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On statistical experiments observing H-diffusions

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1. Introduction

Let Θ be an arbitrary set, $h_\theta(x)$ and $g_\theta(x)$, $x \in (l,r) \subseteq R^1$, $\theta \in \Theta$, be strictly positive measurable functions such that $h_\theta(x)g_\theta(x) =: \sigma^{-2}(x)$, $x \in (l,r)$, is independent of θ and satisfying the following assumptions:

$$\int_l^r h_\theta(x)\,\mathrm{d}\,x = 1, \quad \theta \in \Theta, \tag{1}$$

for all $\theta \in \Theta$, $g_\theta(x)$ is differentiable in x, for all $x \in (l,r)$ and $\theta \in \Theta$ there exists $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \subset (l,r)$, such that

$$\int_{x-\varepsilon}^{x+\varepsilon} |g'_\theta(v)|\,\mathrm{d}\,v < \infty, \tag{2}$$

and for some fixed $x_0 \in (l,r)$ and all $\theta \in \Theta$

$$G_\theta(x) := \int_{x_0}^{x} g_\theta(v)\,\mathrm{d}\,v \to \infty \quad \text{as} \quad x \uparrow r, \quad G_\theta(x) \to -\infty \quad \text{as} \quad x \downarrow l. \tag{3}$$

Let $\Omega := C_{[0,\infty)}(l,r)$ be the space of all continuous mappings $\omega(\cdot) : [0,\infty) \to (l,r)$ with the topology of uniform convergence on compact sets, $\mathcal{F} := \mathcal{B}(\Omega)$ be a σ-algebra of the Borel subsets of Ω, $\mathbb{F} = \{\mathcal{F}_t,\ t \geqslant 0\}$ be the standard filtration of sub-σ-algebras of $\mathcal{F}$ and $X(t,\omega) = \omega(t)$, $t \geqslant 0$, $\omega \in \Omega$.

Denoting $H_\theta(dx) = h_\theta(x)dx$ and

$$\mu_\theta(x) = -\frac{1}{2}(\ln g_\theta(x))'\sigma^2(x), \quad x \in (l,r),\ \theta \in \Theta, \tag{4}$$

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it is known (see, e.g., Mandl [7], Engelbert and Schmidt [1], Karatzas and Shreve [4], Grigelionis [2]) that under the assumptions (1)–(3), for each $\theta \in \Theta$, there exists unique probability measure P_θ on $\mathcal{F}$ such that $P_\theta \circ X^{-1}(0) = H_\theta$,

$$M_\theta(t) := X(t) - \int_0^t \mu_\theta(X(s))\,\mathrm{d}\,s, \quad t \geqslant 0,$$

is a $(P_\theta, \mathbb{F})$-local martingale satisfying

$$\langle M_\theta \rangle_t = \int_0^t \sigma^2(X(s))\,\mathrm{d}\,s, \quad t \geqslant 0,$$

and X is a $(P_\theta, \mathbb{F})$-(H_θ)-diffusion, i.e., a strictly stationary diffusion process on an open interval $(l, r) \subseteq R^1$ with the predetermined stationary distribution H.

For any fixed $\mathbb{F}$-stopping time T we shall consider statistical experiments $\{\Omega, \mathcal{F}_T, (P_{\theta,T}, \theta \in \Theta)\}$, where $P_{\theta,T}$ are restrictions of P_θ to $\mathcal{F}_T$. Using known results on differentiability of probability measures, corresponding to the diffusion processes (see, e.g. Liptser and Shiryaev [6]), in Section 2 we shall derive formulas for the log-likelihood ratios in terms of the stationary density functions and the diffusion coefficients. In Section 3 we shall describe curved exponential families of H-diffusions and related sufficient statistics (cf. Küchler and Sørensen [5]). The results are illustrated by some examples, including well-known diffusion models in natural sciences and mathematical finance (see, e.g., Grigelionis [2], [3]).

2. Log-likelihood ratios for H-diffusions

Proposition 1. a) *Let the assumptions* (1)–(3) *are fulfilled,* $\theta_0 \in \Theta$ *is fixed,*

$$P_\theta\left\{ \int_0^T \left[\left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \right]^2 \mathrm{d}\,s < \infty \right\} = 1, \tag{5}$$

and

$$P_{\theta_0}\left\{ \int_0^T \left[\left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \right]^2 \mathrm{d}\,s < \infty \right\} = 1, \quad \theta \in \Theta. \tag{6}$$

Then $P_{\theta,T} \sim P_{\theta_0,T}$ *and*

$$L_{\theta,T} := \ln \frac{\mathrm{d}\,P_{\theta,T}}{\mathrm{d}\,P_{\theta_0,T}} = \ln \frac{h_\theta(X(0))}{h_{\theta_0}(X(0))} + \frac{1}{2} \int_0^T \left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \mathrm{d}\,M_{\theta_0}(s)$$

$$-\frac{1}{8}\int_0^T \left[\left(\ln\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)'\right]^2 \sigma^2(X(s))\,\mathrm{d}\,s. \tag{7}$$

b) *If, additionally, the functions $g_\theta(x)$, $x \in (l,r)$, $\theta \in \Theta$, are twice continuously differentiable in x, then (cf. Morton* [8]*)*

$$\begin{aligned} L_{\theta,T} &= \frac{1}{2}\ln\frac{h_\theta(X(T))h_\theta(X(0))}{h_{\theta_0}(X(T))h_{\theta_0}(X(0))} \\ &-\frac{1}{4}\int_0^T \Bigg\{\left(\ln\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)'' + \frac{1}{2}\left[\left(\ln\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)'\right]^2 \\ &+\ln\left(\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)'\left[\ln\left(h_{\theta_0}(X(s))\sigma^2(X(s))\right)\right]'\Bigg\}\sigma^2(X(s))\,\mathrm{d}\,s. \end{aligned} \tag{8}$$

Proof. a) Applying the well-known criteria on differentiability of probability measures, corresponding to diffusion processes, and having in mind (4), it is enough to check, that

$$\frac{\mu_\theta(x)-\mu_{\theta_0}(x)}{\sigma^2(x)} = -\frac{1}{2}\left(\ln\frac{g_\theta(x)}{g_{\theta_0}(x)}\right)' = \frac{1}{2}\left(\ln\frac{h_\theta(x)}{h_{\theta_0}(x)}\right)', \quad x\in(l,r).$$

b) Because $\ln g_\theta(x) = -\ln(h_\theta(x)\sigma^2(x))$, from Ito's formula we find that

$$\begin{aligned} \frac{1}{2}\int_0^T \left(\ln\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)'\mathrm{d}\,M_{\theta_0}(s) &= \frac{1}{2}\ln\frac{h_\theta(X(T))h_{\theta_0}(X(0))}{h_{\theta_0}(X(T))h_\theta(X(0))} \\ &-\frac{1}{4}\int_0^T \left(\ln\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)'\left[\ln\left(h_{\theta_0}(X(s))\sigma^2(X(s))\right)\right]'\sigma^2(X(s))\,\mathrm{d}\,s \\ &-\frac{1}{4}\int_0^T \left(\ln\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))}\right)''\sigma^2(X(s))\,\mathrm{d}\,s. \end{aligned} \tag{9}$$

Now from (7) and (9) we obtain (8).

Remark. *If for all $\theta \in \Theta$ $g'_\theta(x)$, $x \in (l,r)$, is continuous in x and*

$$P_\theta\{T<\infty\} = 1, \tag{10}$$

then, obviously, the assumptions (2), (5) *and* (6) *are valid.*

3. Curved exponential families of H-diffusions

Assume that there exist functions $\varphi_j(\theta)$, $\theta \in \Theta$, and $\psi_j(x)$, $x \in (l, r)$, $j = 1, \dots, d$, such that ψ_j, $j = 1, \dots, d$, are continuously differentiable, and

$$\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} = \sum_{j=1}^{d} \varphi_j(\theta)\psi_j(x), \quad x \in (l, r), \quad \theta \in \Theta. \tag{11}$$

Proposition 2. a) *Under the assumptions* (1), (3), (10) *and* (11) *the family* $\{P_{\theta,T}, \theta \in \Theta\}$ *is exponential and the statistics*

$$\psi_j(X(0)) + \frac{1}{2}\int_0^T \psi_j'(X(s))\mathrm{d}\, M_{\theta_0}(s), \quad 1 \leqslant j \leqslant d,$$

$$\int_0^T \psi_j'(X(s))\psi_k'(X(s))\sigma^2(X(s))\mathrm{d}\, s, \quad j \leqslant k,\ 1 \leqslant j, k \leqslant d,$$

are sufficient.

b) *If, additionally, the functions* $\psi_j(x)$, $x \in (l, r)$, $j = 1, \dots, d$, *are twice continuously differentiable, then the family* $\{P_{\theta,T}, \theta \in \Theta\}$ *is exponential and the statistics*

$$\psi_j(X(0)) + \psi_j(X(T)) - \frac{1}{2}\int_0^T \Big\{\psi_j''(X(s)) + \psi_j'(X(s)) \times \big[\ln\big(h_{\theta_0}(X(s))\sigma^2(X(s))\big)\big]'\Big\}\sigma^2(X(s))\mathrm{d}\, s, \quad 1 \leqslant j \leqslant d,$$

$$\int_0^T \psi_j'(X(s))\psi_k'(X(s))\sigma^2(X(s))\mathrm{d}\, s, \quad j \leqslant k,\ 1 \leqslant j, k \leqslant d,$$

are sufficient.

Proof. a) Indeed, from (7) and (11), we have that

$$L_{\theta,T} = \sum_{j=1}^{d} \varphi_j(\theta)\Big(\psi_j(X(0)) + \frac{1}{2}\int_0^T \psi_j'(X(s))\mathrm{d}\, M_{\theta_0}(s)\Big) - \frac{1}{8}\sum_{j,k=1}^{d} \varphi_j(\theta)\varphi_k(\theta)\int_0^T \psi_j'(X(s))\psi_k'(X(s))\sigma^2(X(s))\mathrm{d}\, s,$$

obviously proving a).

b) Analogously, from (8) and (11), we find that

$$L_{\theta,T} = \sum_{j=1}^{d} \varphi_j(\theta)\frac{1}{2}\Big(\psi_j(X(0)) + \psi_j(X(T))$$
$$-\frac{1}{2}\int_0^T \Big\{\psi_j''(X(s)) + \psi_j'(X(s))\left[\ln\left(h_{\theta_0}(X(s))\sigma^2(X(s))\right)\right]'\Big\}\sigma^2(X(s))\mathrm{d}\,s\Big)$$
$$-\frac{1}{8}\sum_{j,k=1}^{d} \varphi_j(\theta)\varphi_k(\theta)\int_0^T \psi_j'(X(s))\psi_k'(X(s))\sigma^2(X(s))\mathrm{d}\,s,$$

proving b).

We conclude by considering several examples of parametric families of H-diffusions, for which the assumptions (1)–(3) are checked in Grigelionis [2], [3]. We shall denote norming constants for the stationary distributions by C and shall take $g_\theta(x) = (h_\theta(x)\sigma^2(x))^{-1}$, $x \in (l,r)$, $\theta \in \Theta$.

Example 1. Let $(l,r) = (0,1)$,

$$h_\theta(x) = Cx^{\beta_1-1}(1-x)^{\beta_2-1}\mathrm{e}^{\lambda x},$$
$$\sigma^2(x) = \sigma^2 x^{\alpha_1}(1-x)^{\alpha_2}\mathrm{e}^{\mu x},$$
$$\theta = (\beta_1,\beta_2,\lambda) \in \Theta := (0,\infty)\cap[2-\alpha_1,\infty)\times(0,\infty)\cap[2-\alpha_2,\infty)\times R^1,$$
$$\alpha_1,\ \alpha_2,\ \sigma^2 > 0,\ \mu \in R^1 \text{ are known constants.}$$

Because

$$\ln\frac{h_\theta(x)}{h_{\theta_0}(x)} = (\beta_1-\beta_{10})\ln x + (\beta_2-\beta_{20})\ln(1-x) + (\lambda-\lambda_0)x,$$

we find that $d=3$, $\varphi_1(\theta) = \beta_1-\beta_{10}$, $\varphi_2(\theta) = \beta_2-\beta_{20}$, $\varphi_3(\theta) = \lambda-\lambda_0$, $\psi_1(x) = \ln x$, $\psi_2(x) = \ln(1-x)$, $\psi_3(x) = x$.

Example 2. a) Let $(l,r) = (0,\infty)$,

$$h_\theta(x) = Cx^{\lambda-1}\exp\left\{-\chi_1 x^{\beta_1} - \chi_2 x^{-\beta_2}\right\},$$
$$\sigma^2(x) = \sigma^2 x^{\gamma},$$
$$\theta = (\chi_1,\chi_2,\lambda) \in \Theta := (0,\infty)\times(0,\infty)\times R^1,$$
$$\beta_1,\ \beta_2,\ \sigma^2 > 0,\ \gamma \in R^1 \text{ are known constants.}$$

Because

$$\ln\frac{h_\theta(x)}{h_{\theta_0}(x)} = (\lambda-\lambda_0)\ln x + (\chi_{10}-\chi_1)x^{\beta_1} + (\chi_{20}-\chi_2)x^{-\beta_2},$$

we have that $d = 3, \varphi_1(\theta) = \lambda - \lambda_0, \varphi_2(\theta) = \chi_{10} - \chi_1, \varphi_3(\theta) = \chi_{20} - \chi_2, \psi_1(x) = \ln x$, $\psi_2(x) = x^{\beta_1}, \psi_3(x) = x^{-\beta_2}$.

b) Let $(l, r) = (0, \infty)$,

$$\begin{aligned} &h_\theta(x) = Cx^{\lambda-1}\exp\left\{-\chi_1 x^{\beta_1}\right\}, \\ &\sigma^2(x) = \sigma^2 x^\gamma, \\ &\theta = (\chi_1, \lambda) \in \Theta := (0, \infty) \times [2-\gamma, \infty) \cap (0, \infty), \\ &\beta_1,\ \sigma^2 > 0,\ \gamma \in R^1 \text{ are known constants.} \end{aligned}$$

From the equality

$$\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} = (\lambda - \lambda_0)\ln x + (\chi_{10} - \chi_1)x^{\beta_1}$$

we find that $d = 2, \varphi_1(\theta) = \lambda - \lambda_0, \varphi_2(\theta) = \chi_{10} - \chi_1, \psi_1(x) = \ln x, \psi_2(x) = x^{\beta_1}$.

c) Let $(l, r) = (0, \infty)$,

$$\begin{aligned} &h_\theta(x) = Cx^{\lambda-1}\exp\left\{-\chi_2 x^{-\beta_2}\right\}, \\ &\sigma^2(x) = \sigma^2 x^\gamma, \\ &\theta = (\chi_2, \lambda) \in \Theta := (0, \infty) \times (-\infty, 2-\gamma] \cap (-\infty, 0), \\ &\beta_2,\ \sigma^2 > 0,\ \gamma \in R^1 \text{ are known constants.} \end{aligned}$$

Because

$$\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} = (\lambda - \lambda_0)\ln x + (\chi_{20} - \chi_2)x^{-\beta_2},$$

we obtain that $d = 2, \varphi_1(\theta) = \lambda - \lambda_0, \varphi_2(\theta) = \chi_{20} - \chi_2, \psi_1(x) = \ln x, \psi_2(x) = x^{-\beta_2}$.

Example 3. Let $(l, r) = (b, \infty)$,

$$\begin{aligned} &h_\theta(x) = Cx^{-\theta}, \\ &\sigma^2(x) = \sigma^2 (x-b)^\gamma, \\ &\theta \in \Theta := (1, \infty) \cap [\gamma - 1, \infty), \\ &b,\ \sigma^2 > 0,\ \gamma \geqslant 1 \text{ are known constants.} \end{aligned}$$

In this case $d = 1, \varphi_1(\theta) = -\theta + \theta_0$ and $\psi_1(x) = \ln x$.

Example 4. Let $(l, r) = (0, \infty)$,

$$\begin{aligned} &h_\theta(x) = Cx^{\lambda}\exp\left\{-\chi(\ln x)^{2k}\right\}, \\ &\sigma^2(x) = \sigma^2 x^\gamma, \\ &\theta = (\chi, \lambda) \in \Theta := (0, \infty) \times R^1, \\ &\sigma^2 > 0,\ \gamma \in R^1,\ k = 1, 2, \dots \text{ are known constants.} \end{aligned}$$

We easily find that in this example $d = 2$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = -\chi + \chi_0$, $\psi_1(x) = \ln x$, $\psi_2(x) = (\ln x)^{2k}$.

Example 5. Let $(l, r) = (-\infty, \infty)$,

$$h_\theta(x) = C(1+x^2)^\lambda \exp\{-\kappa \operatorname{arctg} x\},$$
$$\sigma^2(x) = \sigma^2(1+x^2)^\gamma,$$
$$\theta = (\lambda, \kappa) \in \Theta := \Big(-\infty, -\frac{1}{2}\Big) \cap \Big(-\infty, \frac{1}{2} - \gamma\Big] \times R^1,$$
$$\sigma^2 > 0,\ \gamma \in R^1 \text{ are known constants.}$$

In this case $d = 2$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = -\kappa + \kappa_0$, $\psi_1(x) = \ln(1+x^2)$, $\psi_2(x) = \operatorname{arctg} x$.

References

[1] H.J. Engelbert, W. Schmidt, On solutions of one-dimensional stochastic differential equations without drift, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **68**, 287–314 (1985).

[2] B. Grigelionis, On generalized z-diffusions, In: H.J. Engelbert *et al.* (Eds.), *Stochastic Processes and Applications, Stochastic Monographs*, Gordon& Breach Science Publ. (to appear) (2001).

[3] B. Grigelionis, On point measures of ε-upcrossings for H-diffusions, Preprintas Nr. 2001–25, Matematikos ir informatikos institutas, Vilnius (2001).

[4] I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York (1998).

[5] U. Küchler, M. Sørensen, *Exponential Families of Stochastic Processes*, Springer, Berlin (1997).

[6] R.S. Liptser, A.N. Shiryaev, *Statistics of Random Processes*, Vol. I, Springer, Berlin (2001).

[7] P. Mandl, *Analytical Treatment of One-dimensional Markov Processes*, Springer, New York (1968).

[8] R. Morton, On the optimal control of stationary diffusion processes with inaccessible boundaries and no discounting, *J. Appl. Probab.*, **8**, 551–560 (1971).

Apie statistinius eksperimentus stebint H-difuzijas

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Darbe yra nagrinėjama ergodiškų griežtai stacionarių difuzinių procesų atvirame intervale $(l, r) \subseteq R^1$ su iš anksto duotu stacionariuoju skirstiniu H, vadinamų H-difuzijomis, parametrinė šeima. Yra rastos formulės tikėtinumo santykių logaritmams stacionariųjų skirstinių tankių bei difuzijos koeficientų terminais, o taip pat aprašytos kreivos H-difuzijų eksponentinės šeimos ir atitinkamos pakankamos statistikos. Rezultatai yra iliustruojami keletu pavyzdžių iš gerai žinomų difuzinių modelių gamtos moksluose bei matematinėje finansų teorijoje.