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On integer and fractional parts of some sequences

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1. Introduction

Consider the decomposition of sequence of real numbers b_n into two sequences:

$$b_n = [b_n] + \{b_n\},$$

where $[\cdot]$ denotes the integer and $\{\cdot\}$ the fractional part of a real number. We call the sequence b_n dense if for each infinite arithmetical progression A of natural numbers and each interval $J \subset [0;1)$ the following two sets

$$\{n : [b_n] \in A\}, \quad \{n : \{b_n\} \in J\}$$

are infinite.

For example, the sequences $n\alpha$ with $0 < \alpha < 1$ an irrational number are dense. It may be proved, that if a_n is a sequence of positive numbers increasing unboundedly then for almost all $\alpha > 0$ the sequence $a_n\alpha$ is dense (cnf. [3]). We show that with such a sequence of real numbers a_n the sequences α^{a_n} are also dense for almost all $\alpha > 1$.

In the theorem below λ stands for the Lebesque measure on the real line and $\nu(A)$ denotes the density of a set of natural numbers A (if it exists); for example, $\nu\{n : n \equiv m \pmod M\} = \frac{1}{M}$, where m, M are fixed natural numbers.

Theorem. *Let a_n be a sequence of positive real numbers increasing to infinity, $0 < \epsilon_n < 1$ and*

$$\Delta_n = \Big[\sum_{m \leqslant n} \epsilon_n\Big] \to \infty, \quad \epsilon_n \alpha_0^{a_n - a_{n-\Delta_n}} \gg 1, \tag{1}$$

where $\alpha_0 > 1$ is some fixed number. Let I_n and A_n be two sequences of intervals in $[0;1)$ and arithmetical progressions respectively, such that $\lambda(I_n) \geqslant \epsilon_n$, $\nu(A_n) \geqslant \epsilon_n$. Then for almost all $\alpha > \alpha_0$ both sets

$$\{n : [\alpha^{a_n}] \in A_n\}, \quad \{n : \{\alpha^{a_n}\} \in I_n\} \tag{2}$$

are infinite.

Let $M > 1$ be a natural number, $\epsilon_n = M^{-1}$ for all n and $\alpha_0 > 1$. Then for any increasing sequence $a_n \to \infty$ the condition (1) is satisfied. Let ω be a rational number, $\epsilon_n < \omega < 1 - \epsilon_n$, and m an integer, such that $0 \leqslant m < M$. Let the set $L(\omega, m, M)$ consists of all $\alpha > \alpha_0$ for which both sets (2) are infinite with $I_n = (\omega - \epsilon_n, \omega + \epsilon_n)$, $A_n = \{x : x \equiv m \pmod M\}$. From the theorem we have that almost all $\alpha > \alpha_0$ belong to $L(\omega, m, M)$. Because

$$\{\alpha : \alpha > \alpha_0, \alpha^{a_n} \text{is dense}\} = \bigcap_{\omega,m,M} L(\omega, m, M),$$

then α^{a_n} is dense for almost all $\alpha > \alpha_0$ and then for almost all $\alpha > 1$.

The theorem gives some quantitative characteristic of the denseness of almost all sequences α^{a_n}. For an example, if $a_n = n^\gamma, \gamma \geqslant 1$, then $\epsilon_n = n^{-1}$ may be chosen in the theorem. If $a_n = n^\gamma$ and $0 < \gamma < 1$, then (1) holds with $\epsilon_n = \rho_n \log n / n^\gamma$ with an arbitrary sequence $\rho_n \to \infty$.

2. Proof of the Theorem

For the proof we use the measure-theoretic arguments, explained in details, for example, in [1]. Generally speaking, it is to be shown that for certain subsets $\mathcal{B}_n$ of an interval I the set $\limsup \mathcal{B}_n$ contains almost all $\alpha \in I$. This may be done in two steps: proving that $\lambda(\limsup \mathcal{B}_n \cap J) > \delta\lambda(J)$ for any subinterval $J \subset I$ (with $\delta > 0$ independent of J) and then using the Lebesque density theorem to conclude that the set $\limsup \mathcal{B}_n$ contains almost all $\alpha \in I$. The ready to use tool for proving our statements is the following proposition.

Lemma ([2], Lemma 6.1, p.171). *Let J be a subinterval of the real line and $\mathcal{D}_n$ be a sequence of subsets of J. For each open interval $I \subset J$ suppose that there is a sequence of sets $\mathcal{B}_n \subset \mathcal{D}_n \cap I$ such that*

$$\sum_{n=1}^{\infty} \lambda(\mathcal{B}_n) = +\infty,$$

and

$$\limsup_{N\to\infty} \Big(\sum_{n \leqslant N} \lambda(\mathcal{B}_n) \Big)^2 \Big(\sum_{m,n \leqslant N} \lambda(\mathcal{B}_n \cap \mathcal{B}_m) \Big)^{-1} \geqslant \delta\lambda(I), \tag{3}$$

where δ is a positive constant independent on I. Then almost all $\alpha \in J$ belong to infinitely many $\mathcal{D}_n$.

Note, that some quantitative version of this result can be used (see [1], Theorem 3).

We prove only that the first set in (2) is infinite for almost all $\alpha > \alpha_0$. The arguments for the second one are similar and the calculations are easier.

Let $A_n = \{x : x \equiv m_n (\mathrm{mod}\, M_n)\}$ and $M_n^{-1} \geqslant \epsilon_n$. Then

$$\frac{1}{M_n}\alpha_0^{a_n - a_{n-\Delta_n}} \gg 1.$$

It is straightforward to derive that $[\alpha^{a_n}] \in A_n$ (that is $[\alpha^{a_n}] \equiv m_n(\mathrm{mod}\, M_n)$) holds if and only if there exists some natural number s such that

$$\log\alpha \in I(n,s), \quad I(n,s) = \frac{1}{a_n}\big[\log(sM_n + m_n);\ \log(sM_n + m_n + 1)\big].$$

Hence $[\alpha^{a_n}] \in A_n$ holds for an infinite sequence of n if

$$\log\alpha \in \limsup \mathcal{D}_n, \quad \mathcal{D}_n = \bigcup_{s>0} I(n,s).$$

We have to prove now that almost all α $(\alpha > a_0, a_0 = \log\alpha_0)$ belong to $\limsup \mathcal{D}_n$. Let $I = (a; a+b), a > a_0$, and

$$\mathcal{B}_n = I \bigcap \left(\bigcup_{s>0} I(n,s) \right).$$

It is necessary to show that the condition (3) of Lemma is satisfied. Because of $a_n \to \infty$, we may suppose that $a_n > 1$ for all n. For any interval $I(n,s)$ we have

$$\lambda(I(n,s)) = \frac{1}{a_n}\log\left(1 + \frac{1}{sM_n + m_n}\right) \quad \text{and} \quad \frac{c_1}{a_nM_ns} \leqslant \lambda(I(n,s)) \leqslant \frac{c_2}{a_nM_ns} \tag{4}$$

with some positive and absolute constants c_1, c_2. For $s \neq t$ the intervals $I(n,s)$ and $I(n,t)$ are disjoint. Hence, we shall obtain the bound for $\lambda(\mathcal{B}_n)$ from below if we sum all $\lambda(I(n,s))$ such that $I(n,s) \subset I$. For the upper bound we have to sum all $\lambda(I(n,s))$ with the condition $I \cap I(n,s) \neq \varnothing$. Let us establish the lower bound, the upper can be obtained similarly.

The condition $I(n,s) \subset I$, or equivalently,

$$a < \frac{1}{a_n}\log(sM_n + m_n) < \frac{1}{a_n}\log(sM_n + m_n + 1) < a + b$$

give the following range for s:

$$r(n) < s < R(n), \quad r(n) = \frac{\mathrm{e}^{aa_n} - m_n}{M_n}, \quad R(n) = \frac{\mathrm{e}^{(a+b)a_n} - m_n - 1}{M_n}.$$

Observe that

$$R(n) - r(n) = \frac{\mathrm{e}^{aa_n}}{M_n}\left(\mathrm{e}^{ba_n} - 1 - \mathrm{e}^{-aa_n}\right).$$

Because of (1) we have $M_n^{-1}\mathrm{e}^{aa_n} > M_n^{-1}\alpha_0^{a_n} \gg 1$. Hence $R(n) - r(n) \gg 1$ and we can estimate $\lambda(\mathcal{B}_n)$ as follows:

$$\lambda(\mathcal{B}_n) \geqslant \sum_{r(n)<s<R(n)} \lambda(I(n,s)) \geqslant \frac{c_1}{a_n M_n} \sum_{r(n)<s<R(n)} \frac{1}{s} \geqslant \frac{c_3}{a_n M_n} \int\limits_{r(n)}^{R(n)} \frac{\mathrm{d}s}{s}.$$

Using the expressions for $r(n)$ and $R(n)$ we derive that

$$\lambda(\mathcal{B}_n) \geqslant \frac{c_4 b a_n}{a_n M_n} = c_4 \lambda(I) \frac{1}{M_n}.$$

Hence,

$$\sum_{n\leqslant N} \lambda(\mathcal{B}_n) \geqslant c_4 \lambda(I) \sum_{n\leqslant N} \frac{1}{M_n} \to \infty, \quad N \to \infty. \tag{5}$$

Arguing similarly we obtain the bound

$$\lambda(\mathcal{B}_n) \leqslant c_5 \lambda(I) \frac{1}{M_n}. \tag{6}$$

Now we derive the upper bound for $\lambda(\mathcal{B}_k \cap \mathcal{B}_n)$, where $k < n$. The set $\mathcal{B}_k$ consists of the not-intersecting intervals $I \cap I(k,t)$. It is easy to obtain that $I \cap I(k,t) \neq \varnothing$ for

$$M_k^{-1}\big(\mathrm{e}^{aa_k} - m_k - 1\big) < t < M_k^{-1}\big(\mathrm{e}^{(a+b)a_k} - m_k\big). \tag{7}$$

Fix t from this range and find the bound for

$$\lambda\big(I(k,t) \cap \mathcal{B}_n\big) = \sum_{s>0} \lambda\big(I(k,t) \cap I(n,s)\big) \leqslant \sum \lambda\big(I(n,s)\big),$$

where the last sum is taken over those s, for which $I(k,t) \cap I(n,s) \neq \varnothing$. This condition is satisfied if $a_n^{-1}\log(sM_n + m_n)$ or $a_n^{-1}\log(sM_n + m_n + 1)$ belongs to $I(k,t)$. We get then the range $a(n|k,t) < s < A(n|k,t)$ for s, where

$$a(n|k,t) = M_n^{-1}\big((tM_k + m_k)^{\theta_{n,k}} - m_n - 1\big),$$
$$A(n|k,t) = M_n^{-1}\big((tM_k + m_k + 1)^{\theta_{n,k}} - m_n\big),$$

$\theta_{n,k} = a_n/a_k$. Then according to (4) we obtain

$$\lambda\big(I(k,t) \cap \mathcal{B}_n\big) \leqslant \frac{c_2}{a_n M_n} \sum_{a(n|k,t)<s<A(n|k,t)} \frac{1}{s}.$$

We look for which $k < n$ the sum can be estimated by an integral. Using the elementary inequality $(1+x)^\theta - 1 > \theta x$, valid for $x \geqslant 0, \theta > 1$, in

$$\begin{aligned}A(n|k,t) - a(n|k,t) &= \frac{(tM_k + m_k)^{\theta_{n,k}}}{M_n}\Bigg(\left(1 + \frac{1}{tM_k + m_k}\right)^{\theta_{n,k}} - 1 \\ &\qquad - \frac{1}{(tM_k + m_k)^{\theta_{n,k}}}\Bigg),\end{aligned}$$

we get

$$A(n|k,t) - a(n|k,t) > \frac{\theta_{n,k}(tM_k + m_k)^{\theta_{n,k}-1}}{M_n} - \frac{1}{M_n} > c_6 \frac{\theta_{n,k}(tM_k + m_k)^{\theta_{n,k}-1}}{M_n}.$$

Put the lower bound for t from (7) into the right-hand expression:

$$\begin{aligned}\frac{\theta_{n,k}(tM_k + m_k)^{\theta_{n,k}-1}}{M_n} &> \theta_{n,k}\frac{(e^{aa_k} - 1)^{\theta_{n,k}-1}}{M_n} \geqslant c_7\theta_{n,k}\frac{e^{a(a_n - a_k)}}{M_n} \\ &\geqslant c_7\theta_{n,k}\frac{\alpha_0^{a_n - a_k}}{M_n} \geqslant c_7\frac{\alpha_0^{a_n - a_k}}{M_n}.\end{aligned}$$

According to (1) we have $A(n|k,t) - a(n|k,t) \gg 1$ for $k_0 \leqslant k < n - \Delta_n$. For such pairs of $k < n$ we have

$$\lambda(I(k,t) \cap \mathcal{B}_n) \leqslant \frac{c_8}{a_n M_n}\int\limits_{a(n|k,t)}^{A(n|k,t)} \frac{\mathrm{d}s}{s} = \frac{c_8}{a_n M_n}\log\frac{A(n|k,t)}{a(n|k,t)}.$$

The expression under the logarithm may be reduced to the form

$$\frac{A(n|k,t)}{a(n|k,t)} = B\left(1 + \frac{1}{tM_k + m_k}\right)^{\theta_{n,k}}, \quad 1 < B < c_9.$$

Using now the elementary inequality $\log(Bu) < B\log u$ $(B > 1)$ we get

$$\lambda(I(k,t) \cap \mathcal{B}_n) \leqslant c_{10}\frac{\theta_{n,k}}{a_n M_n}\log\left(1 + \frac{1}{tM_k + m_k}\right) \leqslant c_{10}\frac{1}{M_k M_n a_k}\frac{1}{t}.$$

The range for t is given in (7). It is straightforward to obtain

$$\lambda(\mathcal{B}_k \cap \mathcal{B}_n) \leqslant c_{11}\frac{1}{a_k M_k M_n}\log e^{ba_k} = c_{11}\frac{\lambda(I)}{M_k M_n};$$

this holds for $k_0 < k < n - \Delta_n$. We shall use this inequality for $k_0 < k < n - \Delta_N$. For k not in this range we use the bound (6): $\lambda(\mathcal{B}_k \cap \mathcal{B}_n) \leqslant \lambda(\mathcal{B}_n) \leqslant c_5 \frac{\lambda(I)}{M_n}$. Now we have

$$\sum_{k,l \leqslant N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) \ll \sum_{l=1}^{N} \sum_{k \in [k_0, l-\Delta_N]} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) + \sum_{l=1}^{N} \sum_{\substack{k \notin [k_0, l-\Delta_N] \\ k \leqslant l}} \lambda(\mathcal{B}_k \cap \mathcal{B}_l)$$

$$\ll \lambda(I) \sum_{1 \leqslant k \leqslant l \leqslant N} \frac{1}{M_k M_l} + \lambda(I) \sum_{l=1}^{N} (k_0 + \Delta_N) \frac{1}{M_l} \ll \lambda(I) \Big(\sum_{n \leqslant N} \frac{1}{M_n} \Big)^2. \quad (8)$$

It follows now from (5) and (8), that the condition (3) of Lemma is satisfied. Hence the first set in (2) is infinite for almost all $\alpha > \alpha_0$. As it was mentioned above the statement about the second set can be proved similarly.

References

[1] G. Harman, Variants of the second Borel-Cantelli lemma and their applications in metric number theory, In: *Number Theory, Trends in Math*, Birkhäuser, 121–140 (2000).

[2] G. Harman, *Metric Number Theory*, Clarendon Press, Oxford, (1998).

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Sveikosios ir trupmeninės tam tikrų sekų dalys

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Darbe nagrinėjamos natūraliųjų skaičių sekos α^{a_n} čia $a_n > 0, \alpha > 1$. Įrodytoje teoremoje tvirtinama, kad esant tam tikroms sąlygoms skaičiai $[\alpha^{a_n}]$ tenkina lyginius $[\alpha^{a_n}] \equiv m_n \pmod{M_n}$ be galo daug kartų su beveik visais $\alpha > 1$. Taip pat suformuluotas teiginys apie trupmenines šių sekų dalis.