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On Functional Independence of Beurling Zeta-Functions

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Abstract

Let $\mathscr{P}$ be a system of generalized prime numbers, and $\mathscr{N}_{\mathscr{P}}$ the corresponding system of generalized integers. Assuming that $\sum\limits_{\substack{m\leqslant x\\ m\in\mathscr{N}_{\mathscr{P}}}} 1 - ax \ll x^{\beta}$ with $a>0$ and $0\leqslant\beta<1$, we consider the Beurling zeta-function $\zeta_{\mathscr{P}}(s)$, $s=\sigma+it$. Beurling zeta-functions constitute a wide class of non-standard zeta-functions which pose interesting mathematical problems. Numerous authors are searching for restrictions on the systems $\mathscr{P}$ and $\mathscr{N}_{\mathscr{P}}$ that the corresponding Beurling zeta-functions have some properties similar to those of classical zeta-functions. One of such properties is the functional independence which was initiated by O. Hölder and D. Hilbert, and, in the most general form, by S.M. Voronin. This is a motivation to obtain the functional independence in the Voronin sense for a certain class of Beurling zeta-functions. Under a certain additional condition involving the generalized von Mangoldt function, we obtain the functional independence of the function $\zeta_{\mathscr{P}}(s)$. We prove that the function $\zeta_{\mathscr{P}}(s)$ does not satisfy the equation $\sum\limits_{k=0}^{r} s^k F_k\Big(\zeta_{\mathscr{P}}(s), \zeta'_{\mathscr{P}}(s), \dots, \zeta^{(n-1)}_{\mathscr{P}}(s)\Big)=0$ with continuous functions F_k, $k=0,\dots,r$. The proof is based on the universality property of $\zeta_{\mathscr{P}}(s)$ on approximation of analytic functions by shifts $\zeta_{\mathscr{P}}(s+i\tau)$, $\tau\in\mathbb{R}$.

Keywords: approximation of analytic functions; Beurling zeta-function; functional independence; generalized integers; generalized prime numbers; universality

MSC: 11N80; 11M41

1. Introduction

Throughout the paper, we denote by $s=\sigma+it$ a complex variable. We say that the function $g(s)$ is independent in a certain sense if it does not satisfy any equation of a given type. For example, $g(s)$ is algebraically independent if there is no polynomial $p(s)\not\equiv 0$ such that $p(g(s))\equiv 0$, $s\in\mathbb{C}$. Similarly, algebraic-differential independence of $g(s)$ is understood as non-existence of any polynomial $p(s_0,\dots,s_r)\not\equiv 0$ such that $p(g(s), g'(s),\dots,g^{(r)}(s))\equiv 0$ for $s\in\mathbb{C}$.

The independence problem is usually connected with classical well-known functions. For example, the algebraic-differential independence of functions goes back to O. Hölder, who obtained the algebraic-differential independence of the Euler gamma-function $\Gamma(s)$ defined, for $\sigma>0$, by

$$\Gamma(s)=\int\limits_0^{\infty} \mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x,$$

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and by meromorphic continuation elsewhere: $\Gamma(s)$ has simple poles at the points $s = -k$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with residues $(-1)^k/k!$. Thus, he proved [1] that there does not exist any polynomial $p(s_0, \dots, s_r) \not\equiv 0$ satisfying

$$p\Big(\Gamma(s), \Gamma'(s), \dots, \Gamma^{(r)}(s)\Big) \equiv 0.$$

The Hölder theorem has an interesting continuation. D. Hilbert at the International Congress of Mathematicians of the 1900th in Paris presented the list of 23 problems of mathematics that, in his opinion, are the most important for development of mathematics [2]. In the 18th problem, he observed, that the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

is also algebraically differentially independent, and that this can be proved by applying the Hölder theorem on independence of $\Gamma(s)$, and the functional equation for $\zeta(s)$

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \ = \ \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Moreover, Hilbert conjectured the algebraic-differential independence for a more general function

$$\zeta(s, x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}.$$

Hilbert's conjecture was confirmed independently by D.D. Mordukhai-Boltovskoi [3] and A. Ostrowski [4].

Let $\chi(m)$ be a Dirichlet character modulo q, and $L(s, \chi)$ denote the corresponding Dirichlet L-function:

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1.$$

A.G. Postnikov proved in [5] that all $\varphi(q)$ ($\varphi(q)$ is the Euler totient function) Dirichlet L-functions are algebraically differentially independent. In [6,7], he generalized Hilbert's conjecture for the functions

$$L(x, s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} x^m$$

and successfully confirmed it. We notice that, in the case of $\zeta(s, x)$ and $L(x, s, \chi)$, in the definition of algebraic-differential independence, the derivatives with respect to x are involved as well.

The next progress in the field of independence of functions belongs to S.M. Voronin. In [8] (see also [9–11]), he discovered the universality property of the Riemann zeta-function on approximation of a wide class of analytic functions by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, and applied it in [12] for the proof of the functional independence of $\zeta(s)$. Suppose that, for $k = 0, 1, \dots, r$, $F_k : \mathbb{C}^n \to \mathbb{C}$ is a continuous function, and identically for $s \in \mathbb{C}$

$$\sum_{k=0}^{r} s^k F_k\Big(\zeta(s), \zeta'(s), \dots, \zeta^{(n-1)}(s)\Big) = 0.$$

Then, Voronin obtained that $F_k(\dots) \equiv 0$ for all $k = 0, 1, \dots, r$.

In [13], Voronin extended a notion of the functional independence for a tuple of Dirichlet L-functions. Recall that two Dirichlet characters are said to be equivalent if they are generated by the same primitive Dirichlet character. Suppose that $\chi_1, \dots, \chi_l$

are pairwise non-equivalent Dirichlet characters, for $k = 0, 1, \dots, r$, $F_k : \mathbb{C}^{ln} \to \mathbb{C}$ is a continuous function, and identically for $s \in \mathbb{C}$

$$\sum_{k=0}^{r} s^k F_k\Big(L(s,\chi_1), L'(s,\chi_1), \dots, L^{(n-1)}(s,\chi_1), \dots, L(s,\chi_l), L'(s,\chi_l), \dots, L^{(n-1)}(s,\chi_l)\Big) = 0.$$

Then, Voronin proved in [13] that $F_k(\dots) \equiv 0$ for all $k = 0, 1, \dots, r$. The latter theorem is called a joint functional independence of Dirichlet L-functions.

At the moment, many results on functional independence of zeta- and L-functions are known; see survey paper [14]. They are consequences of universality for the corresponding functions. We recall the Steuding class $\widetilde{\mathcal{S}}$ of Dirichlet series

$$\mathcal{L}(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

which includes the majority of classical functions. This class has been investigated in [15], including the functional independence of its members. The coefficients $a(m)$ and functions $\mathcal{L}(s)$ satisfy the following axioms:

1. Ramanujan hypothesis: $a(m) \ll_\varepsilon m^\varepsilon$ for any $\varepsilon > 0$. We recall that $a \ll_\varepsilon b$, $a \in \mathbb{C}$, $b > 0$, which means that there exists a constant $c = c(\varepsilon)$ such that $|a| \leqslant cb$.
2. Analytic continuation: There exists a number $\sigma_{\mathcal{L}} < 1$ such that the function $\mathcal{L}(s)$ has analytic continuation to the region $\sigma > \sigma_{\mathcal{L}}$, except for at most a pole at the point $s = 1$.
3. Finite order: There exists a number $\mu_{\mathcal{L}} \geqslant 0$ such that, for fixed $\sigma > \sigma_{\mathcal{L}}$ and every $\varepsilon > 0$,
$$\mathcal{L}(\sigma + it) \ll_\varepsilon |t|^{\mu_{\mathcal{L}}+\varepsilon}, \quad |t| \to \infty.$$
4. Euler product: There exists $n \in \mathbb{N}$ and, for primes p, $\alpha_j(p) \in \mathbb{C}$, $j = 1, \dots, n$, such that
$$\mathcal{L}(s) = \prod_p \prod_{j=1}^{n} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}.$$
5. Prime mean-square: There exists $\kappa > 0$ such that
$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \leqslant x} |a(p)|^2 = \kappa \quad \text{with} \quad \pi(x) = \sum_{p \leqslant x} 1.$$

We notice that the class $\widetilde{\mathcal{S}}$ is wide. It includes Dirichlet L-functions, Dedekind zeta-functions, zeta-functions of certain cusp forms, Matsumoto zeta-functions defined in [16], etc. …

Now, we state the Steuding theorem on functional independence of functions from the class $\widetilde{\mathcal{S}}$.

Theorem 1 (See [15], Theorem 10.3)**.** *Suppose that $\mathcal{L}(s) \in \widetilde{\mathcal{S}}$ and $F_k : \mathbb{C}^n \to \mathbb{C}$, $j = 0, 1, \dots, r$, are continuous functions, not all identically zero. Then,*

$$\sum_{k=0}^{r} s^k F_k\Big((\mathcal{L}(s), \mathcal{L}'(s), \dots, \mathcal{L}^{(n-1)}(s)\Big) \neq 0$$

for some $s \in \mathbb{C}$.

The aim of this paper is to obtain the functional independence of Beurling zeta-functions $\zeta_{\mathscr{P}}(s)$ defined by A. Beurling in [17]. In Section 2, we introduce Beurling zeta-functions and recall some results on them. Section 3 is devoted to universality of zeta-functions, including Beurling zeta-functions. In Section 4, we prove denseness of one set connected to Beurling zeta-functions, and in Section 5 obtain the functional independence of Beurling zeta-functions.

Proofs use a standard scheme based on universality for a class of Beurling zeta-functions. The novelty of the main results is connected with involving a linear independence of the set $\{\log p : p \in \mathscr{P}\}$ and a hypothesis for a mean value of the generalized van Mangoldt function. The results obtained extend the class of independent zeta-functions.

2. Beurling Zeta-Functions

For the definition of Beurling zeta-functions, the systems of generalized prime numbers and generalized integers are used. Every system $\mathscr{P}$ of real numbers

$$1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant \cdots, \quad \lim_{n \to \infty} p_n = +\infty,$$

is called generalized prime numbers. The system $\mathscr{P}$ generates a system $\mathscr{N}_{\mathscr{P}}$ of generalized integers of the form

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad \alpha_j \in \mathbb{N}_0,\ j = 1, \ldots, r,\ r \in \mathbb{N}.$$

For example, $\{q + \beta : q \in \mathbb{P}, \beta > 0\}$, where $\mathbb{P}$ is the set of all prime numbers, is a system of generalized primes. We observe that systems $\mathscr{P}$ and $\mathscr{N}_{\mathscr{P}}$ consist of not necessarily distinct numbers, and their multiplicities can be greater than 1. Thus, the numbers

$$2k,\ (2k+1)^2 \text{ with multiplicity } 3$$

for $k \in \mathbb{N}$ form a system of generalized primes.

Generalized numbers (primes and integers) were introduced by Beurling in [17] and, until our days, are studied by numerous authors. The main attention is devoted to asymptotics as $x \to \infty$ of the functions

$$\pi_{\mathscr{P}}(x) = \sum_{\substack{p \leqslant x \\ p \in \mathscr{P}}} 1$$

and

$$\mathfrak{N}_{\mathscr{P}}(x) = \sum_{\substack{m \leqslant x \\ m \in \mathscr{N}_{\mathscr{P}}}} 1,$$

and to connections between $\pi_{\mathscr{P}}(x)$ and $\mathfrak{N}_{\mathscr{P}}(x)$. Similarly to the set $\mathbb{P}$, for the investigation of generalized prime numbers, Beurling introduced zeta-functions, analogues of the function $\zeta(s)$. These functions $\zeta_{\mathscr{P}}(s)$ now are called Beurling zeta-functions, and, for $\sigma > \sigma_{\mathscr{P}}$ with some $\sigma_{\mathscr{P}}$, are defined by

$$\zeta_{\mathscr{P}}(s) = \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or

$$\zeta_{\mathscr{P}}(s) = \sum_{m \in \mathscr{N}_{\mathscr{P}}} \frac{1}{m^s}.$$

For precise definition of $\zeta_{\mathscr{P}}(s)$, some additional information on $\mathscr{P}$ is needed. A convenient hypothesis on $\mathscr{P}$ is the bound

$$\mathfrak{N}_{\mathscr{P}}(x) - ax \ll x^{\beta}, \quad a > 0,\ 0 \leqslant \beta < 1. \tag{1}$$

Then, the series and product defining $\zeta_{\mathscr{P}}(s)$ are absolutely convergent for $\sigma > 1$, and the equality

$$\zeta_{\mathscr{P}}(s) = \sum_{m \in \mathscr{N}_{\mathscr{P}}} \frac{1}{m^s} = \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

holds. Moreover, $\zeta_{\mathscr{P}}(s)$ has analytic continuation to the half-plane $\sigma > \beta$, except for a simple pole at the point $s = 1$ with residue a.

We mention several works devoted to the distribution of generalized numbers and the corresponding Beurling zeta-functions. Beurling himself obtained in [17] that the asymptotics

$$\pi_{\mathscr{P}}(x) \sim \frac{x}{\log x}, \quad x \to \infty,$$

follows from the estimate

$$\mathfrak{N}_{\mathscr{P}}(x) - ax \ll \frac{x}{(\log x)^{\alpha}} \tag{2}$$

with $\alpha > 3/2$. An old result of E. Landau [18] shows that the estimate

$$\pi_{\mathscr{P}}(x) - \int\limits_2^x \frac{\mathrm{d}u}{\log u} \ll x\mathrm{e}^{-c\sqrt{\log x}}$$

with $c > 0$ follows from (1). H. Diamond found [19] that

$$\pi_{\mathscr{P}}(x) - \frac{x}{\log x} \ll x(\log x)^{-\alpha}$$

with $\alpha > 1$ gives

$$\mathfrak{N}_{\mathscr{P}}(x) \sim ax, \quad x \to \infty.$$

The paper [20] of B. Nyman contains an interesting theorem on equivalence of estimates

$$\pi_{\mathscr{P}}(x) - \int\limits_2^x \frac{\mathrm{d}u}{\log u} \ll x(\log x)^{-\alpha_1}$$

and

$$\mathfrak{N}_{\mathscr{P}}(x) - ax \ll x(\log x)^{-\alpha_2}$$

with arbitrary positive α_1 and α_2. More precise results on $\pi_{\mathscr{P}}(x)$ and $\mathfrak{N}_{\mathscr{P}}(x)$ can be found in [21–28].

The value distribution of Beurling zeta-functions is studied, without being mentioned in the above papers, in [29–36].

3. Universality

Approximation of analytic functions by shifts of Beurling zeta-functions began to be studied in [37] and continued in [38]. This property of zeta-functions is called universality, and was discovered by S.M. Voronin in [8]: he proved universality of the Riemann zeta-function $\zeta(s)$. The modern version of the Voronin theorem uses the following notation. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathscr{K}$ denote the class of compact subsets of the strip D with connected complements, and $H^0(K)$ with $K \in \mathscr{K}$ stand for the set of continuous non-vanishing functions on K that are analytic inside of K. Moreover, let

$$\mathfrak{U}_T(\dots) = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \dots\},$$

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where meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and in place of dots, a condition satisfied by τ is to be written. Then the following statement is known; see [14,15,39–42].

Proposition 1. *For every $K \in \mathscr{K}$, $f(s) \in H^0(K)$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathfrak{U}_T\left(\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \mathfrak{U}_T\left(\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right)$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Proposition 1 shows that every analytic non-vanishing function in D can be approximated by shifts of $\zeta(s+i\tau)$. Moreover, there are infinitely many shifts $\zeta(s+i\tau)$ approximating a given analytic function $f(s)$. Later, by numerous authors, it was obtained that some other zeta- and L-functions are universal in the Voronin sense. The Linnik-Ibragimov conjecture says [15] that all functions in some half-plane defined by Dirichlet series, having analytic continuation to a wider half-plane, and satisfying some natural growth conditions, are universal. On the other hand, there are examples of non-universal Dirichlet series. Since universality of zeta-functions has various applications for theoretical and practical problems (functional independence, denseness of sets, zero distribution, problems of quantum mechanics, ...) (see a survey paper [14]), search of new universal functions is an important problem. In this context, Beurling zeta-functions are a very attractive class.

Now, we recall the main result of [38]. Set

$$\mathfrak{M}_{\mathscr{P}}(\sigma, T) = \int_0^T |\zeta_{\mathscr{P}}(\sigma+it)|^2 \, \mathrm{d}t,$$

and define

$$\widehat{\sigma}_{\mathscr{P}} = \inf\left\{\sigma : \mathfrak{M}_{\mathscr{P}}(\sigma, T) \ll_\sigma T,\ \sigma > \max\left(\frac{1}{2}, \beta\right)\right\}.$$

In [35], it was obtained that, under (1),

$$\mathfrak{M}_{\mathscr{P}}(\sigma, T) \ll_\sigma T$$

for $\sigma > (\beta+1)/2$. Therefore, $\widehat{\sigma}_{\mathscr{P}} < 1$ exists. On $\mathbb{C}$, define the strip

$$\mathscr{D}_{\mathscr{P}} = \{s \in \mathbb{C} : \widehat{\sigma}_{\mathscr{P}} < \sigma < 1\}.$$

Introduce the space $\mathscr{H}(\mathscr{D}_{\mathscr{P}})$ of analytic functions on $\mathscr{D}_{\mathscr{P}}$ endowed with the topology of uniform convergence on compact sets of $\mathscr{D}_{\mathscr{P}}$. Moreover, we need the generalized von Mangoldt function

$$\Lambda_{\mathscr{P}}(m) = \begin{cases} \log p & \text{if } m = p^l, l \in \mathbb{N}, p \in \mathscr{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\psi_{\mathscr{P}}(x) = \sum_{\substack{m \leqslant x \\ m \in \mathscr{N}_{\mathscr{P}}}} \Lambda_{\mathscr{P}},$$

and suppose that

$$\psi_{\mathscr{P}}(x) - x \ll x^{\beta_1} \tag{3}$$

with $\beta_1 < 1$. Then, in [38], universality of the function $\zeta_{\mathscr{P}}(s)$ has been obtained.

Lemma 1 (See [38], Theorem 2)**.** *Suppose that the system $\mathscr{P}$ of generalized primes satisfies estimates (1) and (3), and the set $\{\log p : p \in \mathscr{P}\}$ is linearly independent over the field of rational numbers. Then, for every compact set $K \subset \mathscr{D}_{\mathscr{P}}$ with connected complement, every continuous non-vanishing function $g(s)$ on K and analytic inside of K, and every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \mathfrak{U}_T \left(\sup_{s\in K} |\zeta_{\mathscr{P}}(s+i\tau) - g(s)| < \varepsilon \right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \mathfrak{U}_T \left(\sup_{s\in K} |\zeta_{\mathscr{P}}(s+i\tau) - g(s)| < \varepsilon \right)$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Thus, Lemma 1 supports the Linnik-Ibragimov conjecture.

A discrete analogue of Lemma 1 is presented in [43].

4. Denseness

A denseness problem for some sets of values of zeta-functions goes back to H. Bohr and R. Courant. In [44], they obtained that the set

$$\{\zeta(\sigma+it) : t \in \mathbb{R}\}$$

with every fixed $1/2 < \sigma \leqslant 1$ is dense in $\mathbb{C}$. Voronin extended the above result to the space $\mathbb{C}^n$, $n \in \mathbb{N}$. In [45], he proved that, for every fixed $1/2 < \sigma \leqslant 1$, the set

$$\left\{ \left(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(n-1)}(\sigma+it) \right) : t \in \mathbb{R} \right\}$$

is dense in $\mathbb{C}^n$.

This section is devoted to extension of the Bohr-Courant theorem [44] for Beurling zeta-functions.

Theorem 2. *Suppose that the system $\mathscr{P}$ satisfies hypotheses of Lemma 1. Then, for every fixed $\widehat{\sigma}_{\mathscr{P}} < \sigma < 1$, the set*

$$\{\zeta_{\mathscr{P}}(\sigma+it) : t \in \mathbb{R}\}$$

is dense in $\mathbb{C}$.

Proof. We have to show that, for every $c \in \mathbb{C}$, $\widehat{\sigma}_{\mathscr{P}} < \sigma < 1$ and $\varepsilon > 0$, there is $t = t_{c,\varepsilon,\sigma} \in \mathbb{R}$ such that

$$|\zeta_{\mathscr{P}}(\sigma+it) - c| < \varepsilon. \tag{4}$$

Suppose that $c \in \mathbb{C} \setminus \{0\}$. Then, by Lemma 1 with $K = \{\sigma\}$, $g(s) \equiv c$ and every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \mathfrak{U}_T(|\zeta_{\mathscr{P}}(\sigma+i\tau) - c| < \varepsilon) > 0. \tag{5}$$

Now, let $c = 0$. Then, by (5),

$$\left|\zeta_{\mathscr{P}}(\sigma+it_{\varepsilon/2,\varepsilon/2,\sigma}) - 0\right| \leqslant \left|\zeta_{\mathscr{P}}(\sigma+it_{\varepsilon/2,\varepsilon/2,\sigma}) - \frac{\varepsilon}{2}\right| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, there exists $t_{c,\varepsilon,\sigma}$ satisfying (4). $\square$

Theorem 2 has an extension to the space $\mathbb{C}^n$, $n \in \mathbb{N}$.

Theorem 3. *Suppose that the system $\mathscr{P}$ satisfies hypotheses of Lemma 1. Then, for every fixed $\widehat{\sigma}_{\mathscr{P}} < \sigma < 1$, the set*

$$\left\{ \left(\zeta_{\mathscr{P}}(\sigma+it), \zeta'_{\mathscr{P}}(\sigma+it), \dots, \zeta^{(n-1)}_{\mathscr{P}}(\sigma+it) \right) : t \in \mathbb{R} \right\}$$

is dense in $\mathbb{C}^n$.

Proof. Denote by $d_{\mathbb{C}^n}(\underline{c}_1, \underline{c}_2)$, $\underline{c}_j = (c_{j,0}, \dots c_{j,n-1}) \in \mathbb{C}^n$, $j = 1, 2$, the metric in $\mathbb{C}^n$, i.e.,

$$d_{\mathbb{C}^n}(\underline{c}_1, \underline{c}_2) = \max_{0 \leqslant k \leqslant n-1} |c_{1,k} - c_{2,k}|.$$

Let

$$\underline{\zeta}_{\mathscr{P}}(s) = \left(\zeta_{\mathscr{P}}(s), \zeta'_{\mathscr{P}}(s), \dots, \zeta^{(n-1)}_{\mathscr{P}}(s) \right).$$

We will show that, for every $\underline{c} = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{C}^n$, $\widehat{\sigma}_{\mathscr{P}} < \sigma < 1$, and $\varepsilon > 0$, there exists $t_{\underline{c},\varepsilon,\sigma} \in \mathbb{R}$ satisfying inequality

$$d_{\mathbb{C}^n}\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it_{\underline{c},\varepsilon,\sigma}), \underline{c} \right) < \varepsilon. \tag{6}$$

First, we suppose that $c_0 \neq 0$, Then it is well redknown (see, for example, Lemma 6.6.1 of [41]) that there exists a polynomial $p_n(s)$ of degree n such that

$$c_k = (\exp(p_n(s)))^{(k)}\big|_{s=0}, \quad k = 0, 1, \dots, n-1. \tag{7}$$

Let us fix σ_0, $\widehat{\sigma}_{\mathscr{P}} < \sigma_0 < 1$. Take a circle γ with a center σ_0 lying in the strip $\mathscr{D}_{\mathscr{P}}$, and a compact set $K \subset \mathscr{D}_{\mathscr{P}}$ containing γ. Then, an application of Lemma 1 with

$$g(s) = \exp(p_n(s - \sigma_0))$$

shows that there exists $\tau \in \mathbb{R}$ such that

$$\sup_{s \in K} |\zeta_{\mathscr{P}}(s + i\tau) - \exp(p_n(s - \sigma_0))| < \delta \tag{8}$$

with every $\delta > 0$. By (7) and the Cauchy integral formula, we have

$$\zeta^{(k)}_{\mathscr{P}}(\sigma_0 + i\tau) - c_k = \frac{k!}{2\pi i} \int_{\gamma} \frac{\zeta_{\mathscr{P}}(z + i\tau) - \exp(p_n(z - \sigma_0))}{(z - \sigma_0)^{k+1}} \, \mathrm{d}z.$$

Hence, in view of (8),

$$\begin{aligned} \left| \zeta^{(k)}_{\mathscr{P}}(\sigma_0 + i\tau) - c_k \right| &\leqslant \frac{k!}{2\pi} \int_{\gamma} \frac{|\zeta_{\mathscr{P}}(z + i\tau) - \exp(p_n(z - \sigma_0))|}{|z - \sigma_0|^{k+1}} \, |\mathrm{d}z| \\ &\leqslant \frac{n!\delta}{2\pi} \int_{\gamma} \frac{|\mathrm{d}z|}{|z - \sigma_0|^{k+1}} = \frac{n!\delta}{2\pi|\rho|^{k+1}} \int_{\gamma} |\mathrm{d}z| = \frac{n!\delta}{\rho^n}, \quad k = 0, 1, \dots, n-1, \end{aligned}$$

where ρ is the radius of the circle γ. Thus, taking $\delta = (\varepsilon\rho^n)/n!$, we obtain (6). Since σ_0 is arbitrary, this proves the theorem for $c_0 \neq 0$.

The case $c_0 = 0$ reduces to the considered one. Let $\underline{c} \in \mathbb{C}^n$ with $c_0 = \varepsilon/2$ and arbitrary $c_1, \dots, c_{n-1}$. Then, there exists $t_{\underline{c},\varepsilon/2,\sigma}$ such that

$$d_{\mathbb{C}^n}\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it_{\underline{c},\varepsilon/2,\sigma}), \underline{c}\right) < \frac{\varepsilon}{2}.$$

Since

$$\left|\zeta_{\mathscr{P}}(\sigma + it_{\underline{c},\varepsilon/2,\sigma}) - \frac{\varepsilon}{2}\right| < \frac{\varepsilon}{2},$$

we have

$$\left|\zeta_{\mathscr{P}}(\sigma + it_{\underline{c},\varepsilon/2,\sigma})\right| < \varepsilon$$

and

$$\left|\zeta_{\mathscr{P}}^{(k)}(\sigma + it_{\underline{c},\varepsilon/2,\sigma}) - c_k\right| < \frac{\varepsilon}{2}, \quad k = 1, \dots, n-1.$$

Therefore, we have

$$d_{\mathbb{C}^n}\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it_{\widehat{\underline{c}},\varepsilon/2,\sigma}), \underline{c}\right) < \varepsilon$$

for $\widehat{\underline{c}} = (0, c_1, \dots, c_{n-1})$. The theorem is proved. □

Remark 1. *Since, by Lemma 1, the set of* τ *satisfying (8) has a positive lower density, there exists a sequence* $\{t(l)\}$, $t(l) \to \infty$ *as* $l \to \infty$, *such that*

$$d_{\mathbb{C}^n}\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it(l)), \underline{c}\right) < \varepsilon.$$

5. Functional Independence

In this section, we will prove the functional independence of Beurling zeta-functions.

Theorem 4. *Suppose that the system* $\mathscr{P}$ *satisfies hypotheses of Lemma 1, and* $F_0, F_1, \dots, F_r : \mathbb{C}^n \to \mathbb{C}$ *are continuous functions, not all identical by zero. Then,*

$$\sum_{k=0}^{r} s^k F_k\left(\zeta_{\mathscr{P}}(s), \zeta'_{\mathscr{P}}(s), \dots, \zeta_{\mathscr{P}}^{(n-1)}(s)\right) \neq 0$$

for some $s \in \mathbb{C}$.

As it was mentioned in Section 2, by estimate (1), the function $\zeta_{\mathscr{P}}(s)$ has meromorphic continuation to the half-plane $\sigma > \beta$ with $0 \leqslant \beta < 1$, and there is no any information on $\zeta_{\mathscr{P}}(s)$ for $\sigma \leqslant \beta$. Therefore, we understand the phrase "identically in s" concerning $\zeta_{\mathscr{P}}(s)$ as "identically in s with $\sigma > \beta$".

Let, for brevity, $\underline{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$. Theorem 4 can be restated in the following equivalent form:

Theorem 5. *Suppose that the system* $\mathscr{P}$ *satisfies hypotheses of Lemma 1. If* $F_0, F_1, \dots, F_r : \mathbb{C}^n \to \mathbb{C}$ *are continuous functions, and identically in* s *with* $\sigma > \beta$

$$\sum_{k=0}^{r} s^k F_k\left(\underline{\zeta}_{\mathscr{P}}(s)\right) = 0,$$

then $F_k(\underline{s}) \equiv 0$ *for* $k = 0, 1, \dots, r$.

Proof of Theorem 4. Without loss of generality, we suppose that $F_0(\underline{s}) \not\equiv 0$. Then, there is $\underline{c} \in \mathbb{C}^n$ such that $F_0(\underline{c}) \neq 0$. By continuity of F_0, there exists an open bounded set G_0 of $\underline{c}$ and $\delta_0 > 0$ satisfying

$$|F_0(\underline{s})| > \delta_0, \quad \underline{s} \in G_0. \tag{9}$$

Fix σ, $\widehat{\sigma}_{\mathscr{P}} < \sigma < 1$, where $\widehat{\sigma}_{\mathscr{P}}$ is from Theorem 3. Then, in virtue of Theorem 3, there exists $t \in \mathbb{R}$ such that

$$\underline{\zeta}_{\mathscr{P}}(\sigma + it) \in G_0.$$

Then, by (9),

$$\left| F_0\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it)\right) \right| > \delta_0,$$

and this proves the theorem for $r = 0$.

For the case $r \geqslant 1$, the powers s^k play an important role. Let

$$k_0 = \max\left\{ k \geqslant 1 : \sup_{\underline{s} \in G_0} |F_k(\underline{s})| \neq 0 \right\}.$$

Then, there exists a closed set $\mathcal{G} \subset G_0$ and $\delta > 0$ such

$$\inf_{\underline{s} \in \mathcal{G}} \left| F_{k_0}(\underline{s}) \right| > \delta. \tag{10}$$

By Remark 1, there exists a sequence $\{t(l)\}$, $t(l) \to \infty$ as $l \to \infty$, satisfying

$$\underline{\zeta}_{\mathscr{P}}(\sigma + it(l)) \in \mathcal{G}.$$

Hence, in view of (10),

$$\left| F_{k_0}\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it(l))\right) \right| > \delta.$$

Therefore,

$$\left| (\sigma + it(l))^{k_0} F_{k_0}\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it(l))\right) \right| \to \infty$$

as $l \to \infty$. This and the definition of k_0 show that

$$\left| \sum_{k=0}^{k_0} (\sigma + it(l))^k F_k\left(\underline{\zeta}_{\mathscr{P}}(\sigma + it(l))\right) \right| \to \infty$$

as $l \to \infty$. Therefore, we have

$$\sum_{k=0}^{r} s^k F_k\left(\underline{\zeta}_{\mathscr{P}}(s)\right) \neq 0$$

for some $s \in \mathbb{C}$. The theorem is proved. □

Proof of Theorem 5. Suppose that identically in s

$$\sum_{k=0}^{r} s^k F_k\left(\underline{\zeta}_{\mathscr{P}}(s)\right) = 0. \tag{11}$$

If there exists k_0 such that $F_{k_0}(\underline{s}) \not\equiv 0$, then, by Theorem 4,

$$\sum_{k=0}^{r} s^k F_k\left(\underline{\zeta}_{\mathscr{P}}(s)\right) \neq 0$$

for some $s \in \mathbb{C}$. However, this contradicts (11). Thus, $F_k(\underline{s}) \equiv 0$ for $k = 0, 1, \dots, r$. □

Similar arguments show that Theorem 5 implies Theorem 4. Actually, suppose that $F_0, F_1, \dots, F_r$ are not all identically zero, i.e., there exists at least F_{k_0} such that

$$F_{k_0}(\underline{s}) \not\equiv 0. \tag{12}$$

If

$$\sum_{k=0}^{r} s^k F_k\Big(\underline{\zeta}_{\mathscr{P}}(s)\Big) \equiv 0,$$

then, by Theorem 5, $F_k(\underline{s}) \equiv 0$ for all $k = 0, 1, \dots, r$, and this contradicts (12). Thus,

$$\sum_{k=0}^{r} s^k F_k\Big(\underline{\zeta}_{\mathscr{P}}(s)\Big) \not\equiv 0$$

for some $s \in \mathbb{C}$.

6. Conclusions

In the paper, we consider functional properties of the Beurling zeta-function $\zeta_{\mathscr{P}}(s)$, $s = \sigma + it$, of a system of generalized prime numbers. Under hypothesis

$$\sum_{\substack{m \leqslant x \\ m \in \mathscr{N}_{\mathscr{P}}}} 1 - ax \ll x^{\beta}, \quad 0 \leqslant \beta < 1,$$

where $\mathscr{N}_{\mathscr{P}}$ denotes the system of generalized integers, the function $\zeta_{\mathscr{P}}(s)$, for $\sigma > 1$, is defined by

$$\zeta_{\mathscr{P}}(s) = \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{m \in \mathscr{N}_{\mathscr{P}}} \frac{1}{m^s}$$

and has meromorphic continuation to the half-plane $\sigma > \beta$. Beurling zeta-functions are interesting analytic objects having modern applications in quasicrystal theory and fractal membrane theory. Assuming additionally that the set $\{\log p : p \in \mathscr{P}\}$ is linearly independent over the field of rational numbers, and that

$$\sum_{\substack{m \leqslant x \\ m \in \mathscr{N}_{\mathscr{P}}}} \Lambda_{\mathscr{P}}(m) - x \ll x^{\beta_1}, \quad \beta_1 < 1,$$

where $\Lambda_{\mathscr{P}}(m)$ is the generalized von Mangoldt function, we obtained the functional independence of the function $\zeta_{\mathscr{P}}(s)$. This means that the equality

$$\sum_{k=0}^{r} s^k F_k\Big(\zeta_{\mathscr{P}}(s), \zeta'_{\mathscr{P}}(s), \dots, \zeta^{(n-1)}_{\mathscr{P}}(s)\Big) \equiv 0$$

in s implies that $F_k(\dots) \equiv 0$ for all $k = 0, 1, \dots, r$. This result extends the classical ones of O. Hölder, B. Riemann, D.D. Mordukhai-Boltovskoi, A. Ostrowski and S.M. Voronin on independence of the Riemann zeta-function $\zeta(s)$.

In generalized number theory, hypotheses are usually stated in terms of mean estimates. Concrete examples of systems $\mathscr{P}$ with given estimates is a complicated problem. Clearly, if $\mathscr{P}$ is a system of rational primes and the Riemann hypothesis is true, then hypotheses of Theorem 4 are satisfied with $\beta = 0$ and $\beta_1 = 1/2 + \varepsilon$, $\forall \varepsilon > 0$. Moreover, it is known [46] that if

$$\mathscr{P} = \{2, q \equiv 1 (\mathrm{mod}\, 4) \text{ with multiplicity } 2,\ q^2 : q \equiv 3 (\mathrm{mod}\, 4)\}$$

with rational primes q, then, under generalized Riemann hypothesis,

$$\psi_{\mathscr{P}}(x) - x \ll x^{1/2+\varepsilon}, \quad \forall \varepsilon > 0,$$

and

$$\mathfrak{N}_{\mathscr{P}}(x) - \frac{\pi x}{4} \ll x^{23/73}.$$

However, this system $\mathscr{P}$ contains identical numbers; therefore, the set $\{\log p : p \in \mathscr{P}\}$ is not linearly independent over $\mathbb{Q}$. Therefore, one of future problems is a search of theoretical examples of $\mathscr{P}$ with given properties.

Another future problem is to extend the results of the paper (Theorems 3–5) for a collection $\zeta_{\mathscr{P}_1}(s), \dots, \zeta_{\mathscr{P}_r}(s)$ of Beurling zeta-functions. This problem is closely connected to deep joint universality problems.

The main purpose of Beurling zeta-functions, as that of $\zeta(s)$, is the description of asymptotics of the system $\mathscr{P}$. Since there are infinitely many systems $\mathscr{P}$, we have infinitely many functions $\zeta_{\mathscr{P}}(s)$, and it is difficult to provide their possible applications. However, we believe that future applications of Beurling zeta-functions will be connected with non-standard theoretical and practical problems.

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