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Opinion

On Some Open Problems in Spatial Fractional Integration

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Abstract

Some open problems regarding fractional powers of the negative generator of a discrete-time random walk and a Markov process are discussed. The suggested approach combines analytic and probabilistic ideas and may be useful for developing fractional operators with multidimensional and/or abstract discrete arguments.

Keywords: spatial fractional differentiation/integration operators; fractional powers of Markov semigroups; fractional operators on fractional sets; variable order fractional operators; fractionally integrated random fields; random walk; limit theorems; long-range dependence

1. Introduction

Fractional calculus is a large field of mathematical and applied research, as documented in several texts and monographs [1–4]. A considerable part of this research deal with fractional differentiation and integration on the real line or its interval. Multidimensional fractional calculus is closely related to partial differential and singular operators on Euclidean spaces. A general fractional theory of bounded operators was developed in [5]. Discrete one-dimensional fractional calculus is the main topic of the monograph [6]. See [7] for late work in a multi-dimensional setting.

The recent paper [8] discussed a rather straightforward and elementary approach towards fractional operators on regular lattice $\mathbb{Z}^\nu$ defined through the binomial expansion $(1-z)^d = \sum_{j=0}^{\infty} \psi_j(d) z^j$, $z \in \mathbb{C}$, $|z| < 1, |d| < 1$ with

$$Tg(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} g(\boldsymbol{t}+\boldsymbol{u}) p(\boldsymbol{u}) = \mathrm{E} g(S_1 + \boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu \tag{1}$$

the transition operator of a random walk $\{S_j; j \ge 0\}$ on $\mathbb{Z}^\nu$ starting at $S_0 = \mathbf{0}$ with (1-step) probabilities $p(\boldsymbol{u}) := \mathrm{P}(S_1 = \boldsymbol{u}); \boldsymbol{u} \in \mathbb{Z}^\nu$. The operator T in (1) is bounded in $L^\infty(\mathbb{Z}^\nu)$, and $T - I$ is called the generator of the random walk. The operators $T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} g(\boldsymbol{t} + \boldsymbol{u}) p_j(\boldsymbol{u}), j = 0, 1, \cdots$ form a Markov semigroup of transition operators of the random walk, with $p_j(\boldsymbol{u}) := \mathrm{P}(S_j = \boldsymbol{u}); \boldsymbol{u} \in \mathbb{Z}^\nu$. Accordingly, fractional power $(I-T)^d, -1 < d < 1$ of the above T is defined in [8] as the integral operator acting on $g : \mathbb{Z}^\nu \to \mathbb{R}$ by

$$(I-T)^d g(\boldsymbol{t}) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} \tau(\boldsymbol{u}; d) g(\boldsymbol{t}+\boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu \tag{2}$$

with coefficients

$$\tau(\boldsymbol{u}; d) := \sum_{j=0}^{\infty} \psi_j(d) p_j(\boldsymbol{u}), \tag{3}$$

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expressed through the binomial coefficients $\psi_j(d) := \Gamma(j-d)/\Gamma(j+1)\Gamma(-d)$, $j \in \mathbb{N}$ and random walk probabilities $p_j(\boldsymbol{u})$. In [8], the sufficient and necessary conditions were obtained for the existence, invertibility, and square summability of kernels $\tau(\boldsymbol{u}; d)$ in (3) (see Theorem 1 below) and the asymptotic behavior of $\tau(\boldsymbol{u}; d)$ as $|\boldsymbol{u}| \to \infty$. These results are applied to solutions of stochastic fractional equations $(I-T)^d X = \varepsilon$ with white noise on the right-hand side.

The present note discusses some potential extensions of the operators (2) from the lattice $\mathbb{Z}^\nu$ to the Euclidean space $\mathbb{R}^\nu$ and an abstract countable (possibly, fractal) set S. Following [9,10], a new class of invertible variable order fractional operators on $\mathbb{Z}^\nu$ is proposed. We also consider some probabilistic applications for the modeling of fractional random fields.

2. Discrete Fractional Operators on $\mathbb{R}^\nu$

A natural extension of (2) occurs when the (discrete) random walk in (1)–(3) is replaced by an absolutely continuous random walk; viz., the sum in (1) is replaced by an integral

$$Tg(\boldsymbol{t}) = \int_{\mathbb{R}^\nu} g(\boldsymbol{t}+\boldsymbol{u})p(\boldsymbol{u})\mathrm{d}\boldsymbol{u} = \mathrm{E}g(S_1+\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^\nu, \tag{4}$$

where S_1 is a random vector with values in $\mathbb{R}^\nu$ and probability density $p(\boldsymbol{u}), \boldsymbol{u} \in \mathbb{R}^\nu$, which means that $\mathrm{P}(S_1 \in A) = \int_A p(\boldsymbol{u})\mathrm{d}\boldsymbol{u}$ for any Borel set $A \subset \mathbb{R}^\nu$. Then S_j is a sum of j independent identically distributed random vectors with distribution S_1, and

$$(I-T)^d g(\boldsymbol{t}) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(\boldsymbol{t}) = \int_{\mathbb{R}^\nu} \tau(\boldsymbol{u}; d) g(\boldsymbol{t}+\boldsymbol{u})\mathrm{d}\boldsymbol{u}, \quad \boldsymbol{t} \in \mathbb{R}^\nu \tag{5}$$

is an integral operator with kernel $\tau(\boldsymbol{u}; d), \boldsymbol{u} \in \mathbb{R}^\nu$ given by the series in (3), where $p_j(\boldsymbol{u}), \boldsymbol{u} \in \mathbb{R}^\nu$ is the probability density of S_j. The above facts together with $\sum_{j=0}^n \psi_j(d)\psi_{n-j}(-d) = \delta_n$ $(-1 < d < 1)$ suggest the invertibility property $(I-T)^d(I-T)^{-d} = I$ under some regularity conditions on the random walk probabilities or the characteristic function (the Fourier transform) $\widehat{p}(\boldsymbol{x}) = \mathrm{E}\mathrm{e}^{\mathrm{i}\langle \boldsymbol{x}, S_1\rangle} = \int_{\mathbb{R}^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{x}, \boldsymbol{u}\rangle} p(\boldsymbol{u})\mathrm{d}\boldsymbol{u}$.

In the discrete argument case $\boldsymbol{t} \in \mathbb{Z}^\nu$, the invertibility of the operator $(I-T)^d$ in (2) is a consequence of the following theorem.

Theorem 1 ([8])**.** *For $-1 < d < 1$, the following conditions are equivalent:*

$$\int_{[-\pi,\pi]^\nu} |1-\widehat{p}(\boldsymbol{x})|^{-2|d|}\mathrm{d}\boldsymbol{x} < \infty,$$

$$\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} \tau(\boldsymbol{u}; -|d|)^2 < \infty.$$

This leads to the following (realistic) open problem as stated below.

Problem 1. *What are necessary and sufficient conditions for square integrability $\int_{\mathbb{R}^\nu} \tau(\boldsymbol{u}; -|d|)^2\mathrm{d}\boldsymbol{u} < \infty$ in terms of the Fourier transform of density $p(\boldsymbol{u})$?*

Apart from $L^p(\mathbb{R}^\nu)$-boundedness properties of operators $(I-T)^d$, the continuous framework in (5) suggests studying their "continuous" limits as the step $h > 0$ of the random walk tends to zero. Let $T_h g(\boldsymbol{t}) = \mathrm{E}g(hS_1+\boldsymbol{t})$. Then, for "sufficiently good" $g(\boldsymbol{t})$,

$$(I-T_h)^d g(\boldsymbol{t}) = \sum_{j=0}^{\infty} \psi_j(d) \int_{\mathbb{R}^\nu} g(\boldsymbol{t}+h\boldsymbol{u})p_j(\boldsymbol{u})\mathrm{d}\boldsymbol{u} = h^{-\nu} \sum_{j=0}^{\infty} \psi_j(d) \int_{\mathbb{R}^\nu} g(\boldsymbol{t}+\boldsymbol{u})p_j(\boldsymbol{u}/h)\mathrm{d}\boldsymbol{u} \tag{6}$$

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and

$$\int_{\mathbb{R}^\nu}(I-T_h)^d g(\boldsymbol{t})\mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{z}\rangle}\mathrm{d}\boldsymbol{t} \;=\; \widehat{g}(\boldsymbol{z})\sum_{j=0}^{\infty}\psi_j(d)\overline{\widehat{p}^j(h\boldsymbol{z})} \;=\; \widehat{g}(\boldsymbol{z})\big(1-\widehat{p}(-h\boldsymbol{z})\big)^d, \quad \boldsymbol{z}\in\mathbb{R}^\nu. \tag{7}$$

Problem 2. *Identify the limits* $\lim_{h\to 0} h^{-\gamma}(I-T_h)^d g$ *for some* $\gamma>0$ *and a class of test functions* g*, and relate these limits to known multidimensional fractional operators or their generalizations.*

The normalizing exponent $\gamma>0$ in Problem 2 and the limit of the Fourier transform in (7) as $h\to 0$ depend on the behavior of $\widehat{p}(\boldsymbol{z})$ at the origin. It is well-known that $\mathrm{E}|S_1|^2<\infty, \mathrm{E}\langle S_1,\boldsymbol{x}\rangle^2=\langle R\boldsymbol{x},\boldsymbol{x}\rangle, \mathrm{E}\langle S_1,\boldsymbol{x}\rangle=\langle \boldsymbol{m},\boldsymbol{x}\rangle$ imply $\widehat{p}(\boldsymbol{z})=1+\mathrm{i}\langle \boldsymbol{m},\boldsymbol{z}\rangle-(1/2)\langle R\boldsymbol{z},\boldsymbol{z}\rangle+o(|\boldsymbol{z}|^2)$ $(|\boldsymbol{z}|\to 0)$. In the case of zero mean $\boldsymbol{m}=\boldsymbol{0}$ and unit covariance matrix $\langle R\boldsymbol{z},\boldsymbol{z}\rangle=|\boldsymbol{z}|^2$, Equation (7) suggests the limit $\lim_{h\to 0} h^{-2d}\widehat{(I-T_h)^d g}(\boldsymbol{z})=\widehat{g}(\boldsymbol{z})|\boldsymbol{z}|^{2d}$ for $\gamma=2d$ coinciding with the Riesz fractional Laplacian $(-\Delta)^d g$ ([3], Ch. 5). A discussion of the limit $h^{-2d}(I-T_h)^d g(\boldsymbol{t})$ in the spatial domain set-up for the above S_1 may require the local central limit theorem for the rescaled probabilities $h^{-\nu}p_j(\boldsymbol{u}/h)\sim(2\pi t)^{-\nu/2}\mathrm{e}^{-|\boldsymbol{u}|^2/2t}$, $jh^2\to t\in(0,\infty)$ [11,12]. When $\boldsymbol{m}\neq\boldsymbol{0}$ and/or $\mathrm{E}|S_1|^2=\infty$, the behaviors of $\widehat{p}(\boldsymbol{z})$ and $p_j(\boldsymbol{u})$ are generally different, possibly leading to different limits of fractional powers $(I-T_h)^d$.

3. Fractional Powers of Markov Semigroup

Let $M=\{M_j; j=0,1,\cdots\}$ be a homogeneous Markov process on a countable set $S=\{s\}$ with transition operator

$$Tg(t):=\mathrm{E}[g(M_1)|M_0=t]=\sum_{u\in S}g(u)p_1(t,u), \quad p_1(t,u):=\mathrm{P}(M_1=u|M_0=t). \tag{8}$$

The operators $T_j, j=0,1,\cdots$ recursively defined by $T_j:=TT_{j-1}, T_0:=I$ form a Markov semigroup $T_jg(s)=T^jg(s)=\sum_{u\in S}g(u)p_j(s,u)$, $p_j(s,u):=\mathrm{P}(M_j=u|M_0=s)$ following the Kolmogorov–Chapman identity $\sum_{s\in S}p_j(t,s)p_\ell(s,u)=p_{j+\ell}(t,s), t,s\in S, j,\ell=0,1,\cdots$. Fractional powers $(I-T)^d$ of the negative generator $I-T$ of (8) can be defined similarly to (5):

$$(I-T)^d g(t) \;:=\; \sum_{j=0}^{\infty}\psi_j(d)T^jg(t)=\sum_{s\in S}\tau(t,s;d)g(s), \tag{9}$$

with coefficients $\tau(t,s;d):=\sum_{j=0}^{\infty}\psi_j(d)p_j(t,s), t,s\in S$. The (formal) identity $(I-T)^d(I-T)^{-d}=I$ follows similarly as in the case of (3):

$$\begin{aligned}\sum_{s\in S}\tau(t,s;d)\tau(s,u;-d) &= \sum_{j,\ell=0}^{\infty}\psi_j(d)\psi_\ell(-d)\sum_{s\in S}p_j(t,s)p_\ell(s,u)\\ &= \sum_{n=0}^{\infty}p_n(t,u)\sum_{j=0}^{n}\psi_j(d)\psi_{n-j}(-d)\\ &= \mathbb{I}(t=u), \quad t,u\in S.\end{aligned}$$

A rigorous study of (9) is open, relying on properties of the Markov chain in (8). Markov chains is a classical object of the probability theory [13,14]. A special case of Markov chain is a simple random walk on a graph that at each time step moves from a site to a neighbor chosen uniformly at random [15,16]. An interesting class of such processes are random walks on fractal graphs, which may explain how heat and waves propagate on a fractal cluster [15,17].

Problem 3. *Introduce and study fractional powers of the negative discrete Laplacian on a fractal set (e.g., on Sierpinski's gasket).*

A very special case of Markov chain is a random walk in a rectangular domain $D \subset \mathbb{Z}^\nu$ with a boundary condition (e.g., absorbsion or reflection) at the boundary ∂D. The corresponding Markov semigroup T_j and the fractional operator in (9) may be useful in Grünwald type finite difference approximation of fractional differential operators with boundary conditions [18].

4. Fractional Operators of Variable Order (VO)

Integration and differentiation to a variable fractional order was introduced in [19]. See the overviews [20,21] for more recent developments in VO fractional calculus, including the multidimensional case. Ref. [20] notes that the proposed VO derivatives are not left inverses of VO integrals.

Refs. [9,10] studied two classes of VO fractional operators on $\mathbb{Z}$ defined as

$$A_{d(\cdot)}g(t) \;:=\; \sum_{j=0}^{\infty} a_j(t)g(t-j), \quad B_{d(\cdot)}g(t) := \sum_{j=0}^{\infty} b_j(t)g(t-j), \quad t \in \mathbb{Z} \tag{10}$$

with coefficients

$$\begin{aligned} a_j(t) &:= \frac{d_{t-1}}{1} \cdot \frac{d_{t-2}+1}{2} \cdot \frac{d_{t-3}+2}{3} \cdots \frac{d_{t-j}+j-1}{j}, \\ b_j(t) &:= \frac{d_{t-1}}{1} \cdot \frac{d_{t-2}+j-1}{2} \cdot \frac{d_{t-3}+j-2}{3} \cdots \frac{d_{t-j}+1}{j}, \end{aligned} \tag{11}$$

$a_0(t) = b_0(t) := 1, a_1(t) = b_1(t) := d_{t-1}$. Here, $d(\cdot) = \{d_t; t \in \mathbb{Z}\}$ is a given sequence of real numbers (a 'VO fractional parameter'). For constant $d_t \equiv d$, Equation (10) agrees with classical discrete fractional operator in (2) with $Tg(t) = g(t-1)$ as the backward shift, viz., $A_{d(\cdot)}g(t) = B_{d(\cdot)}g(t) = (I-T)^d g(t) = \sum_{j=0}^{\infty} \psi_j(d)g(t-j)$. A nice property of (11) is the orthogonality: for any $t \in \mathbb{Z}, k \in \mathbb{N}$

$$\sum_{j=0}^{k} b_j^-(t)a_{k-j}(t-j) = \sum_{j=0}^{k} a_j^-(t)b_{k-j}(t-j) = \delta_k, \tag{12}$$

where $a_j^-(t), b_j^-(t)$ are defined as in (11) with $d(\cdot) = \{d_t, t \in \mathbb{Z}\}$ replaced by $-d(\cdot) := \{-d_t, t \in \mathbb{Z}\}$. (12) is proved in ([10], Thm. 3.1) by a combinatorial argument and induction on k. Property (12) is essential to the study of the invertibility of VO operators $B_{-d(\cdot)}A_{d(\cdot)} = A_{-d(\cdot)}B_{d(\cdot)} = I$. Using (12) and a Grünwald and Letnikov type approximation, in [22], the authors extended discrete VO operators to continuous argument $t \in \mathbb{R}$, in which case they take the form

$$I^{\alpha(\cdot)}f(t) := \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^{t} f(s)(t-s)^{\alpha(t)-1}\mathrm{e}^{-H_-(s,t)}\mathrm{d}s, \quad D^{\alpha(\cdot)}f(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} \frac{f(s)}{\Gamma(1-\alpha(s))}(t-s)^{\alpha(s)}\mathrm{e}^{-H_+(s,t)}\mathrm{d}s, \tag{13}$$

where, for $s < t$

$$H_-(s,t) := \int_s^t \frac{\alpha(u)-\alpha(t)}{t-u}\mathrm{d}u, \quad H_+(s,t) := \int_s^t \frac{\alpha(s)-\alpha(v)}{v-s}\mathrm{d}v.$$

Under some regularity conditions on the VO functional parameter $\alpha(t) \in (0,1), t \in \mathbb{R}$, in [22], the authors prove that

$$D^{\alpha(\cdot)}I^{\alpha(\cdot)}f(t) = f(t) \tag{14}$$

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for any $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The proof of (14) relies on the surprising integral identity

$$\int_0^1 x^{\alpha(x)-1}(1-x)^{-\alpha(x)} \exp\Big\{\int_0^1 \frac{\alpha(u)-\alpha(x)}{u-x}\mathrm{d}u\Big\} \sin(\alpha(x)\pi)\mathrm{d}x \quad = \quad \pi \tag{15}$$

derived from (12) through finite-difference approximation. Ref. [22] notes that, in the case of a step function $\alpha(x)$ taking a finite number of values, Equation (15) leads to an identity for the (Lauricella) hypergeometric function of many variables. The VO operators in (13) were used to construct new classes of multifractional random processes generalizing fractional Brownian motion [22,23].

Problem 4. *Provide a direct calculus based proof of* (15).

It is of interest to extend VO operators in (10) and (13) to higher dimensions. Let $\boldsymbol{d}(\cdot) = \{d(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ be a sequence of real numbers $d(\boldsymbol{t}) \in \mathbb{R}$ and let $p = \{p(\boldsymbol{u}) := \mathrm{P}(S_1 = \boldsymbol{u}); \boldsymbol{u} \in \mathbb{Z}^\nu\}$ be a one-step random walk probability distribution similar to (4) but concentrated on the lattice as in [8]. Generalizing (10) and the discrete version of $(I-T)^d$ in (5), we may define

$$A_{\boldsymbol{d}(\cdot)}g(\boldsymbol{t}) \quad := \quad \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} a(\boldsymbol{t},\boldsymbol{s})g(\boldsymbol{s}), \quad B_{\boldsymbol{d}(\cdot)}g(\boldsymbol{t}) := \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} b(\boldsymbol{t},\boldsymbol{s})g(\boldsymbol{s}), \tag{16}$$

with coefficients

$$\begin{aligned}
a(\boldsymbol{t},\boldsymbol{s}) \quad &:= \quad \sum_{j=0}^\infty a_j(\boldsymbol{t},\boldsymbol{s}), \quad b(\boldsymbol{t},\boldsymbol{s}) := \sum_{j=0}^\infty b_j(\boldsymbol{t},\boldsymbol{s}), \quad \boldsymbol{t},\boldsymbol{s} \in \mathbb{Z}^\nu, \qquad (17)\\
a_j(\boldsymbol{t},\boldsymbol{s}) \quad &:= \quad \sum_{\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{j-1}\in\mathbb{Z}^\nu} \frac{d(\boldsymbol{u}_1)p(\boldsymbol{u}_1-\boldsymbol{t})}{1}\frac{(d(\boldsymbol{u}_2)+1)p(\boldsymbol{u}_2-\boldsymbol{u}_1)}{2}\cdots\frac{(d(\boldsymbol{s})+j-1)p(\boldsymbol{s}-\boldsymbol{u}_{j-1})}{j},\\
b_j(\boldsymbol{t},\boldsymbol{s}) \quad &:= \quad \sum_{\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{j-1}\in\mathbb{Z}^\nu} \frac{d(\boldsymbol{u}_1)p(\boldsymbol{u}_1-\boldsymbol{t})}{1}\frac{(d(\boldsymbol{u}_2)+j-1)p(\boldsymbol{u}_2-\boldsymbol{u}_1)}{2}\cdots\frac{(d(\boldsymbol{s})+1)p(\boldsymbol{s}-\boldsymbol{u}_{j-1})}{j},
\end{aligned}$$

$a(\boldsymbol{t},\boldsymbol{t}) = b(\boldsymbol{t},\boldsymbol{t}) := 1, a_1(\boldsymbol{t},\boldsymbol{s}) = b_1(\boldsymbol{t},\boldsymbol{s}) := d(\boldsymbol{s})p(\boldsymbol{s}-\boldsymbol{t})$. We see that $A_{\boldsymbol{d}(\cdot)} = A_{d(\cdot)}, B_{\boldsymbol{d}(\cdot)} = B_{d(\cdot)}$ in (10) for $\nu = 1$ and $p(1) = 1$, and $A_{\boldsymbol{d}(\cdot)} = B_{\boldsymbol{d}(\cdot)} = (I-T)^d$ in (5) for constant $d(\boldsymbol{t}) = d$ and any $\nu \geq 2$ and random walk $p = \mathrm{P}(S_1 = \boldsymbol{u}); \boldsymbol{u} \in \mathbb{Z}^\nu\}$.

The following discussion ignores the convergence issues of the infinite series in (16) and (17) and other properties of these operators, aiming at a formal (algebraic) justification of their invertibility, viz.,

$$B_{-\boldsymbol{d}(\cdot)}A_{\boldsymbol{d}(\cdot)} \quad = \quad A_{-\boldsymbol{d}(\cdot)}B_{\boldsymbol{d}(\cdot)} \; = \; I, \tag{18}$$

where $-\boldsymbol{d}(\boldsymbol{t}) := \{-d(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$. In terms of the coefficients, Equation (18) amounts to showing the following identities:

$$\Delta'(\boldsymbol{t},\boldsymbol{w}) := \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} b^-(\boldsymbol{t},\boldsymbol{s})a(\boldsymbol{s},\boldsymbol{w}) = \mathbb{I}(\boldsymbol{t}=\boldsymbol{w}), \quad \Delta''(\boldsymbol{t},\boldsymbol{w}) := \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} a^-(\boldsymbol{t},\boldsymbol{s})b(\boldsymbol{s},\boldsymbol{w}) = \mathbb{I}(\boldsymbol{t}=\boldsymbol{w}), \tag{19}$$

where $a^-(\boldsymbol{t},\boldsymbol{s}), b^-(\boldsymbol{t},\boldsymbol{s})$ are obtained by replacing $\boldsymbol{d}(\cdot)$ by $-\boldsymbol{d}(\cdot)$ in (17). From the definitions in (16) and (17), we have that $\Delta'(\boldsymbol{t},\boldsymbol{w}) = \sum_{k=0}^\infty \Delta'_k(\boldsymbol{t},\boldsymbol{w})$, where $\Delta'_k(\boldsymbol{t},\boldsymbol{w}) := \sum_{j=0}^k \Delta'_{j,k-j}(\boldsymbol{t},\boldsymbol{w})$ and

$$\begin{aligned}\Delta'_{j,k-j}(\boldsymbol{t},\boldsymbol{w}) &:= \sum_{\boldsymbol{s}} b_j^-(\boldsymbol{t},\boldsymbol{s})a_{k-j}(\boldsymbol{s},\boldsymbol{w}) \\ &= \frac{1}{j!(k-j)!}\sum_{\boldsymbol{s}}\sum_{\boldsymbol{v}_1,\cdots,\boldsymbol{v}_{j-1}}\sum_{\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{k-j-1}} \\ &\quad \times(-d(\boldsymbol{v}_1))p(\boldsymbol{v}_1-\boldsymbol{t})(-d(\boldsymbol{v}_2)+j-1)p(\boldsymbol{v}_2-\boldsymbol{v}_1)\cdots(-d(\boldsymbol{s})+1)p(\boldsymbol{s}-\boldsymbol{v}_{j-1}) \\ &\quad \times d(\boldsymbol{u}_1)p(\boldsymbol{u}_1-\boldsymbol{s})(d(\boldsymbol{u}_2)+1)p(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdots(d(\boldsymbol{w})+k-j-1)p(\boldsymbol{w}-\boldsymbol{u}_{k-j-1}).\end{aligned}$$

By renaming the variables in the above sum as $\boldsymbol{v}_1 =: \boldsymbol{z}_1, \boldsymbol{v}_2 =: \boldsymbol{z}_2, \cdots, \boldsymbol{v}_{j-1} =: \boldsymbol{z}_{j-1}, \boldsymbol{s} =: \boldsymbol{z}_j, \boldsymbol{u}_1 =: \boldsymbol{z}_{j+1}, \cdots, \boldsymbol{u}_{k-j-1} =: \boldsymbol{z}_{k-1}$, it writes as

$$\begin{aligned}\Delta'_{j,k-j}(\boldsymbol{t},\boldsymbol{w}) &= \frac{1}{j!(k-j)!}\sum_{\boldsymbol{z}_1,\cdots,\boldsymbol{z}_{k-1}} p(\boldsymbol{z}_1-\boldsymbol{t})p(\boldsymbol{z}_2-\boldsymbol{z}_1)\cdots p(\boldsymbol{z}_{k-1}-\boldsymbol{z}_{k-2})p(\boldsymbol{w}-\boldsymbol{z}_{k-1}) \\ &\quad (-d(\boldsymbol{z}_1))(-d(\boldsymbol{z}_2)+j-1)\cdots(-d(\boldsymbol{z}_j)+1)d(\boldsymbol{z}_{j+1})(d(\boldsymbol{z}_{j+2}+1)\cdots(d(\boldsymbol{w})+k-j-1)\end{aligned}$$

implying

$$\Delta'_k(\boldsymbol{t},\boldsymbol{w}) = \sum_{\boldsymbol{z}_1,\cdots,\boldsymbol{z}_{k-1}} j'_n(\boldsymbol{t},\boldsymbol{w},\boldsymbol{z}_1,\cdots,\boldsymbol{z}_{k-1})p(\boldsymbol{z}_1-\boldsymbol{t})p(\boldsymbol{z}_2-\boldsymbol{z}_1)\cdots p(\boldsymbol{z}_{k-1}-\boldsymbol{z}_{k-2})p(\boldsymbol{w}-\boldsymbol{z}_{k-1}),$$

with

$$\begin{aligned}j'_k(\boldsymbol{t},\boldsymbol{w},\boldsymbol{z}_1,\cdots,\boldsymbol{z}_{k-1}) &:= d(\boldsymbol{z}_1)\sum_{j=0}^{k}\frac{(-1)^j}{j!(k-j)!}(d(\boldsymbol{z}_2)-j+1)\cdots(d(\boldsymbol{z}_j)-1)d(\boldsymbol{z}_{j+1}) \\ &\quad \times(d(\boldsymbol{z}_{j+2})+1)\cdots(d(\boldsymbol{z}_{k-1})+k-j-2)(d(\boldsymbol{w})+k-j-1).\end{aligned} \tag{20}$$

A heuristic proof of (19) reduces to the identity

$$j'_k(\boldsymbol{t},\boldsymbol{w},\boldsymbol{z}_1,\cdots,\boldsymbol{z}_{k-1}) \equiv 0 \quad (k\ge 1)$$

and the observation that the r.h.s. of (20) *does not involve random walk probabilities and applies to an arbitrary sequence* $\boldsymbol{d}(\cdot)$; in other words, Equation (19) follows from (12) with $d_{t-1} = d(\boldsymbol{z}_1), d_{t-2} = d(\boldsymbol{z}_2), \cdots, d_{t-k+1} = d(\boldsymbol{z}_{k-1}), d_{t-k} = d(\boldsymbol{w})$. An informal proof of the 2nd relation in (18) proceeds similarly from the 2nd relation in (12). A rigorous derivation of (18) including summability properties of VO operators in (16) seems to be a realistic open problem.

Problem 5. *What conditions on the VO sequence* $\boldsymbol{d}(\cdot)$ *and random walk probabilities* $p(\cdot)$ *imply the boundedness of the* $L^2(\mathbb{Z}^\nu)$ *norms*

$$\sup_{\boldsymbol{t}}\|a(\boldsymbol{t},\cdot)\|_{L^2(\mathbb{Z}^\nu)} < \infty, \quad \sup_{\boldsymbol{t}}\sum_{\boldsymbol{t}}\|b(\boldsymbol{t},\cdot)\|_{L^2(\mathbb{Z}^\nu)} < \infty \tag{21}$$

of the V0 operators in (16) *and their invertibility in* (18)*?*

The results in [8–10,24] suggest that the conditions on $\boldsymbol{d}(\cdot)$ in Problem 5 may depend only on averaging properties (Cesaro mean) of this sequence, whereas those on $p(\cdot)$ may depend on local behavior of the characteristic function $\widehat{p}(\cdot)$ at the origin [25,26].

A very intriguing but hard question concerns the possibility of an extension of VO operators in (16) to continuous argument $\boldsymbol{t} \in \mathbb{R}^\nu$, generalizing the VO operators in (13).

5. Applications to Modeling of Random Fields

In probability theory and its applications, stochastic fractional equations are used for construction and modeling of random processes with desirable dependence properties. The discrete-time fractional stochastic equation with white noise on the r.h.s. corresponding to backward shift operator $Tg(t) = g(t-1)$ in (1) (the deterministic random walk $S_j = -j$) is the basic model for *long-range dependence*, leading to the important class of ARFIMA(p, d, q) processes [27–29]. Multi-dimensional fractional stochastic processes (random fields) appear in many natural sciences and applications (cosmology, environmetrics, spatial statistics, machine learning, etc.) [30–34]. We end this note with two concrete potential applications of the fractional operators discussed in the previous sections.

Discretely fractionally differenced/integrated random field $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{R}^\nu\}$ can be defined as a solution of stochastic equation

$$(I-T)^d X(\boldsymbol{t}) = Y(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^\nu \tag{22}$$

with operator $(I-T)^d$ in (5) and a given random field $Y = \{Y(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{R}^\nu\}$ on the r.h.s. Among such Y, two parametric classes—*fractional Lévy random field* $Y = L_H$ and *Matern Lévy random field* $Y = M_{c,H}$—are of particular importance in spatial modeling and statistics [32,35–38]. Solving (22) with the above Y and studying path properties and scaling limits of the solution X is an interesting and open problem. See [8,39–43] for related probabilistic research. The importance of these random fields is largely due to the explicit form of their covariance functions $\mathrm{E}\mathcal{L}_H(\boldsymbol{t})\mathcal{L}_H(\boldsymbol{s}) \propto \big(|\boldsymbol{t}|^{2H} + |\boldsymbol{s}|^{2H} - |\boldsymbol{t}-\boldsymbol{s}|^{2H}\big)$, $\mathrm{E}\mathcal{M}_{c,H}(\boldsymbol{s})\mathcal{M}_{c,H}(\boldsymbol{t}) \propto (c|\boldsymbol{t}-\boldsymbol{s}|)^H K_H(c|\boldsymbol{t}-\boldsymbol{s}|)$, where $0 < H < 1, c > 0$ are parameters, and K_H is the modified Bessel function of the 2nd type. One may expect that the solution X of (22) with the above Y and an isotropic archetypal density $p(\boldsymbol{u})$ may lead to a new parametric class of spatial covariance functions useful in spatial statistics (kriging) and machine learning [32,34]. For instance, regarding the standard Gaussian density

$$p(\boldsymbol{u}) = (2\pi)^{-\nu/2} \mathrm{e}^{-|\boldsymbol{u}|^2/2}, \quad \boldsymbol{u} \in \mathbb{R}^\nu$$

($S_j = B_j$ is the discretized standard Brownian motion in $\mathbb{R}^\nu$) the corresponding kernel $\tau(\boldsymbol{u}; d) \equiv \tau_B(\boldsymbol{u}; d)$ in (5) has the Fourier transform

$$\widehat{\tau}_B(\boldsymbol{z}; d) = \sum_{j=0}^{\infty} \psi_j(d) \mathrm{e}^{-j|\boldsymbol{z}|^2/2} = (1 - \mathrm{e}^{-|\boldsymbol{z}|^2/2})^d, \quad \boldsymbol{z} \in \mathbb{R}^\nu.$$

A formal calculation of spectral densities of the random field X in (22) with $Y = \mathcal{L}_H$ and $Y = \mathcal{M}_{c,H}$ yields

$$f_{\mathcal{L}_H}(\boldsymbol{z}) \propto \frac{1}{(1-\mathrm{e}^{-|\boldsymbol{z}|^2/2})^{2d} |\boldsymbol{z}|^{\nu+2H}}, \quad f_{\mathcal{M}_{c,H}}(\boldsymbol{z}) \propto \frac{1}{(1-\mathrm{e}^{-|\boldsymbol{z}|^2/2})^{2d} (c^2 + |\boldsymbol{z}|^2)^{H+\frac{\nu}{2}}}. \tag{23}$$

The functions on the r.h.s. of (23) are non-negative, isotropic and integrable (under appropriate conditions on the parameters), in which case the respective covariance functions are well-defined as their Fourier transforms. Recall that the Fourier transform of an isotropic function is the Hankel transform on $(0, \infty)$ of the radial part [36].

Problem 6. *Determine the conditions on d, ν, c, H for the integrability of* (23) *and the existence of X in* (22). *Find the respective covariance functions (the Fourier transforms).*

An analogous problem can be formulated for some other (non-Gaussian) isotropic probability densities $p(\boldsymbol{u})$.

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Non-stationary long-range dependence. In [8], the authors discussed the existence of a stationary solution X of the fractional equation $(I-T)^d X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$ with T in (1) and white noise $\varepsilon(\boldsymbol{t})$ on the r.h.s., together with long-range dependence and scaling properties of the above solution. Similar problems for nonstationary solutions stochastic equations for one-dimensional VO operators in (10) were studied in [9,10,24,44]. A sensible definition of long-range dependence for a non-stationary random field $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ with covariance $r(\boldsymbol{t}, \boldsymbol{s}) = \mathrm{Cov}(X(\boldsymbol{t}), X(\boldsymbol{s}))$ suggested by [8,45] is

$$\liminf_{\lambda\to\infty} \lambda^{-\nu} \int_{\mathbb{R}^{2\nu}} \phi(\boldsymbol{t}/\lambda)\phi(\boldsymbol{s}/\lambda)|r([\boldsymbol{t}],[\boldsymbol{s}])|\mathrm{d}\boldsymbol{t}\mathrm{d}\boldsymbol{s} = \infty$$

for some test function $0 \le \phi \in L^1(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu)$.

Problem 7. *Find sufficient conditions on the VO sequence $\boldsymbol{d}(\cdot)$ and the random walk probabilities $p(\cdot)$ for the existence and long-range dependence properties of solutions of stochastic equations $A_{\boldsymbol{d}(\cdot)}X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}),\ B_{\boldsymbol{d}(\cdot)}Y(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$ for fractional VO operators in (16) with white noise $\varepsilon(\boldsymbol{t})$ on the r.h.s.*

Problem 7 is related to Problem 5, since the natural solutions of the stochastic equations are written as $X(\boldsymbol{t}) = B_{-\boldsymbol{d}(\cdot)}\varepsilon(\boldsymbol{t}), Y(\boldsymbol{t}) = A_{-\boldsymbol{d}(\cdot)}\varepsilon(\boldsymbol{t})$ in view of (18), and their existence; viz., $\sup_{\boldsymbol{t}} \mathrm{E}X^2(\boldsymbol{t}) = \sup_{\boldsymbol{t}} \sum_{\boldsymbol{s}} (b^-(\boldsymbol{t},\boldsymbol{s}))^2 < \infty, \sup_{\boldsymbol{t}} \mathrm{E}Y^2(\boldsymbol{t}) = \sup_{\boldsymbol{t}} \sum_{\boldsymbol{s}} (a^-(\boldsymbol{t},\boldsymbol{s}))^2 < \infty$ is a consequence of the $L^2(\mathbb{Z}^\nu)$-boundedness in (21). With the existence of X, Y being established, the next step could be studying scaling limits of these random fields, in the spirit of [8], by imposing additional conditions on $\boldsymbol{d}(\cdot)$ and $p(\cdot)$.

6. Conclusions

The paper provides an analytic-probabilistic framework that can serve as a foundation for future rigorous investigations of fractional operators in multidimensional and VO contexts. Although it does not resolve specific open questions, it provides a useful basis for further research in the theory and stochastic modeling of random fields.

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