axioms

Article

The Discrete Value Distribution of the Modified Mellin Transform of the Fourth Power of the Riemann Zeta-Function

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Abstract

Let $\mathcal{Z}_2(s)$ denote the modified Mellin transforms of the modulus of the fourth power of the Riemann zeta-function. This paper is devoted to the probabilistic properties of generalized discrete shifts $\mathcal{Z}_2(s+i\psi(k))$, $k \in \mathbb{N}$, with a certain differentiable function $\psi(\tau)$ satisfying some estimate connected to the mean square of the function $\mathcal{Z}_2(s)$ and such that the sequence $\{\kappa\psi(k) : k \in \mathbb{N}\}$ is uniformly distributed modulo 1 with every $\kappa \in \mathbb{R} \setminus \{0\}$. We propose the condition that $\mathcal{Z}_2(s+i\psi(k))$ in the space of analytic functions has a limit distribution concentrated at the point $g_0(s) \equiv 0$. Such a limit theorem is applied for the approximation of the function $g_0(s)$.

Keywords: Mellin transform; probability measure; Riemann zeta-function; space of analytic functions; weak convergence

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1. Introduction

Let $s = \sigma + it$ denote a complex variable, and let the function $|g(u)|u^{\sigma-1} \in L(0,\infty)$. The Mellin transform $M_g(s)$ of $g(u)$ is defined by

$$M_g(s) = \int_0^\infty g(u)u^{s-1}\,\mathrm{d}u$$

and is a useful tool in function theory. Sometimes, it is more convenient to study the Mellin transform of a considered function $g(u)$ and then, using the inverse formula, continue the investigation of $g(u)$.

In analytic number theory, for the investigation of the moments of zeta-functions, so-called modified Mellin transforms are applied. They were introduced in [1]; see also [2]. Suppose that $g(u)u^{-\sigma} \in L(1,\infty)$. Then the modified Mellin transform $\widetilde{M}_g(s)$ of $g(u)$ is given by

$$\widetilde{M}_g(s) = \int_1^\infty g(u)u^{-s}\,\mathrm{d}u. \quad (1)$$

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Let

$$\widetilde{g}(u) = \begin{cases} g(u^{-1}) & \text{if } 0 < u \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easily seen that [3]

$$\widetilde{M}_g(s) = M_{u^{-1}\widetilde{g}(u)}(s).$$

Moreover, $\widetilde{M}_g(s)$ has a certain advantage against $M_g(s)$ because in (1), a convergence problem at the point $u = 0$ does not exist.

The first applications of modified Mellin transforms were devoted to the Riemann zeta-function $\zeta(s)$. We recall that in the half-plane $\sigma > 1$, the function $\zeta(s)$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

or, equivalently, by the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

over all prime numbers p. Moreover, $\zeta(s)$ has analytic continuation to the whole complex plane, except for a point $s = 1$ which is its simple pole, $\mathrm{Res}_{s=1}\zeta(s) = 1$. We notice that $\zeta(s)$ is a symmetric analytic object: it satisfies the symmetric functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ denotes the Euler gamma-function, and this is reflected in its value distribution.

For modified Mellin transforms of $|\zeta(1/2+it)|^{2k}$, the notation $\mathcal{Z}_k(s)$ is used, i.e.,

$$\mathcal{Z}_k(s) = \int_1^{\infty} \left|\zeta\left(\frac{1}{2} + iu\right)\right|^{2k} u^{-s}\,\mathrm{d}u, \quad \sigma > 1.$$

The function $\mathcal{Z}_2(s)$ was introduced and studied in [1] for the needs of the error term $E_2(T)$ in the formula

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 \mathrm{d}t = TP_4(\log T) + E_2(T),$$

where P_4 is a known polynomial of degree four. There exists a conjecture that

$$E_2(T) \ll_\varepsilon T^{1/2+\varepsilon}.$$

Here and further the notation $a \ll_\theta b$, $a \in \mathbb{C}$, $b > 0$, indicates that there exists a constant $C = C(\theta)$ satisfying $|a| \leqslant Cb$. In [1], for the investigation of $E_2(T)$, the meromorphic continuation of $\mathcal{Z}_2(s)$ was obtained. Let ρ denote the non-trivial zeros of the Riemann zeta-function. The definitions from the theory of automorphic forms can be found in [4]. Thus, in [1], in Theorem 2, it was proven that the function $\mathcal{Z}_2(s)$ is meromorphic over the entire complex plane. More precisely, in the half-plane $\sigma > 0$, $\mathcal{Z}_2(s)$ has a pole of order five at the point $s = 1$ and infinitely many simple poles of the form $1/2 \pm \kappa i$, where $\kappa^2 + 1/4$ is in the discrete spectrum of the hyperbolic Laplacian with respect to the full modular group and of the form $\rho/2$. From this, it was derived in [1] that

$$E_2(T) = \Omega_{\pm}(\sqrt{T}).$$

We recall that the latter equality means that

$$E_2(T) = \Omega_+(\sqrt{T}) \quad \text{and} \quad E_2(T) = \Omega_-(\sqrt{T})$$

are satisfied. The first equality indicates that there exists a constant $C > 0$ such that $E_2(T) > C\sqrt{T}$ holds for a sequence $T = T_n$: $T_n \to \infty$, while the second shows that there exists $C > 0$ such that $E_2(T) < -C\sqrt{T}$ is true for a sequence $T = T_n$: $T_n \to \infty$.

The study of the function $\mathcal{Z}_2(s)$ was continued in [3,5–7], where its estimates and estimates for the mean square

$$J_T(\sigma) \overset{\text{def}}{=} \int\limits_0^T |\mathcal{Z}_2(\sigma + it)|^2 \, \mathrm{d}t$$

were given. Namely, in [7], the bound

$$\mathcal{Z}_2(\sigma + it) \ll t^{2-\sigma} (\log t)^{18-14\sigma} \tag{2}$$

was found for all $1/2 < \sigma < 1$ and $t > t_0 > 0$.

More attention was devoted to the quantity $J_T(\sigma)$. It was conjectured in [3] that the estimate

$$J_T(\sigma) \ll_\varepsilon T^{2-2\sigma+\varepsilon}$$

is true for $1/2 < \sigma < 1$. In [5], for $1/2 < \sigma < 1$, the bound

$$J_T(\sigma) \ll T^{(10-8\sigma)/3} \log^c T$$

is given with a certain $c > 0$. Finally, in [6], it was obtained that the bound

$$J_T(\sigma) \ll_\varepsilon T^{(15-12\sigma)/5+\varepsilon} \tag{3}$$

holds for $5/6 \leqslant \sigma \leqslant 5/4$.

In the present paper, we continue the investigations of [8], where the value distribution of the function $\mathcal{Z}_2(s)$ was characterized by using probabilistic language. H. Bohr was the first person who began to study functions defined by Dirichlet series via probabilistic methods. His idea is very simple: for a given set, consider how often the values of the considered function belong to this set. It turned out that Bohr's idea is productive, and the value distribution of some functions can be described by strong probabilistic laws. The first results of the probabilistic type were obtained for the Riemann zeta-function, and they confirmed Bohr's approach. In [9], it was proven that for $\sigma > 1$, the following limit exists:

$$\lim_{T \to \infty} \frac{1}{T} \mathfrak{J}\{t \in [0, T] : \log \zeta(\sigma + it) \in \mathcal{R}\},$$

where $\mathfrak{J}$ is the Jordan measure on the real line, and $\mathcal{R}$ is a rectangle on $\mathbb{C}$ with edges parallel to the axes. In [10], Bohr and Jessen extended their theorem to the half-plane $\sigma > 1/2$.

In the middle of the 20th century, the theory of the weak convergence of probability measures was built, and now the above results for the function $\zeta(s)$ are stated in a more convenient form. We recall the definition of the weak convergence of probability measures. Let $\mathcal{X}$ be a topological space and $\mathcal{B}(\mathcal{X})$ denote its Borel σ-field. Suppose that P and P_n,

$n \in \mathbb{N}$, be probability measures on the measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. If, for every real bounded continuous function x on $\mathcal{X}$,

$$\int\limits_{\mathcal{X}} x \, \mathrm{d}P_n \xrightarrow[n\to\infty]{} \int\limits_{\mathcal{X}} x \, \mathrm{d}P,$$

then we say that P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$). The application of this terminology leads to the following statement. For $A \in \mathcal{B}(\mathbb{C})$, set

$$P_{T,\sigma}(A) = \frac{1}{T}\mathrm{meas}\{t \in [0,T] : \zeta(\sigma+it) \in A\},$$

where meas$A$ denotes the Lebesgue measure in $\mathbb{R}$. Then the theorem of [10] can be stated in the following form [11,12]: Suppose that $\sigma > 1/2$ is fixed. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there is a probability measure P_σ such that $P_{T,\sigma} \xrightarrow[T\to\infty]{\mathrm{w}} P_\sigma$.

The latter theorem is generalized for the space of analytic functions [13]. In this case, the following probability measure is considered:

$$P_T(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}, \quad A \in \mathcal{H}(D),$$

where $\mathcal{H}(D)$ is the space of analytic functions on $D = \{s \in \mathbb{C} : \sigma > 1/2\}$ endowed with the topology of uniform convergence on compacta. It is obtained that as $T \to \infty$, P_T converges weakly to the distribution of a certain $\mathcal{H}(D)$-valued random element connected to the Euler product for $\zeta(s)$.

Similar limit theorems are also known for other functions defined by Dirichlet series. For the results, see [11,13–15].

For investigations of the functions $\mathcal{Z}_k(s)$, a probabilistic approach can be applied as well. Most attention is devoted to $\mathcal{Z}_1(s)$. The first attempt in this direction was made in [16–19]. This research was continued with respect to the possible approximation of analytic functions by shifts $\mathcal{Z}_k(s+i\tau)$; see [20–22].

We recall the main result of [8]. Let $D = \{s \in \mathbb{C} : 5/6 < \sigma < 1\}$ and $\mathcal{H} = \mathcal{H}(D)$ be the analytic space on D functions equipped with the topology of uniform convergence on compacta. For $A \in \mathcal{B}(\mathcal{H})$, define

$$P_{T,\mathcal{H},\varphi}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [T,2T] : \mathcal{Z}_2(s+i\varphi(\tau)) \in A\}.$$

Theorem 1 (see [8], Theorem 1)**.** *Suppose that $\varphi(\tau)$ is an increasing to $+\infty$ differentiable function on $[T_0, \infty)$, $T_0 > 1$, with a decreasing derivative $\varphi'(\tau)$ such that*

$$\sup_{\sigma\in(5/6,1)} \frac{J_{2\varphi(2\tau)}(\sigma)}{\varphi'(2\tau)} \ll \tau, \quad \tau \to \infty.$$

Then $P_{T,\mathcal{H},\varphi} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathcal{H},0}$, where $P_{\mathcal{H},0}$ is the probability measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ degenerated at the point $g(s) \equiv 0$.

The aim of this paper is to obtain a discrete version of Theorem 1 and other results of [8]. We notice that Theorem 1 is of the continuous type because τ in the definition of $P_{T,\mathcal{H},\varphi}$ runs over the interval $[T, 2T]$. In discrete limit theorems, τ in shifts takes values from a certain discrete set. Therefore, in discrete limit theorems, it is easier to understand encoded information. On the other hand, the proofs of discrete theorems are more complicated.

Let $\#A$ denote the cardinality of a set A, and suppose that N runs over the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In the present paper, we consider the probability measures defined by shifts $\mathcal{Z}_2(\sigma + i\psi(k))$ or $\mathcal{Z}_2(s + i\psi(k))$, $k \in \mathbb{N}_0$, with certain functions $\psi(\tau)$.

2. Case of $\psi(k) = hk$

We start with the simplest case of shifts. For $A \in \mathcal{B}(\mathbb{C})$, set

$$Q^{\mathbb{C}}_{N,\sigma}(A) = \frac{1}{N+1} \#\{N \leqslant k \leqslant 2N : \mathcal{Z}_2(\sigma + ikh) \in A\}, \quad h > 0.$$

Let $P^{\mathbb{C}}_0$ be the probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ degenerated at the point $s = 0$. The weak convergence of $Q^{\mathbb{C}}_{N,\sigma}$ as $N \to \infty$ is based on the discrete mean square estimate for the function $\mathcal{Z}_2(s)$. For this, the following lemma (Gallagher lemma) connecting continuous and discrete mean squares is useful.

Lemma 1 (see [23], Lemma 1.4)**.** *Let $T_0, T \geqslant \delta > 0$, $\mathcal{T}$ be a non-empty finite set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$ and*

$$N_\delta(t) = \sum_{\substack{\tau \in \mathcal{T} \\ |t-\tau|<\delta}} 1, \quad t \in \mathcal{T}.$$

Suppose that the function $S(t)$ is continuous on $[T_0, T_0 + T]$ and has a continuous derivative on $(T_0, T_0 + T)$. Then the inequality

$$\sum_{\tau \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(t)|^2 \, \mathrm{d}t + \left(\int_{T_0}^{T_0+T} |S(t)|^2 \, \mathrm{d}t \int_{T_0}^{T_0+T} |S'(t)|^2 \, \mathrm{d}t \right)^{1/2}$$

holds.

Lemma 2. *Suppose that $\sigma \in (5/6, 1)$ and $h > 0$ are fixed numbers. Then the estimate*

$$\sum_{k=N}^{2N} |\mathcal{Z}_2(\sigma + ikh)|^2 \ll_{h,\varepsilon} N^{(15-12\sigma)/5+\varepsilon}$$

holds for every $\varepsilon > 0$.

Proof. We will apply Lemma 1 with $S(t) = \mathcal{Z}_2(\sigma + it)$; therefore, we need the mean square estimate for the derivative $\mathcal{Z}'(\sigma + it)$. Suppose that γ is a closed suitable contour enclosing the point σ and lying in the strip D. Then the Cauchy integral formula gives

$$\mathcal{Z}_2'(\sigma + it) = \frac{1}{2\pi i} \int_\gamma \frac{\mathcal{Z}_2(z + it)}{(z - \sigma)^2} \, \mathrm{d}z.$$

Hence,

$$|\mathcal{Z}_2'(\sigma + it)|^2 \ll_\gamma \int_\gamma \frac{|\mathrm{d}z|}{|z - \sigma|^4} \int_\gamma |\mathcal{Z}_2(z + it)|^2 \, |\mathrm{d}z|,$$

and

$$\int_T^{2T} |\mathcal{Z}_2'(\sigma+it)|^2 \,\mathrm{d}t \ll_\gamma \int_\gamma \left(\int_T^{2T} |\mathcal{Z}_2(z+it)|^2 \,|\mathrm{d}z| \right) \mathrm{d}t \ll_{\gamma,\varepsilon} T^{(15-12\mathrm{Re}z)/5+\varepsilon} \\ \ll_\varepsilon T^{(15-12\sigma)/5+\varepsilon} \tag{4}$$

in view of estimate (3) after a suitable choice of γ.

Now, we use Lemma 1 with $\delta = 1$, $\mathcal{T} = \{k \in \mathbb{N} : k \in [N, 2N]\}$, $T_0 = N - 1/2$, and $T = N + 1$. This yields

$$\sum_{k=N}^{2N} |\mathcal{Z}_2(\sigma+ikh)|^2 \ll_h \int_{N-1/2}^{2N+1/2} |\mathcal{Z}_2(\sigma+iht)|^2 \,\mathrm{d}t \\ + \left(\int_{N-1/2}^{2N+1/2} |\mathcal{Z}_2(\sigma+iht)|^2 \,\mathrm{d}t \int_{N-1/2}^{2N+1/2} |\mathcal{Z}_2'(\sigma+iht)|^2 \,\mathrm{d}t \right)^{1/2}.$$

Since $(\mathcal{Z}_2(\sigma+iht))_t' = ih\mathcal{Z}_2'(\sigma+iht)$ and

$$\int_{N-1/2}^{2N+1/2} |\mathcal{Z}_2(\sigma+iht)|^2 \,\mathrm{d}t \ll_h \int_{(N-1/2)h}^{(2N+1/2)h} |\mathcal{Z}_2(\sigma+it)|^2 \,\mathrm{d}t \\ + \left(\int_{(N-1/2)h}^{(2N+1/2)h} |\mathcal{Z}_2(\sigma+it)|^2 \,\mathrm{d}t \int_{(N-1/2)h}^{(2N+1/2)h} |\mathcal{Z}_2'(\sigma+it)|^2 \,\mathrm{d}t \right)^{1/2},$$

(3) and (4) prove the lemma. □

Let $P_0^{\mathbb{C}}$ be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with unity mass at the point $z = 0$.

Proposition 1. *Suppose that $\sigma \in (5/6, 1)$ is fixed. Then $Q_{N,\sigma}^{\mathbb{C}}$ converges weakly to $P_0^{\mathbb{C}}$ as $N \to \infty$.*

Proof. It is well known that weak convergence to $P_0^{\mathbb{C}}$ is equivalent to convergence to 0 in probability. Thus, let η_N be a random variable on a certain probability space $(\widehat{\Omega}, \mathcal{B}, \mu)$ with the distribution

$$\mu\{\eta_N = kh\} = \frac{1}{N+1}, \quad k = N, \dots, 2N.$$

Then denoting $X_N = \mathcal{Z}_2(\sigma + i\eta_N)$, we have to show that, for every $\delta > 0$,

$$\lim_{N\to\infty} \mu\{\rho_{\mathbb{C}}(X_n, 0) \geqslant \delta\} = 0, \tag{5}$$

where $\rho_{\mathbb{C}}$ is the metric in $\mathbb{C}$. The application of Lemma 2 gives, for $\delta > 0$ and $\varepsilon > 0$,

$$\frac{1}{N+1} \#\{N \leqslant k \leqslant 2N : \rho_{\mathbb{C}}(\mathcal{Z}_2(\sigma+ikh), 0) \geqslant \delta\} \\ \leqslant \frac{1}{\delta N} \sum_{k=N}^{2N} |\mathcal{Z}_2(\sigma+ikh)| \leqslant \frac{1}{\delta} \left(\frac{1}{N} \sum_{k=N}^{2N} |\mathcal{Z}_2(\sigma+ikh)|^2 \right)^{1/2} \\ \ll_{h,\delta,\varepsilon} N^{(10-12\sigma)/5+\varepsilon} = o(1)$$

as $N \to \infty$ for $\sigma > 5/6$. Hence, (5) is valid, and this proves the proposition. □

Proposition 1 can be generalized for the space $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, set

$$Q_N^{\mathcal{H}}(A) = \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \mathcal{Z}_2(s+ikh) \in A\}.$$

Let denote $P_{g_0}^{\mathcal{H}}$ a probability measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ with unit mass at the point $g_0(s) \equiv 0$, $s \in D$.

Proposition 2. *The probability measure $Q_N^{\mathcal{H}}$ converges weakly to $P_{g_0}^{\mathcal{H}}$ as $N \to \infty$.*

Proof. Let $\rho_{\mathcal{H}}$ denote the metric in $\mathcal{H}$ inducing its topology of uniform convergence on compacta, i.e.,

$$\rho_{\mathcal{H}}(g_1, g_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s\in K_j} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_j} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in \mathcal{H},$$

where $\{K_j : j \in \mathbb{N}\} \subset D$ is a sequence of embedded compact sets satisfying

$$D = \bigcup_{j=1}^{\infty} K_j,$$

and every compact set $K \subset D$ lies in some K_j. By a remark in the proof of Proposition 1, we have to show that

$$\lim_{N\to\infty} \mu\{\rho_{\mathcal{H}}(X_N, g_0) \geqslant \delta\} = 0, \quad \forall \delta > 0,$$

or, in virtue of definitions X_N and $Q_N^{\mathcal{H}}$, that

$$\lim_{N\to\infty} \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \rho_{\mathcal{H}}(\mathcal{Z}_2(s+ikh), g_0(s)) \geqslant \delta\} = 0.$$

Obviously,

$$\begin{aligned} &\frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \rho_{\mathcal{H}}(\mathcal{Z}_2(s+ikh), g_0(s)) \geqslant \delta\} \\ &\qquad \leqslant \frac{1}{\delta(N+1)} \sum_{k=N}^{2N} \rho_{\mathcal{H}}(\mathcal{Z}_2(s+ikh), g_0(s)). \end{aligned} \tag{6}$$

Thus, by the definition of the metric $\rho_{\mathcal{H}}$, it suffices to estimate

$$\sum_{k=N}^{2N} \sup_{s\in K} |\mathcal{Z}_2(s+ikh)|$$

for compact sets $K \subset D$. Let γ_K be a suitable closed contour enclosing the set K and lying in D. Then, by the Cauchy integral formula, for $s \in K$,

$$\mathcal{Z}_2(s+ikh) = \frac{1}{2\pi i} \int_{\gamma_K} \frac{\mathcal{Z}_2(z+ikh)}{(z-s)} \, \mathrm{d}z,$$

and this implies that

$$\sup_{s\in K} |\mathcal{Z}_2(s+ikh)| \ll_{\gamma_K} \int_{\gamma_K} |\mathcal{Z}_2(z+ikh)| \, |\mathrm{d}z|.$$

Therefore,

$$\begin{aligned}\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}_2(s+ikh)| &\ll_{\gamma_K}\int\limits_{\gamma_K}\left(\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}_2(z+ikh)|\right)|\mathrm{d}z|\\ &\ll_{\gamma_K}\int\limits_{\gamma_K}\left(N\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}_2(z+ikh)|^2\right)^{1/2}|\mathrm{d}z|\\ &\ll_{\gamma_K} N^{1/2}N^{(15-12\mathrm{Re}z)/10+\varepsilon}=N^{(20-12\mathrm{Re}z)/10+\varepsilon}=o(N)\end{aligned}$$

as $N\to\infty$, because $\mathrm{Re}z>5/6$. This together with (6) proves the proposition. □

3. General Case

In this section, we consider probability measures defined by means of shifts $\mathcal{Z}_2(s+i\psi(k))$, $k\in\mathbb{N}$, with a certain function $\psi(\tau)$. We will consider the case of the space $\mathcal{H}$ only, because in [24] it was observed that weak convergence to $P_0^{\mathcal{H}}$ implies that to $P_0^{\mathbb{C}}$. Thus, we will consider weak convergence for

$$P_N^{\mathcal{H}}(A)\stackrel{\mathrm{def}}{=}\frac{1}{N+1}\#\{N\leqslant k\leqslant 2N:\mathcal{Z}_2(s+i\psi(k))\in A\},\quad A\in\mathcal{B}(\mathcal{H}),$$

as $N\to\infty$.

We recall that

$$J_T(\sigma)=\int\limits_0^T|\mathcal{Z}_2(s+it)|^2\,\mathrm{d}t.$$

Suppose that $\psi(\tau)$ is an increasing to $+\infty$ differentiable function on $[T,+\infty)$, $T>T_0>1$, with decreasing derivative $\psi'(\tau)$ satisfying the estimate

$$\sup_{5/6<\sigma<1}J_{2\psi(2\tau)}(\sigma)\ll\tau\psi'(2\tau),\quad \tau\to\infty.$$

Additionally, we need one individual distribution property of the function $\psi(\tau)$. Let $\{u\}$ denote the fractional part of $u\in\mathbb{R}$ and I_A the indicator function of a set A. Recall that a sequence $\{x_m:m\in\mathbb{N}\}\subset\mathbb{R}$ is uniformly distributed modulo 1 if, for every interval $(a,b]\subset(0,1]$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}I_{(a,b]}(\{x_m\})=b-a.$$

By the Weyl criterion (see, for example, [25] Theorem 2.1), the sequence $\{x_m:m\in\mathbb{N}\}$ is uniformly distributed modulo 1 if and only if, for every integer $k\neq 0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\mathrm{e}^{2\pi ikx_m}=0.$$

Suppose that the sequence $\{\kappa\psi(m):m\in\mathbb{N}\}$ with every real $\kappa\neq 0$ is uniformly distributed modulo 1. Let W denote the class of the above functions $\psi(\tau)$.

We start with a bound for

$$V_N(K)\stackrel{\mathrm{def}}{=}\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}(s+i\psi(k))|,$$

where $K\subset D$ is a compact set.

Lemma 3. *Suppose that* $\psi(\tau) \in W$. *Then*

$$V_N(K) \ll_K N.$$

Proof. As in the proof of Proposition 2, we have

$$\sup_{s \in K} |\mathcal{Z}_2(s + i\psi(k))| \ll_{\gamma_K} \int_{\gamma_K} |\mathcal{Z}_2(z + i\psi(k))|\,|\mathrm{d}z|,$$

where γ_K is a suitable closed contour in D enclosing the set K. Hence,

$$V_N(K) \ll_K \int_{\gamma_K} \left(N \sum_{k=N}^{2N} |\mathcal{Z}_2(\mathrm{Re} z + i\mathrm{Im} z + i\psi(k))|^2 \right)^{1/2} |\mathrm{d}z|. \tag{7}$$

For the estimation of the discrete mean square in (7), we will apply Lemma 1. Following the proof of Lemma 2, we estimate

$$\begin{aligned}
&\int_{N-1/2}^{2N} |\mathcal{Z}_2(\mathrm{Re} z + i\mathrm{Im} z + i\psi(\tau))|^2 \,\mathrm{d}\tau \\
&\quad = \int_{N-1/2}^{2N} |\mathcal{Z}_2(\mathrm{Re} z + i\mathrm{Im} z + i\psi(\tau))|^2 \frac{\mathrm{d}\psi(\tau)}{\psi'(\tau)} \\
&\quad \leqslant \frac{1}{\psi'(2N)} \int_{\psi(N-1/2)-|\mathrm{Im} z|}^{\psi(2N)+|\mathrm{Im} z|} |\mathcal{Z}_2(\mathrm{Re} z + i\tau)|^2 \,\mathrm{d}\tau \\
&\quad \ll \frac{1}{\psi'(2N)} \int_{\psi(N-1/2)-|\mathrm{Im} z|}^{2\psi(2N)} |\mathcal{Z}_2(\mathrm{Re} z + i\tau)|^2 \,\mathrm{d}\tau \\
&\quad \ll \frac{J_{2\psi(2N)}(\mathrm{Re} z)}{\psi'(2N)} \ll \sup_{5/6<\sigma<1} \frac{J_{2\psi(2N)}(\sigma)}{\psi'(2N)} \ll N,
\end{aligned} \tag{8}$$

in virtue of the definition of the class W.

Along the same lines, we obtain that

$$\int_{N-1/2}^{2N} |\mathcal{Z}_2'(\mathrm{Re} z + i\mathrm{Im} z + i\psi(\tau))|^2 \,\mathrm{d}\tau \ll N. \tag{9}$$

Clearly,

$$(\mathcal{Z}_2(s + i\psi(\tau)))'_\tau = \psi'(\tau)\mathcal{Z}_2'(s + i\psi(\tau)).$$

Hence, by (9),

$$\begin{aligned}
&\int_{N-1/2}^{2N} |\mathcal{Z}_2'(\mathrm{Re} z + i\mathrm{Im} z + i\psi(\tau))|^2 \,\mathrm{d}\tau \\
&\quad \ll \int_{N-1/2}^{2N} (\psi'(\tau))^2 |\mathcal{Z}_2'(\mathrm{Re} z + i\mathrm{Im} z + i\psi(\tau))|^2 \,\mathrm{d}\tau \\
&\quad \ll \psi'(N-1)N \ll N,
\end{aligned}$$

because $\psi'(\tau)$ is bounded. This, (7), (8) and Lemma 1 yield the lemma. □

Let ξ_N be a random variable on the probability space $(\Omega, \mathcal{B}, \mu)$ with the distribution

$$\mu\{\xi_N = k\} = \frac{1}{N+1}, \quad k = N, \dots, 2N,$$

and $Y_N = \mathcal{Z}_2(s + i\psi(\xi_N))$. Then, for $\delta > 0$, we derive from Lemma 3 that

$$\begin{aligned}\mu\{\rho_{\mathcal{H}}(Y_N, g_0) \geqslant \delta\} &= \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \rho_{\mathcal{H}}(\mathcal{Z}_2(s + i\psi(k)), g_0) \geqslant \delta\} \\ &\leqslant \frac{1}{\delta(N+1)} \sum_{k=N}^{2N} \rho_{\mathcal{H}}(\mathcal{Z}_2(s + i\psi(k)), g_0) \\ &= \frac{1}{\delta(N+1)} \sum_{k=N}^{2N} \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s \in K_j} |\mathcal{Z}_2(s + i\psi(k))|}{1 + \sup_{s \in K_j} |\mathcal{Z}_2(s + i\psi(k))|} \\ &\ll_{\delta,K} 1 + \frac{1}{N} \sum_{j \leqslant j_0} 2^{-j} \sum_{k=N}^{2N} \sup_{s \in K_j} |\mathcal{Z}_2(s + i\psi(k))| \ll_{\delta} C_{K_{j_0}}\end{aligned}$$

with certain $C_{K_{j_0}} > 0$. Thus, we cannot obtain from this that

$$\lim_{N \to \infty} \mu\{\rho_{\mathcal{H}}(Y_N, g_0) \geqslant \delta\} = 0.$$

Consequently, in the case of generalized shifts, we have to use a more complicated approach to prove weak convergence for $P_N^{\mathcal{H}}$ to $P_{g_0}^{\mathcal{H}}$.

Notice that discrete limit theorems for various zeta-functions using generalized shifts were investigated in numerous papers; see, for example, [26–47]. In these works, limit theorems in the space of analytic functions are applied for the proofs of theorems on the approximation of classes of analytic functions by shifts of zeta-functions. The considered shifts involve the sequences of Gram points and imaginary parts of the non-trivial zeros of the Riemann zeta-function, as well as sequences uniformly distributed modulo 1.

Now we state the main result of this paper.

Theorem 2. *Suppose that $\psi(\tau) \in W$. Then $P_N^{\mathcal{H}}$ converges weakly to the measure $P_{g_0}^{\mathcal{H}}$ as $N \to \infty$.*

We divide the proof of Theorem 2 into lemmas.

For brevity, we note that, for $x, y \geqslant 1$,

$$\zeta_2(x) = \left|\zeta\left(\frac{1}{2} + ix\right)\right|^4, \quad v(x,y) = \exp\left(-\left(\frac{x}{y}\right)^{\theta}\right),\ \theta > \frac{1}{6},$$

and

$$\mathcal{Z}_{2,a,y}(s) = \int_1^a \zeta_2(x) v(x,y) x^{-s}\,\mathrm{d}x, \quad a > 1.$$

Consider the probability measure

$$P_{N,a,y}^{\mathcal{H}}(A) = \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \mathcal{Z}_{2,a,y}(s + i\psi(k)) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}).$$

Lemma 4. *Suppose that $\psi(\tau) \in W$. Then $P_{N,a,y}^{\mathcal{H}}$ converges weakly to $P_{g_0}^{\mathcal{H}}$ as $N \to \infty$.*

Proof. As seen in Section 2, it suffices to show that, for arbitrary compact set $K \subset D$,

$$\sum_{k=N}^{2N} \sup_{s \in K} |\mathcal{Z}_{2,a,y}(s + i\psi(k))| = o(N)$$

as $N \to \infty$. The Cauchy integral formula shows that the latter estimate follows from

$$\sum_{k=N}^{2N} |\mathcal{Z}_{2,a,y}(\sigma + it + i\psi(k))|^2 = o(N), \quad N \to \infty, \tag{10}$$

with $\sigma \in (5/6, 1)$ and bounded t. Since $|z|^2 = z\bar{z}$, where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$, it follows that

$$\begin{aligned} &|\mathcal{Z}_{2,a,y}(\sigma + it + i\psi(k))|^2 \\ &= \int_1^a \zeta_2(x)v(x,y)x^{-\sigma-it-i\psi(k)}\,\mathrm{d}x \int_1^a \zeta_2(x)v(x,y)x^{-\sigma+it+i\psi(k)}\,\mathrm{d}x \\ &= \int_1^a \int_1^a \zeta_2(x_1)\zeta_2(x_2)v(x_1,y)v(x_2,y)(x_1x_2)^{-\sigma}\left(\frac{x_2}{x_1}\right)^{it}\left(\frac{x_2}{x_1}\right)^{i\psi(k)}\mathrm{d}x_1\,\mathrm{d}x_2. \end{aligned} \tag{11}$$

Since $\psi(\tau) \in W$, the sequence

$$\left\{\frac{1}{2\pi}\log\left(\frac{x_2}{x_1}\right)\psi(k) : k \in \mathbb{N}\right\}$$

with $x_1 \neq x_2$ is uniformly distributed modulo 1. Therefore, by the Weyl criterion,

$$\sum_{k=N}^{2N}\left(\frac{x_2}{x_1}\right)^{i\psi(k)} = \sum_{k=N}^{2N}\exp\left(2\pi i\log\left(\frac{x_2}{x_1}\right)\frac{\psi(k)}{2\pi}\right) = o(N)$$

as $N \to \infty$. This together with (11) proves (10), because the integral in (11) with $x_1 = x_2$ is zero. The lemma is proven. □

Now, we will prove an analog of Lemma 4 for the function

$$\mathcal{Z}_{2,y}(s) = \int_1^\infty \zeta_2(x)v(x,y)x^{-s}\,\mathrm{d}x.$$

Clearly, in virtue of the exponential decrease of $v(x,y)$ with respect to x, the latter integral is absolutely convergent in any half-plane $\sigma \geqslant \sigma_0$.

Define

$$P^{\mathcal{H}}_{N,y}(A) = \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \mathcal{Z}_{2,y}(s + i\psi(k)) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}).$$

We will derive the weak convergence for $P^{\mathcal{H}}_{N,y}$ as $N \to \infty$ from Lemma 4. For this, we will apply one statement on convergence in the distribution ($\xrightarrow{\mathcal{D}}$) of certain random elements. Let X_n, $n \in \mathbb{N}$, and X be random elements on a certain probability space and P_n and P denote their distributions, respectively. We recall that $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$, if $P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$.

Lemma 5 (see Theorem 4.2 of [48])**.** *Suppose that $(\mathcal{X}, \rho)$ is a separable metric space and X_{nm} and Y_n $\mathcal{X}$-valued random elements on the probability space $(\widehat{\Omega}, \mathcal{B}, \mu)$ such that*

$$X_{nm} \xrightarrow[n\to\infty]{\mathcal{D}} X_m, \quad \text{for all } m \in \mathbb{N},$$

and

$$X_m \xrightarrow[m\to\infty]{\mathcal{D}} X.$$

Moreover, let, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mu\{\rho(X_{nm}, Y_n) \geqslant \varepsilon\} = 0.$$

Then

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Lemma 6. *Suppose that $\psi(\tau) \in W$. Then $P^{\mathcal{H}}_{N,y}$ converges weakly to $P^{\mathcal{H}}_{g_0}$ as $N \to \infty$.*

Proof. Let ξ_N be a random variable on the probability space $(\widehat{\Omega}, \mathcal{B}, \mu)$ with the distribution

$$\mu\{\xi_N = k\} = \frac{1}{N+1}, \quad k = N, \dots, 2N.$$

Define the $\mathcal{H}$-valued random element

$$X_{N,a,y} = X_{N,a,y}(s) = \mathcal{Z}_{2,a,y}(s + i\psi(\xi_N)).$$

Then, by Lemma 4, we have

$$X_{N,a,y} \xrightarrow[N\to\infty]{\mathcal{D}} X_{a,y} \tag{12}$$

for all $a > 1$, $y \geqslant 1$, where $X_{a,y}$ is a $\mathcal{H}$-valued random element with distribution $P^{\mathcal{H}}_{g_0}$.

By the above remark on the absolute convergence of the integral for $\mathcal{Z}_{2,y}(s)$, it follows that, for $\sigma > 5/6$,

$$\int_a^\infty \zeta_2(x)v(x,y)x^{-s}\,\mathrm{d}x = o_y(1), \quad a \to \infty.$$

Hence, for a compact set $K \subset D$,

$$\sup_{s\in K} |\mathcal{Z}_{2,y}(s + i\psi(k)) - \mathcal{Z}_{2,a,y}(s + i\psi(k))| \leqslant \int_a^\infty \zeta_2(x)v(x,y)x^{-\min_{s\in K}\sigma}\,\mathrm{d}x = o_y(1)$$

as $a \to \infty$. Thus,

$$\lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=N}^{2N} \sup_{s\in K} |\mathcal{Z}_{2,y}(s + i\psi(k)) - \mathcal{Z}_{2,a,y}(s + i\psi(k))| = 0. \tag{13}$$

Introduce one more $\mathcal{H}$-valued random element

$$Y_{N,y} = Y_{N,y}(s) = \mathcal{Z}_{2,y}(s + i\psi(\xi_N)).$$

Then (13) implies that, for every $\varepsilon > 0$,

$$\begin{aligned}\lim_{a\to\infty} \limsup_{N\to\infty} \mu\{\rho_{\mathcal{H}}(Y_{N,y}, X_{N,a,y}) \geqslant \varepsilon\} &= \lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \\ &\quad \#\{N \leqslant k \leqslant 2N : \rho_{\mathcal{H}}(\mathcal{Z}_{2,y}(s + i\psi(k)), \mathcal{Z}_{2,a,y}(s + i\psi(k))) \geqslant \varepsilon\} \\ &\leqslant \lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=N}^{2N} \rho_{\mathcal{H}}(\mathcal{Z}_{2,y}(s + i\psi(k)), \mathcal{Z}_{2,a,y}(s + i\psi(k))) = 0.\end{aligned} \tag{14}$$

Since

$$X_{a,y} \xrightarrow[a\to\infty]{\mathcal{D}} P^{\mathcal{H}}_{g_0},$$

(12), (14) and Lemma 5 prove that

$$Y_{N,y} \xrightarrow[N\to\infty]{\mathcal{D}} P^{\mathcal{H}}_{g_0},$$

and this gives the assertion of the lemma. □

In order to derive Theorem 2 from Lemma 6, we need the approximation results for $\mathcal{Z}_2(s)$ by $\mathcal{Z}_{2,y}(s)$.

4. Approximation in the Mean

We start with recalling the integral representation for $\mathcal{Z}_{2,y}$. Let θ be the same number from the definition of $v(x,y)$.

Lemma 7 (see Lemma 4 of [8]). *Let* $\Gamma(s)$ *denote the gamma-function and*

$$\kappa_y(s) = \theta^{-1}\Gamma(\theta^{-1}s)y^s. \tag{15}$$

Then, for $s \in D$*, the representation*

$$\mathcal{Z}_{2,y}(s) = \frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty} \mathcal{Z}_2(s+z)\kappa_y(z)\,\mathrm{d}z \tag{16}$$

holds.

Lemma 8. *Suppose that* $\psi(\tau) \in W$*. Then*

$$\lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=N}^{2N}\rho_{\mathcal{H}}(\mathcal{Z}_2(s+i\psi(k)), \mathcal{Z}_{2,y}(s+i\psi(k))) = 0.$$

Proof. In view of the definition of the metric $\rho_{\mathcal{H}}$, the equality of the lemma is a corollary of the statement

$$\lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}_2(s+i\psi(k)) - \mathcal{Z}_{2,y}(s+i\psi(k))| = 0 \tag{17}$$

for arbitrary compact set $K \subset D$.

Thus, we take an arbitrary compact set $K \subset D$. Since K is a bounded closed set, there is $\varepsilon > 0$ such that $5/6 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for all $s = \sigma + it$ lying in K. In Lemma 7, θ is a fixed number greater than $1/6$. Let us take $\theta = 1/6 + \varepsilon > 1/6$. Moreover, we introduce a new parameter $\theta_1 = 5/6 + \varepsilon - \sigma$. Then, clearly, $\theta_1 < 0$ and $\theta_1 \geqslant -1/6 + 2\varepsilon$. From this, it follows that the integrand in (16) has a simple pole at the point $z = 0$ (a pole of $\Gamma(\theta^{-1}z)$) and a pole at the point $z = 1 - s$ of order five (a pole of $\mathcal{Z}_2(s+z)$) lying in the strip $\theta_1 \leqslant \mathrm{Re}z \leqslant \theta$. Therefore, using the well-known exponential estimate

$$\Gamma(\sigma + it) \ll \exp(-c|t|), \quad c > 0, \tag{18}$$

which is uniform for $\sigma \in [\sigma_1, \sigma_2]$ with arbitrary $\sigma_1 < \sigma_2$, and the residue theorem, we find that, for $s \in K$,

$$\mathcal{Z}_{2,y}(s) - \mathcal{Z}_2(s) = \frac{1}{2\pi i}\int\limits_{\theta_1-i\infty}^{\theta_1+i\infty} \mathcal{Z}_2(s+z)\kappa_y(z)\,\mathrm{d}z + R_y(s),$$

where
$$R_y(s) = \operatorname*{Res}_{z=1-s} \mathcal{Z}_2(s+z)\kappa_y(z).$$
This and the definition of θ_1, for $s \in K$, lead to
$$\begin{aligned}
&\mathcal{Z}_{2,y}(s+i\psi(k)) - \mathcal{Z}_2(s+i\psi(k)) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{Z}_2\left(\frac{5}{6}+\varepsilon+it+i\tau+i\psi(k)\right)\kappa_y\left(\frac{5}{6}+\varepsilon-\sigma+i\tau\right)\mathrm{d}\tau \\
&\quad + R_y(s+i\psi(k)) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{Z}_2\left(\frac{5}{6}+\varepsilon+i\tau+i\psi(k)\right)\kappa_y\left(\frac{5}{6}+\varepsilon-s+i\tau\right)\mathrm{d}\tau \\
&\quad + R_y(s+i\psi(k)) \\
&\ll \int_{-\infty}^{\infty} \left|\mathcal{Z}_2\left(\frac{5}{6}+\varepsilon+i\tau+i\psi(k)\right)\right| \sup_{s\in K}\left|\kappa_y\left(\frac{5}{6}+\varepsilon-s+i\tau\right)\right|\mathrm{d}\tau \\
&\quad + \sup_{s\in K}|R_y(s+i\psi(k))|
\end{aligned}$$
after the application of the shift $t+\tau \to \tau$. Hence, we have
$$\begin{aligned}
&\frac{1}{N+1}\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}_2(s+i\psi(k)) - \mathcal{Z}_{2,y}(s+i\psi(k))| \\
&\ll \int_{-\infty}^{\infty} \frac{1}{N+1}\sum_{k=N}^{2N}\left|\mathcal{Z}_2\left(\frac{5}{6}+\varepsilon+i\tau+i\psi(k)\right)\right| \sup_{s\in K}\left|\kappa_y\left(\frac{5}{6}+\varepsilon-s+i\tau\right)\right|\mathrm{d}\tau \\
&\quad + \frac{1}{N+1}\sum_{k=N}^{2N}\sup_{s\in K}|R_y(s+i\psi(k))| \overset{\text{def}}{=} \mathcal{I}+S. \qquad (19)
\end{aligned}$$
Repeating the proof of Lemma 3 with minor changes, we obtain that, for $\tau \in \mathbb{R}$,
$$\begin{aligned}
&\frac{1}{N+1}\sum_{k=N}^{2N}\left|\mathcal{Z}_2\left(\frac{5}{6}+\varepsilon+i\tau+i\psi(k)\right)\right| \\
&\ll \left(\frac{1}{N}\sum_{k=N}^{2N}\left|\mathcal{Z}_2\left(\frac{5}{6}+\varepsilon+i\tau+i\psi(k)\right)\right|^2\right)^{1/2} \ll (1+|\tau|)^{1/2}. \qquad (20)
\end{aligned}$$
Moreover, the definition of $\kappa_y(s)$ and the bound (18) yield, for $s \in K$ and $c_1, c_2 > 0$,
$$\kappa_y\left(\frac{5}{6}+\varepsilon-s+i\tau\right) \ll y^{5/6+\varepsilon-\sigma}\exp(-c_1|\tau-t|) \ll_K y^{-\varepsilon}\exp(-c_2|\tau|).$$
This together with (20) gives the bound
$$\mathcal{I} \ll_K y^{-\varepsilon}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp(-c_2|\tau|)\,\mathrm{d}\tau \ll_K y^{-\varepsilon}. \qquad (21)$$

It remains to estimate S. By the residue formula, for $s \in K$,

$$\begin{aligned} &R_y(s+i\psi(k)) \\ &\quad = \Big(\mathcal{Z}_2(z+s+i\psi(k))(\mathcal{Z}_2(z-1+s+i\psi(k)))^5\kappa_y(z)\Big)^{(4)}\Big|_{z=1-s-i\psi(k)} \\ &\quad \overset{\text{def}}{=} \sum_{j=1}^{5}\binom{j}{5}L^{(5-j)}(z,s,\psi(k))\kappa_y^{(j)}(z)\Big|_{z=1-s-i\psi(k)} \end{aligned}$$

in virtue of the Leibniz formula. Hence, estimate (2), the Cauchy integral formula, and (18) show that, for $s \in K$,

$$\begin{aligned} R_y(s+i\psi(k)) &\ll_K (\psi(k))^{c_3}(y\log y)^{c_4}\exp(-c_5|t+\psi(k)|) \\ &\ll_K (\psi(k))^{c_3}(y\log y)^{c_4}\exp(-c_6\psi(k)). \end{aligned}$$

Thus,

$$\begin{aligned} S &\ll \frac{1}{N}(y\log y)^{c_4}\sum_{k=N}^{2N}(\psi(k))^{c_3}\exp\Big(-\frac{c_6}{2}\psi(k)\Big) \\ &\ll \frac{1}{N}(y\log y)^{c_4}\exp\Big(-\frac{c_6}{2}\psi(N)\Big)\sum_{k=N}^{2N}(\psi(k))^{c_3}\exp\Big(-\frac{c_6}{2}\psi(k)\Big) \\ &\ll (y\log y)^{c_4}\exp\Big(-\frac{c_6}{2}\psi(N)\Big). \end{aligned}$$

This, (19) and (21) show that

$$\begin{aligned} &\frac{1}{N+1}\sum_{k=N}^{2N}\sup_{s\in K}|\mathcal{Z}_2(s+i\psi(k))-\mathcal{Z}_{2,y}(s+i\psi(k))| \\ &\qquad \ll_K y^{-\varepsilon}+(y\log y)^{c_4}\exp\Big(-\frac{c_6}{2}\psi(N)\Big). \end{aligned}$$

Thus, taking $N \to \infty$ and then $y \to \infty$, we obtain (17). The lemma is proven. □

5. The Proof of the Main Theorem

In this section, we will present the proof of Theorem 2. For this, we will apply Lemmas 5, 6 and 8.

Proof of Theorem 2. We preserve the notations used in the above-mentioned lemmas. Thus, by Lemma 6,

$$Y_{N,y} \xrightarrow[N\to\infty]{\mathcal{D}} P_{g_0}^{\mathcal{H}} \tag{22}$$

for all $y > 1$. Introduce one more $\mathcal{H}$-valued random element

$$Y_N = Y_N(s) = \mathcal{Z}_2(s+i\psi(\xi_N)).$$

Then, in view of Lemma 8, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{y\to\infty}\limsup_{N\to\infty}\mu\{\rho_{\mathcal{H}}(Y_N,Y_{N,y}) \geqslant \varepsilon\} = \lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{N+1} \\ &\qquad \#\{N \leqslant k \leqslant 2N : \rho_{\mathcal{H}}(\mathcal{Z}_2(s+i\psi(k)),\mathcal{Z}_{2,y}(s+i\psi(k))) \geqslant \varepsilon\} \\ &\quad \leqslant \lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{\varepsilon(N+1)}\sum_{k=N}^{2N}\rho_{\mathcal{H}}(\mathcal{Z}_2(s+i\psi(k)),\mathcal{Z}_{2,y}(s+i\psi(k))) = 0. \end{aligned}$$

The latter equality together with (22) and Lemma 5 proves that

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P^{\mathcal{H}}_{g_0},$$

and this gives the theorem. □

6. Applications

Some problems of the approximation of analytic functions by shifts in the Mellin transforms of powers of the Riemann zeta-function were discussed in [8,20,49]. Theorem 2 leads to the following statement on the approximation of the function g_0.

Theorem 3. *Suppose that* $\psi(\tau) \in W$. *Then, for every compact set* $K \subset D$ *and* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ N \leqslant k \leqslant 2N : \sup_{s\in K} |\mathcal{Z}_2(s+i\psi(k))| < \varepsilon \right\} > 0.$$

Moreover,

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ N \leqslant k \leqslant 2N : \sup_{s\in K} |\mathcal{Z}_2(s+i\psi(k))| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 3 is a direct corollary of Theorem 2. For this, we recall the equivalents of the weak convergence of probability measures. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. A set $A \in \mathcal{B}(\mathcal{X})$ is called a continuity set of the measure P if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A.

Lemma 9 (a part of Theorem 2.1 of [48])**.** *The following statements are equivalent:*

1° $P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$;

2° *For every open set* $G \subset \mathcal{X}$,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

3° *For every continuity set* A *of the measure* P,

$$\lim_{n\to\infty} P_n(A) = P(A).$$

Proof of Theorem 3. For $\varepsilon > 0$, we set

$$G_\varepsilon = \left\{ g \in \mathcal{H} : \sup_{s\in K} |g(s)| < \varepsilon \right\}$$

Then G_ε is an open neighborhood of the function g_0. Since the support of the measure $P^{\mathcal{H}}_{g_0}$ is g_0, the set G_ε is an open neighborhood of an element of the support of $P^{\mathcal{H}}_{g_0}$. Hence,

$$P^{\mathcal{H}}_{g_0}(G_\varepsilon) > 0. \tag{23}$$

This, Theorem 2 and 2° of Lemma 9 show that

$$\liminf_{N\to\infty} P^{\mathcal{H}}_N(G_\varepsilon) \geqslant P^{\mathcal{H}}_{g_0}(G_\varepsilon) > 0,$$

and we thus have the first statement of the theorem.

To prove the second statement of the theorem, we observe that the set G_ε is a continuity set of $P^{\mathcal{H}}_{g_0}$ for all but at most countably many $\varepsilon > 0$. This remark, Theorem 2 and 3° of Lemma 9 imply that the limit

$$\lim_{N\to\infty} P^{\mathcal{H}}_N G_\varepsilon = P^{\mathcal{H}}_{g_0}(G_\varepsilon)$$

exists and, by (23), is positive for all but at most countably many $\varepsilon > 0$. This gives the second statement of the theorem. □

7. Conclusions

In this paper, we considered the statistical properties of discrete shifts $\mathcal{Z}_2(s + i\psi(k))$ in the modified Mellin transform of the fourth power of the Riemann zeta-function. We obtained that

$$\frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \mathcal{Z}_2(s + i\psi(k)) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}),$$

converges weakly to the measure with mass 1 at the point $g_0(s) \equiv 0$. Here $\mathcal{H}$ is the space of analytic functions on the strip $\{s \in \mathbb{C} : 5/6 < \sigma < 1\}$ and the increasing to $+\infty$ differentiable function $\psi(\tau)$, $\tau \in [T, \infty)$, $T > 1$, whose derivative $\psi'(\tau)$ satisfies the estimate

$$\sup_{\sigma\in(5/6,1)} \int\limits_0^{2\psi(2\tau)} |\mathcal{Z}_2(\sigma + it)|^2 \, \mathrm{d}t \ll \tau\psi'(2\tau), \quad \tau \to \infty.$$

Moreover, we suppose that the sequence $\{\kappa\psi(k) : k \in \mathbb{N}\}$ with every real number $\kappa \neq 0$ is uniformly distributed modulo 1. The limit theorem for $\mathcal{Z}_2(s + i\psi(k))$ is applied to the approximation of the function g_0.

For example, the Gram function t_τ, see [50,51] for a definition, satisfies the above hypotheses.

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