbf{F} = \mathbf{F}^{\mathrm{i}} - \mathbf{A}^{\mathrm{d}}\widetilde{\mathbf{G}}$. After finding the solution of Equation (24), we get values of $\mathbf{U}^{\mathrm{b}}$ by (21).

Remark 4. *Final matrix is defined as* $\mathbf{A} = \mathbf{A}^{\mathrm{i}} - \mathbf{C}$ *in Equation* (24) *and contains a nonlocal correction term* $\mathbf{C}$. *The properties of the matrices* $\mathbf{A}^{\mathrm{i}}$ *and* $\mathbf{C}$ *(for example, commutativity) are very important in the study of FDS and their solution methods. However, the simple spectral structure of the classical Poisson matrix* $\mathbf{A}^{\mathrm{i}}$ *is not necessarily preserved. Solving such a linear system of equations is inefficient, so we propose another method in which the matrix of the linear system of equations is* $\mathbf{A}^{\mathrm{i}}$. *Additionally, we will provide the explicit form of the inverse matrix* $(\mathbf{A}^{\mathrm{i}})^{-1}$.

3. Finite-Difference Scheme

We will investigate FDS for differential problem (5) and (6):

$$L^h U = -\delta_x^2 U - \delta_y^2 U = F, \quad p_l \in \omega = \omega^h, \tag{25}$$

$$U_k = [K, U]_k + G_k, \quad p_k \in \partial\overline{\omega} = \partial\overline{\omega}^h, \tag{26}$$

where $U \in \mathbb{R}^{\overline{\omega}}$. For this, FDS $\partial\overline{\omega} = \partial\omega + \partial\omega^+, \partial\omega = \partial\omega^h, \partial\omega^+ = \partial^2\omega^h$ (see Figure 2b). This scheme provides the local approximation $\mathcal{O}(|h|^2)$. We suppose that $||K||_{\max} \le \varkappa < +\infty$, $h^2||K||_\infty \le \rho < 1$. The last condition is valid for small $|h| \le h_0$ with $\rho \in (\varrho, 1)$. We rewrite BC (26) as

$$U_k^{\mathrm{b}} = h^2 \sum_{p_l \in \omega} K_{k\cdot}^{\cdot l} U_l^{\mathrm{i}} + \sum_{p_l \in \partial\overline{\omega}} r^l K_{k\cdot}^{\cdot l} U_l^{\mathrm{b}} + G_k^{\mathrm{b}}, \quad p_k \in \partial\overline{\omega}.$$

The matrix form of these equations for BCs is (20). If the conditions of Corollary 1 are satisfied, then we rewrite BC (26) as (21):

$$U_k = (\widetilde{K}^{\mathrm{i}}, U)_k + \widetilde{G}_k^{\mathrm{b}}, \quad p_k \in \partial\overline{\omega}, \tag{27}$$

where $\widetilde{K}^{\mathrm{i}} = (h^2)^{-1}\widetilde{B}^{\mathrm{i}\cdot l}_{k\cdot} \in \mathbb{R}^{\partial\overline{\omega}\times\omega}$ is a new kernel, $h^2||\widetilde{K}^{\mathrm{i}}||_\infty = ||\widetilde{B}^{\mathrm{i}}||_\infty \le \tilde{\rho} < 1$, $\widetilde{G}^{\mathrm{b}} = \widetilde{G}_k^{\mathrm{b}} \in \mathbb{R}^{\partial\overline{\omega}}$, $||\widetilde{G}^{\mathrm{b}}||_\infty \le \rho^{-1}||G||_\infty$. From (22), we have

$$||U^{\mathrm{b}}||_\infty \le ||U^{\mathrm{i}}||_\infty + \rho^{-1}||G||_\infty, \tag{28}$$

where $U^{\mathrm{b}} = U|_{\partial\omega}$, $U^{\mathrm{i}} = U|_\omega$.

Instead of FDS (25) and (26), we consider the following FDS′:

$$L^h U = -\delta_x^2 U - \delta_y^2 U = F, \quad p_l \in \omega = \omega^h, \tag{29}$$

$$U_k = (\widetilde{K}, U)_k + \widetilde{G}_k, \quad p_k \in \partial\omega = \partial\omega^h, \tag{30}$$

where $U \in \mathbb{R}^{\widetilde{\omega}}$, $\widetilde{K} := \widetilde{K}^{\mathrm{i}}|_{\partial\omega\times\omega}$, $\widetilde{G} := \widetilde{G}^{\mathrm{b}}|_{\partial\omega}$. Then, $h^2||\widetilde{K}||_\infty = ||\widetilde{B}||_\infty \le \tilde{\rho} < 1$, where $\widetilde{B} := \widetilde{B}^{\mathrm{i}}|_{\partial\omega\times\omega}$, $||\widetilde{G}||_\infty \le \rho^{-1}||G||_\infty$.

For this, FDS′ $\partial\omega^+ = \varnothing$, i.e., $\partial\overline{\omega} = \partial\omega$, $m = m_1$ (see Figure 2c), and the local approximation of BC is $\mathcal{O}(|h|)$ only (see Example 6). The second order local approximation of BC is given for kernels with property $k(x, y, \xi, \eta) = 0$, $(\xi, \eta) \in \partial\Omega$ (see Example 7). If we find a solution of this FDS′, then we know U_l^{i}, $p_l \in \omega$, for FDS (25) and (26). Then, using Formula (27), we calculate values U_k, $p_k \in \partial\omega^+ = \partial^2\omega^h$, for this FDS. Furthermore, the Formula (27) (or (21)) allows us to find all values U_k, $p_k \in \partial\overline{\omega}$, if we know U_l, $p_l \in \omega$.

3.1. The Case of Classical BCs

If $K \equiv 0$, then $\widetilde{K} \equiv 0$ and instead of FDS′ we have FDS with Dirichlet BC:

$$-\delta_x^2 U - \delta_y^2 U = F, \quad p_l \in \omega = \omega^h, \tag{31}$$

$$U_k = G_k, \quad p_k \in \partial\omega = \partial\omega^h, \tag{32}$$

which corresponds to a system of linear Equations (13) and (14). This problem (31) and (32) can be divided into two separate classical problems:

(C1) discrete Poisson equation with homogeneous Dirichlet BC

$$-\delta_x^2 U - \delta_y^2 U = F, \quad p_l \in \omega, \tag{33}$$

$$U_k = 0, \quad p_k \in \partial\omega; \tag{34}$$

(C2) discrete Laplace equation with Dirichlet BC

$$-\delta_x^2 U - \delta_y^2 U = 0, \quad p_l \in \omega, \tag{35}$$

$$U_k = G_k, \quad p_k \in \partial\omega. \tag{36}$$

The following theorem expresses the stability of difference problem (31) and (32) ([27] [Ch. IV, Theorem 1]). Here and below, we will denote different constants in the estimates by the same letter C.

Theorem 1. *For a solution of the Dirichlet difference problem, the estimate*

$$||U||_\infty \leq ||G||_\infty + C(\Omega)||\overset{\circ}{F}||_\infty + |h|^2||\overset{*}{F}||_\infty \tag{37}$$

holds with $||\overset{\circ}{F}||_\infty := \max_{p_k \in \overset{\circ}{\omega}^h} |F_k|$, $||\overset{*}{F}||_\infty := \max_{p_k \in \overset{*}{\omega}^h} |F_k|$.

Remark 5. *Formula (37) is proved for an arbitrary domain Ω. In a five-point stencil, the master point is a strictly interior node when all stencil points are interior; otherwise, it is a near-boundary interior node. Then, $\overset{\circ}{\omega}^h$ is grid of strictly inner nodes, $\overset{*}{\omega}^h$ is grid of near-boundary inner nodes. In our case, the domain is rectangular, all stencils are regular, the local approximation error at near-boundary nodes is $\mathcal{O}(|h|^2)$, and the estimate (37) is simpler*

$$||U||_\infty \leq ||G||_\infty + C||F||_\infty. \tag{38}$$

Partial cases of Problem C2 are

$$-\delta_x^2 U - \delta_y^2 U = 0, \quad p_l \in \omega, \tag{39}$$

$$U_k = \delta_k^\kappa, \quad p_k \in \partial\omega, \tag{40}$$

δ_k^κ is Kronecker symbol, $\kappa = 1, \ldots, m$ (see Figure 3). We have $m = m_1 = |\partial\omega^h|$ linearly independent solutions $U = U^\kappa$, $\kappa = 1, \ \ldots, \ m$, and every solution satisfies discrete Laplace equation. From the Maximum Principle $0 \leq U^\kappa \leq 1$, $0 \leq \sum_{\kappa=1}^m U^\kappa \leq 1$.

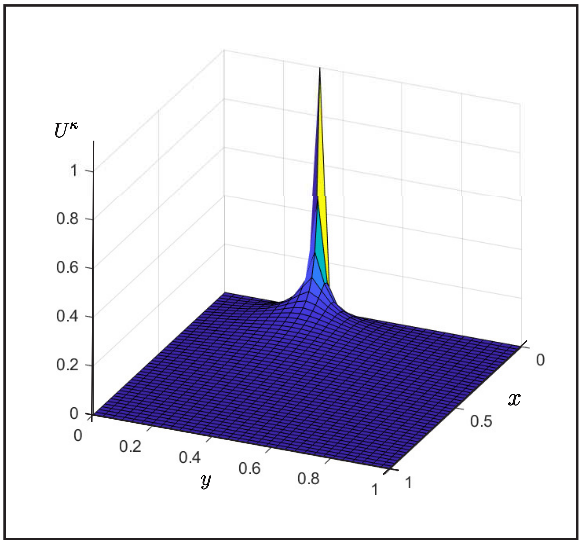


Figure 3. The solution of FDS (39) and (40).

For both Problems C1 and C2, we solve a linear system of equations $\mathbf{A}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} = \mathbf{F}$, where $\mathbf{F} = \mathbf{F}^{\mathrm{i}}$ for the first problem (33) and (34), $\mathbf{F} = -\mathbf{A}^{\mathrm{b}}\mathbf{G}^{\mathrm{b}}$ for the second problem (35) and (36). The choice of solver for the discrete Poisson equation depends strongly on the problem structure and computational constraints. We will note that there are many different methods for solving this system: Direct Methods (Gaussian Elimination/ LU Factorisation), Iterative Methods (Jacoby Method, Gauss–Seidel, Successive Over-Relaxation), Krylov Subspace Methods (Conjugate Gradient, GMRES), Multigrid Methods, Fast Poisson Solver (FFT–based) [45–48]. Multigrid is typically the best solver for the Poisson equation (complexity $\mathcal{O}(n)$). Fast Poisson Solver (complexity $\mathcal{O}(n\log n)$) is the fastest method for rectangles.

Explicit Inverse Matrix

Since we solve the problem with classical boundary conditions many times (for example, FDS (39) and (40), $\kappa = 1,\ldots,m$), we can first find the inverse matrix of the linear system. The further solution reduces to multiplying the inverse matrix by the vector.

The Chebyshev polynomials of the second kind of degree j are defined by formula [49] (although Chebyshev polynomials of the second kind are denoted by U_l in the literature, this article uses this notation for functions defined on a grid)

$$\widetilde{T}_l = \widetilde{T}_l(z) := \frac{(z+\sqrt{z^2-1})^{l+1} - (z-\sqrt{z^2-1})^{l+1}}{2\sqrt{z^2-1}}, \quad l \in \mathbb{Z},$$

where in the general case $z \in \mathbb{C}$, but in this article we conside only $z \in \mathbb{R}$.

These Chebyshev polynomials are solutions to the discrete Cauchy problem:

$$\widetilde{T}_l - 2z\widetilde{T}_{l-1} + \widetilde{T}_{l-2} = 0, \qquad \widetilde{T}_0 = 1,\ \widetilde{T}_1 = 2z. \tag{41}$$

The recurrence equality (41) allow us to find the Chebyshev polynomials $\widetilde{T}_l$ for all $l \in \mathbb{Z}$. For example, $\widetilde{T}_{-1} = 0$, $\widetilde{T}_2 = 4z^2 - 1$. Using Formula (41), we prove that determinant

$$\begin{aligned} d_{l,i} :=& \begin{vmatrix} \widetilde{T}_{l-i} & \widetilde{T}_{l-i-1} \\ \widetilde{T}_{i-1} & \widetilde{T}_i \end{vmatrix} = \begin{vmatrix} 2z\widetilde{T}_{l-i-1} - \widetilde{T}_{l-i-2} & \widetilde{T}_{l-i-1} \\ 2z\widetilde{T}_i - \widetilde{T}_{i+1} & \widetilde{T}_i \end{vmatrix} \\ =& \begin{vmatrix} -\widetilde{T}_{l-i-2} & \widetilde{T}_{l-i-1} \\ -\widetilde{T}_{i+1} & \widetilde{T}_i \end{vmatrix} = \begin{vmatrix} \widetilde{T}_{l-i-1} & \widetilde{T}_{l-i-2} \\ \widetilde{T}_i & \widetilde{T}_{i+1} \end{vmatrix} = d_{l,i+1}. \end{aligned}$$

Thus, this determinant does not depend on i. If $i = l$, then $d_{l,l} = \widetilde{T}_0\widetilde{T}_l - \widetilde{T}_{l-1}\widetilde{T}_{-1} = \widetilde{T}_l$, and we prove the formula

$$\begin{vmatrix} \widetilde{T}_{l-i} & \widetilde{T}_{l-i-1} \\ \widetilde{T}_{i-1} & \widetilde{T}_i \end{vmatrix} = \widetilde{T}_l. \tag{42}$$

In the case $i = 1$, Formula (42) is the same as (41).

The zeros of $\widetilde{T}_l$, $l \in \mathbb{N}$, are

$$z_i = \cos\tfrac{(l-i+1)\pi}{l+1}, \quad i = 1,2,\ldots,l,$$

i.e., all roots $z_i \in (0,1)$.

Let us consider a triangular matrix [18]

$$\mathbf{A}_L = \mathbf{A}_L(z) := \begin{pmatrix} 2z & -1 & & & \\ -1 & 2z & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2z & -1 \\ & & & -1 & 2z \end{pmatrix} \in \mathbb{R}^{L\times L}, \quad z \in \mathbb{R},\ L \geq 1.$$

Lemma 4. $\det \mathbf{A}_L = \widetilde{T}_L$. *If* z *is not zero of* $\widetilde{T}_L$, *then there exists inverse matrix* $(\mathbf{A}_L)^{-1} = \big(\widetilde{T}_L(z)\big)^{-1}\mathbf{T}_L(z)$, *where*

$$\mathbf{T} = \mathbf{T}_L(z) := \begin{pmatrix} \widetilde{T}_{L-1}(z) & \cdots & \widetilde{T}_q(z) & \cdots & \widetilde{T}_k(z) & \cdots & \widetilde{T}_0(z) \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ \widetilde{T}_q(z) & \cdots & \widetilde{T}_q(z)\widetilde{T}_s(z) & \cdots & \widetilde{T}_k(z)\widetilde{T}_s(z) & \cdots & \widetilde{T}_s(z) \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \widetilde{T}_k(z) & \cdots & \widetilde{T}_k(z)\widetilde{T}_s(z) & \cdots & \widetilde{T}_k(z)\widetilde{T}_l(z) & \cdots & \widetilde{T}_l(z) \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ \widetilde{T}_0(z) & \cdots & \widetilde{T}_s(z) & \cdots & \widetilde{T}_l(z) & \cdots & \widetilde{T}_{L-1}(z) \end{pmatrix},$$

matrix $\mathbf{T} = (T_{ij}) \in \mathbb{R}^{L\times L}$ *is symmetric,* $T_{ij} = \widetilde{T}_{L-i}\widetilde{T}_{j-1}$ *for* $j \le i$.

Proof. For a function $D_L = \det\big(\mathbf{A}_L(z)\big)$, we can easily prove the recursive formula $D_L = 2zD_{L-1} - D_{L-2}$. Since $D_1 = 2z$, $D_2 = 4z^2 - 1$, we get $D_L = \widetilde{T}_L$, i.e., $\det \mathbf{A}_L = \widetilde{T}_L$.

If $\mathbf{E} = (E_{ij}) = \mathbf{TA}$, then for $j < i$ we get

$$E_{ij} = \sum_{l=1}^{L} T_{il}A_{lj} = \widetilde{T}_{L-i}(-\widetilde{T}_{j-2} + 2z\widetilde{T}_{j-1} - \widetilde{T}_j) = 0.$$

Similarly, we get $E_{ij} = 0$ for $j > i$. Additionally, we have $\mathbf{AT} = \mathbf{A}^\top\mathbf{T}^\top = (\mathbf{TA})^\top = \mathbf{E}^\top = \mathbf{E}$. Using (41), we find $E_{11} = E_{LL} = 2z\widetilde{T}_{L-1} - \widetilde{T}_{L-2} = \widetilde{T}_L$, and for $i = 2, \dots, L-1$, we derive (using symmetry of the matrix $\mathbf{T}$)

$$\begin{aligned} E_{ii} &= -\widetilde{T}_{L-i}\widetilde{T}_{i-2} + 2z\widetilde{T}_{L-i}\widetilde{T}_{i-1} - \widetilde{T}_{L-i-1}\widetilde{T}_{i-1} \\ &= \widetilde{T}_{L-i}(-\widetilde{T}_{i-2} + 2z\widetilde{T}_{i-1} - \widetilde{T}_i) + \begin{vmatrix} \widetilde{T}_{L-i} & \widetilde{T}_{L-i-1} \\ \widetilde{T}_{i-1} & \widetilde{T}_i \end{vmatrix} \\ &= \widetilde{T}_{L-i}\cdot 0 + \widetilde{T}_L = \widetilde{T}_L. \end{aligned}$$

So, we prove that $\mathbf{T}_L\mathbf{A}_L = \mathbf{A}_L\mathbf{T}_L = \mathbf{I}_L\widetilde{T}_L$. □

Lemma 4 allows us to find the matrix inverse formula for the matrix $\mathbf{Z}$ (see (16)).

Corollary 3. *If* $z \ge 1$, *then* $\mathbf{A}_L(z)$ *is nonsingular.*

Example 2. *If* $z = 1$, *then* $\widetilde{T}_l(1) = l + 1$, $l \in \mathbb{N}$. *So,* $\big(\mathbf{A}_L(1)\big)^{-1} = \frac{1}{L+1}\mathbf{T}_L(1)$,

$$\mathbf{T}_L(1) = \begin{pmatrix} L & L-1 & L-2 & \cdots & 3 & 2 & 1 \\ L-1 & (L-1)2 & (L-2)2 & \cdots & 3\cdot 2 & 2\cdot 2 & 2 \\ L-2 & (L-2)2 & (L-2)3 & \cdots & 3\cdot 3 & 2\cdot 3 & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3 & 3\cdot 2 & 3\cdot 3 & \cdots & 3(L-2) & 2(L-2) & L-2 \\ 2 & 2\cdot 2 & 2\cdot 3 & \cdots & 2(L-2) & 2(L-1) & L-1 \\ 1 & 2 & 3 & \cdots & L-2 & L-1 & L \end{pmatrix}.$$

Now we can find

$$(\mathbf{\Lambda}_x)^{-1} = h_x^3\mathbf{T}_{N-1}(1),\ (\mathbf{\Lambda}_y)^{-1} = h_y^3\mathbf{T}_{M-1}(1).$$

The Kronecker product has the property $(\mathbf{A}\otimes\mathbf{B})^{-1} = \mathbf{A}^{-1}\otimes\mathbf{B}^{-1}$. *Thus,* $(\mathbf{A}_x^{\mathrm{i}})^{-1} := h_x^3\mathbf{I}_{M-1}\otimes\mathbf{T}_{N-1}(1)$, $(\mathbf{A}_y^{\mathrm{i}})^{-1} := h_y^3\mathbf{T}_{M-1}(1)\otimes\mathbf{I}_{N-1}$.

Chebyshev polynomials can be extended to matrix arguments. For a square matrix $\mathbf{Z}$, the Chebyshev polynomials of the second kind $\widetilde{T}_l(\mathbf{Z})$ are defined similarly to the scalar case but using matrix multiplication [50]. The recurrence relation is:

$$\widetilde{T}_l(\mathbf{Z}) = 2\mathbf{Z}\,\widetilde{T}_{l-1}(\mathbf{Z}) - \widetilde{T}_{l-2}(\mathbf{Z}), \quad \widetilde{T}_0(\mathbf{Z}) = \mathbf{I}, \quad \widetilde{T}_1(\mathbf{Z}) = 2\mathbf{Z},$$

where $\mathbf{I}$ denotes the identity matrix of the same size as $\mathbf{Z}$. Then, using $\widetilde{T}_l(\mathbf{Z})$, we can define a symmetric block matrix $\mathbf{T}_L(\mathbf{Z}) = (\mathbf{T}_{ij})$, where $\mathbf{T}_{ij} = \widetilde{T}_{L-i}(\mathbf{Z})\widetilde{T}_{j-1}(\mathbf{Z})$, $j \leq i$. For example, $\widetilde{T}_l(z\mathbf{I}) = \widetilde{T}_l(z) \cdot \mathbf{I}$, $\mathbf{T}_L(z\mathbf{I}) = \mathbf{T}_L(z) \otimes \mathbf{I}$.

Let us consider a triangular block matrix (see (15))

$$\mathbf{A}_L(\mathbf{Z}) := \begin{pmatrix} 2\mathbf{Z} & -\mathbf{I} & & & \\ -\mathbf{I} & 2\mathbf{Z} & -\mathbf{I} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbf{I} & 2\mathbf{Z} & -\mathbf{I} \\ & & & -\mathbf{I} & 2\mathbf{Z} \end{pmatrix}_{L\times L} \in \mathbb{R}^{Ll\times Ll}, \ \mathbf{Z}, \mathbf{I} \in \mathbb{R}^{l\times l}, \ L, l \geq 1.$$

Lemma 5. $\det \mathbf{A}_L(\mathbf{Z}) = \det \widetilde{T}_L(\mathbf{Z})$. *If* $\det \widetilde{T}_L(\mathbf{Z}) \neq 0$, *then*

$$\big(\mathbf{A}_L(\mathbf{Z})\big)^{-1} = \big(\mathbf{I}_L \otimes \mathrm{T}_L(\mathbf{Z})\big)^{-1}\mathbf{T}_L(\mathbf{Z}). \tag{43}$$

Proof. Since $T_1(\mathbf{Z}) = 2\mathbf{Z}$, $T_0(\mathbf{Z}) = \mathbf{I}$ and

$$\begin{vmatrix} \widetilde{T}_1(\mathbf{Z}) & -\mathbf{I} & & & \\ -\widetilde{T}_0(\mathbf{Z}) & 2\mathbf{Z} & -\mathbf{I} & & \\ & -\mathbf{I} & 2\mathbf{Z} & -\mathbf{I} & \\ & & -\mathbf{I} & 2\mathbf{Z} & -\mathbf{I} \\ & & & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} \mathbf{I} & \widetilde{T}_1(\mathbf{Z}) & & & \\ & \widetilde{T}_2(\mathbf{Z}) & -\mathbf{I} & & \\ & -\widetilde{T}_1(\mathbf{Z}) & 2\mathbf{Z} & -\mathbf{I} & \\ & & -\mathbf{I} & 2\mathbf{Z} & -\mathbf{I} \\ & & & \ddots & \ddots \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{I} & & \widetilde{T}_1(\mathbf{Z}) & & \\ & \mathbf{I} & \widetilde{T}_2(\mathbf{Z}) & & \\ & & \widetilde{T}_3(\mathbf{Z}) & -\mathbf{I} & \\ & & -\widetilde{T}_2(\mathbf{Z}) & 2\mathbf{Z} & -\mathbf{I} \\ & & & \ddots & \ddots \end{vmatrix} = \ldots = \begin{vmatrix} \mathbf{I} & & & & \widetilde{T}_1(\mathbf{Z}) \\ & \mathbf{I} & & & \widetilde{T}_2(\mathbf{Z}) \\ & & \ddots & & \vdots \\ & & & \mathbf{I} & \widetilde{T}_{L-1}(\mathbf{Z}) \\ & & & & \widetilde{T}_L(\mathbf{Z}) \end{vmatrix},$$

we have $\det \mathbf{A}_L(\mathbf{Z}) = \det \widetilde{T}_L(\mathbf{Z})$. Repeating the proof of Lemma 4, we get $\mathbf{T}_L(\mathbf{Z})\mathbf{A}_L(\mathbf{Z}) = \mathbf{A}_L(\mathbf{Z})\mathbf{T}_L(\mathbf{Z}) = \mathbf{I}_L \otimes \widetilde{T}_L(\mathbf{Z})$. The explicit Formula (43) for the inverse of a block matrix $\mathbf{A}_L(\mathbf{Z})$ is proved. □

3.2. Examples of Problems with Integral Boundary Conditions

We consider two examples corresponding to different cases of nonlocal boundary conditions.

Example 3 (the case of commutative matrices)**.** *Problems are often studied when NBCs are in only one direction [11]. Let us consider the following BCs:*

$$u(0,y) = \int_0^{l_x} \alpha(x)u(x,y)dx + g^l(y), \quad u(l_x,y) = \int_0^{l_x} \beta(x)u(x,y)dx + g^r(y),$$

where $y \in (0, l_y)$, *and* $u(x,0) = g^b(x)$, $u(x,l_y) = g^t(x)$, $x \in [0,l_x]$. *We approximate one-dimensional integrals using the trapezoidal rule* ($r^{i_1} = h_x/2$ *for* $i_1 = 0, N$, *else* $r^{i_1} = h_x$)*:*

$$U_{0j} = \sum_{i_1=0}^{N} r^{i_1}\alpha^{i_1}U_{i_1 j} + G_j^l, \quad U_{Nj} = \sum_{i_1=0}^{N} r^{i_1}\beta^{i_1}U_{i_1 j} + G_j^r, \quad j \in \mathcal{I}_y, \tag{44}$$

$$U_{i0} = G_i^b, \quad U_{iM} = G_i^t, \quad i \in \overline{\mathcal{I}}_x, \tag{45}$$

where $\alpha^{i_1} = \alpha(x_{i_1})$, $\beta^{i_1} = \beta(x_{i_1})$. *Additionally, we have the compatibility conditions*

$$G_0^b = \sum_{i_1=0}^{N} r^{i_1}\alpha^{i_1}G_{i_1}^b + G_0^l, \quad G_N^b = \sum_{i_1=0}^{N} r^{i_1}\beta^{i_1}G_{i_1}^b + G_0^r,$$
$$G_0^t = \sum_{i_1=0}^{N} r^{i_1}\alpha^{i_1}G_{i_1}^t + G_M^l, \quad G_N^t = \sum_{i_1=0}^{N} r^{i_1}\beta^{i_1}G_{i_1}^t + G_M^r.$$

These conditions are Equation (44) *in the case* $j = 0, M$. *So, we consider the case* $\partial\omega = \varnothing$, $m = m_1$, $\partial\overline{\omega} = \partial\omega = \partial\mathring{\omega} \cup \partial\overset{*}{\omega}$ *(see Figure 4 and Remark 3). We have Dirichlet BCs at the nodes of the* $\partial\mathring{\omega}$ *grid, and BCs* (44) *at the nodes of the* $\partial\overset{*}{\omega}$ *grid. For simplicity, we will take homogeneous BC, i.e.,* $G^l = G^r = G^b = G^t \equiv 0$.

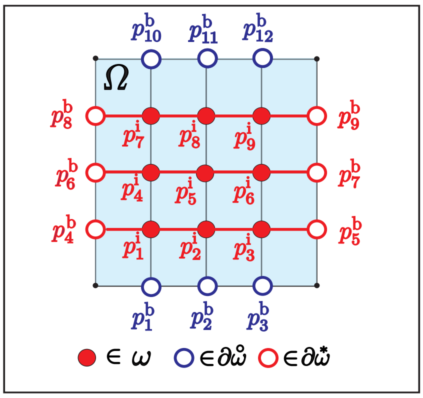


Figure 4. The grid for Example 3, $N = 4$, $M = 4$.

We define matrices

$$\mathbf{P} = \frac{h_x}{2}\begin{pmatrix} \alpha^0 & \alpha^N \\ \beta^0 & \beta^N \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \mathbf{Q} = h_x\begin{pmatrix} \alpha^1 & \dots & \alpha^{N-1} \\ \beta^1 & \dots & \beta^{N-1} \end{pmatrix} \in \mathbb{R}^{2\times(N-1)}$$

$$\mathbf{R} = \mathbf{I}_2 - \mathbf{P} = \begin{pmatrix} 1 - \frac{h_x}{2}\alpha^0 & -\frac{h_x}{2}\alpha^N \\ -\frac{h_x}{2}\beta^0 & 1 - \frac{h_x}{2}\beta^N \end{pmatrix}, \quad \mathbf{S} = \mathbf{R}^{-1} = (\mathbf{I}_2 - \mathbf{P})^{-1}.$$

If $\det \mathbf{R} = 1 - (\alpha^0 + \beta^N)h_x/2 + (\alpha^0\beta^N - \alpha^N\beta^0)h_x^2/4 \neq 0$, *then* $\mathbf{S} = \mathbf{R}^{-1}$ *exists. If* h_x *is small enough, then this condition is fulfilled.*

We can rewrite BC in the form (26), *where*

$$K_{0j}^{i_1 j_1} = \alpha^{i_1}\delta_j^{j_1}h_y^{-1}, \quad K_{Nj}^{i_1 j_1} = \beta^{i_1}\delta_j^{j_1}h_y^{-1}, \quad j \in \mathcal{I}_y,$$

and another $K_{ij}^{i_1 j_1} = 0$. *Then, we have BC in the matrix form* (20), *where*

$$\mathbf{B}^{\mathrm{b}} = \begin{pmatrix} \widetilde{\mathbf{O}} & & & & \\ & \mathbf{P} & & & \\ & & \ddots & & \\ & & & \mathbf{P} & \\ & & & & \widetilde{\mathbf{O}} \end{pmatrix}_{p\times(q+2)} \in \mathbb{R}^{m\times m}, \quad \mathbf{B}^{\mathrm{i}} = \begin{pmatrix} \widetilde{\mathbf{O}} & & \\ \mathbf{Q} & & \\ & \ddots & \\ & & \mathbf{Q} \\ & & \widetilde{\mathbf{O}} \end{pmatrix}_{p\times q} \in \mathbb{R}^{m\times n},$$

$\widetilde{\mathbf{O}} = \mathbf{O}_{(N-1)\times(N-1)}$, $\mathbf{G}^{\mathrm{b}} = \mathbf{0}$, $p = M+1$, $q = M-1$, $n = (N-1)(M-1)$, $m = 2(N+M)-4$. *Then*

$$\mathbf{A}^{\mathrm{dd}} = \begin{pmatrix} \mathbf{I}_{N-1} & & & & \\ & \mathbf{R} & & & \\ & & \ddots & & \\ & & & \mathbf{R} & \\ & & & & \mathbf{I}_{N-1} \end{pmatrix}, \; (\mathbf{A}^{\mathrm{dd}})^{-1} = \begin{pmatrix} \mathbf{I}_{N-1} & & & & \\ & \mathbf{S} & & & \\ & & \ddots & & \\ & & & \mathbf{S} & \\ & & & & \mathbf{I}_{N-1} \end{pmatrix},$$

and we can find the matrix

$$\mathbf{C} = -\mathbf{A}^{\mathrm{d}}(\mathbf{A}^{\mathrm{dd}})^{-1}\mathbf{B}^{\mathrm{i}} = \mathbf{I}_{M-1} \otimes \mathbf{D},$$

where $\mathbf{D} = h_x^{-2}(\tilde{\mathbf{J}}\mathbf{S}\mathbf{Q})$. *So, by Lemma 3, the matrices* $\mathbf{A}_x^{\mathrm{i}} - \mathbf{C} = \mathbf{I}_{M-1} \otimes (\mathbf{\Lambda}_x - \mathbf{D})$ *and* $\mathbf{A}_y^{\mathrm{i}} = \mathbf{\Lambda}_y \otimes \mathbf{I}_{N-1}$ *commute, and we can use many of the same efficient solution methods as in the classical case.*

Example 4. *Let us consider the following BC*

$$u(x,y) = \gamma \int_{\Omega} u(\xi,\eta)d\xi d\eta + g(x,y), \quad (x,y) \in \partial\Omega,$$

and approximate this BC by

$$U_k = \gamma[1,U] + G_k, \quad p_k \in \partial\overline{\omega} = \partial\overline{\omega}^h, \tag{46}$$

and in this case $K \equiv \gamma$ *(see Figure 5). Then, we have BC in the matrix form (20) with*

$$\mathbf{B}^{\mathrm{b}} = \begin{pmatrix} \alpha\mathbf{E}_{11} & \beta\mathbf{E}_{12} \\ \alpha\mathbf{E}_{21} & \beta\mathbf{E}_{22} \end{pmatrix}, \quad \mathbf{B}^{\mathrm{i}} = \overline{\gamma}\begin{pmatrix} \mathbf{E}_{13} \\ \mathbf{E}_{23} \end{pmatrix},$$

where $\mathbf{E}_{11} \in \mathbb{R}^{m_1\times m_1}$, $\mathbf{E}_{12} \in \mathbb{R}^{m_1\times m_2}$, $\mathbf{E}_{21} \in \mathbb{R}^{m_2\times m_1}$, $\mathbf{E}_{22} \in \mathbb{R}^{m_2\times m_2}$, $\mathbf{E}_{13} \in \mathbb{R}^{m_1\times n}$, $\mathbf{E}_{23} \in \mathbb{R}^{m_2\times n}$, *all elements of these matrices are* $e_{ij} = 1$, *and* $\alpha = \gamma h^2/2$, $\beta = \gamma h^2/4$, $\overline{\gamma} = \gamma h^2$, $m_1 = 2(N+M)-4$, $m_2 = 4$, $m = m_1 + m_2 = 2(N+M)$. *We will notice the property* $\mathbf{E}_{lk}\mathbf{E}_{kq} = m_k\mathbf{E}_{lq}$, $l,k = 1,2$, $q = 1,2,3$.

Then, the matrix $\mathbf{A}^{\mathrm{bb}}$ *is of the form*

$$\mathbf{A}_{\alpha,\beta} := \begin{pmatrix} \mathbf{I}_{m_1} - \alpha\mathbf{E}_{11} & -\beta\mathbf{E}_{12} \\ -\alpha\mathbf{E}_{21} & \mathbf{I}_{m_2} - \beta\mathbf{E}_{22} \end{pmatrix}, \quad \alpha,\beta \in \mathbb{R}, \; m_1, m_2 > 0.$$

Additionally, we define $\mathbf{A}_\alpha := \mathbf{I}_{m_1} - \alpha\mathbf{E}_{11}$, *i.e.,* $\mathbf{A}_{\alpha,\beta} = \mathbf{A}_\alpha$ *in the case* $m_2 = 0$. *Note that* $\mathbf{A}_{0,0} = \mathbf{I}_m$, $\mathbf{A}_0 = \mathbf{I}_{m_1}$.

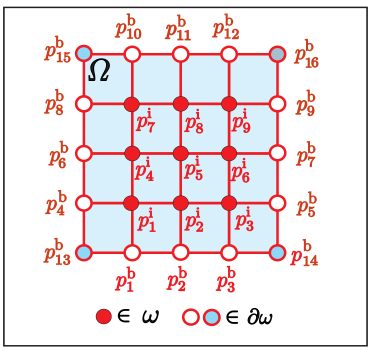


Figure 5. The grid for Example 4, $N = 4$, $M = 4$.

Lemma 6. *The following properties hold:*

$$\det \mathbf{A}_{\alpha,\beta} = d := 1 - m_1\alpha - m_2\beta; \tag{47}$$

$$\mathbf{A}_{\alpha,\beta}\mathbf{B}^{\mathrm{i}} = \det \mathbf{A}_{\alpha,\beta}\mathbf{B}^{\mathrm{i}}; \tag{48}$$

$$\mathbf{A}_{\alpha_1,\beta_1}\mathbf{A}_{\alpha_2,\beta_2} = \mathbf{A}_{\alpha_1+\alpha_2 d_1,\beta_1+\beta_2 d_1}, \quad d_1 = \det \mathbf{A}_{\alpha_1,\beta_1}; \tag{49}$$

if $\det \mathbf{A}_{\alpha,\beta} \neq 0$, *then*

$$(\mathbf{A}_{\alpha,\beta})^{-1} = \mathbf{A}_{-\alpha/d,-\beta/d}, \tag{50}$$

$$(\mathbf{A}_{\alpha,\beta})^{-1}\mathbf{B}^{\mathrm{i}} = (\det \mathbf{A}_{\alpha,\beta})^{-1}\mathbf{B}^{\mathrm{i}}. \tag{51}$$

Proof. First, we find

$$\det \mathbf{A}_\alpha = \begin{vmatrix} 1-\alpha & -\alpha & \dots & -\alpha \\ -\alpha & 1-\alpha & \dots & -\alpha \\ \dots & \dots & \ddots & \dots \\ -\alpha & -\alpha & & 1-\alpha \end{vmatrix} = \begin{vmatrix} 1-\alpha & -\alpha & \dots & -\alpha \\ -1 & 1 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ -1 & 0 & & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 1-m_1\alpha & -\alpha & \dots & -\alpha \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & & 1 \end{vmatrix} = 1 - m_1\alpha.$$

If $\alpha \neq 1/m_1$, then (using property $\mathbf{E}_{11}\mathbf{E}_{11} = m_1\mathbf{E}_{11}$) we get

$$(\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11})(\mathbf{I}_{m_1} + \tfrac{\alpha}{1-m_1\alpha}\mathbf{E}_{11}) = (I_{m_1} + \tfrac{\alpha}{1-m_1\alpha}\mathbf{E}_{11})(\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11}) = \mathbf{I}_{m_1},$$

i.e., we prove (50) in the case $m_2 = 0$ (see, Sherman—Morrison Formula [51]):

$$(\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11})^{-1} = \mathbf{I}_{m_1} + \tfrac{\alpha}{1-m_1\alpha}\mathbf{E}_{11}.$$

If $\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11}$ is invertible, the Schur complement of the block $I_{m_1} - \alpha\mathbf{E}_{11}$ of the matrix $\mathbf{A}_{\alpha,\beta}$ (see [52]) is

$$\begin{aligned} \mathbf{S} &= \mathbf{I}_{m_2} - \beta\mathbf{E}_{22} - \alpha\mathbf{E}_{21}(\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11})^{-1}\beta\mathbf{E}_{12} \\ &= \mathbf{I}_{m_2} - \beta\mathbf{E}_{22} - \alpha\beta\mathbf{E}_{21}(\mathbf{I}_{m_1} + \tfrac{\alpha}{1-m_1\alpha}\mathbf{E}_{11})\mathbf{E}_{12} \\ &= \mathbf{I}_{m_2} - \beta\mathbf{E}_{22} - \alpha\beta m_1\mathbf{E}_{22} - \tfrac{\alpha^2\beta m_1^2}{1-m_1\alpha}\mathbf{E}_{22} \\ &= \mathbf{I}_{m_2} - \beta(1 + m_1\alpha + \tfrac{m_1^2\alpha^2}{1-m_1\alpha})\mathbf{E}_{22} = \mathbf{I}_{m_2} - \tfrac{\beta}{1-m_1\alpha}\mathbf{E}_{22}, \end{aligned}$$

(we use properties $\mathbf{E}_{21}\mathbf{I}_{m_1}\mathbf{E}_{12} = m_1\mathbf{E}_{22}$ and $\mathbf{E}_{21}\mathbf{E}_{11}\mathbf{E}_{12} = m_1^2\mathbf{E}_{22}$) and (see [53])

$$\begin{aligned} \det \mathbf{A}_{\alpha,\beta} &= \det(\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11}) \det \mathbf{S} = \det(\mathbf{I}_{m_1} - \alpha\mathbf{E}_{11}) \det(\mathbf{I}_{m_2} - \tfrac{\beta}{1-m_1\alpha}\mathbf{E}_{22}) \\ &= (1 - m_1\alpha)(1 - \tfrac{m_2\beta}{1-m_1\alpha}) = 1 - m_1\alpha - m_2\beta. \end{aligned}$$

Using continuity, we obtain that the proven Formula (47) is also true when $\alpha = 1/m_1$.

Formula (48) follows from

$$\begin{pmatrix} \mathbf{I}_{m_1} - \alpha\mathbf{E}_{11} & -\beta\mathbf{E}_{12} \\ -\alpha\mathbf{E}_{21} & \mathbf{I}_{m_2} - \beta\mathbf{E}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{E}_{13} \\ \mathbf{E}_{23} \end{pmatrix} = (1 - m_1\alpha - m_2\beta) \begin{pmatrix} \mathbf{E}_{13} \\ \mathbf{E}_{23} \end{pmatrix}.$$

The property (49) can be verified in an elementary manner using the same reasoning. Then, by (49), property (50) follows, and property (51) follows from (48). □

For BC (46) $d = 1 - \gamma(l_x h_y + l_y h_x - h_x h_y) = 1 - \gamma|\partial\Omega^h|$.

Corollary 4. *If* $\gamma|\partial\Omega^h| \neq 1$, *then*

$$\widetilde{\mathbf{B}}^{\mathrm{i}} = (\mathbf{A}^{\mathrm{bb}})^{-1}\mathbf{B}^{\mathrm{i}} = d^{-1}\mathbf{B}^{\mathrm{i}} = (1 - \gamma|\partial\Omega^h|)^{-1}\mathbf{B}^{\mathrm{i}}.$$

Using Corollary 4, we derive

$$\mathbf{C} = -\mathbf{A}^{\mathrm{b}}(\mathbf{A}^{\mathrm{bb}})^{-1}\mathbf{B}^{\mathrm{i}} = \gamma d^{-1}(h_x^{-1}h_y\mathbf{E}_{M-1} \otimes \overline{\mathbf{E}}_{N-1} + h_x h_y^{-1}\overline{\mathbf{E}}_{M-1} \otimes \mathbf{E}_{N-1}),$$

where

$$\mathbf{E}_k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{k\times k}, \quad \overline{\mathbf{E}}_k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{k\times k}.$$

Note that $\mathbf{\Lambda}_k\mathbf{E}_k = \overline{\mathbf{E}}_k$, $[\mathbf{\Lambda}_k, \mathbf{E}_k] = \overline{\mathbf{E}}_k - \overline{\mathbf{E}}_k^\top$, where $\mathbf{\Lambda}_k$ is defined in Section 2.4. Using Formula (17) we get

$$\begin{aligned}[\mathbf{A}^{\mathrm{i}}_y, \mathbf{C}] &= \tfrac{\gamma d^{-1}}{h_x h_y}[\mathbf{\Lambda}_{M-1}, \mathbf{E}_{M-1}] \otimes \overline{\mathbf{E}}_{N-1} + \tfrac{\gamma d^{-1}}{h_x h_y^3}[\mathbf{\Lambda}_{M-1}, \overline{\mathbf{E}}_{M-1}] \otimes \mathbf{E}_{N-1} \\ &= \tfrac{\gamma d^{-1}}{h_x h_y}(\overline{\mathbf{E}}_{M-1} - \overline{\mathbf{E}}^\top_{M-1}) \otimes \overline{\mathbf{E}}_{N-1} + \tfrac{\gamma d^{-1}}{h_x h_y^3}[\mathbf{\Lambda}_{M-1}, \overline{\mathbf{E}}_{M-1}] \otimes \mathbf{E}_{N-1}.\end{aligned}$$

Then, we have

$$[\mathbf{A}^{\mathrm{i}}_y, \mathbf{C}]_{11} = \tfrac{\gamma d^{-1}}{h_x h_y^3}[\mathbf{\Lambda}_{M-1}, \overline{\mathbf{E}}_{M-1}]_{11} = \tfrac{\gamma d^{-1}}{h_x h_y^3}(1 - \delta^1_{M-1})(1 - \delta^2_{M-1}).$$

Analogously, we get $[\mathbf{A}^{\mathrm{i}}_x, \mathbf{C}]_{11} = \frac{\gamma d^{-1}}{h_x^3 h_y}(1 - \delta^1_{N-1})(1 - \delta^2_{N-1})$. So, $[\mathbf{A}^{\mathrm{i}}_y, \mathbf{C}] \neq \mathbf{O}$ and $[\mathbf{A}^{\mathrm{i}}_x, \mathbf{C}] \neq \mathbf{O}$ for $\gamma \neq 0$, $N, M > 3$.

3.3. Solving a Problem with Nonlocal Conditions

We return to FDS′ (29) and (30) (which we obtained by studying FDS (25) and (26)). This problem can be divided into two separate problems:

(P1) classical problem with Dirichlet BC:

$$\begin{aligned} -\delta_x^2 W - \delta_y^2 W &= F, \quad p_l \in \omega, \\ W_k &= \widetilde{G}_k, \quad p_k \in \partial\omega = \partial\omega^h, \end{aligned}$$

$W \in \mathbb{R}^{\widetilde{\omega}}$, $F \in \mathbb{R}^{\omega}$, $\widetilde{G} \in \mathbb{R}^{\partial\omega}$;

(P2) discrete Laplace equation with nonlocal BC:

$$-\delta_x^2 V - \delta_y^2 V = 0, \quad p_l \in \omega, \tag{52}$$

$$V_k = (\widetilde{K}, V)_k + \overline{G}_k, \quad p_k \in \partial\omega = \partial\omega^h, \tag{53}$$

where $V \in \mathbb{R}^{\widetilde{\omega}}$, $\overline{G} = (\widetilde{K}, W) \in \mathbb{R}^{\partial\omega}$.

Then a solution of FDS′ is $U = W + V$, $p_k \in \widetilde{\omega} = \omega + \partial\omega$.

Problem P1 is classical, and we can use various methods to find it (see Section 3.1). From (38) we have

$$||W||_\infty \le ||\widetilde{G}||_\infty + C||F||_\infty. \tag{54}$$

For Problem P2, BC is nonlocal. Using (54) and $||\widetilde{B}||_\infty < 1$, we estimate

$$||\overline{G}||_\infty = ||(\widetilde{K}, W)||_\infty \le ||\widetilde{B}||_\infty ||W||_\infty \le ||\widetilde{G}||_\infty + C||F||_\infty. \tag{55}$$

Lemma 7. *For a solution of FDS* (52) *and* (53)*, the estimate* $||V||_\infty \le C||\overline{G}||_\infty$ *holds.*

Proof. For Equation (52) we use the Maximum Principle: $\max_{p_k \in \widetilde{\omega}} |V_k| \le \max_{p_k \in \partial\omega} |V_k|$. Then for $p_k \in \partial\omega$ we estimate

$$|V_k| \le (|\widetilde{K}|, 1) \max_{p_l \in \omega} |V_l| + |\overline{G}_k| \le ||\widetilde{B}||_\infty \max_{p_l \in \partial\omega} |V_l| + ||\overline{G}||_\infty.$$

Since $||\widetilde{B}||_\infty \le \tilde{\rho} < 1$ we get

$$\max_{p_k \in \partial\omega} |V_k| \le (1 - ||\widetilde{B}||_\infty)^{-1} ||\overline{G}||_\infty$$

and

$$||V||_\infty = \max_{p_k \in \widetilde{\omega}} |V_k| \le \max_{p_k \in \partial\omega} |V_k| \le (1 - ||\widetilde{B}||_\infty)^{-1} ||\overline{G}||_\infty \le (1 - \tilde{\rho})^{-1} ||\overline{G}||_\infty.$$

□

Corollary 5. *For a solution of FDS′* (29)–(30)*, the stability estimate* $||U||_\infty \le C(||\widetilde{G}||_\infty + ||F||_\infty)$ *holds.*

Proof. This follows from Lemma 7, (54) and (55). □

Theorem 2. *For a solution of FDS* (25)–(26)*, the stability estimate*

$$||U||_\infty \le C(||G||_\infty + ||F||_\infty).$$

holds.

Proof. We use Corollary 5 and (28). □

If U^κ, $\kappa = 1, \ldots, m = |\partial\omega|$, are the solutions of the problem C2 (39) and (40), then we search for solution of FDS (52) and (53) in the form $V = \sum_{\kappa=1}^m y_\kappa U^\kappa$. Such V satisfies the discrete Laplace Equation (52). We substitute such V into BC (53) and get

$$y_k = \sum_{\kappa=1}^m a_{k\cdot}^{\cdot\kappa} y_\kappa + \overline{G}_k, \quad a_{k\cdot}^{\cdot\kappa} = (\widetilde{K}, U^\kappa)_k, \;\; k, \kappa = 1, \ldots, m.$$

We have linear system $\mathbf{Y} = \boldsymbol{\mathcal{A}}\mathbf{Y} + \overline{\mathbf{G}}$ for $\mathbf{Y} := (y_1, \ldots, y_m)^\top$ with a matrix $\boldsymbol{\mathcal{A}} := (a_{k\cdot}^{\cdot\kappa}) \in \mathbb{R}^{m \times m}$. Note that the size of this linear system is only $m = 2(N + M) - 4$. We estimate

$$\begin{aligned} ||\boldsymbol{\mathcal{A}}||_\infty &= \max_{k=1,\ldots,m} \sum_{\kappa=1}^m |a_{k\cdot}^{\cdot\kappa}| \le \max_{k=1,\ldots,m} \sum_{\kappa=1}^m \sum_{l=1}^n |\widetilde{K}_{k\cdot}^{\cdot l}| \cdot U_l^\kappa h^2 \\ &\le \max_{k=1,\ldots,m} \sum_{l=1}^n |\widetilde{B}_{k\cdot}^{\cdot l}| \sum_{\kappa=1}^m U_l^\kappa \le \max_{k=1,\ldots,m} \sum_{l=1}^n |\widetilde{B}_{k\cdot}^{\cdot l}| = ||\widetilde{B}||_\infty \le \tilde{\rho} < 1. \end{aligned}$$

From Lemma 2 we get that the matrix $\mathbf{I}_m - \boldsymbol{\mathcal{A}}$ is nonsingular and $||(\mathbf{I}_m - \boldsymbol{\mathcal{A}})^{-1}||_\infty \leq (1-\tilde{\rho})^{-1}$. So, $\mathbf{Y} = (\mathbf{I}_m - \boldsymbol{\mathcal{A}})^{-1}\overline{\mathbf{G}}$ and $||\mathbf{Y}||_\infty \leq (1-\tilde{\rho})^{-1}||\overline{\mathbf{G}}||_\infty$.

3.4. Convergence

If u is the solution of differential problem (see Section 2.1), and U is the solution of FDS (25) and (26), then a grid function Z, $Z_k := U_k - u(x_k)$, $p_k \in \overline{\omega} = \overline{\omega}^h$, satisfies a discrete problem

$$L^h Z = -\delta_x^2 Z - \delta_y^2 Z = \psi, \quad p_l \in \omega = \omega^h,$$
$$Z_k = [K, Z]_k + \phi_k, \quad p_k \in \partial\overline{\omega} = \partial\overline{\omega}^h,$$

where $||\psi||_{\infty,\omega} = \mathcal{O}(|h|^2)$, $||\phi||_{\infty,\partial\overline{\omega}} = \mathcal{O}(|h|^2)$. We will decompose $Z = W + V$, where $W \in \mathbb{R}^{\overline{\omega}}$ and $V \in \mathbb{R}^{\overline{\omega}}$ are solutions to the following problems:

$$L^h W = -\delta_x^2 W - \delta_y^2 W = \psi, \quad p_l \in \omega,$$
$$W_k = 0, \quad p_k \in \partial\overline{\omega},$$

and

$$L^h V = -\delta_x^2 V - \delta_y^2 V = 0, \quad p_l \in \omega, \tag{56}$$
$$V_k = [K, V]_k + \varphi_k, \quad \varphi_k = \phi_k + (K, W)_k, \quad p_k \in \partial\overline{\omega}. \tag{57}$$

From Theorem 1 (see Remark 5), we get $||W||_{\infty,\overline{\omega}} = ||W||_{\infty,\omega} = \mathcal{O}(|h|^2)$. Then, using (12), we estimate

$$||(K, W)||_{\infty,\partial\overline{\omega}} \leq ||B||_\infty \cdot ||W||_{\infty,\omega} \leq ||W||_{\infty,\omega} = \mathcal{O}(|h|^2),$$

and we get $||\varphi||_{\infty,\partial\overline{\omega}} = \mathcal{O}(|h|^2)$. Since $\overline{\omega} = \overline{\omega}^h = \widetilde{\omega}^h \cup \partial^2\omega^h$, we rewrite BC (56) as

$$V_k = \langle K, V\rangle_k^{\widetilde{\omega}^h} + \langle K, V\rangle_k^{\partial^2\omega^h} + \varphi_k, \quad p_k \in \partial\omega, \tag{58}$$
$$V_k = \langle K, V\rangle_k^{\widetilde{\omega}^h} + \langle K, V\rangle_k^{\partial^2\omega^h} + \varphi_k, \quad p_k \in \partial^2\omega. \tag{59}$$

Using Maximum Principle for Equation (56), from (58), we get the estimate

$$\begin{aligned}||V||_{\infty,\partial\omega} &\leq ||B||_\infty \cdot ||V||_{\infty,\widetilde{\omega}^h} + ||K||_{\max}||V||_{\infty,\partial^2\omega}h^2 + C|h|^2 \\ &\leq \rho_1 \cdot ||V||_{\infty,\partial\omega} + \varkappa||V||_{\infty,\partial^2\omega}h^2 + C|h|^2.\end{aligned}$$

Since $\rho_1 < 1$ (see (12)), we have

$$||V||_{\infty,\widetilde{\omega}} \leq ||V||_{\infty,\partial\omega} \leq C|h|^2(1 + ||V||_{\infty,\partial^2\omega^h}). \tag{60}$$

From (59) and (12), we get the estimate

$$\begin{aligned}||V||_{\infty,\partial^2\omega} &\leq \rho_1||V||_{\infty,\partial^2\omega} + ||B||_\infty \cdot ||V||_{\infty,\widetilde{\omega}^h} + C|h|^2 \\ &\leq (\rho_1 + C_1|h|^2)||V||_{\infty,\partial^2\omega} + C_2|h|^2.\end{aligned}$$

We choose h_0 so that $\rho_1 + C_1|h|^2 \leq \rho = (1+\rho_1)/2 < 1$. Then, we have $||V||_{\infty,\partial^2\omega} = \mathcal{O}(|h|^2)$. From (60), it follows that $||V||_{\infty,\widetilde{\omega}} = \mathcal{O}(|h|^2)$, and $||V||_{\infty,\overline{\omega}} = \mathcal{O}(|h|^2)$. Finally, $||Z||_{\infty,\overline{\omega}} = \mathcal{O}(|h|^2)$. So, we proved the following theorem.

Theorem 3. *The FDS (25) and (26) converges uniformly with the rate $\mathcal{O}(|h|^2)$, that is, it is of the second-order accuracy.*

3.5. Solution Flowchart

We will briefly discuss the proposed solution method from the perspective of linear algebra. Instead of a system of linear Equations (19) and (23), we consider $1 + m_1$, $m_1 = |\partial\omega|$, systems of linear equations:

$$\begin{aligned}
&(\text{P1}) \qquad \mathbf{A}^{\rm i}\mathbf{W}^{\rm i} + \mathbf{A}^{\rm d}\mathbf{W}^{\rm d} = \mathbf{F}^{\rm i}, \quad \mathbf{W}^{\rm d} = \widetilde{\mathbf{G}},\\
&(\text{C2}) \qquad \mathbf{A}^{\rm i}\mathbf{U}^{\kappa,{\rm i}} + \mathbf{A}^{\rm d}\mathbf{U}^{\kappa,{\rm d}} = \mathbf{0}, \quad \mathbf{U}^{\kappa,{\rm d}} = \mathbf{I}_{*\kappa}, \quad \kappa = 1, \ldots, m_1,
\end{aligned}$$

where $\mathbf{I}_{*\kappa}$ is the κ-column of the matrix $\mathbf{I}_m$. These equations we rewrite as

$$\mathbf{A}^{\rm i}\mathbf{W}^{\rm i} = \mathbf{F}^{\rm i} - \mathbf{A}^{\rm d}\widetilde{\mathbf{G}}, \tag{61}$$

$$\mathbf{A}^{\rm i}\boldsymbol{\mathcal{U}}^{\rm i} = -\mathbf{A}^{\rm d}, \quad \boldsymbol{\mathcal{U}}^{\rm i} = (\mathbf{U}^{1,{\rm i}}, \ldots, \mathbf{U}^{m_1,{\rm i}}) \in \mathbb{R}^{n \times m_1}. \tag{62}$$

Note that the second equation is matrix equation. The solutions of these equations are

$$\mathbf{W}^{\rm i} = (\mathbf{A}^{\rm i})^{-1}(\mathbf{F}^{\rm i} - \mathbf{A}^{\rm d}\widetilde{\mathbf{G}}), \quad \boldsymbol{\mathcal{U}}^{\rm i} = -(\mathbf{A}^{\rm i})^{-1}\mathbf{A}^{\rm d}.$$

Then, $\mathbf{U} = \mathbf{W} + \boldsymbol{\mathcal{U}}^{\rm i}\mathbf{Y}$, $\mathbf{Y} = (y_k) \in \mathbb{R}^{m_1}$, satisfies Equation (19). Equation (23) gives linear system

$$\mathbf{Y} = \boldsymbol{\mathcal{A}}\mathbf{Y} + \overline{\mathbf{G}}, \tag{63}$$

where $\boldsymbol{\mathcal{A}} = \widetilde{\mathbf{B}}\boldsymbol{\mathcal{U}}^{\rm i} \in \mathbb{R}^{m_1 \times m_1}$, $\overline{\mathbf{G}} = \widetilde{\mathbf{B}}\mathbf{W}^{\rm i} \in \mathbb{R}^{m_1}$.

The following algorithm is employed to solve the discrete system (25) and (26):

Step 1: Construct the matrices and the vectors $\mathbf{A}^{\rm i} \in \mathbb{R}^{n\times n}$, $\mathbf{A}^{\rm d} \in \mathbb{R}^{n \times m_1}$, $\mathbf{F}^{\rm i} \in \mathbb{R}^n$, $\mathbf{B}^{\rm i} \in \mathbb{R}^{m \times n}$, $\mathbf{B}^{\rm b} \in \mathbb{R}^{m \times m_1}$, $\mathbf{G}^{\rm b} \in \mathbb{R}^m$ (see Equations (19) and (20));

Step 2: Construct $\widetilde{\mathbf{B}}^{\rm i} = (\mathbf{I}_m - \mathbf{B}^{\rm b})^{-1}\mathbf{B}^{\rm i} \in \mathbb{R}^{m\times n}$, $\widetilde{\mathbf{G}}^{\rm b} = (\mathbf{I}_m - \mathbf{B}^{\rm b})^{-1}\mathbf{G}^{\rm b} \in \mathbb{R}^m$ (see (21)), we also have $\widetilde{\mathbf{B}} \in \mathbb{R}^{m_1 \times n}$, $\widetilde{\mathbf{G}} \in \mathbb{R}^{m_1}$ (see (23));

Step 3: Solve Equations (61) and (62), and find $\mathbf{W} \in \mathbb{R}^{n+m_1}$, $\boldsymbol{\mathcal{U}}^{\rm i} \in \mathbb{R}^{(n+m_1)\times m_1}$;

Step 4: Construct the matrix $\boldsymbol{\mathcal{A}} = \widetilde{\mathbf{B}}\boldsymbol{\mathcal{U}}^{\rm i} \in \mathbb{R}^{m_1 \times m_1}$ and vector $\overline{\mathbf{G}} = \widetilde{\mathbf{B}}\mathbf{W}^{\rm i} \in \mathbb{R}^{m_1}$;

Step 5: Solve Equation (63), find $\mathbf{Y}$, $\mathbf{Y} = (y_k) \in \mathbb{R}^{m_1}$, and $\mathbf{U}^{\rm i} = \mathbf{W}^{\rm i} + \boldsymbol{\mathcal{U}}^{\rm i}\mathbf{Y}$;

Step 6: Determine the solution on the boundary points $\mathbf{U}^{\rm b}$ (see (21)).

3.6. The Case with Kernel $k = k(\xi, \eta)$

Let us consider FDS$'$ (29) and (30) in the case $K = K_{k\cdot}^{\cdot l} = k^l$, $p_k \in \partial\overline{\omega}$, $p_l \in \overline{\omega}$. This case corresponds to the differential case $k = k(\xi, \eta)$. Matrices

$$\mathbf{B} = (\mathbf{B}^{\rm i}, \mathbf{B}^{\rm b}) = \mathbf{e}_m \otimes \mathbf{b}, \quad \mathbf{B}^{\rm i} = \mathbf{e}_m \otimes \mathbf{b}^{\rm i}, \quad \mathbf{B}^{\rm b} = \mathbf{e}_m \otimes \mathbf{b}^{\rm b},$$

where $\mathbf{e}_m = (1, \ldots, 1)^\top \in \mathbb{R}^m$, $\mathbf{b} = (\mathbf{b}^{\rm i}, \mathbf{b}^{\rm b})$, $\mathbf{b}^{\rm i} = (b^1, \ldots, b^n) = h^2\mathbf{k}^{\rm i}$, $\mathbf{k}^{\rm i} = (k^1, \ldots, k^n)$, $\mathbf{b}^{\rm b} = (b^{n+1}, \ldots, b^{n+m}) = h^2(k^{n+1}/2, \ldots, k^{n+m_1}/2, k^{n+m_1+1}/4, \ldots, k^{n+m}/4)$.

Lemma 8. *The following properties hold:*

$$\det \mathbf{A}^{\rm bb} = \det(\mathbf{I}_m - \mathbf{B}^{\rm b}) = d := 1 - \sum_{l=1}^{m} b^{n+l} = 1 - \langle K, 1\rangle; \tag{64}$$

$$\mathbf{B}^{\rm b}\mathbf{B}^{\rm i} = (1-d)\mathbf{B}^{\rm i}, \quad (\mathbf{B}^{\rm b})^k = (1-d)^{k-1}\mathbf{B}^{\rm b}, \ k \in \mathbb{N}; \tag{65}$$

if $d \in (0,2)$*, then*

$$(\mathbf{A}^{\mathrm{bb}})^{-1} = (\mathbf{I}_m - \mathbf{B}^{\mathrm{b}})^{-1} = \mathbf{I}_m + d^{-1}\mathbf{B}^{\mathrm{b}}, \tag{66}$$

$$\widetilde{\mathbf{B}}^{\mathrm{i}} = (\mathbf{A}^{\mathrm{bb}})^{-1}\mathbf{B}^{\mathrm{i}} = d^{-1}\mathbf{B}^{\mathrm{i}}. \tag{67}$$

Proof. The proof of the first Formula (64) repeats the proof of (47). Since $\mathbf{B}^{\mathrm{b}}\mathbf{B}^{\mathrm{i}} = \sum_{l=1}^{m} b^{n+l}\mathbf{B}^{\mathrm{i}}$, the first Formula (65) and $(\mathbf{B}^{\mathrm{b}})^2 = (1-d)\mathbf{B}^{\mathrm{b}}$ are proved. The second Formula (65) follows from this. If $d \in (0,2)$, then $|1-d| < 1$ and we prove (66) and (67):

$$(\mathbf{I}_m - \mathbf{B}^{\mathrm{b}})^{-1} = \mathbf{I}_m + \sum_{i=0}^{\infty}(\mathbf{B}^{\mathrm{b}})^{i+1} = \mathbf{I}_m + \sum_{i=1}^{\infty}(1-d)^i\mathbf{B}^{\mathrm{b}} = \mathbf{I}_m + d^{-1}\mathbf{B}^{\mathrm{b}},$$

$$\widetilde{\mathbf{B}}^{\mathrm{i}} = (\mathbf{I}_m + d^{-1}\mathbf{B}^{\mathrm{b}})\mathbf{B}^{\mathrm{i}} = \mathbf{B}^{\mathrm{i}} + d^{-1}\mathbf{B}^{\mathrm{b}}\mathbf{B}^{\mathrm{i}} = \mathbf{B}^{\mathrm{i}} + d^{-1}(1-d)\mathbf{B}^{\mathrm{i}} = d^{-1}\mathbf{B}^{\mathrm{i}}.$$

□

Formula (67) shows that the kernels $\widetilde{K}^{\mathrm{i}} := d^{-1}K^{\mathrm{i}}$ (see BC (27)) and $\widetilde{K} := \widetilde{K}^{\mathrm{i}}|_{\partial\omega\times\omega}$ (see BCs (30) and (53)) do not depend on $k \in \partial\overline{\omega}$ and $k \in \partial\omega$, respectively. Moreover, BC (53) shows that $V \equiv const$ at the nodes of the grid $\partial\omega$. From Maximum Principle for discrete Laplace equation $V \equiv const$ at $\widetilde{\omega}$ and from BC (53), we have the equation for constant $V = (\widetilde{K}, 1)V + (\widetilde{K}, W)$. Since

$$1 - (\widetilde{K}, 1) = d^{-1}(d - (K, 1)) = d^{-1}(1 - [K, 1]),$$

we have

$$(1 - [K, 1])V = (K, W). \tag{68}$$

If $[K, 1] \neq 1$, then from (68) we find

$$V = (1 - [K, 1])^{-1}(K, W), \tag{69}$$

where W is the solution of classical Problem P1. Then

$$U_l = W_l + (1 - [K, 1])^{-1}(K, W), \quad p_l \in \omega, \tag{70}$$

and substituting this expression into (27) we get

$$U_k^b = \widetilde{G}_k^{\mathrm{b}} + d^{-1}(K, W) + d^{-1}(K, 1)\frac{(K, W)}{1 - [K, 1]} = \widetilde{G}_k^{\mathrm{b}} + \frac{(K, W)}{1 - [K, 1]}, \quad p_k \in \partial\overline{\omega}. \tag{71}$$

The case $[K, 1] = 1$ is singular: if $(\widetilde{K}, W) = 0$, then all constants satisfy problem P2; if $(\widetilde{K}, W) \neq 0$, then problem P2 has no solutions.

Remark 6. *We have found necessary and sufficient conditions for the existence of a unique solution to a problem with integral BC in the case of a special kernel. If* $||K||_\infty = [|K|, 1] < 1$ *(sufficient condition), then* $|\langle K, 1\rangle|, |(K, 1)|, |[K, 1]| < 1$*, i.e.,* $d \in (0, 2)$ *and* $[K, 1] \neq 1$*.*

Example 5. *Let us consider the case* $K \equiv \gamma$ *(see Example 4 as well). Since*

$$d = 1 - \gamma|\partial\Omega^h| = 1 - \gamma(l_x h_y + l_y h_x - h_x h_y),$$

we can choose $h_0 = h_0(\gamma)$ *such that* $d > 0$ *for* $|h| \leq h_0$*. This condition allows to use FDS′ (29) and (30) instead of FDS (25) and (26). If* $\gamma|\Omega| \neq 1$*, where* $|\Omega| = l_x l_y$*, then we have the exact solution of FDS (25) and (26):*

$$U_l = W_l + V, \quad p_l \in \omega, \quad U_k^b = \widetilde{G}_k^b + V, \quad p_k \in \partial\overline{\omega}, \quad V = \frac{\gamma(1, W)}{1 - \gamma|\Omega|}. \tag{72}$$

4. Numerical Simulations

In this section, we present numerical experiments designed to illustrate and confirm the theoretical results and conclusions obtained above. Three representative examples are considered. In each example, we study the numerical solution of the Poisson equation subject to a double integral boundary condition. We investigate the dependence of the solution on the nonlocality parameter γ and on the properties of the kernel. In addition, we analyze the dependence of the numerical error on the discretization step size and examine the convergence order of the proposed numerical method.

Example 6. *We consider the model problem* (5) *and* (6) *with* $f(x,y) = 0$, $g(x,y) = xy$ *and constant kernel* $k(x,y,\xi,\eta) = \gamma$ *in the domain* $\Omega = (0,1) \times (0,1)$. *The right-hand side function* f *in the differential equation and the boundary conditions were prescribed to satisfy the given exact solution* $u(x,y) = w + v$, $w = xy$, $v = 0.25\gamma/(1-\gamma)$.

We use it for a quantitative assessment of the numerical accuracy. The accuracy of the numerical method applied to this problem was estimated by calculating the maximum absolute error E_h and root-mean-square error $\widehat{E}_h$, defined as:

$$E_h := ||\mathbf{E}||_\infty = \max_{p_l \in \overline{\omega}^h} |e_l|, \quad \widehat{E}_h = ||\mathbf{E}||_{2,r} := \sqrt{\sum_{p_l \in \overline{\omega}^h} r^l |e_l|^2},$$

$$p = \log(E_h/E_{h/2})/\log 2, \quad \widehat{p} = \log(\widehat{E}_h/\widehat{E}_{h/2})/\log 2,$$

where $\mathbf{E} = (e_l)$, $e_l = U_l - u(p_l)$, $p_l \in \overline{\omega}^h$.

Experimental convergence rate $\widetilde{p}$ is estimated by comparing numerical solutions on successively refined meshes. Let U_h and $U_{h/2}$ denote numerical solutions computed on meshes with sizes h and $h/2$. The error is approximated by

$$\widetilde{E}_h := \|U_h - U_{h/2}\|_\infty, \quad \widetilde{p} = \log\Big(\widetilde{E}_h/\widetilde{E}_{h/2}\Big)/\log 2.$$

A comparison between the experimental convergence rates obtained via mesh refinement and the theoretical order of accuracy confirms the reliability of the proposed method.

These results are summarised in Table 1, where V is defined by Formula (72), $|\Delta V|_h = |V - v|$, $\overline{\delta} = |\Delta V|_h/|\Delta V|_{h/2}$, $\overline{p} = \log\overline{\delta}/\log 2$. The numerical data consistently indicate second-order accuracy for $\gamma = 0.5$. The case $\gamma = 0$ corresponds to the Laplace problem with Dirichlet boundary conditions, for which the solution is obtained exactly. As a result, the observed error is small and is dominated by numerical round-off effects. The observed error is small and is mainly influenced by numerical rounding effects.

Table 1. (Example 6) Numerical errors and convergence rates for $\gamma = 0$ and $\gamma = 0.5$.

		$\gamma = 0$	$\gamma = 0.5$			
N	M	E_h	E_h	p	$\widetilde{E}_h$	$\widetilde{p}$
10	10	3.33×10^{-16}	2.22×10^{-3}			
20	20	7.77×10^{-16}	5.92×10^{-4}	1.91	1.62×10^{-3}	
40	40	2.44×10^{-15}	1.52×10^{-4}	1.96	4.39×10^{-4}	1.89
80	80	3.44×10^{-15}	3.86×10^{-5}	1.98	1.13×10^{-4}	1.95
160	160	8.55×10^{-15}	9.70×10^{-6}	1.99	2.89×10^{-5}	1.98
N	M		$\|V - v\|$	$\overline{p}$	$\widehat{E}_h$	$\widehat{p}$
10	10		2.85×10^{-2}		2.09×10^{-3}	
20	20		1.34×10^{-2}	1.08	5.76×10^{-4}	1.86
40	40		6.48×10^{-3}	1.05	1.50×10^{-4}	1.94
80	80		3.18×10^{-3}	1.03	3.83×10^{-5}	1.97
160	160		1.58×10^{-3}	1.01	9.67×10^{-6}	1.99

A numerical convergence study highlighting the influence of the parameter γ, with computations performed on a uniform grid of size $N = M = 160$, is presented in Table 2. Additionally, we compare the constant part $v = 0.25\gamma/(1-\gamma)$ of the exact solution with numerical solution V of Problem P2 (52) and (53) (see Formula (72)). We show how V is approximating v, and how E_h and $|\Delta V|_h = |v - V|$ depend on the γ value. Note that the convergence order $|\Delta V|_h = \mathcal{O}(h)$, but $E_h = \mathcal{O}(h^2)$.

Example 7. *We consider the model problem (5) and (6) with* $f(x,y) = 2\sin x \sin y$, $g(x,y) = 0$ *and kernel* $k = k(\xi,\eta) = \gamma\pi^{-2}\sin(\xi)\sin(\eta)/2$ *in the domain* $\Omega = (0,2\pi)\times(0,2\pi)$*. Such a kernel is considered in Section 3.6 and* $k(\xi,\eta) = 0$ *for* $(\xi,\eta) \in \partial\Omega$*. The right-hand side function* f *and the boundary conditions were chosen to be consistent with the prescribed exact solution* $u(x,y) = w + v$, $w = \sin x \sin y$, $v = \gamma/2$*. In this example*

$$\int_\Omega k(\xi,\eta)d\xi d\eta = 0, \quad \int_\Omega |k(\xi,\eta)|d\xi d\eta = \frac{8|\gamma|}{\pi^2}, \quad \int_\Omega k(\xi,\eta)w(\xi,\eta)d\xi d\eta = \frac{\gamma}{2}.$$

The numerical simulations are presented in Table 3, where V is defined by Formula (69). Note that we get the solution (see (70) and (71)) not only for $|\gamma| < \pi^2/8$, but for all γ. In this example, $\mathbf{B}^{\mathrm{b}} = \mathbf{O}$, $\mathbf{A}^{\mathrm{bb}} = \mathbf{I}$, $\widetilde{K}^{\mathrm{i}} = K$, $\widetilde{G}^{\mathrm{b}} = G$ (see (27)). Therefore, the convergence order $\Delta V_h = \mathcal{O}(h^2)$.

Table 2. (Example 6) Convergence study for the numerical solution ($N = M = 160$) for various values of the parameter γ.

γ	v	V	$\lvert V - v\rvert$	E_h	$\widetilde{E}_h$	$\widehat{E}_h$
−10.0	−0.2273	−0.1996	2.77×10^{-2}	6.22×10^{-5}	1.35×10^{-4}	1.21×10^{-5}
−1.0	−0.125	−0.1219	3.07×10^{-3}	2.38×10^{-6}	6.91×10^{-6}	2.39×10^{-6}
−0.5	−0.0833	−0.0818	1.55×10^{-3}	1.06×10^{-6}	3.12×10^{-6}	1.06×10^{-6}
0	0	0	0	8.55×10^{-15}	1.09×10^{-14}	2.35×10^{-15}
0.5	0.25	0.2484	1.58×10^{-3}	9.70×10^{-6}	2.89×10^{-5}	9.67×10^{-6}
0.95	4.75	4.7435	6.52×10^{-3}	3.52×10^{-3}	1.05×10^{-2}	3.54×10^{-3}
0.99	24.75	24.652	9.85×10^{-2}	9.53×10^{-2}	2.82×10^{-1}	9.59×10^{-2}
1	Inf	6400.0	Inf	Inf	4800	Inf
1.01	−25.25	−25.35	1.03×10^{-1}	1.01×10^{-1}	3.08×10^{-1}	1.01×10^{-1}
1.05	−5.25	−5.258	7.63×10^{-3}	4.53×10^{-3}	1.37×10^{-2}	4.35×10^{-3}
10.0	−0.2778	−0.3134	3.56×10^{-2}	1.53×10^{-4}	6.59×10^{-4}	2.78×10^{-5}

Table 3. (Example 7) Numerical errors and convergence rate.

N	M	γ	E_h	p		$\lvert V - v\rvert$	$\overline{p}$
10	10		4.71×10^{-2}			1.68×10^{-2}	
20	20		1.24×10^{-2}	1.93		4.13×10^{-3}	2.02
40	40	1	3.09×10^{-3}	2.01		1.03×10^{-3}	2.005
80	80		7.71×10^{-4}	2.00		2.57×10^{-4}	2.001
160	160		1.93×10^{-4}	2.00		6.43×10^{-5}	2.0003
N	M	γ	$\lvert V - v\rvert$	E_h	γ	$\lvert V - v\rvert$	E_h
		−10	6.43×10^{-4}	7.71×10^{-4}	1	6.43×10^{-5}	1.93×10^{-4}
		−5	3.21×10^{-4}	4.50×10^{-4}	$\pi^2/8$	7.93×10^{-5}	2.08×10^{-4}
160	160	−2	1.29×10^{-4}	2.57×10^{-4}	2	1.29×10^{-4}	2.57×10^{-4}
		$-\pi^2/8$	7.93×10^{-5}	2.08×10^{-4}	5	3.21×10^{-4}	4.50×10^{-4}
		−1	6.43×10^{-5}	1.93×10^{-4}	10	6.43×10^{-4}	7.71×10^{-4}
		0	0	1.29×10^{-4}			

Example 8. *We consider the model problem* (5) *and* (6) *with*

$$f(x,y) = -2e^{x+y}, \quad g(x,y) = e^{x+y} - \gamma(e-1)\frac{ex\sin x + e\cos x - 1}{x^2+1}$$

and kernel $k = k(x,\xi) = \gamma\cos(x\xi)$ *in the domain* $\Omega = (0,1)\times(0,1)$. *The right-hand side function* f *in the differential equation and the boundary conditions were prescribed to satisfy the given exact solution* $u(x,y) = e^{x+y}$. *In this example,*

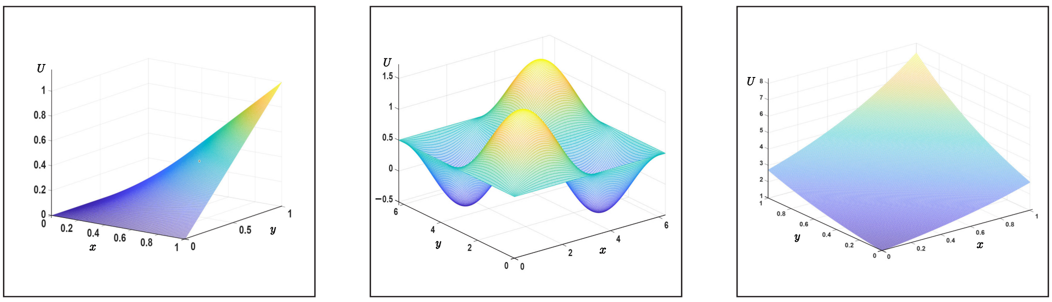
$$\max_{x\in[0,1]}\int_{\Omega}|k(x,\xi)|d\xi d\eta = |\gamma|.$$

Table 4 presents a numerical convergence study illustrating the effect of the parameter γ on a uniform grid with $N = M = 160$, along with the experimental order of convergence obtained from successive mesh refinements.

Numerical solutions for these three examples are presented in Figure 6.

Table 4. (Example 8) Numerical errors and convergence rate.

N	M	γ	E_h	p	γ	E_h	p
10	10		7.74×10^{-3}			3.10×10^{-1}	
20	20		2.13×10^{-3}	1.86		8.76×10^{-2}	1.82
40	40	0.5	5.56×10^{-4}	1.94	1	2.27×10^{-2}	1.95
80	80		1.42×10^{-6}	1.97		5.74×10^{-3}	1.98
160	160		3.58×10^{-5}	1.99		1.44×10^{-3}	1.99
N	M	γ	E_h		γ	E_h	
		-10	6.88×10^{-3}		0.5	3.58×10^{-5}	
		-5	1.75×10^{-3}		1	1.44×10^{-3}	
160	160	-2	2.30×10^{-4}		2	8.77×10^{-4}	
		-1	6.80×10^{-5}		5	3.57×10^{-3}	
		-0.5	2.57×10^{-5}		10	1.43×10^{-2}	
		0	1.41×10^{-6}				



(**a**) Example 6, $\gamma = 0$ (**b**) Example 7, $\gamma = 1$ (**c**) Example 8

Figure 6. Solutions of Problems from Examples 6, 7, and 8 with $N = M = 160$.

5. Conclusions

In this article, we have considered the numerical solution of the two-dimensional (2D) Poisson equation, taking into account the double integral nonlocal boundary condition, using the second-order FDS (25) and (26). We approximated the integral BC by the trapezoidal formula. We divided the problem into two parts: the classical Poisson equation with local Dirichlet BC (see problem P1) and the Laplace equation with a nonlocal condition (see problem P2 (52) and (53)). For problem P2, we found the fundamental system of solutions by solving the discrete Laplace equation with classical Dirichlet BC. We derived the inverse matrix formula for the block matrix, which is used in the classical case. The stability of the FDS was established assuming that the kernel satisfies condition (12). We examined several distinct boundary condition cases and validated the theoretical results using numerical simulations.

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