

An estimate for the Taylor coefficients

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We examine the Taylor coefficients of an analytic in $|z| < 1$ function $F(z)$ having a fairly particular form. Such functions appear frequently in analytic and probabilistic combinatorics as generating functions of the values of mappings defined on assemblies, multisets, selections or additive arithmetic semigroups (see [1–4], [6–8]).

Let

$$\begin{aligned} F(z) &:= \sum_{N \geq 0} m_N z^N = \sum_{k \geq 0} b_k z^k \exp \left\{ \sum_{j \geq 1} \frac{a_j z^j}{j} \right\} \\ &=: H(z) \exp \{U(z)\} =: H(z)G(z), \end{aligned} \quad (1)$$

where $a_j, b_j \in \mathbb{C}$. Typically, the function $H(z)$ satisfies some smoothness conditions on the circumference $|z| = 1$ and is "better" than $U(z)$.

In the case $H(z) \equiv 1$, formal expansion of the series leads to the formula

$$m_N = m_N(a_1, \dots, a_N) = \sum_{L(\bar{k})=N} \prod_{j=1}^n \left(\frac{a_j}{j} \right)^{k_j} \frac{1}{k_j!}, \quad (2)$$

where the summation is extended over vectors $\bar{k} = (k_1, \dots, k_n)$ with nonnegative integer coordinates and satisfying the relation $L(\bar{k}) := 1k_1 + \dots Nk_n = N$. Given some initial information on a_j , it is rather difficult to use (2) to derive asymptotical properties of m_N as $N \rightarrow \infty$.

In [8], we have obtained a few estimates of m_N in terms of the Taylor coefficients $\tilde{m}_N$ of the function

$$D(z) := \sum_{n \geq 0} \tilde{m}_n z^n = \exp \left\{ \sum_{j \leq N} \frac{d_j z^j}{j} \right\} =: \exp \{V(z)\},$$

provided that $|a_j| \leq d_j \leq d < \infty$ for each $1 \leq j \leq N$. This individual bound and a rather strong requirement $\sup_{|z| \leq 1} |H'(z)| \leq H < \infty$ were the main obstacles in some applications of the results. An instance of them is presented at the end of the paper. We now generalize Proposition 2 of our paper [8].

Theorem. *Let a, b, d , and B be positive constants such that $|a_j| \leq a$, $0 \leq d_j \leq d$,*

$$\sum_{j \leq N} |b_j| \leq B, \quad j|b_j| \leq b, \quad (3)$$

and

$$\sum_{j \leq \delta N} \frac{|a_j|}{j} \leq \sum_{j \leq \delta N} \frac{d_j}{j} + C, \quad \sum_{\delta N < j \leq N} \frac{d_j}{j} \geq c \log \frac{1}{\delta} \quad (4)$$

with some $c > 0$, $C \geq 0$, and arbitrary $\delta \in (0, 1]$. Then there exists a positive constant $c_1 = c_1(c, d)$ such that

$$m_N \ll \exp \left\{ \sum_{j \leq N} \frac{d_j - 1}{j} - c_1 \min_{|t| \leq \pi} \sum_{j \leq N} \frac{d_j - \Re(a_j e^{-itj})}{j} \right\}.$$

The constant in $\ll$, the analog of the symbol $O(\cdot)$, depends on a , b , d , B , and C only.

REMARK. The appearance of the first sum under the exponent is natural. To verify this, take $a_j \equiv d_j$ and assume the condition $0 < d_0 \leq d_j \leq d$. By Lemma 1 in [8] we have

$$m_N \asymp \exp \left\{ \sum_{j \leq N} \frac{d_j - 1}{j} \right\}$$

where the constants in $\asymp$ depend on d_0 and d only.

Actually, in the estimate of m_N one can take

$$c_1 = \min \left\{ (\sqrt{1+c} - 1)^2, (\sqrt{1+c} - 1)/2d \right\} \leq \min \{c^2/4, c/2d\}.$$

Note also that $c \leq d$.

Difficulties arising in the case of functions with unbounded coefficients a_j have been discussed in author's paper [6]. This article and remark [7] contain a few asymptotic formulas for m_N obtained under more restrictive conditions than those used in Theorem above.

Proof of Theorem. Without loss of generality, we may assume that $a_j = 0$ and $b_j = 0$ for $j > N$, nevertheless, even after this change, we leave the same notation of $U(z)$ and $H(z)$. Let $0 < \alpha, \delta < 1$ be arbitrary fixed numbers, $K = \delta N \geq 1$,

$$G_1(z) := \exp \left\{ \alpha \sum_{j \leq K} \frac{a_j}{j} z^j \right\}, \quad G_2(z) := \exp \left\{ -\alpha \sum_{K < j \leq N} \frac{a_j}{j} z^j \right\},$$

and $G_3(z) := G^\alpha(z) - G_1(z)$.

By Cauchy's formula

$$m_N = \frac{1}{2\pi i N} \int_{|z|=1} \frac{F'(z)}{z^N} dz$$

$$\begin{aligned}
&= \frac{1}{2\pi i N} \int_{|z|=1} G^{1-\alpha}(z) (H(z)U'(z) + H'(z))G_1(z) \frac{dz}{z^N} \\
&\quad + \frac{1}{2\pi i N} \int_{|z|=1} G^{1-\alpha}(z) (H(z)U'(z) + H'(z))G_3(z) \frac{dz}{z^N} =: J_1 + J_2. \quad (5)
\end{aligned}$$

We have

$$\begin{aligned}
|J_2| &\leq \frac{B}{2\pi N} \max_{|z|=1} |G^{1-\alpha}(z)| \int_{|z|=1} |U'(z)| |G_3(z)| |dz| \\
&\quad + \frac{1}{2\pi N} \max_{|z|=1} |G^{1-\alpha}(z)| \int_{|z|=1} |H'(z)| |G_3(z)| |dz| =: J_{21} + J_{22}. \quad (6)
\end{aligned}$$

Further,

$$\begin{aligned}
J_{21} &\leq \frac{BD^{1-\alpha}(1)}{2\pi N} \exp \left\{ (1-\alpha) \min_{|t| \leq \pi} (\Re U(e^{it}) - V(1)) \right\} \\
&\quad \times \left(\int_{|z|=1} |U'(z)|^2 |dz| \right)^{1/2} \left(\int_{|z|=1} |G_3(z)|^2 |dz| \right)^{1/2}. \quad (7)
\end{aligned}$$

Since $|a_j| \leq a$, by virtue of Parseval's equality, the first integral on the right-hand side does not exceed $2\pi a^2 N$. For the second integral, we apply (2) to get

$$G_3(z) = \sum_{n \geq K} \left(\sum_{\substack{1k_1 + \dots + nk_n = n \\ \exists j > K \text{ with } k_j \geq 1}} \prod_{j=1}^n \left(\frac{\alpha a_j}{j} \right)^{k_j} \frac{1}{k_j!} \right) z^n.$$

The sum in the braces does not exceed q_n defined via

$$\sum_{n \geq 0} q_n z^n = \exp \left\{ \alpha \sum_{j \leq N} \frac{|a_j|}{j} z^j \right\} =: Q(z).$$

Hence by (4)

$$\begin{aligned}
\int_{|z|=1} |G_3(z)|^2 |dz| &\leq \frac{2\pi}{K^2} \sum_{n \geq 1} q_n^2 n^2 = \frac{1}{K^2} \int_{|z|=1} |Q'(z)|^2 |dz| \\
&\leq \frac{2\pi e^{2C} D^{2\alpha}(1)}{K^2} \sum_{j \leq N} |a_j|^2 \leq \frac{2\pi a^2 e^{2C} D^{2\alpha}(1)}{\delta^2 N}.
\end{aligned}$$

Inserting these estimates of integrals into (7) we have

$$J_{21} \leq \frac{a^2 B e^C D(1)}{\delta N} \exp \left\{ (1-\alpha) \min_{|t| \leq \pi} (\Re U(e^{it}) - V(1)) \right\}. \quad (8)$$

Similarly,

$$J_{22} \leq \frac{abe^C D(1)}{\delta N} \exp \left\{ (1 - \alpha) \min_{|t| \leq \pi} (\Re U(e^{it}) - V(1)) \right\}. \quad (9)$$

The estimates (8) and (9) imply the satisfactory bound for J_2 in (6).

Investigating J_1 , we use the convolution arguments. Observe that

$$J_1 = \frac{1}{N} \sum_{\substack{n, s, k \geq 0 \\ n+s+k \leq N-1}} g_n \tilde{g}_s b_k a_{N-s-n-k} + \frac{1}{N} \sum_{\substack{n, s \geq 0 \\ n+s \leq N-1}} g_n \tilde{g}_s (N-s-n) b_{N-n-s}.$$

Here g_s and $\tilde{g}_s$ are the s -th Taylor coefficients of $G^{1-\alpha}(z)$ and $G_1(z)$ respectively. Thus by (4),

$$\sum_{s \leq N} |g_s| \leq \exp \left\{ (1 - \alpha) \sum_{j \leq N} \frac{|a_j|}{j} \right\} \leq \exp \left\{ (1 - \alpha)(C + V(1)) \right\},$$

and

$$\sum_{s \leq N} |\tilde{g}_s| \leq \exp \left\{ C\alpha + \alpha \sum_{j \leq K} \frac{d_j}{j} \right\}.$$

Exploiting the conditions of Theorem, we now obtain

$$\begin{aligned} |J_1| &\leq \frac{a}{N} \sum_{n \leq N} |g_n| \sum_{s \leq N} |\tilde{g}_s| \sum_{k \leq N} |b_k| + \frac{b}{N} \sum_{n \leq N} |g_n| \sum_{s \leq N} |\tilde{g}_s| \\ &\leq \frac{(aB + b)e^C D(1)}{N} \exp \left\{ -\alpha \sum_{K < j \leq N} \frac{d_j}{j} \right\} \leq C_2 \frac{D(1)\delta^{c\alpha}}{N}, \end{aligned} \quad (10)$$

where $C_2 = (aB + b)e^C$.

Set $C_3 = \max\{C_2, (a^2B + ab)e^C\}$ and $E = \exp \left\{ \min_{|t| \leq \pi} (\Re U(e^{it}) - V(1)) \right\}$. It follows from (5), (6), (7), (9), and (10) that

$$m_N \leq \frac{C_3 D(1)}{N} \left(\frac{E^{1-\alpha}}{\delta} + \delta^{c\alpha} \right),$$

provided that $\delta N \geq 1$. The choice

$$\delta = \max \left\{ \min \{1, E^{(1-\alpha)/(1+c\alpha)}\}, \frac{1}{N} \right\}$$

gives the estimate

$$m_N \leq \frac{2C_3 D(1)}{N} \left(\exp \left\{ \frac{c\alpha(1-\alpha)}{1+c\alpha} \log E \right\} + \frac{1}{N^{c\alpha}} \right). \quad (11)$$

The desire to have the first factor under the exponent as large as possible leads to the choice $\alpha = (\sqrt{1+c} - 1)/c$. Further, the conditions of Theorem yield $e^{-2d-C} N^{-2d} \leq E \leq e^C$. This enables us to get rid of the second term in (11). In this way we obtain

$$m_N \leq \frac{C_4 D(1)}{N} \left(\exp \left\{ \min \{ (\sqrt{1+c} - 1)^2, (\sqrt{1+c} - 1)/2d \} \log E \right\} \right),$$

where $C_4 > 0$ depends on C_3 , C , and d .

Theorem is proved.

An application. Let $\mathcal{F}_N$ be the set of all mappings τ of an N set into itself, $k_j = k_j(\tau)$ be the number of the components in the functional digraph of τ , $1 \leq j \leq N$, and $f : \mathcal{F}_N \rightarrow \mathbb{C}$ be a completely multiplicative function, maybe, depending on N or other parameters. It has the following expression

$$f(\tau) = \prod_{j=1}^N f_j^{k_j(\tau)}, \quad 0^0 := 1,$$

where $f_j \in \mathbb{C}$. Assume that $|f(\tau)| \leq 1$. Then (see [1] or [8])

$$1 + \sum_{N=1}^{\infty} \frac{e^{-N} z^N}{N!} \sum_{\tau \in \mathcal{F}_N} f(\tau) = \exp \left\{ \sum_{j=1}^{\infty} \frac{\lambda_j f_j}{j} z^j \right\},$$

where

$$\lambda_j = e^{-j} \sum_{s=0}^{j-1} \frac{j^s}{s!} = \frac{1}{2} + \frac{8\theta}{\sqrt{j}}, \quad |\theta| \leq 1.$$

The last estimate has been proved in [5]. Applying Theorem with $a_j = \lambda_j f_j$ and $d_j = 1/2$ for each $1 \leq j \leq N$ and Stirling's formula we obtain

$$N^{-N} \left| \sum_{\tau \in \mathcal{F}_N} f(\tau) \right| \leq C_5 \exp \left\{ -c_2 \min_{|t| \leq \pi} \sum_{j \leq N} \frac{1 - \Re(f_j e^{itj})}{j} \right\},$$

with absolute positive constants C_5 and c_2 .

In its turn, this inequality could be used to estimate concentration of values of an additive function defined on $\mathcal{F}_N$.

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Tayloro koeficientų įvertis

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