

The sixth power moment of the Riemann zeta-function in the critical strip

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The aim of this paper is to obtain the asymptotics for the quantity

$$\int_1^{\infty} |\zeta(\sigma + it)|^{2k} dt,$$

when $k = 3$ in the region $1/2 < \sigma < 1$. The cases $k = 1, 2$ have been completely investigated earlier and explicit asymptotic formulas have been found.

Generally, we have (see [1])

$$\int_1^T |\zeta(\sigma + it)|^{2k} dt = T \sum_{n=1}^{\infty} d_k^2(n) n^{-2\sigma} + R(k, \sigma; T),$$

were $R(k, \sigma; T) = o(T)$ as $T \rightarrow \infty$ and $1/2 < \sigma < 1$. Some improvements of the estimate for the error term $R(k, \sigma; T)$ were obtained assuming special cases for k in subranges of the critical strip $1/2 < \sigma < 1$. The latest results can be found in [1]. We will consider the case $k = 3$. The best estimate for the error term was obtained recently by A. Ivič [1] and has the form

$$R(3, \sigma; T) \ll_{\epsilon} T^{(17-12\sigma)/10}, \quad \frac{7}{12} < \sigma < 1.$$

Applying the new approximate functional equation for $|\zeta(\sigma + it)|^{2k}$ and the machinery of the paper [2] we may deduce the estimate for the error term $R(3, \sigma; T)$ valid in the whole critical strip.

Lemma 1. *Let $k \geq 1$ be an integer number. There exist constants $c \geq 1$, $\alpha_1(u, v)$, an integer U and the polynomials of U -th degree $\alpha_2(u, v) = P_{u,v}(\sigma)$, all depending only on k , such that*

$$|\zeta(\sigma + it)|^{2k} = \sum_{mn \leq cT^k} d_k(m) d_k(n) (mn)^{-\sigma} \left(\frac{m}{n}\right)^{it} K_1(mn, t)$$

$$+ \left(\frac{t}{2\pi}\right)^{-k(2\sigma-1)} \sum_{mn \leq cT^k} d_k(m)d_k(n)(mn)^{\sigma-1} \left(\frac{m}{n}\right)^{it} K_2(mn, t) \\ + O(T^{-2})$$

uniformly for $T \leq t \leq 2T$ and $1/2 < \sigma \leq \sigma_0 < 1$, where

$$K_j(x, t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \left\{ \frac{(t/2\pi)^k}{x} \right\}^z e^{z^2/T} \left(1 + \sum_{u=0}^U \sum_v \alpha_j(u, v) z^u t^{-2v} \right) \frac{dz}{z},$$

$j = 1, 2$, and $\sum_v$ denotes the summation over

$$\text{Max} \left(1, \frac{u}{3} \right) \leq v \leq u.$$

The proof of this Lemma can be found in [2]. In the cases $k = 1, 2$ this approximate functional equation implies the explicit formulas for the mean values.

Let T_1 and T_2 be any real numbers such that $T \leq T_1 \leq T_2 \leq 2T$. First we consider the integral

$$\int_{T_1}^{T_2} |\zeta(\sigma + it)|^{2k} dt.$$

Substituting the expression of the lemma in the integral we obtain

$$\int_{T_1}^{T_2} |\zeta(\sigma + it)|^{2k} = \sum_{mn \leq cT^k} d_k(m)d_k(n)(mn)^{-\sigma} \int_{T_1}^{T_2} \left(\frac{m}{n}\right)^{it} K_1(mn, t) dt \\ + \sum_{mn \leq cT^k} d_k(m)d_k(n)(mn)^{\sigma-1} \int_{T_1}^{T_2} \left(\frac{m}{n}\right)^{it} \left(\frac{t}{2\pi}\right)^{-2k(\sigma-1/2)} K_2(mn, t) dt + O(1).$$

The calculation of the above sums is complicated and needs serious machinery. All these general calculations are done in the paper [2] for special case $k = 2$. Applying the same method it is not difficult to obtain the corresponding estimates in the case $k = 3$. Splitting the calculations in the case $u = 0$ and $u \geq 1$, and later $m = n$, $m \neq n$, we find that

$$\int_{T_1}^{T_2} |\zeta(\sigma + it)|^6 dt = \sum_{m^2 \leq cT^3} d_3^2(m)(m)^{-2\sigma} \int_{T_1}^{T_2} H(m^2, t) dt \\ + \sum_{m^2 \leq cT^3} d_3^2(m)(m)^{-2+2\sigma} \int_{T_1}^{T_2} \left(\frac{t}{2\pi}\right)^{-6(\sigma-1/2)} H(m^2, t) dt$$

$$\begin{aligned}
& + \sum_{mn \leq cT^3, m \neq n} d_3(m)d_3(n)(mn)^{-\sigma} \left(\frac{m}{n}\right)^{it} \left(i \log \frac{m}{n}\right)^{-1} H(mn, t) \Big|_{T_1}^{T_2} \\
& + \sum_{mn \leq cT^3, m \neq n} d_3(m)d_3(n)(mn)^{\sigma-1} \left(\frac{m}{n}\right)^{it} \left(i \log \frac{m}{n}\right)^{-1} \left(\frac{t}{2\pi}\right)^{-6(\sigma-1/2)} H(mn, t) \Big|_{T_1}^{T_2} \\
& + \sum_{mn \leq cT^3, m \neq n} d_3(m)d_3(n)(mn)^{\sigma-1} \left(i \log \frac{m}{n}\right)^{-1} \left(\frac{m}{n}\right)^{iaT} H(mn, aT) \\
& \times \left[\left(\frac{T_2}{2\pi}\right)^{-6(\sigma-1/2)} - \left(\frac{T_1}{2\pi}\right)^{-6(\sigma-1/2)} \right] + O(T^{3(1-\sigma)-1/2+\varepsilon}) + O(T^\varepsilon),
\end{aligned}$$

uniformly in σ , $1/2 < \sigma \leq \sigma_0 < 1$ where $H(mn, t) = 0$ or $H(mn, t) = 0$ for $mn > (t/2\pi)^3$ or $mn < (t/2\pi)^3$, respectively, and a , $1 < a < 2$, is a constant.

Now putting $T_2 = 2T_0$, $T_1 = T_0$, $T_0 = T/2^n$ and summing over n , we obtain by [2]

$$\int_0^T |\zeta(\sigma + it)|^6 dt = T \sum_{m \leq (T/2\pi)^{3/2}} d_3^2(m) m^{-2\sigma} + O(T^{3(1-\sigma)-1/2+\varepsilon}) + O(T^\varepsilon).$$

The last equality can be rewritten in the following form

$$\int_0^T |\zeta(\sigma + it)|^6 dt = T \sum_{m=1}^{\infty} d_3^2(m) m^{-2\sigma} + O(T^{5/2-3\sigma+\varepsilon}) + O(T^\varepsilon),$$

uniformly in σ , $1/2 < \sigma \leq \sigma_0 < 1$. For $k > 3$ this method does not give sharper estimates for the function $R(k, \sigma; T)$.

References

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- [2] A. Каченас, Асимптотик четвертого степенного момент дзет-функции Риман в критическом поле, *Liet. Matem. Rink.*, 36(1), 39–54 (1996).

Šešetasis Rymano dzeta funkcijos momentas kritinėje juostoje

A. Kačėnas

Straipsnyje nagrinėjamas šešetasis Rymano dzeta funkcijos momentas kritinėje juostoje. Gautas įvertis

$$\int_0^T |\zeta(\sigma + it)|^6 dt = T \sum_{m=1}^{\infty} d_3^2(m) m^{-2\sigma} + O(T^{5/2-3\sigma+\varepsilon}) + O(T^\varepsilon),$$

tolygiai σ atžvilgiu, kai $1/2 < \sigma \leq \sigma_0 < 1$.