mathematics

MDPI

Article

Existence of Solutions to the Nonstationary Stokes System with a Nonlinear Overdetermination Condition

Vytautas Bačianskas and Kristina Kaulakytė *

Institute of Applied Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko Str., 24, LT-03225 Vilnius, Lithuania; vytautas.bacianskas@mif.stud.vu.lt
* Correspondence: kristina.kaulakyte@mif.vu.lt

Abstract

In this paper, we study an inverse problem for the nonstationary Stokes system in a bounded domain Ω with a nonlinear integral overdetermination condition, describing the kinetic energy $E(t)$ of the fluid. We construct two classes of solutions: weak and very weak. In the case where the kinetic energy E belongs to $W_2^1(0,T)$, we construct weak solutions. If E belongs only to $L_2(0,T)$, we construct very weak solutions.

Keywords: inverse problem; nonstationary Stokes system; very weak solution; primitive function; Lebesgue points

MSC: 35R30

1. Introduction

The Stokes system represents a special case of the Navier–Stokes equations in which inertial effects are negligible compared with viscous forces. Although the nonlinear convective term is omitted, the Stokes equations still describe a wide range of physically significant flows and remain widely used in various applications. In this paper, we focus on the nonstationary Stokes system describing the flow of a viscous incompressible fluid.

Suppose that $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded domain with Lipschitz boundary $\partial\Omega$ and T is a finite positive real number. We denote $\Omega_T = \Omega \times (0,T)$ and $S_T = \partial\Omega \times (0,T)$. The classical Stokes problem is to find the velocity vector field $\boldsymbol{u}$ and the pressure p satisfying

$$\begin{cases} \rho \boldsymbol{u}_t(x,t) - \nu\Delta\boldsymbol{u}(x,t) + \nabla p(x,t) = \rho \boldsymbol{f}(x,t), & (x,t) \in \Omega_T, \\ \operatorname{div} \boldsymbol{u} = 0, \quad (x,t) \in \Omega_T, \\ \boldsymbol{u}\big|_{S_T} = 0, \quad \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\boldsymbol{f}$ describes the external forces, $\boldsymbol{u}_0$ is the initial condition, $\nu > 0$ is the constant coefficient of kinematic viscosity, and $\rho > 0$ is the density of the fluid. Since we consider an incompressible fluid, the density ρ is constant, and without loss of generality, we assume that $\rho \equiv 1$. Notice that in the classical formulation, both $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are prescribed functions. In this case, the solvability of problem (1) is well known and can be found in [1].

In addition to the classical Stokes problem, it is possible to formulate inverse problems where $\boldsymbol{f}$ is treated as an unknown and must be determined as part of the solution. For such a problem to be well posed, an additional condition is required. Most existing results on inverse problems involve a linear integral condition given by

Check for updates

Academic Editor: Dhananjay Gopal

Received: 9 March 2026
Revised: 14 April 2026
Accepted: 17 April 2026
Published: 22 April 2026

$$\int_{\Omega} \boldsymbol{u}(x,t)\cdot\boldsymbol{\omega}(x)dx = F(t) \quad \forall t\in[0,T],\ F(0)=\int_{\Omega} \boldsymbol{u}_0(x)\cdot\boldsymbol{\omega}(x)dx, \tag{2}$$

where $\boldsymbol{\omega}$ and F are known functions. In many papers (see [2–13]), the authors assume that the function F is sufficiently smooth (for instance, $F\in W_2^1(0,T)$). However, in [14,15], the solvability of a Poiseuille type inverse problem (1) with a linear integral side condition (similar to (2)) was also proved under minimal regularity on F, i.e., assuming that $F\in L_2(0,T)$ only.

In this work, we consider the Stokes problem (1) together with a nonlinear condition as follows:

$$\int_{\Omega} |\boldsymbol{u}(x,t)|^2 dx = E^2(t) \quad \forall t\in[0,T],\ E(0)=\|\boldsymbol{u}_0\|_{L_2(\Omega)}, \tag{3}$$

where E is a given function. In [16], the inverse problem for the heat equation with nonlinear condition (3) was considered assuming that the function E belongs to the Sobolev space $W_2^1(0,T)$. Problem (1), (3) can be seen as a generalization of [16] to the vector-valued case where nonlinear condition (3) becomes a specification of kinetic energy $\frac{1}{2}\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega)}$ at time t. Thus, this formulation has a clear physical interpretation: we look for the flow velocity $\boldsymbol{u}$, pressure p, and external forces $\boldsymbol{f}$ such that the described flow has the prescribed kinetic energy. Therefore, the recovered force is constructed within a specific admissible class tailored to the prescribed energy profile. In [16], the authors constructed solutions (u,f) such that u and u_t belong to $L_2(0,T;L_2(\Omega))$, while $f\in L_2(0,T;W_2^{-2}(\Omega))$. In the present paper, we investigate the Stokes problem in two cases. For the case when $E\in W_2^1(0,T)$, we construct the classical weak solutions, while for the case when E belongs only to $L_2(0,T)$, we use a different approach compared with previous works and construct very weak solutions (defined in a similar way as in [15]) such that $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$ belong to $L_2(0,T;L_2(\Omega))$, while $\boldsymbol{f}$ belongs to a certain dual space.

The novelty of the paper lies in the combination of several aspects. While linear overdetermination conditions for the Stokes system and nonlinear energy-type conditions for scalar equations have been studied previously, the present paper treats a nonlinear integral overdetermination condition in the context of the nonstationary Stokes system. In particular, we develop a framework for constructing weak and very weak solutions under low regularity assumptions on the prescribed energy function E, which, to the best of our knowledge, has not been studied in this setting.

Throughout most of the paper, it is assumed that $\boldsymbol{u}_0$ is not identically equal to 0 so that the norm $\|\boldsymbol{u}_0\|_{L_2(\Omega)}$ is positive. The case when $\boldsymbol{u}_0\equiv 0$ is treated in a similar way, so it will be discussed briefly at the end of this paper.

2. Notation and Auxiliary Results

2.1. Function Spaces

We introduce some notations that are used in this paper. First, suppose that $G\subset\mathbb{R}^d$, $d=2,3$, is a bounded open domain with a Lipschitz boundary. Let $C^\infty(G)$ denote the space of all infinitely differentiable (smooth) functions defined on a set G, and let $C_0^\infty(G)\subset C^\infty(G)$ be the subspace of compactly supported functions, whose supports are contained in G. For $p\geq 1$, we use the usual notation for Lebesgue spaces $L_p(G)$, consisting of measurable functions u with a finite norm $\|u\|_{L_p(G)}=\left(\int_G |u(x)|^p dx\right)^{\frac{1}{p}}$, and Sobolev spaces $W_p^k(G)$, consisting of functions having all weak derivatives of order $|\alpha|\leq k$, such that the norm $\|u\|_{W_p^k(G)}=\left(\sum_{|\alpha|\leq k}\|D^\alpha u(x)\|^p_{L_p(G)}\right)^{\frac{1}{p}}$ is finite. Here, α is a multi-index and D^α is the weak partial derivative of order $|\alpha|$. Additionally, the subspace $\mathring{W}_p^k(G)$ is defined as the closure of $C_0^\infty(G)$ in the norm of $W_p^k(G)$. In the case when $p=2$, the spaces $L_2(G)$, $W_2^k(G)$, and $\mathring{W}_2^k(G)$ are also Hilbert spaces.

https://doi.org/10.3390/math14091402

Vector-valued functions are denoted by bold letters. For a normed space V, we write $\boldsymbol{u} \in V$ to indicate that each component of $\boldsymbol{u} = (u_1, u_2, \dots, u_d)$ belongs to V. Moreover, we define

$$\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} = \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}.$$

If $\boldsymbol{u} \in V$, then $\|\boldsymbol{u}\|_V^2 = \sum_{i=1}^d \|u_i\|_V^2$ and $\|\nabla \boldsymbol{u}\|_V^2 = \sum_{i=1}^d \sum_{j=1}^d \left\| \frac{\partial u_i}{\partial x_j} \right\|_V^2$. The dual space of V is denoted as V^* and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and V^*. If H is a Hilbert space, the scalar product between $u \in H$ and $v \in H$ is written as $\langle u, v \rangle_H$. It will be clear from the context whether the angle brackets refer to the scalar product or to the pairing of a normed space and its dual. The (weak) divergence of $\boldsymbol{u} = (u_1, u_2, \dots, u_d)$ is defined as $\operatorname{div} \boldsymbol{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}$. We use the following spaces that consist of vector-valued functions $\boldsymbol{u} : G \to \mathbb{R}^d$:

$D(G)$ is the space $\mathring{W}_2^1(G)$ with the equivalent norm $\|\boldsymbol{u}\|_{D(G)} = \|\nabla \boldsymbol{u}\|_{L_2(G)}$;
$J_0^\infty(G)$ is the space of functions $\boldsymbol{u} \in C_0^\infty(G)$ such that $\operatorname{div} \boldsymbol{u} = 0$;
$J(G)$ is the closure of the set $J_0^\infty(G)$ with respect to the $L_2(G)$ norm;
$H(G)$ is the closure of the set $J_0^\infty(G)$ with respect to the $D(G)$ norm.

Lastly, we use spaces that incorporate both time and space. Let $0 < T < \infty$ be a real number and denote $G_T = G \times (0, T)$, and $S_T = \partial G \times (0, T)$. The spaces $W_2^{0,1}(G_T)$, $W_2^{1,0}(G_T)$ and $W_2^{1,1}(G_T)$ are defined similarly to Sobolev spaces, i.e.,

$$W_2^{0,1}(G_T) = \{u \in L_2(G_T) : u_t \in L_2(G_T)\}, \ \|u\|_{W_2^{0,1}(G_T)} = \left(\|u\|_{L_2(G_T)}^2 + \|u_t\|_{L_2(G_T)}^2 \right)^{\frac{1}{2}};$$

$$W_2^{1,0}(G_T) = \{u \in L_2(G_T) : u_{x_i} \in L_2(G_T), 1 \le i \le d\}, \ \|u\|_{W_2^{1,0}(G_T)} = \left(\|u\|_{L_2(G_T)}^2 + \|\nabla u\|_{L_2(G_T)}^2 \right)^{\frac{1}{2}};$$

$$W_2^{1,1}(G_T) = W_2^{0,1}(G_T) \cap W_2^{1,0}(G_T), \ \|u\|_{W_2^{1,1}(G_T)} = \left(\|u\|_{L_2(G_T)}^2 + \|\nabla u\|_{L_2(G_T)}^2 + \|u_t\|_{L_2(G_T)}^2 \right)^{\frac{1}{2}}.$$

Spaces $\mathring{W}_2^{1,0}(G_T)$ and $\mathring{W}_2^{1,1}(G_T)$ are subspaces of $W_2^{1,0}(G_T)$ and $W_2^{1,1}(G_T)$, respectively, consisting of functions u such that $u\big|_{S_T} = 0$ in the sense of traces.

We mainly work with solenoidal functions with zero boundary conditions defined on the surface S_T. Let $\mathcal{J}_0^\infty(G_T)$ be defined as the space of functions $\boldsymbol{u} \in C^\infty(\overline{G_T})$ such that $\boldsymbol{u}\big|_{S_T} = 0$ and $\operatorname{div} \boldsymbol{u} = 0$. Then the closure of $\mathcal{J}_0^\infty(G_T)$ in the space $W_2^{1,0}(G_T)$ is denoted as $\mathring{V}_2^{1,0}(G_T)$, while the closure in the space $W_2^{1,1}(G_T)$ is denoted as $\mathring{V}_2^{1,1}(G_T)$.

2.2. Eigenvalues and Eigenfunctions of the Stokes Operator

Let $\boldsymbol{v}_k \in H(\Omega)$ and $\lambda_k \in \mathbb{R}$, $k \in \mathbb{N}$, be the eigenfunctions and eigenvalues of the Stokes operator as follows:

$$\begin{cases} -\nu \Delta \boldsymbol{v}_k(x) + \nabla p(x) = \lambda_k \boldsymbol{v}_k(x), \quad x \in \Omega, \\ \operatorname{div} \boldsymbol{v}_k = 0, \quad x \in \Omega, \\ \boldsymbol{v}_k\big|_{\partial\Omega} = 0, \end{cases}$$

which satisfy the following integral identity:

$$\nu \int_\Omega \nabla \boldsymbol{v}_k(x) \cdot \nabla \boldsymbol{\eta}(x) dx = \lambda_k \int_\Omega \boldsymbol{v}_k(x) \cdot \boldsymbol{\eta}(x) dx, \quad \forall \boldsymbol{\eta} \in H(\Omega).$$

It is known that the eigenvalues λ_k are real, positive, and $\lambda_k \to \infty$ as $k \to \infty$ (see, e.g., [17], Chapter 2.6). Additionally, the eigenfunctions $\{\boldsymbol{v}_k\}_{k=1}^\infty$ form an orthogonal basis in $H(\Omega)$ and $J(\Omega)$, and can be normalized in $L^2(\Omega)$ so that $\|\boldsymbol{v}_k\|_{L_2(\Omega)} = 1$. These eigenfunctions have the following properties:

$$\int_\Omega \boldsymbol{v}_k(x)\cdot\boldsymbol{v}_l(x)dx = \begin{cases}1, \text{ if } k=l,\\ 0, \text{ if } k\neq l,\end{cases} \quad \int_\Omega \nabla\boldsymbol{v}_k(x)\cdot\nabla\boldsymbol{v}_l(x)dx = \begin{cases}\frac{\lambda_k}{\nu}, \text{ if } k=l,\\ 0, \text{ if } k\neq l.\end{cases} \tag{4}$$

3. Weak Solution

3.1. Definition of a Weak Solution

In order to derive the definition of a weak solution to problem (1), (3), we first suppose that $\boldsymbol{u}$, $\boldsymbol{f}$, and p are sufficiently smooth. Multiply Equation $(1)_1$ by a test function $\boldsymbol{\eta}\in C^\infty(\overline{\Omega_T})$ such that $\boldsymbol{\eta}\big|_{S_T}=0$, $\operatorname{div}\boldsymbol{\eta}=0$ and apply integration by parts with respect to x to get that

$$\int_\Omega \boldsymbol{u}_t(x,t)\cdot\boldsymbol{\eta}(x,t)dx+\nu\int_\Omega \nabla\boldsymbol{u}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dx=\int_\Omega \boldsymbol{f}(x,t)\cdot\boldsymbol{\eta}(x,t)dx. \tag{5}$$

There are no surface integrals over $\partial\Omega$ because $\boldsymbol{\eta}\big|_{S_T}=0$. The integral term containing the pressure function p vanishes because $\operatorname{div}\boldsymbol{\eta}=0$, and so

$$\int_\Omega \nabla p(x,t)\cdot\boldsymbol{\eta}(x,t)dx=-\int_\Omega p(x,t)\operatorname{div}\boldsymbol{\eta}\,dx=0.$$

Next, we integrate (5) with respect to time from 0 to t. Thus,

$$\int_0^t\int_\Omega \boldsymbol{u}_t(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau+\nu\int_0^t\int_\Omega \nabla\boldsymbol{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)dxd\tau=\int_0^t\int_\Omega \boldsymbol{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau \tag{6}$$

for all $t\in(0,T]$. By density arguments, this identity remains valid for all $\boldsymbol{\eta}\in\mathring{V}_2^{1,0}(\Omega_T)$.

Definition 1. *Suppose that $E\in W_2^1(0,T)$, $\boldsymbol{u}_0\in W_2^2(\Omega)\cap H(\Omega)$, and E satisfies the compatibility condition $E(0)=\|\boldsymbol{u}_0\|_{L_2(\Omega)}$. A pair of functions $(\boldsymbol{u},\boldsymbol{f})\in\mathring{V}_2^{1,1}(\Omega_T)\times L_2(\Omega_T)$ is called a weak solution of problem (1), (3) if it satisfies the integral identity (6) for all $t\in(0,T]$ and for all test functions $\boldsymbol{\eta}\in\mathring{V}_2^{1,0}(\Omega_T)$, and as well satisfies side condition (3) and initial condition $\boldsymbol{u}(x,0)=\boldsymbol{u}_0(x)$ for a.e. $x\in\Omega$.*

The first goal of this paper is to prove the following theorem.

Theorem 1. *Suppose that $\boldsymbol{u}_0\in W_2^2(\Omega)\cap H(\Omega)$, $E\in W_2^1(0,T)$ are given functions and E satisfies the compatibility condition $E(0)=\|\boldsymbol{u}_0\|_{L_2(\Omega)}$. Then there exists at least one weak solution $(\boldsymbol{u},\boldsymbol{f})$ to problem (1), (3).*

3.2. Approximate Solution

For every $N\in\mathbb{N}$, we will find approximate solutions $(\boldsymbol{u}^{(N)},\boldsymbol{f}^{(N)})$ of problem (1), (3) represented in the following forms:

$$\boldsymbol{u}^{(N)}(x,t)=\sum_{k=1}^N w_k(t)\boldsymbol{v}_k(x),\quad \boldsymbol{f}^{(N)}(x,t)=\sum_{k=1}^N q_k(t)\boldsymbol{v}_k(x), \tag{7}$$

where $\boldsymbol{v}_k$ are eigenfunctions of the Stokes operator. We require $\boldsymbol{u}^{(N)}$ and $\boldsymbol{f}^{(N)}$ to satisfy for a.e. $t\in(0,T)$ and $j=1,\ldots,N$ in the following system:

https://doi.org/10.3390/math14091402

$$\begin{cases} \int_\Omega (\boldsymbol{u}^{(N)})_t(x,t)\cdot \boldsymbol{v}_j(x)dx + \nu\int_\Omega \nabla \boldsymbol{u}^{(N)}(x,t)\cdot\nabla \boldsymbol{v}_j(x)dx = \int_\Omega \boldsymbol{f}^{(N)}(x,t)\cdot \boldsymbol{v}_j(x)dx, \\ \boldsymbol{u}^{(N)}(x,0) = \sum_{k=1}^{N}\beta_k \boldsymbol{v}_k(x), \quad x\in\Omega, \\ \int_\Omega \left|\boldsymbol{u}^{(N)}(x,t)\right|^2 dx = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}}E^2(t), \end{cases} \tag{8}$$

where β_k are the Fourier coefficients of the function $\boldsymbol{u}_0$ in terms of the orthogonal basis $\{\boldsymbol{v}_k\}_{k=1}^\infty$ in $H(\Omega)$ so that $\boldsymbol{u}_0 = \sum_{k=1}^\infty \beta_k \boldsymbol{v}_k$.

Substituting expressions (7) into $(8)_1$ and using the fact that eigenfunctions are orthonormal in $L_2(\Omega)$ and orthogonal in $H(\Omega)$ (see (4)), we get an ordinary differential equation as follows:

$$w_k'(t) + \lambda_k w_k(t) = q_k(t). \tag{9}$$

Together with the initial condition $w_k(0) = \beta_k$, Equation (9) has a unique solution as follows:

$$w_k(t) = \int\limits_0^t e^{-\lambda_k(t-\tau)}q_k(\tau)\,d\tau + \beta_k.$$

Substituting $(7)_1$ into $(8)_3$, we get

$$\int_\Omega |\boldsymbol{u}^{(N)}(x,t)|^2dx = \left\|\sum_{k=1}^N w_k(t)\boldsymbol{v}_k\right\|^2_{L_2(\Omega)} = \sum_{k=1}^N \|w_k(t)\boldsymbol{v}_k\|^2_{L_2(\Omega)} = \sum_{k=1}^N (w_k(t))^2 = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}}E^2(t), \tag{10}$$

where the last equality follows from $(8)_3$.

To satisfy condition (10) for all $t\in[0,T]$, we choose

$$w_k(t) = \frac{\beta_k}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}}E(t), \tag{11}$$

and to satisfy Equation (9), we set

$$q_k := w_k' + \lambda_k w_k \in L_2(0,T). \tag{12}$$

Since the function E is from the space $W_2^1(0,T)$, it is continuous. Therefore, $\boldsymbol{u}^{(N)}$ belongs to the space $C([0,T];L_2(\Omega))$ (see, e.g., [18], Chapter 5.9.2) and $\boldsymbol{u}^{(N)}(\cdot,0)$ is well defined. Since $E(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$, we also have that

$$\boldsymbol{u}^{(N)}(x,0) = \sum_{k=1}^N w_k(0)\boldsymbol{v}_k(x) = \frac{E(0)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}}\sum_{k=1}^N \beta_k\boldsymbol{v}_k(x) = \sum_{k=1}^N \beta_k \boldsymbol{v}_k(x).$$

Next, we will get estimates for $\boldsymbol{u}^{(N)}$ and $\boldsymbol{f}^{(N)}$. Recall that β_k are Fourier coefficients of $\boldsymbol{u}_0$, so $\sum_{k=1}^\infty \beta_k^2 = \|\boldsymbol{u}_0\|^2_{L_2(\Omega)}$. Thus,

$$\|\boldsymbol{u}^{(N)}\|^2_{L_2(\Omega_T)} = \int_0^T\int_\Omega |\boldsymbol{u}^{(N)}(x,t)|^2dxdt \overset{(10)}{=} \frac{\sum_{k=1}^N\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}}\int_0^T E^2(t)dt \le \|E\|^2_{L_2(0,T)}. \tag{13}$$

Similarly, by using properties of eigenfunctions (see (4)), we obtain

$$\begin{aligned} \|\nabla \boldsymbol{u}^{(N)}\|^2_{L_2(\Omega_T)} &= \int_0^T\int_\Omega |\nabla\boldsymbol{u}^{(N)}(x,t)|^2dxdt = \int_0^T\int_\Omega\left|\sum_{k=1}^N w_k(t)\nabla\boldsymbol{v}_k(x)\right|^2dxdt \\ &= \int_0^T\sum_{k=1}^N |w_k(t)|^2\frac{\lambda_k}{\nu}dt = \frac{\|E\|^2_{L_2(0,T)}}{\nu\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}}\sum_{k=1}^N\beta_k^2\lambda_k. \end{aligned} \tag{14}$$

For the time derivative $\boldsymbol{u}_t^{(N)}$, we have

$$\|\boldsymbol{u}_t^{(N)}\|^2_{L_2(\Omega_T)} = \int_0^T \left\| \sum_{k=1}^N w_k'(t)\boldsymbol{v}_k \right\|^2_{L_2(\Omega)} dt = \int_0^T \sum_{k=1}^N (w_k'(t))^2 dt = \int_0^T \sum_{k=1}^N \frac{\beta_k^2 |E'(t)|^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} dt$$
$$= \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^T |E'(t)|^2 dt \leq \|E\|^2_{W_2^1(0,T)}. \tag{15}$$

Thus, functions $\boldsymbol{u}^{(N)}$ belong to the space $\mathring{V}_2^{1,1}(\Omega_T)$. Finally, using Equation (12), we get an estimate for $\boldsymbol{f}^{(N)}$ in the space $L_2(\Omega_T)$ as follows:

$$\int_0^T |q_k(t)|^2 dt = \int_0^T |w_k'(t) + \lambda_k w_k(t)|^2 dt \leq 2\int_0^T \left(|w_k'(t)|^2 + |\lambda_k w_k(t)|^2 \right) dt$$
$$= 2\int_0^T \left| \frac{\beta_k E'(t)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \right|^2 dt + 2\int_0^T \left| \frac{\lambda_k \beta_k E(t)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \right|^2 dt$$
$$= \frac{2\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \left(\|E'\|^2_{L_2(0,T)} + \lambda_k^2 \|E\|^2_{L_2(0,T)} \right)$$
$$\leq \frac{2\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} (1+\lambda_k^2) \|E\|^2_{W_2^1(0,T)}, \tag{16}$$

and

$$\|\boldsymbol{f}^{(N)}\|^2_{L_2(\Omega_T)} = \int_0^T \int_\Omega \left| \sum_{k=1}^N q_k(t) \boldsymbol{v}_k(x) \right|^2 dx dt = \int_0^T \left\| \sum_{k=1}^N q_k(t) \boldsymbol{v}_k \right\|^2_{L_2(\Omega)} dt$$
$$= \sum_{k=1}^N \int_0^T |q_k(t)|^2 dt \overset{(16)}{\leq} \sum_{k=1}^N \frac{2\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} (1+\lambda_k^2) \|E\|^2_{W_2^1(0,T)}$$
$$\leq 2\|E\|^2_{W_2^1(0,T)} \left(1 + \frac{\sum_{k=1}^N \beta_k^2 \lambda_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \right). \tag{17}$$

Bounds (14) and (17) are independent of N because we have assumed that $\boldsymbol{u}_0 \in W_2^2(\Omega) \cap H(\Omega)$. Therefore,

$$\sum_{k=1}^\infty \beta_k^2 \lambda_k^2 \leq \nu^2 \|\Delta \boldsymbol{u}_0\|^2_{L_2(\Omega)} \leq \nu^2 \|\boldsymbol{u}_0\|^2_{W_2^2(\Omega)}, \tag{18}$$
$$\sum_{k=1}^\infty \beta_k^2 \lambda_k \leq \nu \|\boldsymbol{u}_0\|^2_{H(\Omega)}. \tag{19}$$

From (17), we get that $\boldsymbol{f}$ is bounded in the space $L_2(\Omega_T)$ and

$$\|\boldsymbol{f}^{(N)}\|^2_{L_2(\Omega_T)} \leq 2\|E\|^2_{W_2^1(0,T)} \left(1 + \frac{\nu^2 \|\boldsymbol{u}_0\|^2_{W_2^2(\Omega)}}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \right).$$

Using (13), (14), and (15) together with (19), we can estimate $\boldsymbol{u}^{(N)}$ in the space $\mathring{V}_2^{1,1}(\Omega_T)$ as follows:

$$\|\boldsymbol{u}^{(N)}\|^2_{\mathring{V}_2^{1,1}(\Omega_T)} = \|\boldsymbol{u}^{(N)}\|^2_{L_2(\Omega_T)} + \|\nabla \boldsymbol{u}^{(N)}\|^2_{L_2(\Omega_T)} + \|\boldsymbol{u}_t^{(N)}\|^2_{L_2(\Omega_T)}$$
$$\leq \|E\|^2_{W_2^1(0,T)} \left(2 + \frac{\|\boldsymbol{u}_0\|^2_{H(\Omega)}}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \right).$$

3.3. Convergence

Since the sequence $\{\boldsymbol{u}^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $\mathring{V}_2^{1,1}(\Omega_T)$ and the sequence $\{\boldsymbol{f}^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $L_2(\Omega_T)$, there exists a subsequence $\{N_l\}_{l=1}^{\infty}$ such that $\boldsymbol{u}^{(N_l)}$ weakly converges to a function $\boldsymbol{u} \in \mathring{V}_2^{1,1}(\Omega_T)$ and $\boldsymbol{f}^{(N_l)}$ weakly converges to some $\boldsymbol{f} \in L_2(\Omega_T)$. Importantly, this means that $\boldsymbol{u}^{(N_l)} \rightharpoonup \boldsymbol{u}$, $\boldsymbol{u}_t^{(N_l)} \rightharpoonup \boldsymbol{u}_t$ and $\nabla \boldsymbol{u}^{(N_l)} \rightharpoonup \nabla \boldsymbol{u}$ in $L_2(\Omega_T)$. Next, consider arbitrary functions $d_j \in C[0,T]$, $j \in \mathbb{N}$. Multiply Equation (8)$_1$ by d_j, and then sum them over all $j \in \{1,2,\dots,M\}$, where $M \le N$. Denote $\boldsymbol{\eta}(x,t) = \sum_{k=1}^{M} d_k(t)\boldsymbol{v}_k(x)$, and replace N by N_l in the integral identity as follows:

$$\int_\Omega \boldsymbol{u}_t^{(N_l)}(x,t)\cdot\boldsymbol{\eta}(x,t)dx + \nu\int_\Omega \nabla\boldsymbol{u}^{(N_l)}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dx = \int_\Omega \boldsymbol{f}^{(N_l)}(x,t)\cdot\boldsymbol{\eta}(x,t)dx.$$

By integrating this identity with respect to time from 0 to t, we get

$$\int_0^t\!\!\int_\Omega \boldsymbol{u}_t^{(N_l)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau + \nu\int_0^t\!\!\int_\Omega \nabla\boldsymbol{u}^{(N_l)}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)dxd\tau = \int_0^t\!\!\int_\Omega \boldsymbol{f}^{(N_l)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau. \tag{20}$$

For the first integral in (20), we notice that it can be written as an inner product in the space $L_2(\Omega_T)$. Then, by using the fact that $\boldsymbol{u}_t^{(N_l)} \rightharpoonup \boldsymbol{u}_t$ in this space, we get

$$\int_0^t\!\!\int_\Omega \boldsymbol{u}_t^{(N_l)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau = \langle \boldsymbol{u}_t^{(N_l)}, \boldsymbol{\eta}\mathbb{1}_{\{\tau\le t\}}\rangle_{L_2(\Omega_T)} \xrightarrow{l\to\infty} \langle \boldsymbol{u}_t, \boldsymbol{\eta}\mathbb{1}_{\{\tau\le t\}}\rangle_{L_2(\Omega_T)} = \int_0^t\!\!\int_\Omega \boldsymbol{u}_t(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau.$$

For the other integrals, we can similarly write them as inner products in $L_2(\Omega_T)$ and, by using weak convergence, we obtain that

$$\int_0^t\int_\Omega \nabla\boldsymbol{u}^{(N_l)}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)dxd\tau \xrightarrow{l\to\infty} \int_0^t\int_\Omega \nabla\boldsymbol{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)dxd\tau,$$
$$\int_0^t\int_\Omega \boldsymbol{f}^{(N_l)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau \xrightarrow{l\to\infty} \int_0^t\int_\Omega \boldsymbol{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau.$$

Thus, after passing to the limit as $l\to\infty$ in (20), we get integral identity (6) for the pair of limit functions $(\boldsymbol{u},\boldsymbol{f})$. However, it holds only for those $\boldsymbol{\eta}$ that are finite linear combinations of functions $\boldsymbol{v}_k$. These linear combinations are dense in the space $\mathring{V}_2^{1,0}(\Omega_T)$. The proof is similar to the proof of Theorem 2.3.8 in the book [19]. Additionally, since $N_l\to\infty$ when $l\to\infty$, the number M in the sum $\boldsymbol{\eta}(x,t) = \sum_{k=1}^{M} d_k(t)\boldsymbol{v}_k(x)$ can be arbitrarily large. Therefore, by approximation, we have that the constructed solution $(\boldsymbol{u},\boldsymbol{f})$ satisfies (6) for all $\boldsymbol{\eta} \in \mathring{V}_2^{1,0}(\Omega_T)$.

Side condition

Next, we need to show that the function $\boldsymbol{u}$ satisfies the side condition as follows:

$$\int_\Omega |\boldsymbol{u}(x,t)|^2 dx = E^2(t)$$

for all $t\in[0,T]$. By the definition of $\boldsymbol{u}^{(N)}$ (see (7)) and properties of eigenfunctions (see (4)), for $N,M\in\mathbb{N}$ with $N\le M$, we obtain

$$\int_\Omega \boldsymbol{u}^{(N)}(x,t)\cdot\boldsymbol{u}^{(M)}(x,t)dx = \int_\Omega \left(\sum_{k=1}^{N} w_k(t)\boldsymbol{v}_k(x)\right)\cdot\left(\sum_{k=1}^{M} w_k(t)\boldsymbol{v}_k(x)\right)dx$$
$$= \sum_{k=1}^{N} w_k^2(t) \overset{(11)}{=} \frac{\sum_{k=1}^{N}\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} E^2(t).$$

After integrating this equality from 0 to an arbitrary (fixed) $t \in [0, T]$, we get

$$\int_0^t \int_\Omega \boldsymbol{u}^{(N)}(x,\tau) \cdot \boldsymbol{u}^{(M)}(x,\tau) dx d\tau = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t E^2(\tau) d\tau.$$

Now, we replace M by the subsequence N_l which we got earlier and take the limit as $l \to \infty$. For any $N \in \mathbb{N}$, the inequality $N \le N_l$ holds for sufficiently large l. Additionally, $\boldsymbol{u}^{(N_l)} \rightharpoonup \boldsymbol{u}$ in $L_2(\Omega_T)$. Thus, after passing to the limit, we obtain

$$\int_0^t \int_\Omega \boldsymbol{u}^{(N)}(x,\tau) \cdot \boldsymbol{u}(x,\tau) dx d\tau = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t E^2(\tau) d\tau.$$

The last equality is true for all $N \in \mathbb{N}$, so we can replace N by N_l and take the limit as $l \to \infty$. Recalling that $\sum_{k=1}^\infty \beta_k^2 = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$, we get

$$\int_0^t \|\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\Omega)} d\tau = \int_0^t E^2(\tau) d\tau. \tag{21}$$

The function $\boldsymbol{u}$ is from $C([0,T]; L_2(\Omega))$ due to the embedding $W_2^{0,1}(\Omega_T) \subset C([0,T]; L_2(\Omega))$ (see [18], Chapter 5.9.2). Therefore, $\|\boldsymbol{u}(\cdot,\tau)\|_{L_2(\Omega)}$ is a continuous function. Similarly, the function E is continuous on $[0,T]$ since $E \in W_2^1(0,T) \subset C[0,T]$. Hence, we can differentiate both sides of (21) by t and obtain

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega)} = E^2(t), \tag{22}$$

for all $t \in [0,T]$. Notice that (22) is equivalent to $\int_\Omega |\boldsymbol{u}(x,t)|^2 dx = E^2(t)$ for all $t \in [0,T]$.

Initial condition

Let $\boldsymbol{u}_0^{(N)} = \sum_{k=1}^N \beta_k \boldsymbol{v}_k$ denote the partial sum of the Fourier series of $\boldsymbol{u}_0$ with respect to the basis $\{\boldsymbol{v}_k\}$ in $J(\Omega)$. Clearly, $\boldsymbol{u}_0^{(N)} \to \boldsymbol{u}_0$ in $L_2(\Omega)$. Let $N, M \in \mathbb{N}$ with $N \le M$. Then, by (7) and the orthonormality of $\boldsymbol{v}_k$ in $L_2(\Omega)$, we obtain

$$\begin{aligned}
&\int_\Omega \left(\boldsymbol{u}^{(N)}(x,t) - \boldsymbol{u}_0^{(N)}(x)\right) \cdot \left(\boldsymbol{u}^{(M)}(x,t) - \boldsymbol{u}_0^{(M)}(x)\right) dx \\
&\quad = \int_\Omega \left(\sum_{k=1}^N w_k(t)\boldsymbol{v}_k(x) - \sum_{k=1}^N \beta_k \boldsymbol{v}_k(x)\right) \cdot \left(\sum_{k=1}^M w_k(t)\boldsymbol{v}_k(x) - \sum_{k=1}^M \beta_k \boldsymbol{v}_k(x)\right) dx \\
&\quad \overset{(11)}{=} \int_\Omega \left(\sum_{k=1}^N \frac{\beta_k \boldsymbol{v}_k(x)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \left(E(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}\right)\right) \cdot \left(\sum_{k=1}^M \frac{\beta_k \boldsymbol{v}_k(x)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \left(E(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}\right)\right) dx \\
&\quad = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} (E(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})^2.
\end{aligned}$$

Integrating both sides of the last equality from 0 to an arbitrary but fixed $t \in [0,T]$ yields

$$\begin{aligned}
\int_0^t \int_\Omega \left(\boldsymbol{u}^{(N)}(x,\tau) - \boldsymbol{u}_0^{(N)}(x)\right) \cdot \left(\boldsymbol{u}^{(M)}(x,\tau) - \boldsymbol{u}_0^{(M)}(x)\right) dx d\tau \\
= \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t (E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})^2 d\tau.
\end{aligned} \tag{23}$$

Now, we have that $\boldsymbol{u}^{(N_l)} \rightharpoonup \boldsymbol{u}$ in $L_2(\Omega_T)$; therefore, $\boldsymbol{u}^{(N_l)} - \boldsymbol{u}_0^{(N_l)} \rightharpoonup \boldsymbol{u} - \boldsymbol{u}_0$ in the space $L_2(\Omega_T)$. Therefore, if we replace M by N_l in (23) and pass to the limit as $l \to \infty$, we get that

https://doi.org/10.3390/math14091402

$$\int_0^t \int_\Omega \left(\boldsymbol{u}^{(N)}(x,\tau) - \boldsymbol{u}_0^{(N)}(x) \right) \cdot (\boldsymbol{u}(x,\tau) - \boldsymbol{u}_0(x)) dx d\tau = \frac{\sum_{k=1}^{N} \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t (E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})^2 d\tau.$$

Similarly, if we replace N by N_l and pass to the limit again as $l \to \infty$, we get

$$\int_0^t \int_\Omega (\boldsymbol{u}(x,\tau) - \boldsymbol{u}_0(x)) \cdot (\boldsymbol{u}(x,\tau) - \boldsymbol{u}_0(x)) dx d\tau = \int_0^t (E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})^2 d\tau,$$

where we have used that $\sum_{k=1}^{\infty} \beta_k^2 = \|\boldsymbol{u}_0\|^2_{L_2(\Omega)}$. Rewriting the final expression, we obtain

$$\int_0^t \|\boldsymbol{u}(\cdot,\tau) - \boldsymbol{u}_0\|^2_{L_2(\Omega)} d\tau = \int_0^t (E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})^2 d\tau.$$

Differentiating by t yields

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0\|^2_{L_2(\Omega)} = (E(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})^2, \quad \forall t \in [0,T]. \tag{24}$$

By taking $t = 0$ in (24) and using the condition $E(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$, we finally obtain that

$$\|\boldsymbol{u}(\cdot,0) - \boldsymbol{u}_0\|^2_{L_2(\Omega)} = 0$$

which is equivalent to $\boldsymbol{u}(x,\ 0) = \boldsymbol{u}_0(x)$ for almost all $x \in \Omega$.

4. Very Weak Solution

4.1. Definition of a Very Weak Solution

In this section, we consider problem (1), (3) in the case where the function E does not possess a weak derivative and belongs only to $L_2(0,T)$. Since a function in L_2 is defined only almost everywhere, the compatibility condition $E(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$ must be generalized. Specifically, it is understood in the following sense: $t = 0$ is a Lebesgue point of $E(t)$ with the value of $\|\boldsymbol{u}_0\|_{L_2(\Omega)}$. Equivalently,

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t |E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}| d\tau = 0. \tag{25}$$

Similarly, $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$ is understood to mean that $t = 0$ is a Lebesgue point of $\boldsymbol{u}(\cdot,t)$ with the value of $\boldsymbol{u}_0$, where $\boldsymbol{u}_0 \in L_2(\Omega)$ and $\boldsymbol{u}(\cdot,t) \in L_2(\Omega)$ for a.e. $t \in (0,T)$. Equivalently,

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t \|\boldsymbol{u}(\cdot,\tau) - \boldsymbol{u}_0\|_{L_2(\Omega)} d\tau = 0.$$

For the derivation of the very weak solution, we also need a generalized version of a primitive function. For any function $\boldsymbol{g} \in L_2(0,T)$, define its primitive $S_g(t) = \int_0^t \boldsymbol{g}(\tau) d\tau$. Then $\frac{\mathrm{d}S_g}{\mathrm{d}t} = \boldsymbol{g}$ almost everywhere and $S_g(0) = 0$. We extend this notion to a larger class of functions in the same way as in [15]. Denote by $\hat{W}_2^{0,1}(\Omega_T)$ the subspace $\{\boldsymbol{\phi} \in W_2^{0,1}(\Omega_T) : \boldsymbol{\phi}(\cdot,\ T) = 0\}$.

Lemma 1. *If $h \in (\hat{W}_2^{0,1}(\Omega_T))^*$, then there exists a unique $\boldsymbol{H} \in L_2(\Omega_T)$ such that*

$$\langle h, \boldsymbol{\eta} \rangle = \int_0^T \int_\Omega \boldsymbol{H}(x,t) \cdot \boldsymbol{\eta}_t(x,t) dx dt \quad \forall \boldsymbol{\eta} \in \hat{W}_2^{0,1}(\Omega_T). \tag{26}$$

Proof. The functional defined in (26) is bounded since, for any $\boldsymbol{H} \in L_2(\Omega_T)$,

$$|\langle h, \boldsymbol{\eta} \rangle| \le \|\boldsymbol{H}\|_{L_2(\Omega_T)} \|\boldsymbol{\eta}_t\|_{L_2(\Omega_T)} \le \|\boldsymbol{H}\|_{L_2(\Omega_T)} \|\boldsymbol{\eta}\|_{\hat{W}_2^{0,1}(\Omega_T)}.$$

Let $D : \hat{W}_2^{0,1}(\Omega_T) \to L_2(\Omega_T)$ be the derivative operator defined by $D\boldsymbol{\eta} = \boldsymbol{\eta}_t$. To show that D is surjective, let $\boldsymbol{\phi} \in L_2(\Omega_T)$ and set

$$\boldsymbol{\psi}(x,t) := -\int_t^T \boldsymbol{\phi}(x,\tau)d\tau.$$

Then $\boldsymbol{\psi} \in \hat{W}_2^{0,1}(\Omega_T)$, and since $\boldsymbol{\psi}$ is absolutely continuous in t,

$$\boldsymbol{\psi}_t(x,t) = \frac{\partial}{\partial t}\left(\int_0^t \boldsymbol{\phi}(x,\tau)d\tau - \int_0^T \boldsymbol{\phi}(x,\tau)d\tau\right) = \boldsymbol{\phi}(x,t).$$

The operator D is injective. Indeed, if $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \hat{W}_2^{0,1}(\Omega_T)$ satisfy $(\boldsymbol{\eta}_1)_t = (\boldsymbol{\eta}_2)_t$, then $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)_t = 0$, so $\boldsymbol{\eta}_1(x,t) - \boldsymbol{\eta}_2(x,t)$ is constant in time. Since, by definition of the space $\hat{W}_2^{0,1}(\Omega_T)$, we know that $\boldsymbol{\eta}_1(x,T) = \boldsymbol{\eta}_2(x,T) = 0$ for almost all $x \in \Omega$, it follows that $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$. Operator D is linear and bounded; hence, it is an isomorphism between $\hat{W}_2^{0,1}(\Omega_T)$ and $L_2(\Omega_T)$.

Let $h \in (\hat{W}_2^{0,1}(\Omega_T))^*$. Consider the linear functional F on $L_2(\Omega_T)$ defined by $F(\boldsymbol{\phi}) = \langle h, D^{-1}\boldsymbol{\phi}\rangle = (h \circ D^{-1})(\boldsymbol{\phi})$. Then F is bounded as D^{-1} is bounded. Therefore, by the Riesz representation theorem on $L_2(\Omega_T)$, there exists a unique $\boldsymbol{H} \in L_2(\Omega_T)$ such that

$$F(\boldsymbol{\phi}) = \langle h, D^{-1}\boldsymbol{\phi}\rangle = \int_0^T \int_\Omega \boldsymbol{H}(x,t) \cdot \boldsymbol{\phi}(x,t)dxdt.$$

For any $\boldsymbol{\eta} \in \hat{W}_2^{0,1}(\Omega_T)$, we have $\boldsymbol{\eta} = D^{-1}\boldsymbol{\eta}_t$, which implies

$$F(\boldsymbol{\eta}_t) = \langle h, \boldsymbol{\eta}\rangle = \int_0^T \int_\Omega \boldsymbol{H}(x,t) \cdot \boldsymbol{\eta}_t(x,t)dxdt.$$

□

Remark 1. *If $h \in (\hat{W}_2^{0,1}(\Omega_T))^*$, define $S_h := -\boldsymbol{H}$, where $\boldsymbol{H} \in L_2(\Omega_T)$ is the function associated with h by Lemma 1. This definition extends the classical notion of a primitive function. Indeed, if h corresponds via the Riesz representation theorem to a function $\boldsymbol{h} \in L_2(\Omega_T)$, i.e., $\langle h, \boldsymbol{\eta}\rangle = \int_0^T\int_\Omega \boldsymbol{h}(x,t) \cdot \boldsymbol{\eta}(x,t)dxdt$, then integration by parts gives*

$$\langle h, \boldsymbol{\eta}\rangle = \int_0^T \int_\Omega \frac{\partial S_{\boldsymbol{h}}(x,t)}{\partial t} \cdot \boldsymbol{\eta}(x,t)dxdt = -\int_0^T \int_\Omega S_{\boldsymbol{h}}(x,t) \cdot \boldsymbol{\eta}_t(x,t)dxdt,$$

using $S_{\boldsymbol{h}}(x,0) = 0$ and $\boldsymbol{\eta}(x,T) = 0$.

Let us define a very weak solution to problem (1), (3). Suppose that $\boldsymbol{u}$, $\boldsymbol{f}$, and p are sufficiently smooth. Multiplying Equation $(1)_1$ by an arbitrary function $\boldsymbol{\eta} \in C^\infty(\overline{\Omega_T})$ such that $\boldsymbol{\eta}\big|_{S_T} = 0$, $\operatorname{div}\boldsymbol{\eta} = 0$, $\boldsymbol{\eta}(\cdot,\ T) = 0$ and applying integration by parts with respect to x, we get identity (5). Unlike in the case of a weak solution, we integrate (5) with respect to time over $(0,T)$ and apply integration by parts to the first and last integrals. Thus, we get

$$\begin{aligned}
&\int_0^T\int_\Omega \boldsymbol{u}(x,t) \cdot \boldsymbol{\eta}_t(x,t)dxdt + \int_\Omega \boldsymbol{u}_0(x) \cdot \boldsymbol{\eta}(x,0)dx \\
&\qquad - \nu \int_0^T\int_\Omega \nabla\boldsymbol{u}(x,t) \cdot \nabla\boldsymbol{\eta}(x,t)dx = \int_0^T\int_\Omega S_{\boldsymbol{f}}(x,t) \cdot \boldsymbol{\eta}_t(x,t)dx.
\end{aligned} \tag{27}$$

By density arguments, (27) remains valid for all $\boldsymbol{\eta} \in \hat{V}_2^{1,1}(\Omega_T)$, where $\hat{V}_2^{1,1}(\Omega_T)$ is defined as the closure of the space $\{\boldsymbol{v} \in C^\infty(\overline{\Omega_T}) : \boldsymbol{v}\big|_{S_T} = 0,\ \boldsymbol{v}(\cdot,T) = 0,\ \operatorname{div}\boldsymbol{v} = 0\}$ in the $W_2^{1,1}(\Omega_T)$-norm.

Definition 2. *Suppose that $E \in L_2(0,T)$, $\boldsymbol{u}_0 \in W_2^2(\Omega) \cap H(\Omega)$, and $E(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$ in the sense of definition (25). A pair of functions $(\boldsymbol{u}, f) \in \mathring{V}_2^{1,0}(\Omega_T) \times (\hat{W}_2^{0,1}(\Omega_T))^*$ is called a very weak solution of problem (1), (3) if the integral identity (27) holds for all test functions $\boldsymbol{\eta} \in \hat{V}_2^{1,1}(\Omega_T)$ and if the initial and side conditions are satisfied as follows:*

$$\int_\Omega |\boldsymbol{u}(x,t)|^2 dx = E^2(t), \ \ \text{for a.e. } t \in [0,T], \tag{28}$$

$$\boldsymbol{u}(\cdot,\ 0) = \boldsymbol{u}_0 \text{ in the sense of Lebesgue points.} \tag{29}$$

The second goal of this paper is to prove the following theorem.

Theorem 2. *Suppose that $\boldsymbol{u}_0 \in W_2^2(\Omega) \cap H(\Omega)$, $E \in L_2(0,T)$ are given functions and E satisfies the compatibility condition $E(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$ in the sense of Lebesgue points. Then there exists at least one very weak solution $(\boldsymbol{u}, f)$ to problem (1), (3).*

4.2. Approximate Solution

We cannot define the approximate solution as in Section 3.1 since definition (12) involves the derivative of E. Therefore, we approximate E by a sequence $E_n \in W_2^1(0,T)$ such that $E_n \to E$ in $L_2(\Omega)$ and $E_n(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$. For every $N \in \mathbb{N}$ and $n \in \mathbb{N}$, we define approximate solutions $(\boldsymbol{u}_n^{(N)}, \boldsymbol{f}_n^{(N)})$ to problem (1), (3) by

$$\boldsymbol{u}_n^{(N)}(x,t) = \sum_{k=1}^N w_{k,n}(t)\boldsymbol{v}_k(x), \quad \boldsymbol{f}_n^{(N)}(x,t) = \sum_{k=1}^N q_{k,n}(t)\boldsymbol{v}_k(x), \tag{30}$$

where $\{\boldsymbol{v}_k\}_{k=1}^\infty$ is a basis in $J(\Omega)$ and $H(\Omega)$.

The coefficients $w_{k,n}$ and $q_{k,n}$ are determined by the system, which holds for a.e. $t \in (0,T)$ and $j = 1, \dots, N$, as follows:

$$\begin{cases} \displaystyle\int_\Omega (\boldsymbol{u}_n^{(N)})_t(x,t) \cdot \boldsymbol{v}_j(x)dx + \nu \int_\Omega \nabla \boldsymbol{u}_n^{(N)}(x,t) \cdot \nabla \boldsymbol{v}_j(x)dx = \int_\Omega \boldsymbol{f}_n^{(N)}(x,t) \cdot \boldsymbol{v}_j(x)dx, \\ \boldsymbol{u}_n^{(N)}(x,0) = \displaystyle\sum_{k=1}^N \beta_k \boldsymbol{v}_k(x), \quad x \in \Omega, \\ \displaystyle\int_\Omega \left|\boldsymbol{u}_n^{(N)}(x,t)\right|^2 dx = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} E_n^2(t), \end{cases} \tag{31}$$

where β_k, $k \in \mathbb{N}$, are the Fourier coefficients of the function $\boldsymbol{u}_0$. As in Section 3.2, we get the same expressions for $w_{k,n}$ and $q_{k,n}$, namely,

$$w_{k,n}(t) = \frac{\beta_k}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} E_n(t), \tag{32}$$

$$q_{k,n} = (w_{k,n})' + \lambda_k w_{k,n} \in L_2(0,T). \tag{33}$$

The function $E_n \in W_2^1(0,T)$ is continuous. Therefore, $\boldsymbol{u}_n^{(N)}$ is from the space $C([0,T]; L_2(\Omega))$ and $\boldsymbol{u}_n^{(N)}(\cdot,0)$ is well defined. Using that $E_n(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$, we obtain

$$\boldsymbol{u}_n^{(N)}(x,0) = \sum_{k=1}^N w_{k,n}(0)\boldsymbol{v}_k(x) = \frac{E_n(0)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \sum_{k=1}^N \beta_k \boldsymbol{v}_k(x) = \sum_{k=1}^N \beta_k \boldsymbol{v}_k(x).$$

Then the pair of functions $(\boldsymbol{u}_n^{(N)}, \boldsymbol{f}_n^{(N)})$ satisfies system (31).

Next, we get suitable estimates for $\boldsymbol{u}_n^{(N)}$ and $f_n^{(N)}$. The bounds for $\boldsymbol{u}_n^{(N)}$ and $\nabla \boldsymbol{u}_n^{(N)}$ we get by the same arguments as in the previous section (see (13) and (14)) as follows:

$$\|\boldsymbol{u}_n^{(N)}\|^2_{L_2(\Omega_T)} \le \|E_n\|^2_{L_2(0,T)}, \tag{34}$$

$$\|\nabla \boldsymbol{u}_n^{(N)}\|^2_{L_2(\Omega_T)} \le \frac{\|E_n\|^2_{L_2(0,T)}}{\nu \|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \sum_{k=1}^{N} \beta_k^2 \lambda_k. \tag{35}$$

Estimate (17) involves the derivative of E; thus, the sequence $\{f_n^{(N)}\}_{N=1}^{\infty}$ cannot be estimated in $L_2(\Omega_T)$. Instead, we get an estimate of the primitive function $S_{f_n^{(N)}}$ in the space $L_2(\Omega_T)$. First, integrate Equation (33) from 0 to t to get

$$\begin{aligned} S_{q_{k,n}}(t) &= \int_0^t (w_{k,n})'(\tau)d\tau + \lambda_k \int_0^t w_{k,n}(\tau)d\tau = w_{k,n}(t) - w_{k,n}(0) + \lambda_k \int_0^t w_{k,n}(\tau)d\tau \\ &\overset{(32)}{=} \frac{\beta_k}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} E_n(t) - \beta_k + \frac{\lambda_k \beta_k}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \int_0^t E_n(\tau)d\tau. \end{aligned}$$

We use the following estimate that we get by applying the Cauchy–Schwarz inequality:

$$\int_0^T \left| \int_0^t E_n(\tau)d\tau \right|^2 dt \le \int_0^T \left(\int_0^T |E_n(\tau)|d\tau \right)^2 dt \le T^2 \|E_n\|^2_{L_2(0,T)}.$$

This, together with the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, $a,b,c \in \mathbb{R}$, yields

$$\begin{aligned} \|S_{q_{k,n}}\|^2_{L_2(0,T)} &= \int_0^T |S_{q_{k,n}}(t)|^2 dt = \int_0^T \left(\frac{\beta_k}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} E_n(t) - \beta_k + \frac{\lambda_k \beta_k}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \int_0^t E_n(\tau)d\tau \right)^2 dt \\ &\le 3\left(\frac{\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^T E_n^2(t)dt + \beta_k^2 T + \frac{\lambda_k^2 \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^T \left| \int_0^t E_n(\tau)d\tau \right|^2 dt \right) \\ &\le \frac{3\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \|E_n\|^2_{L_2(0,T)} + 3\beta_k^2 T + \frac{3\lambda_k^2 \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} T^2 \|E_n\|^2_{L_2(0,T)}. \end{aligned}$$

Therefore, by the last inequality,

$$\begin{aligned} \|S_{f_n^{(N)}}\|^2_{L_2(\Omega_T)} &= \int_0^T \left\| \sum_{k=1}^{N} S_{q_{k,n}}(\tau) \boldsymbol{v}_k \right\|^2_{L_2(\Omega)} d\tau = \sum_{k=1}^{N} \int_0^T |S_{q_{k,n}}(\tau)|^2 d\tau \\ &\le \sum_{k=1}^{\infty} \left(\frac{3\beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \|E_n\|^2_{L_2(0,T)} + 3\beta_k^2 T + \frac{3\lambda_k^2 \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} T^2 \|E_n\|^2_{L_2(0,T)} \right) \\ &= 3\|E_n\|^2_{L_2(0,T)} + 3T\|\boldsymbol{u}_0\|^2_{L_2(\Omega)} + 3T^2 \frac{\|E_n\|^2_{L_2(0,T)}}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \sum_{k=1}^{\infty} \beta_k^2 \lambda_k^2. \end{aligned}$$

Inequality (18) gives the following bound:

$$\|S_{f_n^{(N)}}\|^2_{L_2(\Omega_T)} \le 3\|E_n\|^2_{L_2(0,T)} + 3T\|\boldsymbol{u}_0\|^2_{L_2(\Omega)} + 3T^2 \nu^2 \frac{\|\boldsymbol{u}_0\|^2_{W_2^2(\Omega)}}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}} \|E_n\|^2_{L_2(0,T)}. \tag{36}$$

Combining Equations (34), (35), and (19), we have

$$\begin{aligned} \|\boldsymbol{u}_n^{(N)}\|^2_{\mathring{V}_2^{1,0}(\Omega_T)} &= \|\boldsymbol{u}_n^{(N)}\|^2_{L_2(\Omega_T)} + \|\nabla \boldsymbol{u}_n^{(N)}\|^2_{L_2(\Omega_T)} \\ &\le \|E_n\|^2_{L_2(0,T)} + \frac{\|\boldsymbol{u}_0\|^2_{H(\Omega)}}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \|E_n\|^2_{L_2(0,T)}. \end{aligned} \tag{37}$$

By using these estimates, we conclude that the sequence $\{\boldsymbol{u}_n^{(N)}\}_{N=1}^{\infty}$ is bounded in the space $\mathring{V}_2^{1,0}(\Omega_T)$, while the sequence $\{S_{\boldsymbol{f}_n^{(N)}}\}_{N=1}^{\infty}$ is bounded in the space $L_2(\Omega_T)$.

4.3. Convergence

Since the sequence $\{\boldsymbol{u}_n^{(N)}\}_{N=1}^{\infty}$ is bounded in $\mathring{V}_2^{1,0}(\Omega_T)$ and the sequence $\{S_{\boldsymbol{f}_n^{(N)}}\}_{N=1}^{\infty}$ is bounded in $L_2(\Omega_T)$, there exists for each fixed $n \in \mathbb{N}$ a subsequence $\{N_l\}_{l=1}^{\infty}$ such that $\boldsymbol{u}_n^{(N_l)}$ weakly converges to $\boldsymbol{u}_n \in \mathring{V}_2^{1,0}(\Omega_T)$ and $S_{\boldsymbol{f}_n^{(N_l)}}$ weakly converges to $S_{f_n} \in L_2(\Omega_T)$ for some $f_n \in (\hat{W}_2^{0,1}(\Omega_T))^*$. We also have that $\boldsymbol{u}_n^{(N_l)} \rightharpoonup \boldsymbol{u}_n$ and $\nabla \boldsymbol{u}_n^{(N_l)} \rightharpoonup \nabla \boldsymbol{u}_n$ in $L_2(\Omega_T)$. Next, consider arbitrary functions $d_j \in C^1[0,T]$ with $d_j(T) = 0$, $k \in \mathbb{N}$. Multiply Equation (31)$_1$ by d_j and sum over all $j \in \{1, 2, \ldots, M\}$, where $M \leq N$. If we also denote $\boldsymbol{\eta}(x,t) = \sum_{k=1}^{M} d_k(t)\boldsymbol{v}_k(x)$ and integrate by t from 0 to T, we get the following integral identity:

$$\int_0^T\int_\Omega (\boldsymbol{u}_n^{(N)})_t(x,t)\cdot\boldsymbol{\eta}(x,t)dxdt + \nu\int_0^T\int_\Omega \nabla\boldsymbol{u}_n^{(N)}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt = \int_0^T\int_\Omega \boldsymbol{f}_n^{(N)}(x,t)\boldsymbol{\eta}(x,t)dxdt. \tag{38}$$

Then, we integrate by parts on the first and third integrals in Equation (38). Recalling that $\boldsymbol{u}_n^{(N)}(x,0) = \sum_{k=1}^{N}\beta_k\boldsymbol{v}_k(x)$ and that $\boldsymbol{\eta}(\cdot,\ T) \equiv 0$, we get

$$\begin{aligned}\int_\Omega \boldsymbol{u}_n^{(N)}(x,0)\cdot\boldsymbol{\eta}(x,0)dx + \int_0^T\int_\Omega \boldsymbol{u}_n^{(N)}(x,t)\boldsymbol{\eta}_t(x,t)dxdt \\ -\nu\int_0^T\int_\Omega \nabla\boldsymbol{u}_n^{(N)}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt = \int_0^T\int_\Omega S_{\boldsymbol{f}_n^{(N)}}(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt.\end{aligned} \tag{39}$$

We interpret the last three integrals in (39) as inner products in the space $L_2(\Omega_T)$ and use weak convergence to get that

$$\begin{aligned}&\int_0^T\int_\Omega \boldsymbol{u}_n^{(N_l)}(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T\int_\Omega \boldsymbol{u}_n(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt,\\ &\int_0^T\int_\Omega \nabla\boldsymbol{u}_n^{(N_l)}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T\int_\Omega \nabla\boldsymbol{u}_n(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt,\\ &\int_0^T\int_\Omega S_{\boldsymbol{f}_n^{(N_l)}}(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt \xrightarrow{l\to\infty} \int_0^T\int_\Omega S_{f_n}(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt.\end{aligned}$$

Since $\sum_{k=1}^{N}\beta_k\boldsymbol{v}_k \to \boldsymbol{u}_0$ in $L_2(\Omega)$ as $N \to \infty$, we can pass to the limit in the first integral on the left-hand side of (39), and we get

$$\begin{aligned}\int_\Omega \boldsymbol{u}_0(x)\cdot\boldsymbol{\eta}(x,0)dx + \int_0^T\int_\Omega \boldsymbol{u}_n(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt \\ -\nu\int_0^T\int_\Omega \nabla\boldsymbol{u}_n(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt = \int_0^T\int_\Omega S_{f_n}(x,t)\cdot\boldsymbol{\eta}_t(x,t)dxdt,\end{aligned} \tag{40}$$

where $\boldsymbol{\eta}$ is a linear combination $\sum_{k=1}^{M} d_k(t)\boldsymbol{v}_k(x)$. The number M in the sum can be arbitrarily large because $N_l \to \infty$ when $l \to \infty$. Since the set of all these linear combinations is dense in the space $\hat{V}_2^{1,1}(\Omega_T)$ (see Theorem 2.3.8 in [19]), by approximation, we get that integral identity (40) holds for all $\boldsymbol{\eta} \in \hat{V}_2^{1,1}(\Omega_T)$.

Next, we have to pass to the limit as $n \to \infty$. For each $n \in \mathbb{N}$, we have

$$\|\boldsymbol{u}_n\|_{\mathring{V}_2^{1,0}(\Omega_T)} \leq \liminf_{l\to\infty}\|\boldsymbol{u}_n^{(N_l)}\|_{\mathring{V}_2^{1,0}(\Omega_T)}, \quad \|S_{f_n}\|_{L_2(\Omega_T)} \leq \liminf_{l\to\infty}\|S_{\boldsymbol{f}_n^{(N)}}\|_{L_2(\Omega_T)}.$$

Additionally, $E_n \to E$ in $L_2(0,T)$. Therefore, from (36) and (37), we get that $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$ is bounded in $\mathring{V}_2^{1,0}(\Omega_T)$ and $\{S_{f_n}\}_{n=1}^{\infty}$ is bounded in $L_2(\Omega_T)$. Thus, there exists a subsequence

$\{n_l\}_{l=1}^{\infty}$ such that $\boldsymbol{u}_{n_l} \rightharpoonup \boldsymbol{u}$ and $S_{f_{n_l}} \rightharpoonup S_f$ for some $\boldsymbol{u} \in \mathring{V}_2^{1,0}(\Omega_T)$ and $f \in (\hat{W}_2^{0,1}(\Omega_T))^*$. Finally, after passing to the limit in (40) as $n_l \to \infty$, we get

$$\int_\Omega \boldsymbol{u}_0(x) \cdot \boldsymbol{\eta}(x,0)dx + \int_0^T\int_\Omega \boldsymbol{u}(x,t) \cdot \boldsymbol{\eta}_t(x,t)dxdt \\ - \nu \int_0^T\int_\Omega \nabla \boldsymbol{u}(x,t) \cdot \nabla \boldsymbol{\eta}(x,t)dxdt = \int_0^T\int_\Omega S_f(x,t) \cdot \boldsymbol{\eta}_t(x,t)dxdt$$

for all $\boldsymbol{\eta} \in \hat{V}_2^{1,1}(\Omega_T)$.

Side condition

By the definition of $\boldsymbol{u}_n^{(N)}$ (see (30)) and properties of eigenfunctions (see (4)), we obtain that for $N, M \in \mathbb{N}$, $N \leq M$, for all $n, m \in \mathbb{N}$, and for almost all $t \in (0,T)$ holds the following equality:

$$\int_\Omega \boldsymbol{u}_n^{(N)}(x,t) \cdot \boldsymbol{u}_m^{(M)}(x,t)dx = \int_\Omega \left(\sum_{k=1}^N w_{k,n}(t)\boldsymbol{v}_k(x)\right) \cdot \left(\sum_{k=1}^M w_{k,m}(t)\boldsymbol{v}_k(x)\right)dx \\ = \sum_{k=1}^N w_{k,n}(t)w_{k,m}(t) = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} E_n(t)E_m(t).$$

Integrate this last equality from 0 to an arbitrary but fixed $t \in [0,T]$ to get

$$\int_0^t \int_\Omega \boldsymbol{u}_n^{(N)}(x,\tau) \cdot \boldsymbol{u}_m^{(M)}(x,\tau)dxd\tau = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t E_n(\tau)E_m(\tau)d\tau. \tag{41}$$

Since there exists a subsequence M_l such that $\boldsymbol{u}_m^{(M_l)} \rightharpoonup \boldsymbol{u}_m$ in $L_2(\Omega_T)$, after replacing M by M_l in (41), we can pass to the limit as $l \to \infty$, provided that $N \leq M_l$ for sufficiently large l. Then, for all $N \in \mathbb{N}$,

$$\int_0^t \int_\Omega \boldsymbol{u}_n^{(N)}(x,\tau) \cdot \boldsymbol{u}_m(x,\tau)dxd\tau = \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t E_n(\tau)E_m(\tau)d\tau. \tag{42}$$

Similarly, there exists a subsequence N_l such that $\boldsymbol{u}_n^{(N_l)} \rightharpoonup \boldsymbol{u}_n$ in $L_2(\Omega_T)$, so after replacing N by N_l in (42) and passing to the limit as $l \to \infty$, we have

$$\int_0^t \int_\Omega \boldsymbol{u}_n(x,\tau) \cdot \boldsymbol{u}_m(x,\tau)dxd\tau = \frac{\sum_{k=1}^\infty \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t E_n(\tau)E_m(\tau)d\tau = \int_0^t E_n(\tau)E_m(\tau)d\tau, \tag{43}$$

since $\sum_{k=1}^\infty \beta_k^2 = \|\boldsymbol{u}_0\|^2_{L_2(\Omega)}$. Next, we take the limit as $n, m \to \infty$ in a similar way. It does not matter in which order we take the limit as the result will be the same. First, we already have that there is a subsequence n_l such that $\boldsymbol{u}_{n_l} \rightharpoonup \boldsymbol{u}$ in $L_2(\Omega_T)$. We also have that $E_{n_l} \rightharpoonup E$ in $L_2(0,T)$. Thus, if we replace n by n_l in (43) and pass to the limit as $l \to \infty$,

$$\int_0^t \int_\Omega \boldsymbol{u}(x,\tau) \cdot \boldsymbol{u}_m(x,\tau)dxd\tau = \int_0^t E(\tau)E_m(\tau)d\tau. \tag{44}$$

Lastly, if we replace m by n_l in (44) and pass to the limit as $l \to \infty$,

$$\int_0^t \int_\Omega \boldsymbol{u}(x,\tau) \cdot \boldsymbol{u}(x,\tau)dxd\tau = \int_0^t E(\tau)E(\tau)d\tau.$$

Thus, by differentiating this equality by t, we see that, for a.e. $t \in (0,T)$, side condition (28) is satisfied as follows:
$$\int_\Omega |\boldsymbol{u}(x,t)|^2 dx = E^2(t).$$

Initial condition

Denote $\boldsymbol{u}_0^{(N)} = \sum_{k=1}^N \beta_k \boldsymbol{v}_k$ to be the partial sum of the Fourier series for $\boldsymbol{u}_0$ in $H(\Omega)$. Then it is clear that $\boldsymbol{u}_0^{(N)} \to \boldsymbol{u}_0$ in $L_2(\Omega)$ and $\boldsymbol{u}_0^{(N)} \to \boldsymbol{u}_0$ in $L_2(\Omega_T)$ as well. Suppose that $N, M \in \mathbb{N}$, $n, m \in \mathbb{N}$ and that $N \le M$. Then from (30) and from properties of eigenfunctions, we obtain

$$\begin{aligned}
&\int_\Omega \left(\boldsymbol{u}_n^{(N)}(x,t) - \boldsymbol{u}_0^{(N)}(x)\right) \cdot \left(\boldsymbol{u}_m^{(M)}(x,t) - \boldsymbol{u}_0^{(M)}(x)\right) dx \\
&= \int_\Omega \left(\sum_{k=1}^N w_{k,n}(t)\boldsymbol{v}_k(x) - \sum_{k=1}^N \beta_k \boldsymbol{v}_k(x)\right) \cdot \left(\sum_{k=1}^M w_{k,m}(t)\boldsymbol{v}_k(x) - \sum_{k=1}^M \beta_k \boldsymbol{v}_k(x)\right) dx \\
&= \int_\Omega \left(\sum_{k=1}^N \frac{\beta_k \boldsymbol{v}_k(x)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}}\left(E_n(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}\right)\right) \cdot \left(\sum_{k=1}^M \frac{\beta_k \boldsymbol{v}_k(x)}{\|\boldsymbol{u}_0\|_{L_2(\Omega)}}\left(E_m(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}\right)\right) dx \\
&= \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}}(E_n(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})(E_m(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}).
\end{aligned}$$

Integrating the last identity from 0 to an arbitrary t gives us

$$\begin{aligned}
&\int_0^t \int_\Omega \left(\boldsymbol{u}_n^{(N)}(x,\tau) - \boldsymbol{u}_0^{(N)}(x)\right) \cdot \left(\boldsymbol{u}_m^{(M)}(x,\tau) - \boldsymbol{u}_0^{(M)}(x)\right) dx d\tau \\
&= \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t (E_n(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})(E_m(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}) d\tau.
\end{aligned} \tag{45}$$

Now, pick a subsequence M_l, $l \in \mathbb{N}$, such that $\boldsymbol{u}_m^{(M_l)} \rightharpoonup \boldsymbol{u}_m$ in $L_2(\Omega_T)$. We also know that $\boldsymbol{u}_0^{(M_l)}$ strongly converges to $\boldsymbol{u}_0$ in $L_2(\Omega_T)$; thus, it also weakly converges. Using this, we can replace M with M_l in Equation (45) and pass to the limit as $l \to \infty$ to get

$$\begin{aligned}
&\int_0^t \int_\Omega (\boldsymbol{u}_n^{(N)}(x,\tau) - \boldsymbol{u}_0^{(N)}(x)) \cdot (\boldsymbol{u}_m(x,\tau) - \boldsymbol{u}_0(x)) dx d\tau \\
&= \frac{\sum_{k=1}^N \beta_k^2}{\|\boldsymbol{u}_0\|^2_{L_2(\Omega)}} \int_0^t (E_n(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})(E_m(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}) d\tau.
\end{aligned} \tag{46}$$

Similarly, we can find a subsequence N_l, $l \in \mathbb{N}$, such that $\boldsymbol{u}_n^{(N_l)} \rightharpoonup \boldsymbol{u}_n$ in $L_2(\Omega_T)$. Then $\boldsymbol{u}_n^{(N_l)} - \boldsymbol{u}_0^{(N_l)} \rightharpoonup \boldsymbol{u}_n - \boldsymbol{u}_0$ in $L_2(\Omega_T)$. Replace N with N_l in (46) and pass to the limit as $l \to \infty$ to get

$$\begin{aligned}
&\int_0^t \int_\Omega (\boldsymbol{u}_n(x,\tau) - \boldsymbol{u}_0(x)) \cdot (\boldsymbol{u}_m(x,\tau) - \boldsymbol{u}_0(x)) dx d\tau \\
&\qquad = \int_0^t (E_n(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})(E_m(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}) d\tau.
\end{aligned}$$

Finally, choose a subsequence n_l, $l \in \mathbb{N}$, such that $\boldsymbol{u}_{n_l} \rightharpoonup \boldsymbol{u}$ in $L_2(\Omega_T)$. Then since we also have that $E_{n_l} \rightharpoonup E$ in $L_2(0,T)$, we can replace n with n_l and pass to the limit as follows:

$$\begin{aligned}
&\int_0^t \int_\Omega (\boldsymbol{u}(x,\tau) - \boldsymbol{u}_0(x)) \cdot (\boldsymbol{u}_m(x,\tau) - \boldsymbol{u}_0(x)) dx d\tau \\
&\qquad = \int_0^t (E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})(E_m(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}) d\tau.
\end{aligned}$$

https://doi.org/10.3390/math14091402

Similarly, replacing m by n_l and passing to the limit, we get

$$\int_0^t \int_\Omega (\boldsymbol{u}(x,\tau) - \boldsymbol{u}_0(x)) \cdot (\boldsymbol{u}(x,\tau) - \boldsymbol{u}_0(x)) dx d\tau$$
$$= \int_0^t (E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)})(E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}) d\tau.$$

Differentiating this equality by t and taking the square root of both sides shows that

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0\|_{L_2(\Omega)} = |E(t) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}| \tag{47}$$

for a.e. $t \in (0,T)$. Recall that we have assumed that $E(0) = \|\boldsymbol{u}_0\|_{L_2(\Omega)}$ in terms of Lebesgue points; thus,

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t |E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}| d\tau = 0.$$

Hence, if we integrate Equation (47) from 0 to an arbitrary but positive t, divide both sides by t and take the limit as $t \to 0^+$ as follows:

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t \|\boldsymbol{u}(\cdot,\tau) - \boldsymbol{u}_0\|_{L_2(\Omega)} d\tau = \lim_{t\to 0^+} \frac{1}{t} \int_0^t |E(\tau) - \|\boldsymbol{u}_0\|_{L_2(\Omega)}| d\tau = 0.$$

This shows that initial condition (29) is satisfied in the sense of definition (25).

5. Discussion

We show that when $\boldsymbol{u}_0 \equiv 0$, there exist multiple weak and very weak solutions. Since the arguments are similar, we only show the existence of very weak solutions.

Suppose that $\boldsymbol{u}_0 \equiv 0$, $E \in L_2(0,T)$ and $\{E_n\}_{n=1}^\infty$ is a sequence of $W_2^1(0,T)$ functions converging to E in L^2-norm. We find approximate solutions $(\boldsymbol{u}^{(N)}, \boldsymbol{f}^{(N)})$ represented in the forms (30), i.e.,

$$\boldsymbol{u}_n^{(N)}(x,t) = \sum_{k=1}^N w_{k,n}(t)\boldsymbol{v}_k(x), \quad \boldsymbol{f}_n^{(N)}(x,t) = \sum_{k=1}^N q_{k,n}(t)\boldsymbol{v}_k(x),$$

that satisfy the following system for a.e. $t \in (0,T)$ and $k \le N$:

$$\begin{cases} \int_\Omega (\boldsymbol{u}_n^{(N)})_t(x,t) \cdot \boldsymbol{v}_k(x) dx + \nu \int_\Omega \nabla \boldsymbol{u}_n^{(N)}(x,t) \cdot \nabla \boldsymbol{v}_k(x) dx = \int_\Omega \boldsymbol{f}_n^{(N)}(x,t)\boldsymbol{v}_k(x) dx, \\ \boldsymbol{u}_n^{(N)}(x,0) = 0, \quad x \in \Omega, \\ \int_\Omega \left|\boldsymbol{u}_n^{(N)}(x,t)\right|^2 dx = \gamma_k^2 E_n^2(t), \end{cases} \tag{48}$$

where $\{\gamma_k\}_{k=1}^\infty$ is any sequence, such that $\sum_{k=1}^\infty \gamma_k = 1$ and such that the series $\sum_{k=1}^\infty \gamma_k^2 \lambda_k^2$ converges. From (48), we get the following expressions for $w_{k,n}$ and $q_{k,n}$:

$$w_{k,n}(t) = \gamma_k E_n(t), \quad q_{k,n}(t) = w'_{k,n} + \lambda_k w_{k,n}.$$

The estimates for $\boldsymbol{u}_n^{(N)}$ are similar to (34) and (35) as follows:

$$\|\boldsymbol{u}_n^{(N)}\|^2_{L_2(\Omega_T)} \le \|E_n\|^2_{L_2(0,T)}, \quad \|\nabla \boldsymbol{u}_n^{(N)}\|^2_{L_2(\Omega_T)} \le \frac{\|E_n\|^2_{L_2(0,T)}}{\nu} \sum_{k=1}^N \gamma_k^2 \lambda_k \le C\|E_n\|^2_{L_2(0,T)}$$

for some positive constant C depending on ν and the sequence $\{\gamma_k\}_{k=1}^{\infty}$. This shows that $\boldsymbol{u}_n^{(N)}$ is from the space $\mathring{V}_2^{1,0}(\Omega_T)$. As for $\boldsymbol{f}_{k,n}$, we get

$$S_{q_{k,n}}(t) = w_{k,n}(t) - \overbrace{w_{k,n}(0)}^{0} + \lambda_k \int_0^t w_{k,n}(\tau)d\tau = \gamma_k E_n(t) + \lambda_k \gamma_k \int_0^t E_n(\tau)d\tau$$

and

$$\begin{aligned}\|S_{q_{k,n}}\|^2_{L_2(0,T)} &= \int_0^T |S_{q_{k,n}}(t)|^2 dt = \int_0^T \left(\gamma_k E_n(t) + \lambda_k \gamma_k \int_0^t E_n(\tau)d\tau\right)^2 dt \\ &\leq 2\left(\gamma_k^2 \int_0^T E_n^2(t)dt + \lambda_k^2 \gamma_k^2 \int_0^T \left|\int_0^t E_n(\tau)d\tau\right|^2 dt\right) \\ &\leq 2\gamma_k^2 \|E_n\|^2_{L_2(0,T)} + 2\lambda_k^2 \gamma_k^2 T^2 \|E_n\|^2_{L_2(0,T)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|S_{\boldsymbol{f}_n^{(N)}}\|^2_{L_2(\Omega_T)} &= \sum_{k=1}^{N} \int_0^T |S_{q_{k,n}}(\tau)|^2 d\tau \leq \sum_{k=1}^{\infty} \left(2\gamma_k^2 \|E_n\|^2_{L_2(0,T)} + 2\lambda_k^2 \gamma_k^2 T^2 \|E_n\|^2_{L_2(0,T)}\right) \\ &= 2(1 + C^* T^2)\|E_n\|^2_{L_2(0,T)},\end{aligned}$$

for some positive constant C^* depending on the sequence $\{\gamma_k\}_{k=1}^{\infty}$. This shows that for a fixed $n \in \mathbb{N}$, the sequence $\{S_{\boldsymbol{f}_n^{(N)}}\}_{N=1}^{\infty}$ is bounded in $L_2(\Omega_T)$. Once we have these bounds, we can prove the existence of very weak solutions in a similar way as in Section 3.3.

Thus, in the case $\boldsymbol{u}_0 \equiv 0$, the sequence $\{\gamma_k\}_{k=1}^{\infty}$ may be chosen arbitrarily, whereas for $\boldsymbol{u}_0 \neq 0$, the sequence $\{\gamma_k\}_{k=1}^{\infty}$ is fixed. Therefore, when $\boldsymbol{u}_0 \equiv 0$, there exist infinitely many solutions, and uniqueness does not hold in this case. For nonhomogeneous initial data, the uniqueness of solution remains an open problem.

Remark 2. *The assumption $\boldsymbol{u}_0 \in W_2^2(\Omega) \cap H(\Omega)$ is required to obtain estimates for the approximate solutions. In particular, this regularity condition allows us to control the series associated with the Fourier coefficients β_k. Moreover, it guarantees that* $\operatorname{div} \boldsymbol{u}_0 = 0$ *and* $\boldsymbol{u}_0|_{\partial\Omega} = 0$.

6. Conclusions

In this paper, we constructed weak and very weak solutions to the nonstationary Stokes problem with a nonlinear integral overdetermination condition as follows:

$$\int_\Omega |\boldsymbol{u}(x,t)|^2 dx = E^2(t).$$

We have considered two cases, depending on the regularity of the function E, i.e., when the kinetic energy E belongs to $W_2^1(0,T)$ and when E is only from $L_2(0,T)$. For the case when $E \in W_2^1(0,T)$, we constructed the classical weak solutions, while when E belongs only to $L_2(0,T)$, we used a different approach and constructed very weak solutions.
Due to the low regularity of the energy function E and the nonlinearity of the integral overdetermination condition, the question of uniqueness in the case of nonhomogeneous initial data remains open. In the case of zero initial data, uniqueness does not hold.

Author Contributions: Conceptualization, V.B. and K.K.; methodology, V.B. and K.K.; formal analysis, V.B. and K.K.; investigation, V.B. and K.K.; writing—original draft, V.B.; writing—review and editing, K.K.; supervision, K.K. All authors have read and agreed to the published version of the manuscript.

Funding: The research of K. Kaulakytė has received funding from the Research Council of Lithuania (LMTLT), Agreement No. S-MIP-23-43.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ladyzhenskaya, O.A. *The Mathematical Theory of Viscous Incompressible Flow*; Gordon and Breach: New York, NY, USA, 1969.
2. Cannon, J.R. *The One-Dimensional Heat Equation*; Encyclopedia of Mathematics and Its Applications; Cambridge University Press: Cambridge, UK, 1984.
3. Cannon, J.R. The solution of the heat equation subject to the specification of energy. *Q. Appl. Math.* **1963**, *21*, 155–160. [CrossRef]
4. Galdi, G.P.; Pileckas, K.; Silvestre, A.L. On the unsteady Poiseuille flow in a pipe. *Z. Angew. Math. Phys.* **2007**, *58*, 994–1007. [CrossRef]
5. Kamynin, V.L.; Vasin, I.A. Inverse problems for linearized Navier-Stokes equations with integral overdetermination. Unique solvability and passage to the limit. *Ann. Univ. Ferrara* **1992**, *38*, 229–247. [CrossRef]
6. Kumar, P.; Mohan, M.T. Existence, uniqueness and stability of an inverse problem for two-dimensional convective Brinkman–Forchheimer equations with the integral overdetermination. *Banach J. Math. Anal.* **2022**, *16*, 58. [CrossRef]
7. Pileckas, K.; Keblikas, V. Existence of a Nonstationary Poiseuille Solution. *Sib. Math. J.* **2005**, *46*, 514–526. [CrossRef]
8. Pileckas, K. Existence of Solutions with the Prescribed Flux of the Navier–Stokes System in an Infinite Cylinder. *J. Math. Fluid Mech.* **2006**, *8*, 542–563. [CrossRef]
9. Pileckas, K. On the Behavior of a Nonstationary Poiseuille Solution as $t \to \infty$. *Sib. Math. J.* **2005**, *46*, 707–716. [CrossRef]
10. Pileckas, K. Solvability in weighted spaces of the three-dimensional Navier-Stokes problem in domains with cylindrical outlets to infinity. *Topol. Methods Nonlinear Anal.* **2007**, *29*, 333–360.
11. Pileckas, K. The Navier–Stokes System in Domains with Cylindrical Outlets to Infinity. Leray's Problem. In *Handbook of Mathematical Fluid Dynamics*; Friedlander, S., Serre, D., Eds.; Elsevier: Amsterdam, The Netherlands, 2007; Volume 4, pp. 445–647. [CrossRef]
12. Vasin, I.A. The existence and uniqueness of the generalized solution of the inverse problem for the nonlinear nonstationary Navier-Stokes system in the case of integral overdetermination. *Math. Notes* **1993**, *54*, 1002–1009. [CrossRef]
13. Hazanee, A.; Ismailov, M.I.; Lesnic, D.; Kerimov, N.B. An inverse time-dependent source problem for the heat equation. *Appl. Numer. Math.* **2013**, *69*, 13–33. [CrossRef]
14. Kaulakytė, K.; Kozulinas, N.; Pileckas, K. Time-periodic Poiseuille-type solution with minimally regular flow rate. *Nonlinear Anal. Model. Control* **2021**, *26*, 947–968. [CrossRef]
15. Pileckas, K.; Čiegis, R. Existence of nonstationary Poiseuille-type solutions under minimal regularity assumptions. *Z. Angew. Math. Phys.* **2020**, *71*, 192. [CrossRef]
16. Belickas, T.; Kaulakytė, K.; Puriuškis, G. Nonstationary heat equation with nonlinear side condition. *Math. Model. Anal.* **2025**, *30*, 109–119. [CrossRef]
17. Pileckas, K.; Korobkov, M.; Russo, R. *The Steady Navier-Stokes System*; Advances in Mathematical Fluid Mechanics; Birkhäuser Cham: Cham, Switzerland, 2024. [CrossRef]
18. Evans, L.C. *Partial Differential Equations*, 2nd ed.; American Mathematical Society: Providence, RI, USA, 2010.
19. Pileckas, K.; Panasenko, G. *Multiscale Analysis of Viscous Flows in Thin Tube Structures*; Advances in Mathematical Fluid Mechanics; Birkhäuser: Cham, Switzerland, 2024. [CrossRef]