Liet. matem. rink., T. 41, spec. nr., 632–636

On $H(D)$-valued random elements

Antanas LAURINČIKAS (VU, ŠU)
e-mail: antanas.laurincikas@maf.vu.lt

For any simply connected region D on the complex plane $\mathbb{C}$, by $H(D)$ denote the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Investigations of the universality of zeta-functions use some properties of $H(D)$-valued random elements. In this paper we consider the support of the series of such random elements.

Let $G_1, \ldots, G_n$ be simply connected regions on $\mathbb{C}$, and $H(G_1, \ldots, G_n) = H(G_1) \times \ldots \times H(G_n)$. Let $\{K_{jm}\}$ be a sequence of compact subsets of G_j such that

$$G_j = \bigcup_{m=1}^{\infty} K_{jm},$$

$K_{jm} \subset K_{j,m+1}$, and if K_j is a compact and $K_j \subset G_j$, then $K_j \subseteq K_{jm}$ for some m, $j = 1, \ldots, n$. The existence of the sequence $\{K_{jm}\}$ is given in [1]. For $f_j, g_j \in H(G_j)$ we put

$$\varrho(f_j, g_j) = \sum_{m=1}^{\infty} 2^{-m} \frac{\varrho_{jm}(f_j, g_j)}{1 + \varrho_{jm}(f_j, g_j)},$$

where

$$\varrho_{jm}(f_j, g_j) = \sup_{s \in K_{jm}} \left| f_j(s) - g_j(s) \right|, \quad j = 1, \ldots, n.$$

Then ϱ_j is a metric on $H(G_j)$ which induces its topology, $j = 1, \ldots, n$. For $\underline{f} = (f_1, \ldots, f_n)$, $\underline{g} = (g_1, \ldots, g_n) \in H(G_1, \ldots, G_n)$ we take

$$\varrho(\underline{f}, \underline{g}) = \max_{1 \leqslant j \leqslant n} \varrho_j(f_j, g_j).$$

Then we have that ϱ is a metric on $H(G_1, \ldots, G_n)$ which induces its topology.

Let S be a separable metric space, and let $\mathcal{B}(S)$ stand for the class of Borel sets of S. We recall that a minimal closed set $S_P \subseteq S$ such that $P(S_P) = 1$ is called the support of a probability measure P on $(S, \mathcal{B}(S))$. Note that S_P consists of all $x \in S$ such that for every neighbourhood G of x the inequality $P(G) > 0$ is satisfied.

Let X be a S-valued random element defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the support of the distribution $\mathbb{P}(X \in A)$, $A \in \mathcal{B}(S)$, is called the support of the random element X. We will denote the support of X by S_X.

Theorem. *Let $\{X_m\}$ be a sequence of independent $H(G_1,\dots,G_n)$-valued random elements such that the series*

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely. Then the support of the sum of the latter series is the closure of the set of all $\underline{f} \in H(G_1,\dots,G_n)$ which may be written as a convergent series

$$\underline{f} = \sum_{m=1}^{\infty} \underline{f}_m, \quad \underline{f}_m \in S_{X_m}.$$

We divide the proof of the theorem into three lemmas.

Lemma 1. *Let X and Y be two independent $H(G_1,\dots,G_n)$-valued random elements with distributions P and Q, respectively. Then the distribution of the sum $X+Y$ is the convolution $P * Q$ of P and Q.*

Proof. Suppose that X and Y are defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $A \in \mathcal{B}(H(G_1,\dots,G_n))$. From the independence of X and Y we have that the distribution $\mathbb{P}(X+Y \in A)$ of $X+Y$ is equal to the product

$$(P \times Q)\big((\underline{x}\underline{y})\colon \underline{x}+\underline{y} \in A\big), \quad \underline{x}, \underline{y} \in H(G_1,\dots,G_n).$$

However, denoting by I_A the indicator function of the set A, by the Fubini theorem we find

$$(P \times Q)\big((\underline{x}\underline{y})\colon \underline{x}+\underline{y} \in A\big) = \int\limits_{H(G_1,\dots,G_n)\times H(G_1,\dots,G_n)} I_A(\underline{x}+\underline{y})\mathrm{d}(P\times Q)$$

$$\int\limits_{H(G_1,\dots,G_n)} \Bigg(\int\limits_{H(G_1,\dots,G_n)} I_A(\underline{x}+\underline{y})(P(\mathrm{d}\,\underline{x})\Bigg) Q(\mathrm{d}\,\underline{y})$$

$$\int\limits_{H(G_1,\dots,G_n)} P(A-\underline{y})Q(\mathrm{d}\,\underline{y}) = (P * Q)(A).$$

Lemma 2. *Let X and Y be two independent $H(G_1,\dots,G_n)$-valued random elements. Then the support S_{X+Y} of the sum $X+Y$ is the closure of the set*

$$\mathcal{S} = \big\{\underline{f} \in H(G_1,\dots,G_n)\colon \underline{f} = \underline{x}+\underline{y} \quad \textit{with} \quad \underline{x} \in S_X, \underline{y} \in S_Y\big\}.$$

Proof. Let $\underline{x} \in S_X$, $\underline{y} \in S_Y$, and $\underline{f} = \underline{x}+\underline{y}$. We take an arbitrary positive number δ and put

$$A = \big\{\underline{g} \in H(G_1,\dots,G_n)\colon \varrho(\underline{f},\underline{g}) < \delta\big\}.$$

Moreover, let P and Q be the distributions of random elements X and Y, respectively. Then we have

$$(P * Q)(A) = \int\limits_{H(G_1,\dots,G_n)} P(A-\underline{g})Q(\mathrm{d}\,\underline{g}) > \int\limits_{\{\underline{g}:\ \varrho(\underline{y},\underline{g})<\delta/2\}} P(A-\underline{g})Q(\mathrm{d}\,\underline{g})$$
$$\geqslant P\left(\left\{\underline{g}:\ \varrho(\underline{x},\underline{g}) < \frac{\delta}{2}\right\}\right) \int\limits_{\{\underline{g}:\ \varrho(\underline{x},\underline{g})<\delta/2\}} Q(\mathrm{d}\,\underline{g})$$
$$= P\left(\left\{\underline{g}:\ \varrho(\underline{x},\underline{g}) < \frac{\delta}{2}\right\}\right) Q\left(\left\{\underline{g}:\ \varrho(\underline{x},\underline{g}) < \frac{\delta}{2}\right\}\right) > 0,$$

since by the definition of the support each multiplier is positive. This and Lemma 1 show that $S \subseteq S_{X+Y}$. But the set S_{X+Y} is closed, hence $\bar{S} \subseteq S_{X+Y}$.

It remains to show that $\bar{S} \supseteq S_{X+Y}$. Suppose that there exists a point $\underline{f}$ such that $\underline{f} \in S_{X+Y}$ but $\underline{f} \notin \bar{S}$. Since $\underline{f} \in S_{X+Y}$, for any $\delta > 0$ and A defined above we have

$$(P * Q)(A) = \int\limits_{H(G_1,\dots,G_n)} P(A-\underline{g})Q(\mathrm{d}\,\underline{g}) > 0.$$

The later inequality is possible only it there exists a point $\underline{u} \in S_Y$ such that $P(A-\underline{u}) > 0$. Therefore there exists a point $\underline{v} \in S_X$ in the sphere $\{\underline{g}:\ \varrho(\underline{f}-\underline{u},\underline{g}) < \delta\}$. Then $\varrho(\underline{f},\underline{u}+\underline{v}) < \delta$ and $\underline{f}' = \underline{u}+\underline{v} \in S$. Moreover, if $\delta \to 0$ then $\underline{f}' \to \underline{f}$. Thus, $\underline{f} \in \bar{S}$, and this contradiction proves that $\bar{S} \supseteq S_{X+Y}$. The lemma is proved.

Now let $\{A_m\}$ be a sequence of sets on $H(G_1,\dots,G_n)$. By $\operatorname{Lim} A_m$ denote a set of such $\underline{f} \in H(G_1,\dots,G_n)$ that every neighbourhood of $\underline{f}$ contains at least one point belonging to almost all sets A_m.

Lemma 3. *Let P_n and P be probability measures on $(H(G_1,\dots,G_n), \mathcal{B}(H(G_1,\dots,G_n)))$ and let P_n converge weakly to P as $n \to \infty$. Then $S_P \subseteq \operatorname{Lim} S_{P_n}$.*

Proof. Let $\underline{f} \in S_P$, and, for $\varepsilon > 0$, $A_\varepsilon = \{\underline{g}:\ \varrho(\underline{f},\underline{g}) < \varepsilon\}$. For a fixed $\underline{f}$ the boundaries of the spheres $\varrho(\underline{f},\underline{g}) < \varepsilon$ do not intersect for different ε. Therefore we can choose ε such that A_ε should be the continuity set of P. Then the properties of the weak convergence yield

$$\lim_{n\to\infty} P_n(A_\varepsilon) = P(A_\varepsilon) > 0.$$

So, we have $P_n(A_\varepsilon) > 0$ for $n > n_0(\underline{f},\varepsilon)$. For these $n > n_0$ the distance of $\underline{f}$ from S_{P_n} does not exceed ε. Hence, since ε is an arbitrary number, we find that $f \in \operatorname{Lim} S_{P_n}$. Therefore $S_P \subseteq \operatorname{Lim} P_n$.

Proof of the Theorem. Let $\{X_m\}$ be given on $(\Omega, \mathcal{F}, \mathbb{P})$, and

$$X = \sum_{m=1}^{\infty} X_m = L_n + R_n,$$

where

$$L_n = \sum_{m=1}^{m} X_m, \quad R_n = \sum_{m=n+1}^{\infty} X_m.$$

Since the series of the theorem converges almost surely, for any $\varepsilon > 0$

$$\mathbb{P}\big(\omega \in \Omega\colon \varrho(R_n, \underline{0}) \geqslant \varepsilon\big) \to 0, \qquad n \to \infty. \tag{1}$$

Let

$$P_n(A) = \mathbb{P}(L_n \in A), \quad P(A) = \mathbb{P}(X \in A), \quad A \in \mathcal{B}(H(G_1, \ldots, G_n)).$$

Then the above relations imply the weak convergence of P_n to P as $n \to \infty$. Therefore in view of Lemma 3

$$S_X \subseteq \operatorname{Lim} S_{L_n}. \tag{2}$$

Now let $\underline{f}_0 \in \operatorname{Lim} S_{L_n}$, and, for any $\delta > 0$,

$$A_\delta = \big\{\underline{f}\colon \varrho(\underline{f}, \underline{f}_0) < \delta\big\}.$$

Then there exists n_1 such that for $n > n_1$

$$\mathbb{P}(L_n \in A_\varepsilon) = P_n(A_\varepsilon) > 0. \tag{3}$$

Define $B_\delta = \big\{\underline{f}\colon \varrho(\underline{f}, \underline{0}) < \delta\big\}$. Then by (1) for $n > n_2$

$$\mathbb{P}(R_n \in B_\delta) > 0. \tag{4}$$

Let $Q_n(A) = \mathbb{P}(R_n \in A)$, $A \in \mathcal{B}(H(G_1, \ldots, G_n))$. Then we have that $P = P_n * Q_n$. Hence, from (3), (4) and Lemma 1 we obtain

$$P(A_{2\delta}) \int\limits_{H(G_1,\ldots,G_n)} P_n(A_{2\delta} - \underline{g}) Q_n(\mathrm{d}\,\underline{g})$$
$$\geqslant \int\limits_{B_\delta} P_n(A_{2\delta} - \underline{g}) Q_n(\mathrm{d}\,\underline{g}) \geqslant P_n(A_\delta \int\limits_{B_\delta} Q_n(\mathrm{d}\,\underline{g}) = P_n(A_\delta) Q_n(B_\delta) > 0$$

for $n \geqslant n_3 = \max(n_1, n_2)$. This means that $f_0 \in S_X$. Therefore $S_X \supseteq \lim S_{L_n}$. This and (2) imply

$$S_X = \operatorname{Lim} S_{L_n}. \tag{5}$$

Since $X_1, \ldots, X_n$ are independent, by Lemma 2 we have that S_{L_n} is the closure of the set of all $\underline{f} \in H(G_1, \ldots, G_n)$ which can be written as a sum

$$\underline{f} = \sum_{m=1}^{n} \underline{f}_m, \quad \underline{f}_m \in S_{X_m}.$$

Now if $f \in S_X$, then there exists a sequence $\{g_n\colon \underline{g}_n \in S_{L_n}\}$ and $\lim\limits_{n\to\infty} \underline{g}_n = \underline{f}$. This together with (5) yield the assertion of the lemma.

References

[1] J.B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York (1973).

Apie $H(D)$-reikšmius atsitiktinius elementus

A. Laurinčikas

Straipsnyje nagrinėjamas nepriklausomų $H(G_1) \times \ldots \times H(G_n)$-reikšmių atsitiktinių elementų eilutės nešėjas. Čia $H(D)$ yra funkcijų, analizinių srityje D, erdvė su tolygaus konvergavimo ant kompaktų topologija.