

On the mean square for the periodic zeta–function on the critical line

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Let $s = \sigma + it$ be a complex variable, and let $\mathfrak{A} = \{a_m, m \in \mathbb{Z}\}$ be a sequence of complex numbers with period $k > 0$. The periodic zeta-function $\zeta(s; \mathfrak{A})$, for $\sigma > 1$, is defined by

$$\zeta(s; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and by analytic continuation elsewhere. It was introduced and studied by B.C. Berndt and L. Schoenfeld in [1]. The papers [2] and [3] are devoted to asymptotics of the mean square

$$\int_0^T |\zeta(\sigma + it; \mathfrak{A})|^2 dt, \quad T \rightarrow \infty,$$

in the critical strip. For this aim in [2] an approximation of $\zeta(s; \mathfrak{A})$ by a finite sum was used, while in [3] an approximate functional equation for $\zeta(s; \mathfrak{A})$ was applied. An asymptotic formula of [3] is more precise than in [2]. The aim of this note is to apply an approximate functional equation in the case $\sigma = \frac{1}{2}$.

Let

$$\begin{aligned} K(k) &= \sum_{q=1}^k |a_q|^2; \\ K_1(k) &= \sum_{q=1}^k q |a_q|^2 \sum_{m=1}^{\infty} \frac{1}{m(mk + q)}; \\ K_2(k) &= k \sum_{q=1}^k \frac{|a_q|^2}{q}. \end{aligned}$$

Theorem. *Let $T \rightarrow \infty$. Then*

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it; \mathfrak{A}\right) \right|^2 dt = k^{-1} K(k) T \log T + k^{-1} K(k) T (2\gamma - \log \pi - 1)$$

$$-k^{-1}T(K_1(k) - K_2(k)) + Bk^{\frac{1}{2}}K(k)T^{\frac{1}{2}}\log T + BkK(k).$$

Proof. Using the notation of [3], we have Using the notation of [3], we have

$$\begin{aligned} & \int_{\frac{T}{2}}^T \left| \zeta\left(\frac{1}{2} + it; \mathfrak{A}\right) \right|^2 dt = k^{-1} \int_{\frac{T}{2}}^T \left| \sum_{q=1}^k a_q \sum_{0 \leq m \leq r} \frac{1}{(m + q/k)^{1/2+it}} \right|^2 dt \\ & + k^{-1} \int_{\frac{T}{2}}^T \left| \sum_{q=1}^k a_q \sum_{1 \leq m \leq n} e^{-2\pi imq/k} \frac{1}{m^{1/2-it}} \right|^2 dt \\ & + Bk^{-1} \left| \sum_{q=1}^k a_q \right|^2 \int_{\frac{T}{2}}^T t^{-1/2} dt + BK(k) \int_{\frac{T}{2}}^T t^{-3/2} dt \\ & + Bk^{-1} \left| \int_{\frac{T}{2}}^T \left(\frac{2\pi}{t} \right)^{it} e^{-it} \sum_{q_1=1}^k a_{q_1} \sum_{q_2=1}^k \bar{a}_{q_2} \right. \\ & \quad \times \sum_{0 \leq m_1 \leq r} \frac{1}{(m_1 + \frac{q_1}{k})^{1/2+it}} \sum_{1 \leq m_2 \leq n} e^{2\pi im_2/k} \frac{1}{m_2^{1/2+it}} dt \Big| \\ & + Bk^{-1} \left| \int_{\frac{T}{2}}^T \left(\frac{2\pi}{t} \right)^{1/4} \sum_{q_1=1}^k \bar{a}_{q_1} \sum_{q_2=1}^k a_{q_2} \sum_{0 \leq m \leq r} \frac{1}{(m + \frac{q}{k})^{1/2-it}} \right. \\ & \quad \times \exp \left\{ i f \left(\frac{q_2}{k}, t \right) \right\} \psi \left(2y - 2n + l - \frac{q_2}{k} \right) dt \Big| \\ & + Bk^{-1} \left| \int_{\frac{T}{2}}^T \left(\frac{2\pi}{t} \right)^{1/4-it} e^{-it} \sum_{q_1=1}^k \bar{a}_{q_1} \sum_{q_2=1}^k a_{q_2} \sum_{1 \leq m \leq n} e^{2\pi imq_1/k} \frac{1}{m^{1/2+it}} \right. \\ & \quad \times \exp \left\{ i f \left(\frac{q_2}{k}, t \right) \right\} \psi \left(2y - 2n + l - \frac{q_2}{k} \right) dt \Big| \\ & + Bk^{-1} \left| \int_{\frac{T}{2}}^T \sum_{q=1}^k a_q \sum_{0 \leq m \leq r} \frac{1}{(m + \frac{q}{k})^{1/2+it}} \bar{R}(s, k) dt \right| \\ & + Bk^{-1} \left| \int_{\frac{T}{2}}^T \left(\frac{2\pi}{t} \right)^{it} \sum_{q=1}^k a_q \sum_{1 \leq m \leq n} e^{-2\pi imq/k} \frac{1}{m^{1/2-it}} \bar{R}(s, k) dt \right| \end{aligned}$$

$$+ Bk^{-1} \left| \int_{\frac{T}{2}}^T \left(\frac{2\pi}{t} \right)^{1/4} \sum_{q=1}^k a_q e^{if(\frac{q}{k}, t)} \psi \left(2y - 2n + l - \frac{q}{k} \right) \bar{R}(s, k) dt \right| \stackrel{\text{def}}{=} \sum_{j=1}^{10} I_j. \quad (1)$$

Let $T_1 = \max \left(\frac{T}{2}, 2\pi(m_1 + \frac{q_1}{k})^2, 2\pi(m_2 + \frac{q_2}{k})^2 \right)$. Then, taking $M_q = \left[\left(\frac{T}{2\pi} \right)^{1/2} - \frac{q}{k} \right]$, we find

$$\begin{aligned} I_1 &= k^{-1} \sum_{q=1}^k |a_q|^2 \sum_{0 \leq m \leq M_q} \int_{T_1}^T \frac{dt}{m + \frac{q}{k}} \\ &+ k^{-1} \sum_{q=1}^k |a_q|^2 \sum_{0 \leq m_1 \leq M_q} \sum_{0 \leq m_2 \leq M_q} \frac{1}{(m_1 + \frac{q_1}{k})^{1/2} (m_2 + \frac{q_2}{k})^{1/2}} \\ &\times \int_{T_1}^T \left(\frac{m_1 + \frac{q}{k}}{m_2 + \frac{q}{k}} \right)^{it} dt + k^{-1} \sum_{q_1=1}^k a_{q_1} \sum_{q_2=1}^k \bar{a}_{q_2} \\ &\times \sum_{\substack{0 \leq m_1 \leq M_{q_1} \\ m_1 \neq m_2}} \sum_{0 \leq m_2 \leq M_{q_2}} \frac{1}{(m_1 + \frac{q_1}{k})^{1/2} (m_2 + \frac{q_2}{k})^{1/2}} \int_{T_1}^T \left(\frac{m_2 + \frac{q_2}{k}}{m_1 + \frac{q_1}{k}} \right)^{it} dt \\ &+ k^{-1} \sum_{q_1=1}^k a_{q_1} \sum_{\substack{q_2=1 \\ q_1 \neq q_2}}^k \bar{a}_{q_2} \sum_{0 \leq m \leq \max(M_{q_1}, M_{q_2})} \frac{1}{(m + \frac{q_1}{k})^{1/2} (m + \frac{q_2}{k})^{1/2}} \\ &\times \int_{T_1}^T \left(\frac{m + \frac{q_2}{k}}{m + \frac{q_1}{k}} \right)^{it} dt \stackrel{\text{def}}{=} \sum_{j=1}^4 I_{1j}. \end{aligned} \quad (2)$$

It is not difficult to see that for sufficiently large T

$$\begin{aligned} I_{11} &= k^{-1} \sum_{q=1}^k |a_q|^2 \sum_{0 \leq m \leq M_q} \frac{1}{m + \frac{q}{k}} \left(T - \max \left(\frac{T}{2}, 2\pi \left(m + \frac{q}{k} \right)^2 \right) \right) \\ &= k^{-1} T \sum_{q=1}^k |a_q|^2 \sum_{0 \leq m \leq M_q} \frac{1}{m + \frac{q}{k}} - \frac{T}{2} \sum_{q=1}^k \frac{|a_q|^2}{q} \\ &- \frac{T}{2k} \sum_{q=1}^k |a_q|^2 \sum_{1 \leq m \leq \frac{\sqrt{T}}{2\sqrt{\pi}} - \frac{q}{k}} \frac{1}{m + \frac{q}{k}} - 2\pi k^{-1} \sum_{q=1}^k |a_q|^2 \sum_{\substack{1 \leq m \leq M_q \\ m > \frac{\sqrt{T}}{2\sqrt{\pi}} - \frac{q}{k}}} \left(m + \frac{q}{k} \right). \end{aligned} \quad (3)$$

By the formula

$$\sum_{m \leq x} \frac{1}{m} = \log x + \gamma + \frac{B}{x},$$

where γ is the Euler constant, we have

$$\begin{aligned} \sum_{0 \leq m \leq M_q} \frac{1}{m + \frac{q}{k}} &= \frac{k}{q} + \sum_{1 \leq m \leq M_q} \frac{1}{m + \frac{q}{k}} = \frac{k}{q} + \sum_{1 \leq m \leq M_q} \frac{1}{m} - q \sum_{1 \leq m \leq M_q} \frac{1}{m(mk + q)} \\ &= \frac{k}{q} + \frac{1}{2} \log T - \log \sqrt{2\pi} + \gamma - q \sum_{m=1}^{\infty} \frac{1}{m(mk + q)} + \frac{B}{\sqrt{T}}. \end{aligned}$$

Similarly

$$\sum_{1 \leq m \leq \frac{\sqrt{T}}{2\sqrt{\pi}} - \frac{q}{k}} \frac{1}{m + \frac{q}{k}} = \frac{1}{2} \log T - \log \sqrt{4\pi} + \gamma - q \sum_{m=1}^{\infty} \frac{1}{m(mk + q)} + \frac{B}{\sqrt{T}}.$$

Moreover,

$$\begin{aligned} \sum_{\substack{1 \leq m \leq M_q \\ m > \frac{\sqrt{T}}{2\sqrt{\pi}} - \frac{q}{k}}} \left(m + \frac{q}{k} \right) &= \frac{\left[\sqrt{\frac{T}{2\pi}} - \frac{q}{k} \right] \left(\left[\sqrt{\frac{T}{2\pi}} - \frac{q}{k} \right] + 1 \right)}{2} \\ &\quad - \frac{\left[\sqrt{\frac{T}{4\pi}} - \frac{q}{k} \right] \left(\left[\sqrt{\frac{T}{4\pi}} - \frac{q}{k} \right] + 1 \right)}{2} \\ &\quad + B\sqrt{T} = \frac{T}{4\pi} - \frac{T}{8\pi} + B\sqrt{T} = \frac{T}{8\pi} + B\sqrt{T}. \end{aligned}$$

All these estimates together with (3) yield

$$\begin{aligned} I_{11} &= \frac{K(k)}{4k} T \log T + \frac{T}{2k} \left(K(k) \left(\gamma - \log \sqrt{\pi} - \frac{1}{2} \right) \right. \\ &\quad \left. - K_1(k) + K_2(k) \right) + Bk^{-1} K(k) T^{1/2}. \end{aligned} \quad (4)$$

For the integrals I_{1j} , $j \geq 2$, we can apply the estimates of [3] with $\sigma = \frac{1}{2}$. This gives

$$I_{12} = Bk^{-1/2} K(k) T^{1/4} + Bk^{-1} K(k) T^{1/2} \log T, \quad (5)$$

$$I_{13} = Bk^{1/2} K(k) T^{1/4} + BK(k) T^{1/2} \log T, \quad (6)$$

$$I_{14} = BkK(k). \quad (7)$$

Therefore (2) and (4)–(7) show that

$$\begin{aligned} I_1 &= \frac{K(k)}{4k} T \log T + \frac{T}{2k} \left(K(k) \left(\gamma - \log \sqrt{\pi} - \frac{1}{2} \right) \right. \\ &\quad \left. - K_1(k) + K_2(k) \right) + BK(k) T^{1/2} \log T + Bk^{1/2} K(k) T^{1/4} + BkK(k). \end{aligned} \quad (8)$$

Now let $T_2 = \max\left(\frac{T}{2}, 2\pi m_1^2, 2\pi m_2^2\right)$. Then

$$\begin{aligned}
 I_2 = & k^{-1} \sum_{q=1}^k |a_q|^2 \sum_{1 \leq m \leq M_0} \frac{1}{m} \left(T - \max\left(\frac{T}{2}, 2\pi m^2\right) \right) \\
 & + Bk^{-1} \sum_{q=1}^k |a_q|^2 \sum_{1 \leq m_1 \leq m_2 \leq M_0} \frac{m_2}{(m_1 m_2)^{1/2} (m_2 - m_1)} \\
 & + Bk^{-1} \sum_{\substack{q_1=1 \\ q_1 \neq q_2}}^k \sum_{q_2=1}^k |a_{q_1} \bar{a}_{q_2}| \sum_{1 \leq m_1 \leq m_2 \leq M_0} \frac{m_2}{(m_1 m_2)^{1/2} (m_2 - m_1)} \\
 & + Bk^{-1} \sum_{\substack{q_1=1 \\ q_1 \neq q_2}}^k \sum_{q_2=1}^k |a_{q_1} \bar{a}_{q_2}| \sum_{1 \leq m \leq M_0} \frac{1}{m} \sum_{j=1}^4 I_{2j}.
 \end{aligned} \tag{9}$$

Similarly to the case of I_{11} we find

$$\begin{aligned}
 I_{21} = & k^{-1} K(k) T \left(\log M_0 + \gamma + \frac{B}{\sqrt{T}} \right) - k^{-1} K(k) \frac{T}{2} \sum_{1 \leq m \leq \sqrt{\frac{T}{4\pi}}} \frac{1}{m} \\
 & - 2\pi k^{-1} K(k) \sum_{\substack{1 \leq m \leq M_0 \\ m > \sqrt{\frac{T}{4\pi}}}} m = \frac{K(k)}{4k} T \log T \\
 & + \frac{T}{2k} K(k) \left(\gamma - \log \sqrt{\pi} - \frac{1}{2} \right) + Bk^{-1} K(k) T^{1/2}.
 \end{aligned} \tag{10}$$

Moreover,

$$I_{22} = Bk^{-1} K(k) T^{1/2} \log T,$$

$$I_{23} = BK(k) T^{1/2} \log T,$$

$$I_{24} = BK(k) \log T.$$

This and (9), (10) give

$$I_2 = \frac{K(k)}{4k} T \log T + \frac{T}{2k} K(k) \left(\gamma - \log \sqrt{\pi} - \frac{1}{2} \right) + BK(k) T^{1/2} \log T. \tag{11}$$

The integrals I_j , $j \geq 3$, are estimated by the same way like in [3]. We have

$$I_3 = BK(k) T^{1/2},$$

$$I_4 = BK(k) T^{-1/2},$$

$$I_5 = Bk^{1/2} K(k) T^{1/2} + BK(k) T^{1/2} \log T,$$

$$I_6 = Bk^{1/2} K(k) T^{1/4} \log T + BK(k) T^{1/2} \log T,$$

$$I_7 = BK(k)T^{1/2} \log T,$$

$$I_8 + I_9 + I_{10} = Bk^{1/2}K(k)T^{1/2} \log T.$$

Now hence and from (1), (8) and (11) we deduce that

$$\int_{\frac{T}{2}}^T \left| \zeta\left(\frac{1}{2} + it; \mathfrak{A}\right) \right|^2 dt = \frac{1}{2k} K(k) T \log T + \frac{T}{k} K(k) \left(\gamma - \log \sqrt{\pi} - \frac{1}{2} \right) \\ - \frac{T}{2k} (K_1(k) - K_2(k)) + Bk^{1/2} K(k) T^{1/2} \log T + Bk K(k).$$

Taking in the later formula $2^{-\alpha}T$ instead of T and summing over all non-negative integers α , we obtain the assertion of the theorem.

References

- [1] B.C. Berndt, L. Schoenfeld, Periodic analogues of the Euler-Maclaurin and Poisson summation formulas with applications to number theory, *Acta Arith.*, **28**, 23–68 (1975).
- [2] A. Kačėnas, A. Laurinčikas, On the periodic zeta-function, *Liet. Matem. Rink.* (to appear).
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Periodinės dzeta funkcijos antrojo momento kritinėje tiesėje asimptotika

D. Šiaučiūnas

Straipsnyje gauta periodinės dzeta funkcijos antrojo momento asimptotika kritinėje tiesėje.