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Large sample properties of the tire wear rate and failure intensities estimates

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1. Linear path model of tire wear

Let t be the tire run (measured, say, in thousands of kilometers) and $X(t)$ denote the tire wear at the moment t (measured, say, in milimeters). We suppose that, for all $t \geqslant 0$,

$$X(t) = t/A, \tag{1.1}$$

where A is a positive random variable with unknown distribution function π.

Also suppose that tire failures are divided into $m+1$ groups and denote by T^k ($k = 0, \ldots, m$) the moment of failure of the kth type. The number $k = 0$ is assigned to the non-traumatic failure which arises when tire wear attains some critical value x_0. Hence, by (1.1),

$$T^0 = x_0 A.$$

Other failures are traumatic. We suppose that $T^1, \ldots, T^m$ are conditionally independent (given $A = a$) and have intensities which depend only on tire wear, i.e.,

$$P(T^k > t \mid A = a) = \exp\Bigl(-\int_0^t \lambda^k(s/a)\,\mathrm{d}s\Bigr) = \mathrm{e}^{-a\Lambda^k(t/a)},$$

where λ^k, $k = 1, \ldots, m$, are positive measurable functions and

$$\Lambda^k(x) = \int_0^x \lambda^k(y)\,\mathrm{d}y.$$

Set

$$f_k(t \mid a) = \lambda^k(t/a)\mathrm{e}^{-a\Lambda^k(t/a)}.$$

Then $f_1(t_1 \mid a) \ldots f_k(t \mid a)$ is a density of the joint distribution of random variables $A, T^1, \ldots, T^m$ with respect to the product of the measure π and of the Lebesgue measure on $(0; \infty)^m$.

In practice we observe only the earliest failure moment

$$T = \min(T^0, \ldots, T^m), \tag{1.2}$$

and the random variable

$$V = \begin{cases} 0 & \text{for} \quad T = T^0, \\ 1 & \text{for} \quad T = T^1, \\ \vdots & \\ m & \text{for} \quad T = T^m, \end{cases} \tag{1.3}$$

which indicates the type of the failure observed. Now we can formulate the statistical problem which is considered in this paper. Suppose that for each of n tires the failure moment T_i, the failure type V_i and the tire wear level at the failure time X_i are observed. Then $X_i = T_i/A_i$, where A_i is the wear rate of the ith tire. Hence the values of the A_is also are known, i.e., our inferences can be based on the data

$$(A_1, X_1, V_1), \ldots, (A_n, X_n, V_n),$$

which are independent copies of the random vector $(A, T/A, V)$. We have to estimate the unknown distribution function π and cumulative failure intensities Λ^k, $k = 1, \ldots, m$.

2. Non-parametric estimation of the tire wear rate and cumulative failures intensities

The distribution function π is estimated by the empirical distribution function

$$\hat{\pi}_n(a) = \frac{1}{n} \sum_{A_i \leqslant a} 1.$$

It is well-known that this estimate is uniformly consistent, i.e., almost surely

$$\sup_a \left|\hat{\pi}_n(a) - \pi(a)\right| \to 0,$$

as $n \to \infty$. Moreover, if the function π is continuous then the random function

$$\sqrt{n}(\hat{\pi}_n - \pi)$$

tends in distribution in the Skorokhod space $D[0; \infty]$ to a zero mean Gaussian process W^0 with the covariation function

$$EW^0(a)W^0(a') = \pi(a \wedge a') - \pi(a)\pi(a').$$

The estimates of cumulative intensities will be based on the counting processes

$$N_n^k(x) = \sum_{X_i \leqslant x, V_i = k} 1. \tag{2.1}$$

The following theorem shows that these processes satisfy the so-called *multiplicative intensities model* (see [1]).

Theorem 2.1. *Let $\mathcal{F}_x$ denote the σ-field generated by the random variables $A_1, \ldots, A_n$ and $N_n^1(y), \ldots, N_n^m(y)$ with $y \leqslant x$. Then the process N_n^k can be written in the form*

$$N_n^k(x) = \int_0^x \lambda^k(y) H_n(y)\,\mathrm{d}y + M_n^k(x),$$

where

$$H_n(x) = \sum_{X_i \geqslant x} A_i, \tag{2.2}$$

and $(M_n^k(x) \mid 0 \leqslant x \leqslant x_0)$ is a martingale with respect to the filtration $(\mathcal{F}_x \mid 0 \leqslant x \leqslant x_0)$. Moreover, the predictable covariation of the processes M_n^k and M_n^l is given by

$$< M_n^k, M_n^l > (x) = \delta_{kl} \int_0^x \lambda^k(y) H_n(y)\,\mathrm{d}y, \tag{2.3}$$

where δ_{kl} stands for the Kronecker symbol.

In the multiplicative intensities model cumulative intensities are estimated by the Nelson-Aalen estimates. In our case they are given by

$$\hat{\Lambda}_n^k(x) = \int_0^x H_n^{-1}(y)\,\mathrm{d}N_n^k(y) = \sum_{X_i \leqslant x, V_i = k} \left[\sum_{X_j \geqslant X_i} A_j \right]^{-1}. \tag{2.4}$$

Theorem 2.2. *If $EA < \infty$ then the estimate $\hat{\Lambda}_n^k$ is uniformly consistent, i.e., almost surely*

$$\sup_{0 \leqslant x \leqslant x_0} \left| \hat{\Lambda}_n^k(x) - \Lambda^k(x) \right| \to 0, \quad \text{as} \quad n \to \infty.$$

Notice that this theorem is stronger than Theorem IV.1.1 of [1], where only convergence in probability is proved.

Set $\Lambda(x) = \Lambda^1(x) + \ldots + \Lambda^m(x)$ and $h(x) = E\left[A\mathrm{e}^{-A\Lambda(x)}\right]$. If $EA < \infty$ then, by the Strong Law of Large Numbers, almost surely

$$n^{-1} H_n(x) \to h(x),$$

as $n \to \infty$. Therefore Theorem IV.1.2 of [1] (see the remark below the proof of this theorem) implies that the random function

$$\sqrt{n}(\hat{\Lambda}_n^1 - \Lambda^1, \ldots, \hat{\Lambda}^m - \Lambda^m)$$

tends in distribution in the space $D^m[0; x_0]$ to $(W^1, \ldots, W^m)$, the vector of independent zero mean Gaussian processes with covariation function

$$EW^k(x)W^k(x') = \sigma^k(x \wedge x'),$$

where

$$\sigma^k(x) = \int_0^x \frac{\lambda^k(y)}{h(y)}\, \mathrm{d}y.$$

Our last theorem shows that the estimates $\hat{\pi}_n$ and $\hat{\Lambda}_n^k$ are asymptotically independent.

Theorem 2.3. *If the function π is continuous and $EA < \infty$ then the random function*

$$\sqrt{n}(\hat{\pi}_n - \pi, \hat{\Lambda}_n^1 - \Lambda^1, \ldots, \hat{\Lambda}^m - \Lambda^m)$$

tends in distribution in the space $D[0; \infty] \times D^m[0; x_0]$ to a random vector function with independent components.

3. The proofs

Proof of Theorem 2.1. Fix arbitrary $0 \leqslant y < x \leqslant x_0$ and prove that

$$E\big[N_n^k(x) - N_n^k(y) \mid \mathcal{F}_y\big] = E\bigg[\int_y^x \lambda^k(u) H_n(u) du \mid \mathcal{F}_y\bigg].$$

By linearity, it suffices to consider the case $n = 1$. If $A = a$ and $X \leqslant y$ then the random variable $N^k(x)$ almost surely coincides with $N^k(y)$ (for short we omit the lower index 1). If $A = a$ and $X > y$ then $N^k(x)$ takes two values, 0 with probability $1 - p$ and 1 with probability p, where

$$\begin{aligned} p &= P(ya < T^k \leqslant xa, T^1, \ldots, T^m \mid T > ya, A = a) \\ &= \mathrm{e}^{a\Lambda(y)} \int_{ya}^{xa} \lambda^k(t/a) \mathrm{e}^{-a\Lambda(t/a)}\, \mathrm{d}t = a\mathrm{e}^{a\Lambda(a)} \int_y^x \lambda^k(u) \mathrm{e}^{-a\Lambda(u)}\, \mathrm{d}u. \end{aligned}$$

Hence

$$E\big[N^k(x) | \mathcal{F}_y\big] - N^k(y) = 1_{\{X > x\}} A \int_y^x \lambda^k(u) \mathrm{e}^{-A[\Lambda(u) - \Lambda(y)]}\, \mathrm{d}u. \tag{3.1}$$

In the same way we get

$$E\big[1_{\{X>x\}}|\mathcal{F}_y\big] = 1_{\{X>y\}}e^{-A\left[\Lambda(x)-\Lambda(y)\right]}.$$

Therefore the right-hand side of (3.1) equals

$$A\int_y^x \lambda^k(u)E\big[1_{\{X>u\}}|\mathcal{F}_y\big]\,\mathrm{d}u = E\left[\int_y^x \lambda^k(u)H(u)\,\mathrm{d}u|\mathcal{F}_y\right].$$

We proved that M_n^k is a martingale. This means that $\int \lambda^k H_n$ is a compensator of the counting process N_n^k. Therefore (2.3) follows from the continuity of the compensator (see [1], Section II. 4).

Proof of Theorem 2.2. First prove that $\hat{\Lambda}_n^k(x)$ is a consistent estimate of $\Lambda^k(x)$ for each fixed $x \in [0; x_0]$. Fix an integer $q \geqslant 1$ and for $r = 0, 1, \ldots, q$ define $y_r = rx/q$. Also set

$$K_{nr} = \sum_{y_{r-1}<X_i\leqslant y_r, V_i=k} \big[H_n(y_r)\big]^{-1}.$$

Then

$$\hat{\Lambda}_n^k(x) \leqslant \sum_{r=1}^{q} K_{nr}.$$

By the Strong Law of Large Numbers, K_{nr} tends almost surely to

$$c_{qr} = \frac{E\left[A\int_{y_{r-1}}^{y_r} \lambda^k(y)e^{-A\Lambda(y)}\,\mathrm{d}y\right]}{E\left[Ae^{-A\Lambda(y_r)}\right]}.$$

Therefore almost surely

$$\limsup_{n\to\infty} \hat{\Lambda}_n^k(x) \leqslant \lim_{q\to\infty} \sum_{r=1}^{q} c_{qr} = \Lambda^k(x).$$

In the same way we prove that almost surely

$$\liminf_{n\to\infty} \hat{\Lambda}_n^k(x) \geqslant \Lambda^k(x).$$

Hence $\hat{\Lambda}_n^k(x)$ is a consistent estimate of $\Lambda^k(x)$. Now uniform consistency is proved analogously as the Glivenko-Cantelli theorem.

Proof of Theorem 2.3. Define a probability space $(\Omega, P) = \prod_{i=1}^{\infty}(\Omega_i, P_i)$, where, for all i, $\Omega_i = (0;\infty) \times (0;1)^m$ and P_i is a product of the measure π and of the Lebesgue measure on $(0;1)^m$. The components of the vector $\omega_i \in \Omega_i$ will be denoted by $(\tilde{a}_i, u_{i1}, \dots, u_{im})$. For all $\omega = (\omega_i \mid i \geqslant 1)$ and $a = (a_i \mid i \geqslant 1)$ define

$$\tilde{A}_i(\omega) = \tilde{a}_i, \quad \tilde{T}_i^0(a, \omega) = x_0 a_i,$$

and, for $k = 1, \dots, m$,

$$\tilde{T}_i^k(a, \omega) = a_i \Psi^k(-\log(1 - u_{ik})/a_i),$$

where Ψ^k denotes the inverse function of Λ^k. If $\tilde{A} = (\tilde{A}_i \mid i \geqslant 1)$ then $(\tilde{A}_i, \tilde{T}_i^0(\tilde{A}), \dots, \tilde{T}_i^m(\tilde{A}))$ are independent copies of the vector $(A, T^0, \dots, T^m)$.

Define the random variables $\tilde{T}_i(a)$ and $\tilde{V}_i(a)$ by formulae (1.2) and (1.3) with $T^0, \dots, T^m$ replaced by $\tilde{T}^0(a), \dots, \tilde{T}^m(a)$. Set $\tilde{X}_i(a) = \tilde{T}_i(a)/a_i$ and define the processes $\tilde{N}_n^k(a, x)$ and $\tilde{H}_n(a, x)$ by equalities (2.1) and (2.2) with X_i, V_i, A_i replaced by $\tilde{X}_i(a), \tilde{V}_i(a), a_i$. Repeating the proof of Theorem 2.1 shows that $\tilde{N}_n^k(a) - \int \lambda^k \tilde{H}_n(a)$ is a martingale with respect to the filtration $(\tilde{\mathcal{F}}_x(a) \mid 0 \leqslant x \leqslant x_0)$, where $\tilde{\mathcal{F}}_x(a)$ is the σ-field generated by the random variables $\tilde{N}_n^1(a, y), \dots, \tilde{N}_n^m(a, y)$ with $y \leqslant x$.

Define the processes $\tilde{\Lambda}_n^k(a)$ by formula (2.4) with H_n, N_n^k replaced by $\tilde{H}_n(a), \tilde{N}_n^k(a)$. By the Weak Law of Large Numbers,

$$n^{-1}\tilde{H}_n(a, x) - n^{-1}\sum_{i=1}^n a_i e^{-a_i \Lambda(x)}$$

tends to 0 in probability as $n \to \infty$. Hence we can repeat the proof of Theorem IV.1.2 of [1] to get the following result: *if, for all* $x \in [0; x_0]$,

$$n^{-1}\sum_{i=1}^n a_i e^{-a_i \Lambda(x)} \to h(x), \tag{3.2}$$

then the random function

$$\sqrt{n}(\tilde{\Lambda}_n^1(a, \cdot) - \Lambda^1(\cdot), \dots, \tilde{\Lambda}_n^m(a, \cdot) - \Lambda^m(\cdot))$$

tends in distribution in the space $D^m[0; x_0]$ *to* $(W^1, \dots, W^m)$.

Fix arbitrary bounded continuous function ψ from $D^m[0;1]$ to $\mathbb{R}$. Then

$$E\psi(\tilde{\Lambda}_n^1(a, \cdot) - \Lambda^1(\cdot), \dots, \tilde{\Lambda}_n^m(a, \cdot) - \Lambda^m(\cdot)) \to E\psi(W^1, \dots, W^m),$$

for all sequences a satisfying (3.2). The expectation on the left-hand side is a variant of the conditional expectation $E[\psi(\hat{\Lambda}_n^1 - \Lambda^1, \dots, \hat{\Lambda}_n^m - \Lambda^m)|\mathcal{A}]$, where $\mathcal{A}$ denotes the σ-field generated by the random variables $A_1, A_2, \dots$. Since almost surely

$$\sup_{0 \leqslant x \leqslant x_0}\left|n^{-1}\sum_{i=1}^n A_i e^{-A_i \Lambda(x)} - h(x)\right| \to 0$$

(this is proved analogously as the Glivenko-Cantelli theorem),

$$E\left[\psi(\hat{\Lambda}_n^1 - \Lambda^1, \dots, \hat{\Lambda}_n^m - \Lambda^m) | \mathcal{A}\right] \to E\psi(W^1, \dots, W^m)$$

almost surely. This yields, for any bounded continuous function φ from $D[0;\infty]$ to $\mathbb{R}$,

$$E\left[\varphi(\hat{\pi}_n - \pi)\psi(\hat{\Lambda}_n^1 - \Lambda^1, \dots, \hat{\Lambda}_n^m - \Lambda^m)\right] \to E\left[\varphi(W^0)\right] E\left[\psi(W^1, \dots, W^m)\right].$$

References

[1] P.K. Andersen, O. Borgan, R.D. Gill, N. Keiding, *Statistical Models Based on the Counting Processes*, Springer-Verlag, New York, Inc. (1993).

Padangų dylimo greičio pasiskirstymo ir sprogimų intensyvumų įverčių asimptotinės savybės

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Nagrinėjamas padangos eksploatavimo modelis, kuriame padangų sprogimų intensyvumai priklauso nuo padangos nusidėvėjimo laipsnio. Sukonstruoti neparametriniai sprogimų intensyvumų iverčiai ir įrodyta, kad jie asimptotiškai nepriklauso nuo padangos dylimo greičio pasiskirstymo neparametrinio įverčio.