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The investigations of zeta-functions in Lithuania

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Functions of complex variable $s = \sigma + it$ defined in a certain half-plane by Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a(m)}{m^s} \quad \text{or} \quad \sum_{m=1}^{\infty} a(m) \mathrm{e}^{-\lambda_m s}$$

with complex coefficients $a(m)$ and real exponents λ_m, and having applications in number theory usually are called the zeta-functions. The most of them are generalizations of the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

Zeta-functions play an important role in number theory and in mathematics in general, therefore they are extensively studied by many mathematicians in all the world.

The investigations of zeta-functions in Lithuania can be divided into three parts. We ascribe to the first part the results on classical zeta-functions obtained for needs of the distribution of prime numbers. Essentially they consern the Hecke zeta-function. The investigations of the value-distribution itself of classical zeta-functions form the second part. The remainder part consists of permanent investigations of zeta-functions with multiplicative coefficients for needs of probabilistic number theory.

1. The Hecke Z-functions were introduced by E. Hecke in [5], [6] as a tool in studying the distribution of prime ideal numbers. Let $\mathbb{K}$ be an algebraic number field of a degree n, and let $\mathbb{K}_0$ stand for the extension of $\mathbb{K}$ to an ideal number system. Denote by α and γ an integer ideal number and a prime ideal number from $\mathbb{K}_0$, respectively. Moreover, let $\mu \neq (0)$ be a fixed integer ideal of $\mathbb{K}$. Denote by $\lambda(\alpha)$ and $\Xi(\alpha)$ the Hecke characters of the first and the second kind, respectively, for the definition see [5], [6]. For example, in the case of the quadratic field $\mathbb{Q}(\sqrt{d})$, $d < 0$,

$$\lambda^m(\alpha) = \mathrm{e}^{img \arg \alpha},$$

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where $m \in \mathbb{Z}$ and g is the number of roots from 1 which are units mod μ of the field $\mathbb{Q}(\sqrt{d})$. If χ is a character of the multiplicative group of the reduced residue system mod μ and

$$\chi(\varepsilon)\lambda^m(\alpha) = 1$$

for all units mod μ, then

$$\Xi(\alpha) = \lambda(\alpha)\chi(\alpha).$$

Denote by $N\alpha$ the norm of the number α, and let the asterisk in $\sum^*$ mean that the sum is taken over the set of nonassociated nonzero numbers.

The Hecke Z-functions $Z(s,m)$ and $Z(s,\Xi)$ are defined by the Dirichlet series

$$\sum_{\alpha}{}^{*}\lambda^m(\alpha)(N\alpha)^{-s} \quad \text{and} \quad \sum_{\alpha}{}^{*}\Xi(\alpha)(N\alpha)^{-s}, \quad \sigma > 1.$$

E. Hecke [5], [6] obtained the analytic continuation of the functions $Z(s,m)$ and $Z(s,\Xi)$, and also proved functional equations for them. He also applied these functions for the distribution of prime ideal numbers.

The investigations of the Hecke Z-functions in Lithuania are inseparably related to the name of Professor Jonas Kubilius. During the first period of his scientific activity J. Kubilius devoted much attention to multidimensional geometry of numbers and obtained fundamental results in the field. For this aim he developed the analytic machinery of Hecke's Z-functions.

The first results on Z-functions were obtained in the case of the Gaussian number field $\mathbb{Q}(i)$ [11]. Here an estimate for $Z(s,m)$ in the strip $-1/2 \leqslant \sigma \leqslant 4$ was given, a zero-free region was found and the logarithmic derivative of $Z(s,m)$ in this region was estimated. All these results alloved for J. Kubilius to find an asymptotic formula with a good error term for

$$\sum_{\substack{N\gamma \leqslant x \\ \varphi_1 \leqslant \arg \leqslant \varphi_2}} 1,$$

as $x \to \infty$.

In [12] J. Kubilius considered the function $Z(s,m)$ in the case of the quadratic imaginary number field and proved the following inportant zero-density theorem. Let $Q(\sigma)$ denote the number of functions $Z(s,m)$, $|m| \leqslant M$, $M \geqslant 2$, having at least one zero $\varrho = \beta + i\gamma$, $\sigma \leqslant \beta \leqslant 1$, $|\gamma| \leqslant \log^2 M$. Then, for any $\varepsilon > 0$, the estimate

$$Q(\sigma) = BM^{(16+\varepsilon)(1-\sigma)}$$

is valid. From the latter result J. Kubilius derived that there are infinitely many prime numbers of the form

$$p = a^2 + b^2,$$

with $|b| \leqslant p^{7/24+\varepsilon}$, $\varepsilon > 0$. The function $Z(s, \Xi)$ was studied systematically in the papers [13], [14]. The first part of the paper [13] is devoted to general number field. Here the estimates for $Z(s, \Xi)$ and its logarithmic derivative are given as well as a zero-free region is indicated. To define this region we need some notation. The Hecke characters of the first kind form an infinite Abelian group with basis consisting from $n-1$ elements. We fix a such basis $\lambda_1, \dots, \lambda_{n-1}$. Then the numbers $m_1, \dots, m_{n-1}$ such that

$$\lambda(\alpha) = \lambda_1^{m_1}(\alpha) \dots \lambda_{n-1}^{m_{n-1}}(\alpha)$$

are called the exponents of the character λ. We put

$$U(\Xi) = \prod_{j=1}^{n-1} (|m_j| + 3).$$

Then in [13] it is proved that the function $Z(s, \Xi)$ has no zeros in the region

$$\sigma \geqslant 1 - \frac{c}{\log\left((|t|+3)U(\Xi)\right)}, \quad c > 0.$$

The mentioned results were applied to obtain the asymptotic distribution law for prime ideal numbers. The results obtained improve and generalize these of E. Hecke and H. Rademacher. The second part of [13] and the paper [14] contain results for $Z(s, \Xi)$ in the case of the quadratic imaginary number field. Here the principal role is played by the zero-density method. This method in some applications replace the generalized Riemann hypothesis which asserts that all the zeros of $Z(s, \Xi)$ lie in the region $\sigma \leqslant 1/2$.

Later, the students of Professor J. Kubilius J. Urbelis, J. Vaitkevičius, K. Bulota, A. Matuliauskas, M. Maknys and E. Gaigalas continued the investigations of their scientific supervisor in the theory of Hecke's Z-functions. J. Urbelis considered a general number field [25], and also a purely real field and a quadratic real field. J. Vaitkevičius improved [26] some results of [11] in the case of the Gaussian number field. K. Bulota [1] derived an approximate functional equation for $Z(s, \Xi)$ in the case of $\mathbb{Q}(\sqrt{d})$, $d < 0$, and applied it for the distribution of prime numbers. It is of interest to note that Bulota's functional eqaution was used by S.M. Voronin to prove the functional independence of Hecke's Z-functions as well as to estimate the number of zeros in some regions for zeta-functions attached to certain quadratic forms. Some years later an approximate functional equation of Bulota was improved by A.F. Lavrik [19]. A. Matuliauskas dealt with the field $\mathbb{Q}(\sqrt{d})$, $d > 0$. He obtained an approximate functional equation for the Hecke Z-function [21] and applied it to estimate $Z(1/2 + it, \Xi)$. M. Maknys studied both imaginary and the real quadratic number fields and developed the Kubilius zero-density method [20]. E. Gaigalas studied [2] the scalar product of the Hecke Z-functions of two quadratic imaginary number fields. He obtained the analytic continuation to the region $\sigma > 1/2$ for this product and proved zero-density theorems. Later E. Gaigalas investigated the Poincaré series and extended results of R.A. Rankin and J. Lehner on the nonvanishing of these series. Also he considered the zeta-function of a binary Hermitian form over the ring of integers of the extensions $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(i, \sqrt{d})/\mathbb{Q}(\sqrt{d})$ with $d \equiv 2 (\mathrm{mod}\, 8)$.

Note that the results of A. Matuliauskas, some results of K. Bulota, M. Maknys and E. Gaigalas are purely devoted to zeta-functions, however we ascribed them to the first part because of their relation to algebraic number theory.

E. Manstavičius jointly with K.-H.Indlekofer and R. Warlimont [7] investigated the zeta-function associated with some semigroup introduced by J. Knopfmacher.

2. A new research period for zeta-functions starts in Lithuania in the 8th decade of the last century. Zeta-functions were began to study themselves, but not only having in mind their applications. The first results in this field were obtained by E. Stankus. In [22], [23] he proved limit theorems for Dirichlet L-functions with increasing modulus on the complex plane. The results obtained improved those of P.D.T.A. Elliott who proved theorems of a such kind for the modulo and the argument separately. Moreover, E. Stankus investigated a discrete version of moments for Dirichlet L functions including complex moments. The latter investigations continue those of M. Jutila and A.F. Lavrik. Last time E. Stankus deals with zeta-functions of generalized prime numbers.

Probably in 1977 the author of this paper also began to study the value-distribution of zeta-function. The first result in this direction was a limit theorem in the sense of the weak convergence of probability measures for Dirichlet L-functions. Note that an idea of application of probabilistic methods in the investigations of the value-distribution of zeta-functions comes back to H. Bohr and B. Jessen. In the third decade of the last century they proved the first probabilistic limit theorems for $\log \zeta(s)$. In [15] a generalization of the Voronin theorem on the universality of $\zeta(s)$ was given for a class of zeta-functions with multiplicative coefficients. The latter paper also contains a theorem on the functional independence of the mentioned zeta-functions. Later the results of [15] were extended to other classes of zeta functions with multiplicative coefficients.

The next results of the author are related to the value-distribution of $\zeta(s)$. Limit theorems for this function were proved in the region $\sigma \geqslant 1/2$ in $\mathbb{R}$ and $\mathbb{C}$ as well as in the space of analytic and continuous functions. Note that in the case $\sigma = 1/2$ or $\sigma \to 1/2+0$ some norming for $\zeta(s)$ is necessary. The most of the results obtained can be found in the monograph [16].

The second problem of $\zeta(s)$ investigated in Lithuania is the moment problem. We obtained the asymptotic for moments of the normalized Riemann zeta-function on and near the critical line and proved the estimates of Heath-Brown's type near the critical line. For the mean square the Atkinson formula near the critical line also was obtained. A. Kačėnas found the asyptotics for the mean square and for the fourth moment of $\zeta(s)$ near the critical line [9] as well as he gave an estimate for the twelfth moment. R. Šleževičienė obtained [24] multidimensional limit theorem for $\zeta(s)$.

Much attention in Lithuania was and is devoted to the Lerch zeta-function $L(\lambda, \alpha, s)$ introduced and studied by M. Lerch in 1887. For $\sigma > 1$ the function $L(\lambda, \alpha, s)$ is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. Here $\lambda \in \mathbb{R}$ and α, $0 < \alpha \leqslant 1$, are fixed parameters. First, the author jointly with R. Garunkštis obtained an approximation of $L(\lambda, \alpha, s)$ by a finite sum and limit theorems on $\mathbb{C}$ and in the space of analytic functions. Later the universality of $L(\lambda, \alpha, s)$ for some parameters was also obtained. R. Garunkštis dealt with weighted limit theorems for $L(\lambda, \alpha, s)$ [3]. The results obtained on zero-distribution of $L(\lambda, \alpha, s)$ concern zero-free regions, an asymptotic formula for the number of non-trivial zeros, estimates for the number of zeros in various regions. The papers [17] and [18] are devoted to joint value-distribution theorems for $L(\lambda, \alpha, s)$. Discrete limit theorems for $L(\lambda, \alpha, s)$ were proved [8] by J. Ignatavičiūtė.

The first results on the mean square of $L(\lambda, \alpha, s)$ were obtained by D. Klush in 1987–1989. To obtain more precise results in this field an approximate functional equation for $L(\lambda, \alpha, s)$ was obtained in [4].

Also we studied the Matsumoto zeta-function defined by some polynomial Euler product in 1990 by K. Matsumoto. Limit theorems for this functions were proved in the spaces of analytic and meromorphic functions as well as the universality and some results on zero-distribution were given. R. Kačinskaitė obtained [10] discrete limit theorems for the Matsumoto zeta-function.

The last period zeta-functions attached to certain cusp forms as well as to finite Abelian group were studied. The author himself and jointly with K. Matsumoto proved limit theorems and the universality for these functions.

3. Zeta-functions with multiplicative coefficients in connection with probabilistic number theory were and are studied by all numbers of the Kubilius probabilistic number theory school. We recall the names of theses mathematicians: J. Kubilius, R. Uždavinys, A. Bakštys, Z. Juškys, E. Manstavičius, R. Skrabutėnas, A. Laurinčikas, Z. Kryžius, G. Stepanauskas, A. Mačiulis, V. Nakutis, I. Orlov, V. Stakėnas, J. Šiaulys, G. Bareikis.

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Dzeta funkcijų tyrimai Lietuvoje

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