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Extremes of Product of Gaussian Random Variables

Džiugas Chvoinikov [1], Svyatoslav Novikov [2] and Jonas Šiaulys [1,*]

[1] Institute of Mathematics, Vilnius University, Naugarduko 24, 03225 Vilnius, Lithuania; dziugas.chvoinikov@mif.stud.vu.lt
[2] Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland; svyatoslav.novikov@unil.ch
* Correspondence: jonas.siaulys@mif.vu.lt

Abstract

We investigate the asymptotic behavior of the product of Gaussian random variables, with a focus on tail probabilities. Although sums of normal variables are well understood through classical limit theorems, their products exhibit significantly more complex behavior and have no simple closed-form distributions. We analyze the extremes of such products by deriving precise asymptotic expressions for the tail probabilities as the threshold tends to infinity. The study covers both centered and shifted but independent Gaussian variables, as well as cases with heterogeneous variances and nonzero expectations. Using transformation techniques, geometric arguments in high-dimensional spaces, and asymptotic analysis based on Gaussian measures, we establish general results describing the decay rate of tail probabilities. The main theorems provide explicit asymptotic formulas that depend on the number of variables, their shifts, and the variance structure. Several corollaries present results for important particular cases, including the case of identically distributed random variables. Three individual examples are provided at the end of the paper to illustrate the resulting asymptotic formulas and compare them with Monte Carlo estimates.

Keywords: Gaussian distribution; product of random variables; tail function; asymptotic formula

MSC: 60E05; 60E15

1. Introduction

In probability theory and statistical modeling, understanding the behavior of random variables and their interactions is fundamental to a wide range of applications. Among the many operations involving random variables, the products of independent random variables are an important and complex topic of study. Although the sums of random variables have been extensively analyzed in various contexts—often due to their central role in the *Central Limit Theorem*—the products of random variables, particularly those that are normally distributed, have received comparatively less attention. Theory for the products of independent random variables is far less developed than the mature theory for the sums of independent random variables. We mention the books [1,2]. The asymptotic behavior of the product of random variables poses unique challenges and opportunities for understanding the long-term behavior of multiplicative processes.

We say that a random variable (r.v.) ξ has a *normal distribution* if its density function has the form

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$$f_\xi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2 \right\}, \ x \in \mathbb{R},$$

where the parameter $\mu \in \mathbb{R}$ is the expectation, and $\sigma > 0$ is the standard deviation of ξ. The distribution function (d.f.) of a normal r.v. ξ with parameters μ and σ is

$$F_\xi(x) = \mathbb{P}(\xi \leqslant x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{ -\frac{1}{2}\Big(\frac{y-\mu}{\sigma}\Big)^2 \right\} \mathrm{d}y.$$

In the case where $\mathbb{E}\xi = \mu = 0$ and $\mathbb{E}\xi^2 = \sigma^2 = 1$, this distribution is called the standard normal distribution. In such a case,

$$f_\xi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\}, \ F_\xi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{ -\frac{y^2}{2} \right\} \mathrm{d}y.$$

The normal distribution, or *Gaussian distribution*, along with various modifications of this distribution is a cornerstone of probability theory and is widely used across disciplines such as economics, finance, physics, machine learning, etc.; see, for instance, the classical [3–6] or recent sources [7–15]. For instance, in [7], the inverse Gaussian distribution is employed to model the behavior of high-frequency cryptocurrency returns. In [8], the Gaussian distribution serves as the reference error model, under which the proposed modal regression approach retains efficiency while providing additional robustness against outliers and non-normal data. In [10], the Gaussian distribution is used to represent process noise, measurement noise, and uncertainty in equality constraints, enabling probabilistic modeling and state estimation. In [12], a novel one-class domain adaptation framework is proposed that leverages only normal-class data from the target domain at test-time to adapt ensemble models for anomaly detection. In [13], a generalized normal distribution is introduced, which is constructed as a mixture of two generalized Gaussian distributions, and which is suitable for modeling skewed data with strong skewness and high kurtosis. While in [15], the asymptotic normality is considered for a mode-adaptive factor model.

The ubiquity of the Gaussian distribution is largely due to the *Central Limit Theorem*, which states that, under certain conditions, the sums of independent and identically distributed random variables approach a normal distribution as the number of variables increases. However, when we consider the product of random variables, the situation is more complicated. The product of normally distributed random variables, especially as the number of factors increases, yields a complex distribution that cannot generally be expressed in a simple closed form. Understanding the asymptotic behavior of such products is therefore crucial in both theoretical and applied contexts, where multiplicative processes often arise in the modeling of dynamic systems.

In particular, the asymptotic behavior of the products of normal distributions has important implications in areas such as finance, signal processing, and statistical mechanics. For instance, in the modeling of asset prices or stochastic processes that evolve multiplicatively over time, the products of random variables can capture the behavior of systems under the influence of independent normally distributed shocks. The asymptotic analysis of these products is critical for understanding long-term trends, volatility, and distributional convergence.

The products of normal random variables pose several theoretical challenges. Unlike the sums of normal variables, which yield normal distributions, the products of independent normal variables do not yield normal distributions, nor do they follow any simple standard distributions. Moreover, multiplying normal distributions yields distributions with increasingly heavier tails.

We recall that a r.v. X together with its distribution function F_X is called heavy-tailed if $\mathbb{E}e^{\delta X} = \infty$ for all $\delta > 0$. In the opposite case, a r.v. X together with its d.f. is called light-tailed.

It is obvious that the Gaussian distribution ξ is light-tailed because

$$\mathbb{E}e^{\delta\xi} = \exp\left\{\mu\delta + \frac{\sigma^2}{2}\delta^2\right\}$$

for all $\delta > 0$ and all values of the parameters μ and $\sigma > 0$. From the main results presented in Section 2 it follows that by multiplying two independent standard normal distributions ξ_1 and ξ_2 we obtain a light-tailed distribution $\xi = \xi_1\xi_2$ with tail asymptotics

$$\mathbb{P}(\xi > x) \underset{x\to\infty}{\sim} \frac{1}{\sqrt{2\pi x}}\,e^{-x},$$

whereas by multiplying three or more independent standard normal distributions $\xi_1, \xi_2, \ldots, \xi_n$, $n \geqslant 3$, we obtain a heavy-tailed Weibull-type distribution with tail asymptotics

$$\mathbb{P}\big(\xi_1\xi_2\ldots\xi_n > x\big) \underset{x\to\infty}{\sim} \frac{2^{n/2-1}}{\sqrt{\pi n}}\,x^{-1/n}\exp\Big\{-\frac{n}{2}x^{2/n}\Big\}.$$

If we multiply nonstandard normal distributions, then the distribution of the product also becomes heavier, but this phenomenon is not as clearly visible as when multiplying standard normal distributions. The fact that the distribution obtained by multiplying independent r.v.s becomes heavier has been observed in previous works as well [16–21].

Various published studies have derived exact formulas for the density of the product or the distribution function of the product of independent normal distributions. Usually, such formulas are written using special functions related to the set of Meijer G-functions; see, e.g., [22–29]. For example, it was observed that for independent standard normal r.v.s ξ_1 and ξ_2, the density of the product $\xi_1\xi_2$ is

$$f_{\xi_1\xi_2}(x) = \frac{1}{\pi}K_0\big(|x|\big),\ x \in \mathbb{R},$$

where

$$K_0(x) = \frac{1}{2}\int_0^\infty y^{-1}\exp\Big\{-y - \frac{x^2}{4y}\Big\}\mathrm{d}y,\ x \geqslant 0.$$

is the modified Bessel function of the second kind of order zero.

Special functions related to Meijer G-functions are usually expressed by rather complicated integrals and do not have an explicit analytical expression. Therefore, it makes sense to look for relatively simple analytical expressions that would sufficiently well approximate the density or distribution functions of the product of normal distributions. In this work, we find relatively simple analytical formulas that approximate the tails of the distributions of products of independent normal random variables with sufficient accuracy. In the case of independence, we obtain asymptotic formulas in the most general case, where the elements of the product may have different means and variances.

To obtain asymptotic formulas for the tail of the product of independent random variables, the so-called saddle-point method is usually applied. This method is described in detail in books [30–32] and is applied to the asymptotics of products of various distributions in [33–39]. For instance, in [33] (see Lemma 2.1), it is derived that for two independent non-negative r.v.s with asymptotically Weibullian tail distributions

$$\mathbb{P}(\xi_i > x) = D_i x^{\gamma_i}\exp\big\{-\beta_i x^{\alpha_i}\big\}(1 + o(1)),$$

where $D_i, \beta_i, \alpha_i > 0$, $\gamma_i \in \mathbb{R}$, $i \in 1, 2$, we have

$$\mathbb{P}(\xi_1\xi_2 > x) = Dx^{\gamma} \exp\big\{ -\beta x^{\alpha} \big\}(1 + o(1))$$

with constants D, α, β, γ expressed in terms of $D_1, D_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$. In [35] (see Example 3), it is obtained that the density function φ_n of the product of n independent r.v.s with standard normal distributions has the following asymptotic expression

$$\varphi_n(x) = \frac{2^{(n-1)/2}}{\sqrt{2\pi n}}\, x^{1/n-1} \exp\Big\{ -\frac{n}{2} x^{2/n} \Big\} \Big(1 + O\big(x^{-2/n}\big)\Big).$$

Unfortunately, when multiplying a large number of normally distributed random variables with nonzero means, the saddle point method becomes complicated to apply. The reason for this is that normally distributed random variables with nonzero means are not symmetric with respect to zero. There are currently no asymptotic formulas for the distribution of the product of normal random variables when at least three random variables with nonzero means are multiplied, whether dependent or independent. In the case where only two normal random variables are multiplied, in the paper [40], an asymptotic formula was obtained in the most general case, when the random variables are dependent and have possibly different means and variances. However, the authors did not use the saddle-point method in their work.

Another way to obtain asymptotic formulas for the tail of a product of normally distributed random variables is to use exact expressions. As we have already mentioned, those exact expressions are closely related to special functions, such as Bessel or Meijer G-functions of appropriate order; see, for instance, [27,28,41,42]. In [40] (Theorem 2.3), using the exact expression for the density of the product of two normally distributed r.v.s, the following asymptotic formula is obtained

$$\mathbb{P}(\xi_1\xi_2 > x) = \frac{C(1+\rho)\sqrt{\sigma_1\sigma_2}}{\sqrt{2\pi x}} \exp\Bigg\{ \frac{1}{8}\Big(\frac{1+\rho}{1-\rho}\Big)\Big(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\Big)^2 - \frac{x}{\sigma_1\sigma_2(1+\rho)} \Bigg\}$$
$$\times \cosh\Bigg(\Big|\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\Big| \frac{\sqrt{x}}{(1+\rho)\sqrt{\sigma_1\sigma_2}}\Bigg)\Big(1 + O\big(x^{-1/2}\big)\Big),$$

where

$$C = \exp\Bigg\{ -\frac{1}{2(1-\rho^2)}\Big(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho\mu_1\mu_2}{\sigma_1\sigma_2}\Big) \Bigg\},$$

and ξ_1, ξ_2 are normally distributed r.v.s with expectations $\mu_1, \mu_2 \in \mathbb{R}$, standard deviations σ_1, σ_2, and correlation coefficient $\rho \in (-1, 1)$. Unfortunately, when the number of multipliers n is large, the derivation of asymptotic formulas in the described way becomes very complicated due to the complex structure of the special functions.

In such a situation, we use an alternative method to determine the asymptotics of the tail probabilities of the product of normal random variables. Using transformation techniques, geometric arguments in high-dimensional spaces, and asymptotic analysis based on Gaussian measures, we establish general results describing the decay rate of tail probabilities. Our method is suitable for the tail of the product of normal random variables due to the specific density form, because for independent normally distributed random variables $\xi_1, \ldots, \xi_n$, we have

$$\mathbb{P}(\xi_1 \ldots \xi_n > u) = \frac{1}{(2\pi)^{n/2}} \Bigg(\prod_{k=1}^{n} \sigma_k\Bigg)^{-1} \int_{x_1 \ldots x_n > u} \cdots \int \exp\Bigg\{ -\frac{1}{2} \sum_{k=1}^{n} \Big(\frac{x_k - \mu_k}{\sigma_k}\Big)^2 \Bigg\} \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where $\mathbb{E}\xi_k = \mu_k$ and $\mathrm{Var}(\xi_k) = \sigma_k^2$ for $k \in \{1, 2, \ldots, n\}$. Our proposed "geometric" method may also be suitable in more general situations, when considering products of other types of random variables, perhaps even dependent ones. Nevertheless, in this work, we limit ourselves to independent normally distributed random variables, but possibly with different means and variances.

The rest of the paper is organized as follows. In Section 2, we state the main results. Section 3 is devoted to auxiliary statements. Detailed proofs of the two main theorems are given in Sections 4 and 5. Section 6 is intended to show how to apply the resulting asymptotic formulas to particular cases. In Section 7, we provide a brief review of the main results and discuss possible further research.

2. Main Results

The main general result on the distribution tail of the product of independent normally distributed r.v.s is formulated in Corollary 4. Its statement follows directly from Theorem 2, which in turn follows easily from Theorem 1. Thus, the main result of our work is Theorem 1, which describes the asymptotic behavior of the product of the shifted standard Gaussian random variables with an additional positivity condition.

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent normal r.v.s such that $\mathbb{E}\xi_k = 0$ and $\mathbb{E}\xi_k^2 = 1$ for all $k \in \{1, 2, \ldots, n\}$, and let $\{c_1, c_2, \ldots, c_n\}$ be a collection of real constants. Then*

$$\mathbb{P}\bigg(\prod_{k=1}^{n} (\xi_k - c_k) > u^n; \xi_k > c_k, k \in \{1, 2, \ldots, n\} \bigg) \underset{u\to\infty}{\sim} \frac{1}{u\, 2^{n/2}\sqrt{\pi n}} \exp\bigg\{ -\frac{n}{2}(u+\bar{c})^2 - \frac{1}{4}\sum_{k=1}^{n}(c_k - \bar{c})^2 \bigg\}, \tag{1}$$

where $\bar{c} = \frac{1}{n}\sum_{k=1}^{n} c_k$.

In the case that all shift constants are zero, the following result follows directly from Theorem 1.

Corollary 1. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent standard normal r.v.s. Then*

$$\mathbb{P}\bigg(\prod_{k=1}^{n} \xi_k > u; \xi_k > 0, k \in \{1, 2, \ldots, n\} \bigg) \underset{u\to\infty}{\sim} \frac{1}{2^{n/2}\sqrt{\pi n}}\, u^{-1/n} \exp\bigg\{ -\frac{n}{2}u^{2/n} \bigg\}.$$

For the standard normal r.v.s, we have

$$\mathbb{P}\bigg(\prod_{k=1}^{n} \xi_k > x \bigg) = 2^{n-1}\mathbb{P}\bigg(\prod_{k=1}^{n} \xi_k^+ > x \bigg),\ x > 0,$$

where $a^+ = \max\{0, a\}$ denotes the positive part of a. According to this equality, the following statement follows from Corollary 1.

Corollary 2. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent standard normal r.v.s. Then*

$$\mathbb{P}\bigg(\prod_{k=1}^{n} \xi_k > u \bigg) \underset{u\to\infty}{\sim} \frac{2^{n/2-1}}{\sqrt{\pi n}}\, u^{-1/n} \exp\bigg\{ -\frac{n}{2}u^{2/n} \bigg\}.$$

We observe here that the asymptotic equalities of Corollaries 1 and 2 are aligned with the results presented in [17,18,28,33,34,36,37,43].

The following statement can also be easily obtained from the main Theorem 1. In the statement, the asymptotics of the tails of normally distributed random variables with possibly different variances are obtained.

Corollary 3. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent normally distributed r.v.s with $\mathbb{E}\xi_k = 0$ and $\mathbb{E}\xi_k^2 = \sigma_k^2 > 0$ for $k \in \{1, 2, \ldots, n\}$. Then*

$$\mathbb{P}\left(\prod_{k=1}^{n} \xi_k > u\right) \underset{u\to\infty}{\sim} \frac{2^{n/2-1}}{\sqrt{\pi n}}\, u^{-1/n} \left(\prod_{k=1}^{n} \sigma_k\right)^{1/n} \exp\left\{-\frac{n}{2} u^{2/n} \left(\prod_{k=1}^{n} \sigma_k\right)^{-2/n}\right\}.$$

We observe that the last asymptotic relation of Corollary 3 follows from Corollary 2 because the random variables ξ_k/σ_k are distributed according to the standard normal law for $k \in \{1, 2, \ldots, n\}$.

The following statement can also be derived from Theorem 1. This statement yields an asymptotic formula for the tail probability of the product of centered standard normal random variables, but without additional positivity conditions.

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent normal r.v.s such that $\mathbb{E}\xi_k = 0$ and $\mathbb{E}\xi_k^2 = 1$ for all $k \in \{1, 2, \ldots, n\}$, and let $\{c_1, c_2, \ldots, c_n\}$ be a collection of real constants. Then*

$$\mathbb{P}\left(\prod_{k=1}^{n} (\xi_k - c_k) > u^n\right)$$
$$\underset{u\to\infty}{\sim} \frac{N_{\alpha^*}}{u\, 2^{n/2}\sqrt{\pi n}} \exp\left\{-\frac{n}{2}\left(u + \bar{c}_{\alpha^*}\right)^2 - \frac{1}{4}\sum_{k=1}^{n} \left((-1)^{\alpha_k^*} c_k - \bar{c}_{\alpha^*}\right)^2\right\},$$

where $\bar{c}_{\alpha^*} = \frac{1}{n}\sum_{k=1}^{n} (-1)^{\alpha_k^*} c_k$, $\alpha^* = \{\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*\}$ is a collection of elements from $\{0, 1\}$ such that the sum $\sum_{k=1}^{n} \alpha_k^*$ is even and the sum $\sum_{k=1}^{n} (-1)^{\alpha_k^*} c_k$ is as small as possible, and N_{α^*} is the number of collections α^* such that number $\sum_{k=1}^{n} (-1)^{\alpha_k^*} c_k$ is the same. Equivalently, if

$$\mathcal{A}_* := \arg \min_{\substack{\alpha\in\{0,1\}^n \\ \sum_{k=1}^{n} \alpha_k \text{ even}}} \sum_{k=1}^{n} (-1)^{\alpha_k} c_k,$$

then

$$N_{\alpha^*} = \#\mathcal{A}_*.$$

Remark 1. *We observe that the number N_{α^*} can be found according to the following rules:*

(i) *If some of the numbers c_k are equal to zero, then*

$$N_{\alpha^*} = 2^{\#\{k:\, c_k = 0\} - 1}.$$

Indeed , in such a case, we must choose $\alpha_k^* = 1$ for $c_k > 0$, $\alpha_k^* = 0$ for $c_k < 0$, and for $c_k = 0$, we can choose α_k^* arbitrarily up to the condition that $\sum_{k=1}^{n} \alpha_k^*$ is divisible by 2.

(ii) *If there are no zeros among $\{c_1, c_2, \ldots, c_n\}$ and $\#\{k :\, c_k > 0\}$ is even, then*

$$N_{\alpha^*} = 1.$$

Indeed, in such a case, we must choose $\alpha_k^* = 1$ for $c_k > 0$ and $\alpha_k^* = 0$ for $c_k < 0$.

(iii) *If there are no zeros among $\{c_1, c_2, \ldots, c_n\}$ and $\#\{k :\, c_k > 0\}$ is odd, then*

$$N_{\alpha^*} = \#\{k :\, |c_k| \leqslant |c_j|, \text{ for all } j \in \{1, 2, \ldots, n\}\}.$$

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Indeed, the optimal choice of α_k^* is obtained as follows: firstly, we set $\alpha_k^* = 1$ for $c_k > 0$ and $\alpha_k^* = 0$ for $c_k < 0$, and then we change one such α_k^* under the condition $|c_k| \leqslant |c_j|$ for all $j \in \{1, 2, \ldots, n\}$.

Remark 2. *The rules in Remark 1 give a simple procedure for computing N_{α^*}. Indeed, it suffices to scan the collection $\{c_1, c_2, \ldots, c_n\}$ once to count the zero and positive elements and, if needed, find the smallest value of $|c_k|$ and the number of times it is attained. Hence, the computational complexity of this procedure is $O(n)$, in contrast to checking all admissible sign vectors directly, which would require considering $O(2^n)$ possibilities.*

The following statement can be directly derived from Theorem 2. In this statement, the asymptotics of the tails of random variables are obtained for normally distributed random variables with different expectations and variances.

Corollary 4. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent normal r.v.s such that $\mathbb{E}\xi_k = \mu_k$ and $\mathrm{Var}(\xi_k) = \sigma_k^2$ for all $k \in \{1, 2, \ldots, n\}$. Then*

$$\mathbb{P}\Big(\prod_{k=1}^{n} \xi_k > u\Big)$$

$$\underset{u\to\infty}{\sim} \frac{N_\beta \Big(\prod_{k=1}^{n} \sigma_k\Big)^{1/n}}{u^{1/n}\, 2^{n/2}\sqrt{\pi n}} \exp\left\{ -\frac{n}{2}\Big(u^{1/n}\Big(\prod_{k=1}^{n}\sigma_k\Big)^{-1/n} - \overline{\mu}_\beta\Big)^2 - \frac{1}{4}\sum_{k=1}^{n}\Big((-1)^{\beta_k}\frac{\mu_k}{\sigma_k} - \overline{\mu}_\beta\Big)^2\right\},$$

where $\overline{\mu}_\beta = \frac{1}{n}\sum_{k=1}^{n}(-1)^{\beta_k}\frac{\mu_k}{\sigma_k}$, $\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is a collection of elements from $\{0, 1\}$ such that the sum $\sum_{k=1}^{n}\beta_k$ is even and the sum $\sum_{k=1}^{n}(-1)^{\beta_k}\frac{\mu_k}{\sigma_k}$ is as large as possible, and N_β is the number of collections β such that the sum $\sum_{k=1}^{n}(-1)^{\beta_k}\frac{\mu_k}{\sigma_k}$ is the same. Equivalently, if

$$\mathcal{B}_* := \arg \max_{\substack{\beta\in\{0,1\}^n \\ \sum_{k=1}^{n}\beta_k \text{ even}}} \sum_{k=1}^{n}(-1)^{\beta_k}\frac{\mu_k}{\sigma_k},$$

then

$$N_\beta = \#\,\mathcal{B}_*.$$

The statement follows from Theorem 2 by noting that

$$\begin{aligned}\mathbb{P}\Big(\prod_{k=1}^{n}\xi_k > u\Big) &= \mathbb{P}\bigg(\prod_{k=1}^{n}\frac{\xi_k - \mu_k + \mu_k}{\sigma_k} > u\Big(\prod_{k=1}^{n}\sigma_k\Big)^{-1}\bigg) \\ &= \mathbb{P}\bigg(\prod_{k=1}^{n}\Big(\eta_k - \Big(-\frac{\mu_k}{\sigma_k}\Big)\Big) > \Big(u^{1/n}\Big(\prod_{k=1}^{n}\sigma_k\Big)^{-1/n}\Big)^n\bigg),\end{aligned}$$

where η_k are independent standard normal r.v.s.

In the last corollary, we consider the product of independent and identically distributed normal random variables. Note that the form of the asymptotic formulas depends on whether the common mean of the product terms is positive or negative. The corollary directly follows from Corollary 4.

Corollary 5. *Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $n \geqslant 1$, be a collection of independent identically distributed normal r.v.s. If $\mathbb{E}\xi_1 = \mu > 0$ and $\mathrm{Var}(\xi_1) = \sigma^2$, then*

$$\mathbb{P}\Big(\prod_{k=1}^{n}\xi_k > u\Big) \underset{u\to\infty}{\sim} \frac{\sigma}{u^{1/n}2^{n/2}\sqrt{\pi n}}\exp\left\{-\frac{n}{2\sigma^2}\big(u^{1/n} - \mu\big)^2\right\}$$

If $\mathbb{E}\xi_1 = \mu < 0$ *and* $\mathrm{Var}(\xi_1) = \sigma^2$, *then*

$$\mathbb{P}\Big(\prod_{k=1}^{n} \xi_k > u\Big) \underset{u\to\infty}{\sim} \frac{\sigma}{u^{1/n} 2^{n/2} \sqrt{\pi n}} \exp\Big\{ -\frac{n}{2\sigma^2}\big(u^{1/n} + \mu\big)^2 \Big\}$$

in the case of even n, *and*

$$\mathbb{P}\Big(\prod_{k=1}^{n} \xi_k > u\Big) \underset{u\to\infty}{\sim} \frac{\sigma\sqrt{n}}{u^{1/n} 2^{n/2} \sqrt{\pi}} \exp\Big\{ -\frac{n}{2\sigma^2}\Big(u^{1/n} + \mu\,\frac{n-2}{n}\Big)^2 - \frac{n-1}{n}\frac{\mu^2}{\sigma^2} \Big\}$$

in the case of odd n.

3. Auxiliary Lemmas

In this section, we present and prove two auxiliary assertions of great importance for the proofs of the main theorems.

Let

$$\mathcal{B}_R := \Big\{ \vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n) : \|\vec{\mathbf{x}}\| = \Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2} < R \Big\}$$

be an open ball of radius R in $\mathbb{R}^n$, and let

$$\mathcal{H}_0 := \Big\{ \vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n) : \sum_{k=1}^{n} x_k = 0 \Big\}$$

be a hyperplane in $\mathbb{R}^n$. In addition, let

$$\mathcal{B}_R^* := \mathcal{B}_R \cap \mathcal{H}_0.$$

Lemma 1. *Suppose that* $n \geqslant 2$, $\varepsilon \in (0,1)$, *and* $\lambda > 0$ *are fixed numbers. Then, for sufficiently large* u,

$$\mathcal{B}^*_{(1-\varepsilon)\sqrt{2\lambda}\, n^{1/4}} \subset \mathcal{E}_{u,\lambda} \subset \mathcal{B}^*_{(1+\varepsilon)\sqrt{2\lambda}\, n^{1/4}},$$

where

$$\begin{aligned}\mathcal{E}_{u,\lambda} &:= \Big\{ \vec{\mathbf{y}} = (y_1, \ldots, y_n) : y_k > 0, \prod_{k=1}^{n} y_k > u^n, \sum_{k=1}^{n} y_k = un + \frac{\lambda\sqrt{n}}{u} \Big\} - \Big(u + \frac{\lambda}{u\sqrt{n}}\Big)\mathbf{1}_n \\ &= \Big\{ \vec{\mathbf{x}} = (x_1, \ldots, x_n) : x_k + \Big(u + \frac{\lambda}{u\sqrt{n}}\Big) > 0, \prod_{k=1}^{n} \Big(x_k + \big(u + \frac{\lambda}{u\sqrt{n}}\big)\Big) > u^n, \sum_{k=1}^{n} x_k = 0 \Big\}\end{aligned}$$

with $\mathbf{1}_n := (1, 1, \ldots, 1)$.

Proof. We divide the proof of the lemma into several steps.

(i) For large u, uniformly for $\vec{\mathbf{x}}$ in bounded subsets of $\mathbb{R}^n$, we have

$$\prod_{k=1}^{n} \Big(x_k + \big(u + \frac{\lambda}{u\sqrt{n}}\big)\Big) = u^n + \Big(\sum_{k=1}^{n} x_k\Big) u^{n-1} + \Big(\lambda\sqrt{n} + \sum_{1\leqslant i<j\leqslant n} x_i x_j\Big) u^{n-2} + O_{\lambda,n}\big(u^{n-3}\big).$$

If, in addition, $\sum_{k=1}^{n} x_k = 0$, then

$$\prod_{k=1}^{n} \Big(x_k + \big(u + \frac{\lambda}{u\sqrt{n}}\big)\Big) = u^n + \Big(\lambda\sqrt{n} - \frac{1}{2}\sum_{k=1}^{n} x_k^2\Big) u^{n-2} + O_{\lambda,n}\big(u^{n-3}\big). \tag{2}$$

(ii) First, let us prove that

$$\mathcal{E}_{u,\lambda} \subset \mathcal{B}^*_{(1+\varepsilon)\sqrt{2\lambda}\, n^{1/4}} \tag{3}$$

for large u.

If $\vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n) \in \mathcal{E}_{u,\lambda}$, then

$$x_k + u + \frac{\lambda}{u\sqrt{n}} > 0 \text{ for } k \in \{1, \ldots, n\}, \quad \sum_{k=1}^{n} x_k = 0, \ \text{ and } \prod_{k=1}^{n} \Big(x_k + \big(u + \frac{\lambda}{u\sqrt{n}} \big) \Big) > u^n.$$

Before applying expansion (2), we first show that the components x_k are uniformly bounded. Set temporarily

$$a := u + \frac{\lambda}{u\sqrt{n}}, \quad t_k := \frac{x_k}{a}, \quad k \in \{1, \ldots, n\}.$$

Then

$$t_k > -1 \text{ for } k \in \{1, \ldots, n\}, \quad \sum_{k=1}^{n} t_k = 0, \ \text{ and } \prod_{k=1}^{n} (1 + t_k) > \left(\frac{u}{a} \right)^n,$$

implying that

$$\sum_{k=1}^{n} \log(1 + t_k) > n \log(u/a).$$

Let us define two sets of indices $\mathcal{I}_+ := \{k : t_k \geqslant 0\}$ and $\mathcal{I}_- := \{k : t_k < 0\}$. Using the inequalities

$$\log(1+t) \leqslant t - \frac{t^2}{2(1+t)}, \ t \geqslant 0,$$
$$\log(1+t) \leqslant t, \ -1 < t < 0,$$

we obtain

$$\sum_{k=1}^{n} \log(1 + t_k) \leqslant \sum_{k=1}^{n} t_k - \frac{1}{2} \sum_{k \in \mathcal{I}_+} \frac{t_k^2}{1 + t_k} = -\frac{1}{2} \sum_{k \in \mathcal{I}_+} \frac{t_k^2}{1 + t_k}.$$

The derived estimates imply that

$$\sum_{k \in \mathcal{I}_+} \frac{t_k^2}{1 + t_k} \leqslant -2n \log\Big(\frac{u}{a} \Big) \leqslant -2n \log\Big(1 - \frac{\lambda}{u^2\sqrt{n}} \Big) \leqslant \frac{6\lambda\sqrt{n}}{u^2}$$

for large u because of the inequality

$$-\log(1 - x) \leqslant 3x, \ 0 \leqslant x \leqslant 0.9.$$

Since $t_k > -1$ and $\sum_{k=1}^{n} t_k = 0$, we have $t_k \leqslant n - 1$, implying that $1 + t_k \leqslant n$. Thus, for $k \in \mathcal{I}_+$,

$$t_k^2 \leqslant n \frac{t_k^2}{1 + t_k},$$

and therefore

$$\sum_{k \in \mathcal{I}_+} t_k^2 = O_{\lambda,n}(u^{-2}). \tag{4}$$

Set temporarily

$$S := \sum_{k \in \mathcal{I}_+} t_k = -\sum_{k \in \mathcal{I}_-} t_k.$$

By the Cauchy–Schwarz inequality,

$$|S| \leqslant \sum_{k \in \mathcal{I}_+} |t_k| \leqslant \Big(\sum_{k \in \mathcal{I}_+} t_k^2 \Big)^{1/2} \Big(\sum_{k \in \mathcal{I}_+} 1 \Big)^{1/2} = O_{\lambda,n}(u^{-1}).$$

Consequently,

$$\sum_{k\in\mathcal{I}_-} t_k^2 \leqslant \Big(\sum_{k\in\mathcal{I}_-}(-t_k)\Big)^2 = S^2 = O_{\lambda,n}(u^{-2}). \tag{5}$$

The derived estimates (4) and (5) imply that

$$\sum_{k=1}^n x_k^2 = a^2\sum_{k=1}^n t_k^2 = O_{\lambda,n}(1),$$

uniformly for $\vec{\mathbf{x}} \in \mathcal{E}_{u,\lambda}$ if u is sufficiently large. These relations and the asymptotic equality (2) imply that

$$\Big(\lambda\sqrt{n} - \frac{1}{2}\sum_{k=1}^n x_k^2\Big)u^{n-2} + O_{\lambda,n}\big(u^{n-3}\big) > 0$$

if $\vec{\mathbf{x}} \in \mathcal{E}_{u,\lambda}$. Therefore, for some positive $C_{\lambda,n}$,

$$\sum_{k=1}^n x_k^2 < 2\lambda\sqrt{n} + C_{\lambda,n}\frac{1}{u},$$

and for large u,

$$\sum_{k=1}^n x_k^2 \leqslant (1+\varepsilon)^2 2\lambda\sqrt{n},$$

implying the desired relation (3).

(iii) Now let us prove that

$$\mathcal{B}^*_{(1-\varepsilon)\sqrt{2\lambda}\,n^{1/4}} \subset \mathcal{E}_{u,\lambda}. \tag{6}$$

If $\vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n) \in \mathcal{B}^*_{(1-\varepsilon)\sqrt{2\lambda}\,n^{1/4}}$, then

$$\sum_{k=1}^n x_k = 0 \ \text{ and } \ \sum_{k=1}^n x_k^2 \leqslant (1-\varepsilon)^2 2\lambda\sqrt{n}.$$

According to (2), we have that

$$\prod_{k=1}^n\Big(x_k + \big(u + \frac{\lambda}{u\sqrt{n}}\big)\Big) \geqslant u^n\Big(1+\big(1-(1-\varepsilon)^2\big)\lambda\sqrt{n}\,u^{-2} + O_{\lambda,n}\big(u^{-3}\big)\Big) > u^n$$

for sufficiently large u. In addition, the condition $\vec{\mathbf{x}} \in \mathcal{B}^*_{(1-\varepsilon)\sqrt{2\lambda}\,n^{1/4}}$ implies that

$$x_k \geqslant -(1-\varepsilon)\sqrt{2\lambda}n^{1/4} > -\Big(u + \frac{\lambda}{u\sqrt{n}}\Big)$$

for all $k \in \{1, 2, \ldots, n\}$ and for large u. Therefore, the inclusion (6) holds, and the lemma is proved. □

Lemma 2. *Let b_u and b be positive numbers such that $\lim\limits_{u\to\infty}\frac{b_u}{u} = b$. Let $\mathcal{G}_u \subset \{\vec{\mathbf{x}} \in \mathbb{R}^n : x_1 \geqslant b_u\}$, $n \geqslant 2$, be a family of Borel subsets such that $\mathbb{1}_{\{\vec{\mathbf{x}}\in\mathcal{G}_u^*\}} \underset{u\to\infty}{\to} \mathbb{1}_{\{\vec{\mathbf{x}}\in\mathcal{G}\}}$ almost everywhere with respect to the Lebesgue measure, where $\mathcal{G}$ is a Borel subset of $\mathbb{R}^n$ of nonzero Lebesgue measure, and*

$$\begin{aligned}\mathcal{G}_u^* &= (u,1,\ldots,1) * \big(\mathcal{G}_u - b_u\vec{\mathbf{e}}_1\big)\\ &= (u,1,\ldots,1) * \big\{\big((x_1-b_u), x_2, \ldots, x_n\big) : (x_1, x_2, \ldots, x_n) \in \mathcal{G}_u\big\}\\ &= \big\{\big(u(x_1-b_u), x_2, \ldots, x_n\big) : (x_1, x_2, \ldots, x_n) \in \mathcal{G}_u\big\}\end{aligned}$$

with $\vec{\mathbf{e}}_1 = (1,0,\ldots,0)$*, the symbol* $*$ *denoting the Hadamard product (see Chapter 5 in [44], for instance). If* $\vec{\boldsymbol{\xi}} = (\xi_1, \xi_2, \ldots, \xi_n)$ *is a centered n-dimensional Gaussian vector with identity covariance matrix, then*

$$\mathbb{P}(\vec{\boldsymbol{\xi}} \in \mathcal{G}_u) \underset{u\to\infty}{\sim} \frac{\mathrm{e}^{-b_u^2/2}}{u(2\pi)^{n/2}} \int_{\mathcal{G}} \exp\left\{-bx_1 - \frac{1}{2}\sum_{k=2}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}}.$$

Proof. For $u > 0$, we have

$$\begin{aligned}\mathbb{P}(\vec{\boldsymbol{\xi}} \in \mathcal{G}_u) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathcal{G}_u} \exp\left\{-\frac{1}{2}x_1^2 - \frac{1}{2}\sum_{k=2}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}} \\ &= \frac{1}{u(2\pi)^{n/2}} \int_{\mathcal{G}_u^*} \exp\left\{-\frac{1}{2}\left(\frac{y_1}{u} + b_u\right)^2 - \frac{1}{2}\sum_{k=2}^{n} y_k^2\right\} \mathrm{d}\vec{\mathbf{y}} \\ &= \frac{\mathrm{e}^{-b_u^2/2}}{u(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{-\frac{b_u}{u}y_1 - \frac{y_1^2}{2u^2} - \frac{1}{2}\sum_{k=2}^{n} y_k^2\right\} \mathbb{1}_{\{\vec{\mathbf{y}}\in\mathcal{G}_u^*\}} \mathrm{d}\vec{\mathbf{y}}.\end{aligned}$$

Since $\lim_{u\to\infty} b_u/u = b > 0$, there exists u_0 such that for all $u \geqslant u_0$,

$$\exp\left\{-\frac{b_u}{u}y_1 - \frac{y_1^2}{2u^2} - \frac{1}{2}\sum_{k=2}^{n} y_k^2\right\} \mathbb{1}_{\{\vec{\mathbf{y}}\in\mathcal{G}_u^*\}} \leqslant \exp\left\{-\frac{b}{2}y_1 - \frac{1}{2}\sum_{k=2}^{n} y_k^2\right\} \mathbb{1}_{\{\vec{\mathbf{y}}\in\mathbb{R}^n : y_1\geqslant 0\}}.$$

The dominating function is integrable since

$$\int_{\mathbb{R}^n} \exp\left\{-\frac{b}{2}y_1 - \frac{1}{2}\sum_{k=2}^{n} y_k^2\right\} \mathbb{1}_{\{y_1\geqslant 0\}} \mathrm{d}\vec{\mathbf{y}} < \infty.$$

Therefore, the dominated convergence theorem applies since the limit integral is assumed to be positive, and the integrand converges almost everywhere to

$$\exp\left\{-by_1 - \frac{1}{2}\sum_{k=2}^{n} y_k^2\right\} \mathbb{1}_{\{\vec{\mathbf{y}}\in\mathcal{G}\}},$$

and the asymptotic equivalence follows. The lemma is proved. □

4. Proof of Theorem 1

If $n = 1$, then the statement follows immediately from the following well-known bounds:

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{u} - \frac{1}{u^3}\right)\mathrm{e}^{-u^2/2} \leqslant \mathbb{P}(\xi_1 > u) \leqslant \frac{1}{\sqrt{2\pi}}\frac{1}{u}\mathrm{e}^{-u^2/2},\ u > 0; \tag{7}$$

see, for instance, [45] (Section 7.1).

Therefore, we can assume that $n \geqslant 2$ in the remaining part of the proof. Let us denote

$$\begin{aligned}\vec{\mathbf{c}} &= (c_1, c_2, \ldots, c_n), \\ \mathcal{E} &= \left\{\vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n) : x_k > 0, \prod_{k=1}^{n} x_k > 1\right\}.\end{aligned}$$

Choose an orthonormal transformation $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\Gamma\,\vec{\mathbf{e}}_1 = \frac{1}{\sqrt{n}}\mathbf{1}_n.$$

If $\vec{\mathbf{c}} \neq \bar{c}\,\mathbf{1}_n$, then in addition we choose Γ so that

$$\Gamma\,\vec{\mathbf{e}}_2 = \frac{\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n}{\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|},$$

where $\vec{\mathbf{e}}_1 = (1, 0, \ldots, 0)$, $\vec{\mathbf{e}}_2 = (0, 1, \ldots, 0)$, and $\bar{c}$ defined in the statement of the theorem. If $\vec{\mathbf{c}} = \bar{c}\,\mathbf{1}_n$, then all terms below involving $\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|$ are interpreted as zero.

Now we have

$$\begin{aligned}
&\mathbb{P}\bigg(\prod_{k=1}^{n}(\xi_k - c_k) > u^n : \xi_k > c_k, k \in \{1, 2, \ldots, n\}\bigg) \\
&= \mathbb{P}\bigg(\prod_{k=1}^{n}\frac{\xi_k - c_k}{u} > 1 : \frac{\xi_k - c_k}{u} > 0, k \in \{1, 2, \ldots, n\}\bigg) \\
&= \mathbb{P}\Big(\frac{\vec{\boldsymbol{\xi}} - \vec{\mathbf{c}}}{u} \in \mathcal{E}\Big) = \mathbb{P}\big(\vec{\boldsymbol{\xi}} \in u\,\mathcal{E} + \vec{\mathbf{c}}\big) = \mathbb{P}\big(\vec{\boldsymbol{\xi}} \in \Gamma^{-1}(u\,\mathcal{E} + \vec{\mathbf{c}})\big),
\end{aligned} \tag{8}$$

where the last equality follows from the invariance property of the multivariate standard normal distribution; see, for instance, [46] (pp. 383–385), [47] (Chapter 1), or [48] (Proposition 3.3.1). We divide the rest of the proof into several parts.

(i) To obtain the asymptotic formula of the theorem, we will apply Lemma 2 with

$$\mathcal{G}_u = \Gamma^{-1}(u\mathcal{E} + \vec{\mathbf{c}}) \text{ and } b_u = (u + \bar{c})\sqrt{n}.$$

To do this, we first get a suitable expression of the set $\mathcal{G}_u^*$ for this special case. Due to the basic properties of the orthonormal transformation Γ^{-1}, we get

$$\begin{aligned}
\mathcal{G}_u - b_u\vec{\mathbf{e}}_1 &= \Gamma^{-1}(u\mathcal{E} + \vec{\mathbf{c}}) - (u + \bar{c})\sqrt{n}\,\vec{\mathbf{e}}_1 \\
&= \Gamma^{-1}(u\mathcal{E} + \vec{\mathbf{c}}) - \Gamma^{-1}\big((u + \bar{c})\mathbf{1}_n\big) \\
&= \Gamma^{-1}\big(u\mathcal{E} - u\mathbf{1}_n\big) + \Gamma^{-1}\big(\mathbf{c} - \bar{c}\mathbf{1}_n\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{G}_u^* &= (u, 1, \ldots, 1) * \big(\mathcal{G}_u - b_u\mathbf{e}_1\big) \\
&= (u, 1, \ldots, 1) * \Gamma^{-1}\big(u\mathcal{E} - u\mathbf{1}_n\big) + \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2.
\end{aligned} \tag{9}$$

We observe that

$$\begin{aligned}
\vec{\mathbf{y}} \in u\mathcal{E} &\Leftrightarrow \vec{\mathbf{y}} = (ux_1, ux_2, \ldots, ux_n) : x_k > 0, \prod_{k=1}^{n} x_k > 1 \\
&\Leftrightarrow \vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n) : y_k > 0, \prod_{k=1}^{n} y_k > u^n.
\end{aligned}$$

By the AM-GM inequality, we have

$$\sum_{k=1}^{n} y_k \geqslant n\sqrt[n]{y_1 \ldots y_n} > un.$$

Hence

$$\vec{\mathbf{y}} \in u\mathcal{E} \Leftrightarrow \begin{cases} \vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n) : y_k > 0, \prod_{k=1}^{n} y_k > u^n, \\ \sum_{k=1}^{n} y_k = nu + \frac{\lambda\sqrt{n}}{u} \text{ for some } \lambda > 0, \end{cases}$$

and, consequently,

$$u\mathcal{E} - u\mathbf{1}_n = \bigcup_{\lambda>0}\left\{\mathcal{E}_{u,\lambda} + \frac{\lambda}{u\sqrt{n}}\mathbf{1}_n\right\}$$

according to the definition of the set $\mathcal{E}_{u,\lambda}$ in Lemma 1. Substituting this expression into (9), we get

$$\mathcal{G}_u^* = (u,1,\ldots,1) * \Gamma^{-1}\left(\bigcup_{\lambda>0}\left\{\mathcal{E}_{u,\lambda} + \frac{\lambda}{u\sqrt{n}}\mathbf{1}_n\right\}\right) + \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2. \tag{10}$$

Now we observe that

$$\Gamma^{-1}\mathcal{E}_{u,\lambda} \subset \{\vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n) : y_1 = 0\}. \tag{11}$$

Namely, from the definition of $\mathcal{E}_{u,\lambda}$

$$\mathcal{E}_{u,\lambda} \subset \left\{\vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n) : \sum_{k=1}^{n} x_k = 0\right\}.$$

So for all vectors $\vec{\mathbf{x}} \in \mathcal{E}_{u,\lambda}$, we have $\langle\vec{\mathbf{x}}, \mathbf{1}_n\rangle = 0$, which yields the following equality for the vector $\vec{\mathbf{y}} = \Gamma^{-1}\vec{\mathbf{x}}$:

$$y_1 = \langle\vec{\mathbf{y}}, \vec{\mathbf{e}}_1\rangle = \langle\Gamma^{-1}\vec{\mathbf{x}}, \vec{\mathbf{e}}_1\rangle = \langle\vec{\mathbf{x}}, \Gamma\,\vec{\mathbf{e}}_1\rangle = \frac{1}{\sqrt{n}}\langle\vec{\mathbf{x}}, \mathbf{1}_n\rangle = 0$$

because the transformation Γ is orthonormal with the special requirements.

Equality (10) and inclusion (11) imply that

$$\mathcal{G}_u^* = \bigcup_{\lambda>0}\left\{\lambda\vec{\mathbf{e}}_1 + \Gamma^{-1}\mathcal{E}_{u,\lambda}\right\} + \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2. \tag{12}$$

(ii) Inclusion (11) and the last expression (12) imply that for all

$$\vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n) \in \mathcal{G}_u^*,$$

we have $y_1 = \lambda > 0$. Consequently,

$$\mathcal{G}_u^* \subset \{\vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n); y_1 \geqslant 0\},$$

which is equivalent to the condition $\mathcal{G}_u \subset \{\vec{\mathbf{y}} \in \mathbb{R}^n : y_1 \geqslant b_u\}$ of Lemma 2.

(iii) By Lemma 1 we get

$$\Gamma^{-1}\mathcal{E}_{u,\lambda} \subset \left\{\vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n) : y_1 = 0, \sum_{k=2}^{n} y_k^2 < (1+\varepsilon)^2 2\lambda\sqrt{n}\right\} \tag{13}$$

for large u and for any fixed $\lambda > 0$ and $\varepsilon \in (0,1)$. Indeed, according to relation (11),

$$\vec{\mathbf{y}} \in \Gamma^{-1}\mathcal{E}_{u,\lambda} \;\Rightarrow\; y_1 = 0.$$

In addition, by Lemma 1 we have

$$\vec{\mathbf{x}} \in \mathcal{E}_{u,\lambda} \;\Rightarrow\; \sum_{k=1}^{n} x_k^2 < (1+\varepsilon)^2 2\lambda\sqrt{n}$$

for large u. Since the transformation Γ^{-1} preserves the norm, we have that $\|\vec{\mathbf{y}}\| = \|\vec{\mathbf{x}}\|$ for $\vec{\mathbf{y}} = \Gamma^{-1}\vec{\mathbf{x}}$. Hence

$$\vec{\mathbf{y}} \in \Gamma^{-1}\mathcal{E}_{u,\lambda} \;\Rightarrow\; \sum_{k=1}^{n} y_k^2 = \sum_{k=2}^{n} y_k^2 < (1+\varepsilon)^2 2\lambda\sqrt{n},$$

and thus inclusion (13) holds. Similarly, we can derive that

$$\Gamma^{-1}\mathcal{E}_{u,\lambda} \supset \left\{\vec{\mathbf{y}} = (y_1, y_2, \ldots, y_n) : y_1 = 0, \sum_{k=2}^{n} y_k^2 < (1-\varepsilon)^2 2\lambda\sqrt{n}\right\} \tag{14}$$

for large u and for any fixed $\lambda > 0$ and $\varepsilon \in (0,1)$.

(iv) The derived relations (13) and (14) and expression (12) imply that

$$\mathbb{1}_{\mathcal{G}_u^*}(\vec{\mathbf{x}}) \underset{u\to\infty}{\to} \mathbb{1}_{\widehat{\mathcal{G}}}(\vec{\mathbf{x}}) \tag{15}$$

almost everywhere with respect to the Lebesgue measure, where

$$\begin{aligned}\widehat{\mathcal{G}} &= \bigcup_{\lambda>0} \left\{\lambda\vec{\mathbf{e}}_1 + \left\{\vec{\mathbf{y}} \in \mathbb{R}^n : y_1 = 0, \sum_{k=2}^{n} y_k^2 \leqslant 2\lambda\sqrt{n}\right\}\right\} + \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2 \\ &= \left\{\vec{\mathbf{x}} \in \mathbb{R}^n : x_1 \geqslant \frac{1}{2\sqrt{n}} \sum_{k=2}^{n} x_k^2\right\} + \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2.\end{aligned}$$

Indeed, it is enough to check that

$$\mathbb{1}_{\mathcal{G}_u^* - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2}(\vec{\mathbf{x}}) \underset{u\to\infty}{\to} \mathbb{1}_{\widehat{\mathcal{G}} - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2}(\vec{\mathbf{x}})$$

for almost every $\vec{\mathbf{x}} \in \mathbb{R}^n$. It is clear that

$$\widehat{\mathcal{G}} - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2 = \left\{\vec{x} \in \mathbb{R}^n : x_1 \geqslant \frac{1}{2\sqrt{n}} \sum_{k=2}^{n} x_k^2\right\}.$$

If

$$x_1 > \frac{1}{2\sqrt{n}} \sum_{k=2}^{n} x_k^2,$$

then we can choose $\varepsilon \in (0,1)$ such that

$$\sum_{k=2}^{n} x_k^2 < (1-\varepsilon)^2 2x_1\sqrt{n}.$$

Taking $\lambda = x_1$ and $\vec{y} = (0, x_2, \ldots, x_n)$ in (14) and using expression (12) we get that

$$\vec{\mathbf{x}} \in \mathcal{G}_u^* - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2$$

for all sufficiently large u. Hence, the corresponding indicators tend to 1 as u tends to infinity. If

$$x_1 < \frac{1}{2\sqrt{n}} \sum_{k=2}^{n} x_k^2,$$

then either $x_1 \leqslant 0$, in which case

$$\vec{\mathbf{x}} \notin \mathcal{G}_u^* - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2$$

for every u, or $x_1 > 0$ and we can choose $\varepsilon \in (0,1)$ such that

$$\sum_{k=2}^{n} x_k^2 > (1+\varepsilon)^2 2x_1\sqrt{n}.$$

In the latter case, if

$$\vec{\mathbf{x}} \in \mathcal{G}_u^* - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}}_2,$$

then necessarily $\lambda = x_1$ and $\vec{y} = (0, x_2, \dots, x_n)$ in (12), and (13) excludes this for all sufficiently large u. Hence, the corresponding indicators tend to 0 as u tends to infinity. The remaining points of $\widehat{\mathcal{G}} - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\vec{\mathbf{e}_2}$ satisfy

$$x_1 = \frac{1}{2\sqrt{n}} \sum_{k=2}^{n} x_k^2,$$

which is a graph over $(x_2, \dots, x_n)$ and therefore has zero Lebesgue measure. The fixed translation gives the limiting relation (15). Therefore, applying Lemma 2 to probability (8), we get

$$\mathbb{P}\Big(\vec{\boldsymbol{\xi}} \in \Gamma^{-1}\big(u\mathcal{E} + \vec{\mathbf{c}}\big)\Big) \underset{u\to\infty}{\sim} \frac{\mathrm{e}^{-\frac{n}{2}(u+\bar{c})^2}}{u(2\pi)^{n/2}} \int\limits_{\widehat{\mathcal{G}}} \exp\left\{-\sqrt{n}\, x_1 - \frac{1}{2}\sum_{k=2}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}}.$$

(v) To finish the proof of Theorem 1, it suffices to observe that

$$\begin{aligned}
&\int\limits_{\widehat{\mathcal{G}}} \exp\left\{-\sqrt{n}\, x_1 - \frac{1}{2}\sum_{k=2}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}} \\
&= \int\limits_{\left\{\vec{\mathbf{x}}\in\mathbb{R}^n:\, x_1 \geqslant \frac{1}{2\sqrt{n}}\sum_{k=2}^{n} x_k^2\right\}} \exp\left\{-\sqrt{n}x_1 - \frac{1}{2}x_2^2 - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|x_2 - \frac{1}{2}\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|^2 - \frac{1}{2}\sum_{k=3}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}} \\
&= \int\limits_{\left\{\vec{\mathbf{x}}\in\mathbb{R}^n:\, \sqrt{n}x_1 \geqslant \frac{1}{2}\sum_{k=2}^{n} x_k^2\right\}} \exp\left\{-\left(\sqrt{n}x_1 - \frac{1}{2}\sum_{k=2}^{n} x_k^2\right) - \|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|x_2 - \frac{1}{2}\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|^2 - \sum_{k=2}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}} \\
&= \frac{1}{\sqrt{n}} \int\limits_{\mathbb{R}^{n-1}} \exp\left\{-\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|x_2 - \frac{1}{2}\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|^2 - \sum_{k=2}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}} \\
&= \frac{1}{\sqrt{n}} \mathrm{e}^{-\frac{1}{4}\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|^2} \int\limits_{\mathbb{R}^{n-1}} \exp\left\{-\left(x_2 + \frac{1}{2}\|\vec{\mathbf{c}} - \bar{c}\mathbf{1}_n\|\right)^2 - \sum_{k=3}^{n} x_k^2\right\} \mathrm{d}\vec{\mathbf{x}} \\
&= \frac{\pi^{\frac{n-1}{2}}}{\sqrt{n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^{n} (c_k - \bar{c})^2\right\}.
\end{aligned}$$

Theorem 1 is proved.

5. Proof of Theorem 2

If $n = 1$, then the statement of the theorem follows immediately from estimates (7). So we can suppose that $n \geqslant 2$. First, we observe that

$$\begin{aligned}
&\mathbb{P}\left(\prod_{k=1}^{n} (\xi_k - c_k) > u^n\right) \\
&= \sum_{\alpha_1,\alpha_2,\dots,\alpha_n \in\{0,1\}} \mathbb{P}\left(\prod_{k=1}^{n} (\xi_k - c_k) > u^n;\ (-1)^{\alpha_k}(\xi_k - c_k) > 0, k \in \{1,2,\dots,n\}\right).
\end{aligned}$$

Since we consider only large $u > 0$,

$$\begin{aligned}&\mathbb{P}\left(\prod_{k=1}^{n}(\xi_k - c_k) > u^n\right)\\&= \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_n\in\{0,1\}\\ 2\left|\sum_{k=1}^{n}\alpha_k\right.}} \mathbb{P}\left(\prod_{k=1}^{n}(\xi_k - c_k) > u^n;\, (-1)^{\alpha_k}(\xi_k - c_k) > 0, k\in\{1,2,\ldots,n\}\right),\end{aligned}$$

where the notation $2\big|\sum_{k=1}^{n}\alpha_k$ means that the sum $\sum_{k=1}^{n}\alpha_k$ is even. If a r.v. ξ has a centered normal distribution, then

$$\mathbb{P}\big((-1)^{\alpha}(\xi - c) > 0\big) = \mathbb{P}\big(\xi - (-1)^{\alpha}c > 0\big)$$

for all $\alpha \in \{0,1\}$ and $c \in \mathbb{R}$. Therefore, by Theorem 1 we get

$$\begin{aligned}&\mathbb{P}\left(\prod_{k=1}^{n}(\xi_k - c_k) > u^n\right)\\&= \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_n\in\{0,1\}\\ 2\left|\sum_{k=1}^{n}\alpha_k\right.}} \mathbb{P}\left(\prod_{k=1}^{n}\big(\xi_k - (-1)^{\alpha_k}c_k\big) > u^n;\, \xi_k > (-1)^{\alpha_k}c_k,\, k\in\{1,2,\ldots,n\}\right)\\&\underset{u\to\infty}{\sim} \frac{1}{u\,2^{n/2}\sqrt{\pi n}} \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_n\in\{0,1\}\\ 2\left|\sum_{k=1}^{n}\alpha_k\right.}} \exp\left\{-\frac{n}{2}(u+\bar{c}_\alpha)^2 - \frac{1}{4}\sum_{k=1}^{n}\big((-1)^{\alpha_k}c_k - \bar{c}_\alpha\big)^2\right\}\\&\underset{u\to\infty}{\sim} \frac{N_{\alpha^*}}{u\,2^{n/2}\sqrt{\pi n}} \exp\left\{-\frac{n}{2}(u+\bar{c}_{\alpha^*})^2 - \frac{1}{4}\sum_{k=1}^{n}\big((-1)^{\alpha_k^*}c_k - \bar{c}_{\alpha^*}\big)^2\right\},\end{aligned}$$

where $\bar{c}_\alpha = \frac{1}{n}\sum_{k=1}^{n}(-1)^{\alpha_k}c_k$, and the quantities $\alpha_k^*, k\in\{1,2,\ldots,n\}$, $\bar{c}_{\alpha^*}$, and N_{α^*} are defined in the formulation of the theorem. Theorem 2 is proved.

6. Examples

In this section, we present three examples that illustrate the asymptotic formulas derived above and compare them with Monte Carlo estimates for several parameter choices. All computations, figures, and tables in this section were produced using *Wolfram Mathematica*. Monte Carlo simulations were performed with `SeedRandom[1234]` and a sample size of 10^7. Since the probability of large values of normal random variables is very small, a small number of simulation results can lead to excessively large errors.

The 95% confidence intervals in Tables 1 and 2 were computed by the normal approximation

$$\widehat{p}_M(u) = \frac{1}{M}\sum_{j=1}^{M}\mathbf{1}_{\{X_j>u\}}, \qquad \widehat{p}_M(u) \pm 1.96\sqrt{\frac{\widehat{p}_M(u)(1-\widehat{p}_M(u))}{M}},$$

where X_j is the j-th simulated value of the product considered in the corresponding example, and $M = 10^7$ is the sample size. This approximation becomes less reliable as the number of exceedances decreases.

Table 1. Numerical comparison of the Monte Carlo estimates and the asymptotic approximation in Example 1.

u	Monte Carlo	Asymptotic	Absolute Difference	Relative Error	95% CI Lower	95% CI Upper
10	0.0020319	0.0023534	0.0003215	0.1582461	0.0020040	0.0020598
11	0.0014391	0.0016744	0.0002353	0.1634850	0.0014156	0.0014626
12	0.0010506	0.0012056	0.0001550	0.1475511	0.0010305	0.0010707
13	0.0007714	0.0008773	0.0001059	0.1372703	0.0007542	0.0007886
14	0.0005602	0.0006444	0.0000842	0.1502672	0.0005455	0.0005749
15	0.0004175	0.0004773	0.0000598	0.1432143	0.0004048	0.0004302
16	0.0003099	0.0003562	0.0000463	0.1494691	0.0002990	0.0003208
17	0.0002347	0.0002677	0.0000330	0.1406072	0.0002252	0.0002442
18	0.0001781	0.0002025	0.0000244	0.1367306	0.0001698	0.0001864
19	0.0001384	0.0001540	0.0000156	0.1127067	0.0001311	0.0001457
20	0.0001062	0.0001178	0.0000116	0.1089709	0.0000998	0.0001126
21	0.0000821	0.0000905	0.0000084	0.1025418	0.0000765	0.0000877
22	0.0000664	0.0000699	0.0000035	0.0526448	0.0000613	0.0000715
23	0.0000522	0.0000542	0.0000020	0.0384329	0.0000477	0.0000567
24	0.0000414	0.0000422	0.0000008	0.0195722	0.0000374	0.0000454
25	0.0000317	0.0000330	0.0000013	0.0408636	0.0000282	0.0000352

Table 2. Numerical comparison of the Monte Carlo estimates and the asymptotic approximation in Example 2.

u	Monte Carlo	Asymptotic	Absolute Difference	Relative Error	95% CI Lower	95% CI Upper
10	0.0008726	0.0002357	0.0006369	0.7298950	0.0008543	0.0008909
11	0.0005911	0.0001653	0.0004258	0.7203362	0.0005760	0.0006062
12	0.0004077	0.0001172	0.0002905	0.7125056	0.0003952	0.0004202
13	0.0002841	0.0000839	0.0002002	0.7045857	0.0002737	0.0002945
14	0.0002004	0.0000606	0.0001398	0.6974428	0.0001916	0.0002092
15	0.0001436	0.0000442	0.0000994	0.6924674	0.0001362	0.0001510
16	0.0001010	0.0000324	0.0000686	0.6791391	0.0000948	0.0001072
17	0.0000735	0.0000239	0.0000496	0.6742023	0.0000682	0.0000788
18	0.0000550	0.0000178	0.0000372	0.6762143	0.0000504	0.0000596
19	0.0000414	0.0000133	0.0000281	0.6781885	0.0000374	0.0000454
20	0.0000313	0.0000100	0.0000213	0.6797668	0.0000278	0.0000348
21	0.0000230	0.0000076	0.0000154	0.6704150	0.0000200	0.0000260
22	0.0000162	0.0000058	0.0000104	0.6443643	0.0000137	0.0000187
23	0.0000114	0.0000044	0.0000070	0.6141144	0.0000093	0.0000135
24	0.0000078	0.0000034	0.0000044	0.5674669	0.0000061	0.0000095
25	0.0000057	0.0000026	0.0000031	0.5441767	0.0000042	0.0000072

Example 1. *Let $n = 3$, and let ξ_1, ξ_2, ξ_3 be independent normal r.v.s with parameters*

$$(\mu_1, \mu_2, \mu_3) = (0, 0, 0) \quad \textit{and} \quad (\sigma_1, \sigma_2, \sigma_3) = (1, 1.2, 1.5).$$

We present an asymptotic formula for $\mathbb{P}(\xi_1\xi_2\xi_3 > u)$.

In this case,

$$\prod_{k=1}^{3} \sigma_k = \frac{9}{5}, \quad \text{and} \quad \frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2} = \frac{\mu_3}{\sigma_3} = 0.$$

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To derive an asymptotic formula for $\mathbb{P}(\xi_1\xi_2\xi_3 > u)$, we can use Corollary 3 or 4. To use Corollary 4, we observe that for every admissible $\beta = (\beta_1, \beta_2, \beta_3)$ such that $\beta_1 + \beta_2 + \beta_3$ is divisible by 2,

$$\sum_{k=1}^{3} (-1)^{\beta_k} \frac{\mu_k}{\sigma_k} = 0.$$

Therefore $\overline{\mu}_\beta = 0$ and $N_\beta = 4$ in Corollary 4. Consequently, by Corollary 3 or 4 we get that

$$\mathbb{P}\left(\prod_{k=1}^{3} \xi_k > u\right) \underset{u\to\infty}{\sim} \frac{\sqrt{2}\left(\frac{9}{5}\right)^{1/3}}{u^{1/3}\sqrt{3\pi}} \exp\left\{-\frac{3}{2}u^{2/3}\left(\frac{9}{5}\right)^{-2/3}\right\}.$$

In Figure 1, we can see how much the asymptotic formula values differ from the values of $\mathbb{P}(\xi_1\xi_2\xi_3 > u)$ obtained by the Monte Carlo method. Figure 2 presents the corresponding relative error, and Table 1 gives a numerical comparison of the Monte Carlo estimates, asymptotic values, absolute differences, relative errors, and 95% confidence intervals for selected values of u.

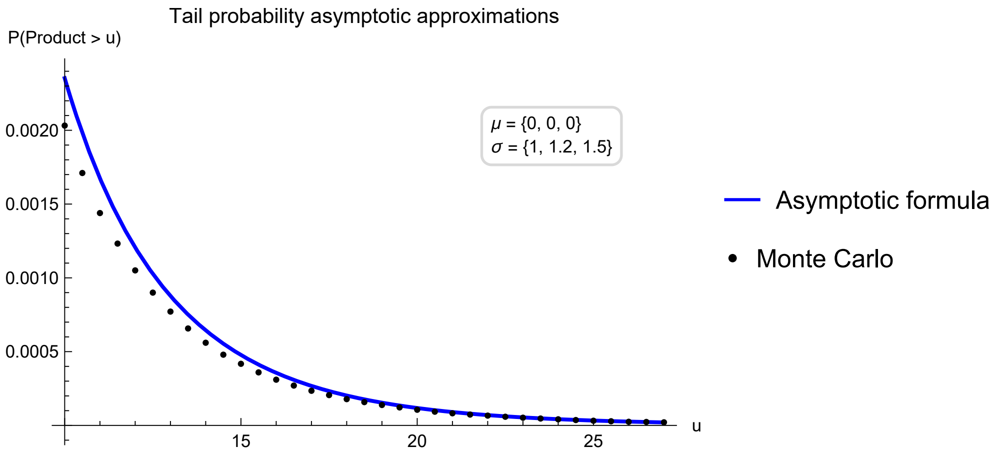

Figure 1. Monte Carlo estimates and the asymptotic approximation for probability in Example 1.

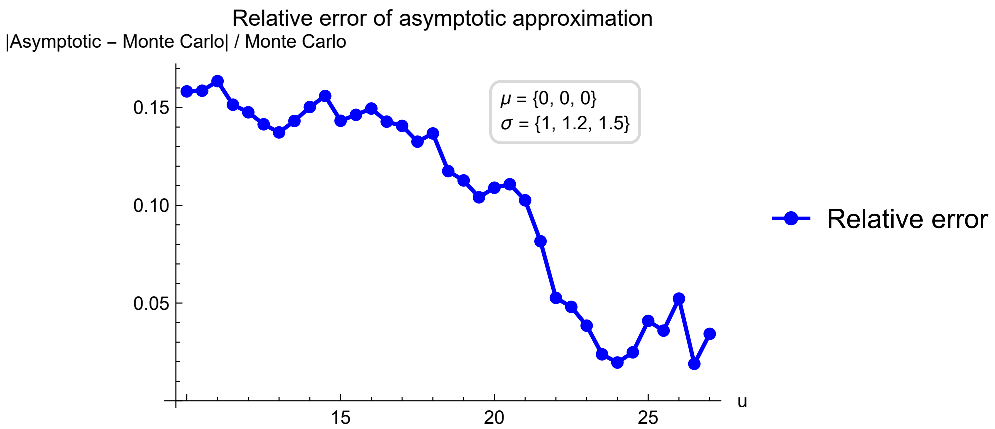

Figure 2. Relative error of the asymptotic approximation in Example 1.

Example 2. *Let $n = 4$, and let $\xi_1, \xi_2, \xi_3, \xi_4$ be independent normal r.v.s with parameters*

$$(\mu_1, \mu_2, \mu_3, \mu_4) = (1, 0, -1, 0) \quad \textit{and} \quad (\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.8, 1.2, 0.3, 0.9).$$

We get the asymptotic formula for the probability $\mathbb{P}(\xi_1\xi_2\xi_3\xi_4 > u)$.

In this case, we can use only Corollary 4. We have

$$\prod_{k=1}^{4}\sigma_k = 0.2592 = \frac{162}{625}.$$

Moreover,

$$\frac{\mu_1}{\sigma_1} = \frac{5}{4}, \quad \frac{\mu_2}{\sigma_2} = 0, \quad \frac{\mu_3}{\sigma_3} = -\frac{10}{3}, \quad \frac{\mu_4}{\sigma_4} = 0.$$

The maximal value of

$$\sum_{k=1}^{4}(-1)^{\beta_k}\frac{\mu_k}{\sigma_k}$$

under the condition that $\beta_1+\beta_2+\beta_3+\beta_4$ is divisible by 2 is obtained for

$$\beta = (0,0,1,1) \quad \text{and} \quad \beta = (0,1,1,0).$$

For both choices,

$$\sum_{k=1}^{4}(-1)^{\beta_k}\frac{\mu_k}{\sigma_k} = \frac{5}{4}+\frac{10}{3} = \frac{55}{12}.$$

Therefore

$$\overline{\mu}_\beta = \frac{1}{4}\cdot\frac{55}{12} = \frac{55}{48}, \quad N_\beta = 2.$$

In addition,

$$\sum_{k=1}^{4}\left((-1)^{\beta_k}\frac{\mu_k}{\sigma_k} - \overline{\mu}_\beta\right)^2 = \left(\frac{5}{4}-\frac{55}{48}\right)^2 + \left(0-\frac{55}{48}\right)^2 + \left(\frac{10}{3}-\frac{55}{48}\right)^2 + \left(0-\frac{55}{48}\right)^2 = \frac{475}{64}.$$

By Corollary 4 we get

$$\mathbb{P}\left(\prod_{k=1}^{4}\xi_k > u\right) \underset{u\to\infty}{\sim} \frac{\left(\frac{162}{625}\right)^{1/4}}{u^{1/4}4\sqrt{\pi}}\exp\left\{-2\left(u^{1/4}\left(\frac{162}{625}\right)^{-1/4} - \frac{55}{48}\right)^2 - \frac{475}{256}\right\}.$$

In Figures 3 and 4, we can see how much the asymptotic values differ from the values of $\mathbb{P}(\xi_1\xi_2\xi_3\xi_4 > u)$ obtained by the Monte Carlo method. The corresponding relative error and numerical values are reported in Figure 5 and Table 2. To achieve more stable relative errors for larger values of u, the size of the Monte Carlo simulations should be increased. Unfortunately, we are limited by technical possibilities. In this case, it remains to rely on rigorously proven asymptotic formulas.

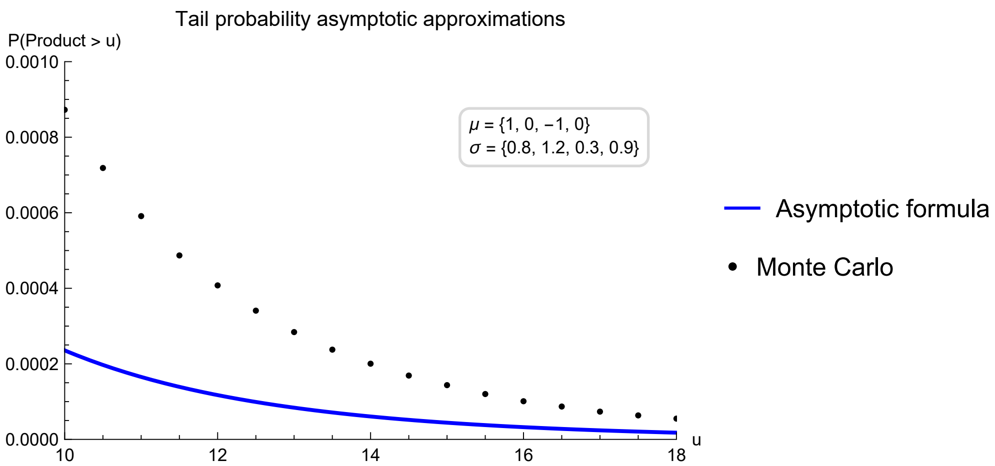

Figure 3. Monte Carlo estimates and the asymptotic approximation in Example 2, small u.

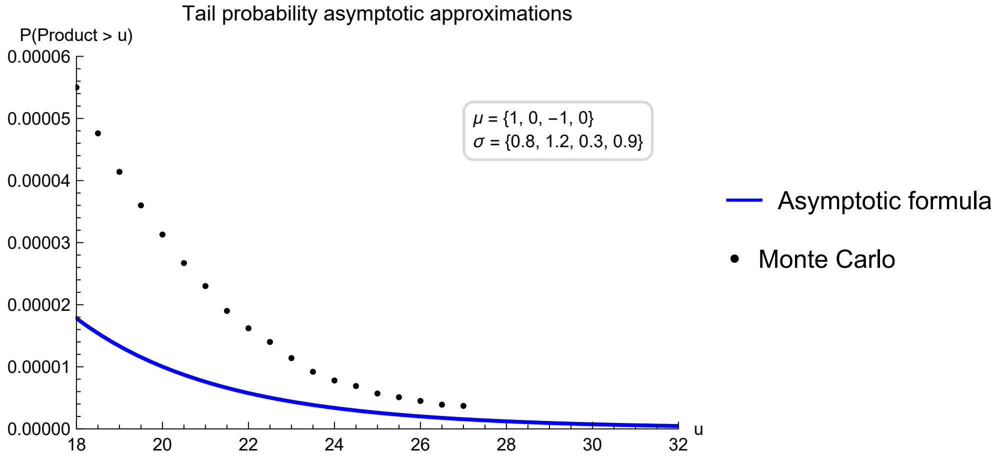

Figure 4. Monte Carlo estimates and the asymptotic approximation in Example 2, large u.

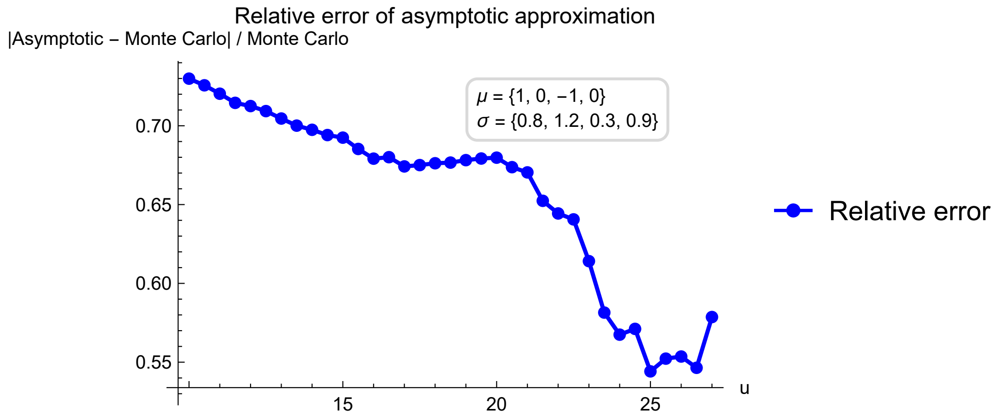

Figure 5. Relative error of the asymptotic approximation in Example 2.

Example 3. *Let $n \in \{2,4,6\}$, and let $\xi_1, \xi_2, \dots, \xi_n$ be independent normal r.v.s with common parameters*

$$\mu_k = 1, \quad \sigma_k = 1, \quad k = 1, \dots, n.$$

We compare the corresponding asymptotic approximations for different dimensions n.

In this case, Corollary 5 gives, for each $n \in \{2,4,6\}$,

$$\mathbb{P}\left(\prod_{k=1}^{n} \xi_k > u\right) \underset{u\to\infty}{\sim} \frac{1}{u^{1/n} 2^{n/2} \sqrt{\pi n}} \exp\left\{-\frac{n}{2}\left(u^{1/n} - 1\right)^2\right\}.$$

Figures 6 and 7 compare the Monte Carlo estimates with the corresponding asymptotic approximations. Figure 6 contains the cases $n = 4$ and $n = 6$, whereas Figure 7 presents the case $n = 2$ separately. Figure 8 presents the relative errors for all three dimensions. For larger values of u, the Monte Carlo sample size needs to be increased. However, we can see that for fixed parameters in this example, the relative error increases as n increases; this suggests that the remainder term in the asymptotic approximation may depend on the number of multipliers n.

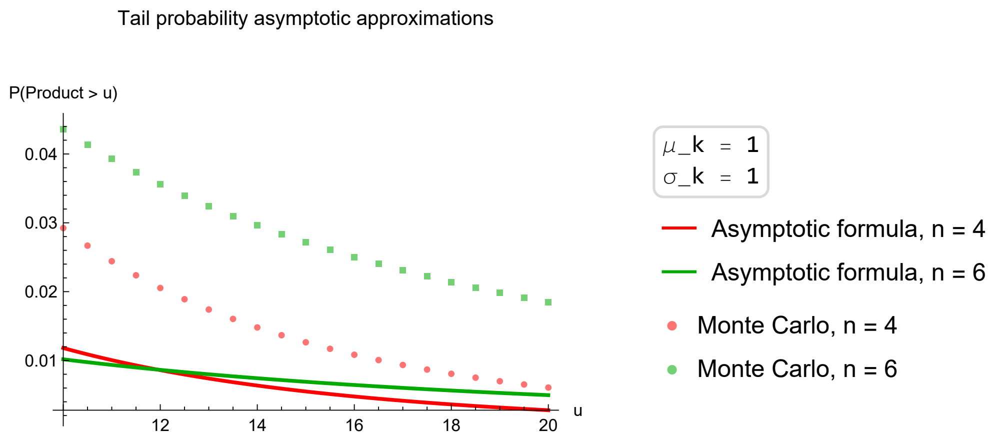

Figure 6. Monte Carlo estimates and asymptotic approximations for dimensions $n = 4$ and $n = 6$.

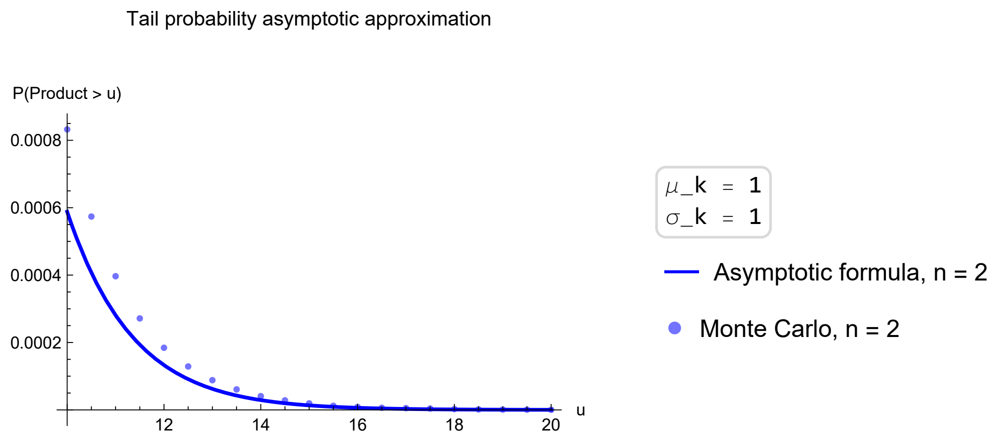

Figure 7. Monte Carlo estimates and asymptotic approximation for dimension $n = 2$.

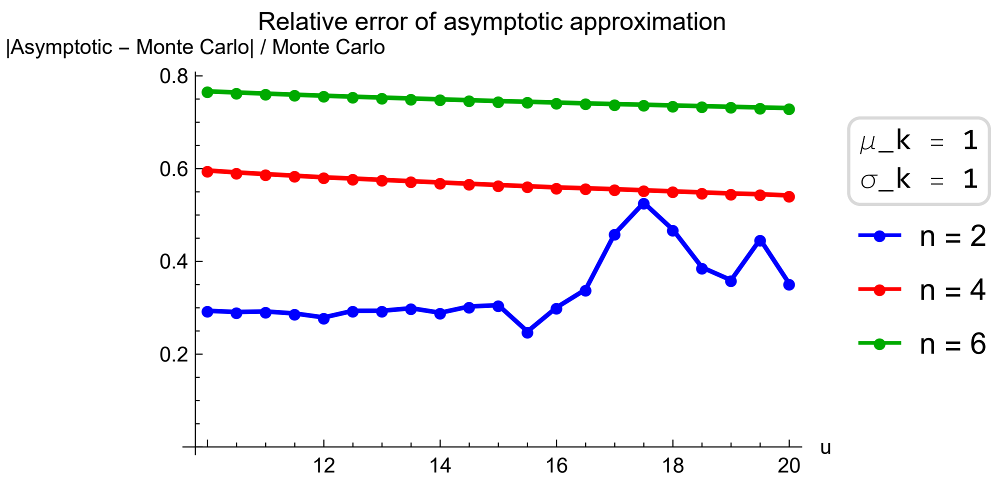

Figure 8. Relative errors of the asymptotic approximations for dimensions $n = 2, 4, 6$.

7. Conclusions

In this paper, we investigated the asymptotic behavior of the products of independent Gaussian random variables, with particular emphasis on tail probabilities. Unlike sums of normal variables, whose properties are well understood, products exhibit significantly more complex distributional features, especially in extreme regimes. By employing transformation techniques, geometric arguments, and asymptotic analysis based on Gaussian measures, we derived explicit asymptotic formulas describing the decay of tail probabilities. Several corollaries highlight important particular cases, including identically distributed variables and products without sign constraints. The theoretical findings were further supported by numerical examples, which demonstrated the accuracy of the derived approximations. To determine the accuracy of the obtained formulas, we compared the asymptotic values of the tail of the product distribution $\mathbb{P}(\xi_1 \dots \xi_n > u)$ of normally distributed independent r.v.s $\{\xi_1, \dots, \xi_n\}$ with the corresponding values obtained by the Monte Carlo method. We assumed that the tail probability values obtained by the Monte Carlo method are accurate. Unfortunately, for large values of the argument, the tail values of the distributions of a normally distributed random variable are very small. Therefore, to find the exact tail values of $\mathbb{P}(\xi_1 \dots \xi_n > u)$, an exceptionally large number of simulations are required. In our work, we limited ourselves to 10^7 simulations. Therefore, in the examples, we did not consider very large values of u in the expression $\mathbb{P}(\xi_1 \dots \xi_n > u)$. The results obtained in Section 6 additionally confirm that our rigorously proven asymptotic formulas are true for any number of multipliers, with any means and any variances. It is only important that the normally distributed random variables being multiplied are independent. At the same time, we note that our work is theoretical, and the tables and graphs are provided solely to illustrate the theoretical result and the limitations of the Monte Carlo method. The used Monte Carlo simulations are acceptable for illustration, but not as strong numerical evidence of accuracy in the far tail.

Future research directions may include extending these results to dependent settings, exploring higher-order asymptotic refinements, and applying the findings to practical problems in finance, physics, and related fields where multiplicative processes play a key role. Of course, to obtain more accurate or general asymptotic formulas, it may be necessary to use not only our "geometric" method but also other methods, such as the saddle-point method or expressions of product distributions of random variables via the Meijer G-functions.

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