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Almost Extraspecial Structures and Pseudofermionic Operators

Daniele Ettore Otera [1,*] and Francesco G. Russo [2,3,4]

[1] Institute of Data Science and Digital Technologies, Vilnius University, LT-08412 Vilnius, Lithuania
[2] School of Science and Technology, University of Camerino, Via Madonna delle Carceri 9, 62032 Camerino, Italy; francesco.russo@unicam.it or frusso@uwc.ac.za or francesco.russo@uct.ac.za
[3] Department of Mathematics and Applied Mathematics, University of the Western Cape, Private Bag X17, Bellville, Cape Town 7535, South Africa
[4] Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X1, Rondebosch, Cape Town 7701, South Africa
* Correspondence: daniele.otera@mif.vu.lt

Abstract

We survey some recent combinatorial properties, which have been found in the context of the algebras of ladder operators in quantum mechanics. More specifically, we review dynamical systems which have nonselfadjoint Hamiltonians and are subject to a formalization in terms of pseudofermionic operators. For these systems, we detect structural analogies between algebras of pseudofermionic operators and the abstract notion of central product, which was originally studied for finite groups.

Keywords: heisenberg groups; hilbert spaces; pseudofermionic operators; extraspecial groups; pauli groups

1. Why Pseudofermionic Operators?

Algebras of pseudofermionic operators are largely involved in the complex dynamical systems. The works [1–4] illustrate the presence of certain products at the level of group structure (the so-called *central products*), which reflect quite naturally at the level of algebras of operators. The usual formalization of the quantum mechanics is then possible.

Looking at the recent literature in mathematical physics and operator theory, one can see that *pseudofermionic operators* generalize the well known *fermionic operators*, which play a fundamental role in many quantum mathematical models. The formalization involves some classical functional analysis and is modelled on a complex Hilbert space $\mathcal{H}$ (endowed of the usual scalar product) via *lowering* and *raising operators*, which indeed lower or raise the eigenvalues of corresponding eigenstates. Pseudofermionic operators are then formalized as a pair of operators, say $\mathbf{a}$ and $\mathbf{b}$, where the lowering operator is $\mathbf{a}$ and the raising operator is $\mathbf{b}$, satisfying the *canonical anticommutation relations* (CAR)

$$\{\mathbf{a},\mathbf{b}\} = \mathbf{ab} + \mathbf{ba} = \mathbb{I}, \qquad \text{and} \qquad \{\mathbf{a},\mathbf{a}\} = \{\mathbf{b},\mathbf{b}\} = 0. \tag{1}$$

Here $\mathbb{I}$ denotes the identity operator and $\mathbf{b} \neq \mathbf{a}^\dagger$ a priori, that is, $\mathbf{b}$ is not necessarily the *adjoint operator* $\mathbf{a}^\dagger$ of $\mathbf{a}$. In particular, if this happens, that is, if $\mathbf{b} = \mathbf{a}^\dagger$ and (1) are satisfied, $\mathbf{a}$ is a *fermionic operator* (of adjoint $\mathbf{a}^\dagger$). The terminology is quite standard in [5,6], but see also [7] for recent generalizations in the literature.

Note that the relations $\{\mathbf{a},\mathbf{a}\} = \{\mathbf{b},\mathbf{b}\} = 0$ from the CAR turn out to be significant from a physical perspective. We sketch a construction with ladder operators which is very

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popular, see [1,2,4]. First of all, we formulate the existence of a nonzero vector $\varphi_0 \in \mathcal{H}$ such that $\mathbf{a}\varphi_0 = 0$, as well as a nonzero vector $\Psi_0 \in \mathcal{H}$ such that $\mathbf{b}^\dagger \Psi_0 = 0$. We then introduce

$$\varphi_1 = \mathbf{b}\varphi_0, \qquad \text{and} \qquad \Psi_1 = \mathbf{a}^\dagger \Psi_0, \tag{2}$$

and the nonselfadjoint operators (a priori)

$$\mathbf{N} = \mathbf{ba}, \qquad \text{and} \qquad \mathbf{N}^\dagger = \mathbf{a}^\dagger \mathbf{b}^\dagger. \tag{3}$$

These two operators generalize the Hamiltonian of a dynamical system, considering the possible case that it is not selfadjoint. Bender [8] produced a series of significant mathematical models, which are based on the nonselfadjoint Hamiltonians, influencing the theoretical research in the area during the last decades. At this point, we may introduce the following selfadjoint operators $\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ where $f \in \mathcal{H}$:

$$\mathbf{S}_\varphi f = \langle \varphi_0, f \rangle \varphi_0 + \langle \varphi_1, f \rangle \varphi_1, \;\; \mathbf{S}_\Psi f = \langle \Psi_0, f \rangle \Psi_0 + \langle \Psi_1, f \rangle \Psi_1. \tag{4}$$

We find the following equations which are satisfied:

$$\mathbf{a}\varphi_1 = \varphi_0, \qquad \mathbf{b}^\dagger \Psi_1 = \Psi_0 \tag{5}$$

$$\mathbf{N}\varphi_n = n\varphi_n, \qquad \mathbf{N}^\dagger \Psi_n = n\Psi_n, \text{ for } n = 0, 1, \tag{6}$$

finding in addition that

$$\langle \varphi_k, \Psi_n \rangle = \delta_{k,n} \text{ for } k, n = 0, 1. \tag{7}$$

From these conditions it turns out that $\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ are strictly positive, selfadjoint and invertible. Moreover

$$\|\mathbf{S}_\varphi\| \leq \|\varphi_0\|^2 + \|\varphi_1\|^2, \qquad \|\mathbf{S}_\Psi\| \leq \|\Psi_0\|^2 + \|\Psi_1\|^2, \tag{8}$$

$$\mathbf{S}_\varphi \Psi_n = \varphi_n, \qquad \mathbf{S}_\Psi \varphi_n = \Psi_n, \tag{9}$$

for $n = 0, 1$, as well as $\mathbf{S}_\varphi = \mathbf{S}_\Psi^{-1}$ and the following intertwining relations are true:

$$\mathbf{S}_\Psi \mathbf{N} = \mathbf{N}^\dagger \mathbf{S}_\Psi, \qquad \mathbf{S}_\varphi \mathbf{N}^\dagger = \mathbf{N}\mathbf{S}_\varphi. \tag{10}$$

With this construction in mind, we have just shown that it is possible to introduce two *fermionic operators* $\mathbf{N}$ and $\mathbf{N}^\dagger$, having eigenvalues 0 and 1, and eigenvectors respectively $\mathcal{F}_\varphi = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_\Psi = \{\Psi_0, \Psi_1\}$. This is a well known approach in mathematical physics, where Hilbert spaces and corresponding notions of functional analysis provide the natural framework for the quantum mechanics.

Moreover the aforementioned operators $\mathbf{a}$ and $\mathbf{b}^\dagger$ are lowering operators for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ respectively; while $\mathbf{b}$ and $\mathbf{a}^\dagger$ are raising operators for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ respectively. In fact the operators $\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ map $\mathcal{F}_\varphi$ in $\mathcal{F}_\Psi$ and viceversa, intertwining as per (10).We omit further details, which are of independent interest in several contexts of functional analysis, topology, operator theory and Lie theory, see [1–4,9].

Now we pass to recall some notions of group theory. The (2×2) nonsingular complex matrices of the general linear group $\mathrm{GL}_2(\mathbb{C})$ with coefficients in the complex field $\mathbb{C}$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

are usually called *Pauli matrices*, see [1,2,10–12]. They satisfy the well known identities

$$X = iZY, \;\; Y = iXZ, \;\; Z = iYX, \;\; X^2 = Y^2 = Z^2 = I. \tag{12}$$

and generate the nonabelian group of order 16 with respect to the usual matrix product

$$\mathsf{P}_{1,2} = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}. \tag{13}$$

Denoting by p a prime and by $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$ the abelian group of the integers modulo p, we know from [13,14] that $\mathbb{Z}(p)$ is a *cyclic group* of order p and it is 1-generated. For $p = 2$, we find the group with two elements and this allows us to introduce more general version of $\mathsf{P}_{1,2}$, namely the *Pauli group* $\mathsf{P}_{n,2}$ *of order* 4^{n+1} *on n-qubits* (sometimes called *large Pauli groups*) where n is a positive integer, see [4,11,15].

It has been largely discussed in [4,11,12,16,17] that it is useful to investigate large Pauli groups and their structure in terms of abelian subgroups and abelian quotients. On the other hand, abelian subgroups and abelian quotients of the large Pauli groups find applications in quantum information theory: notable examples are quantum error correcting codes [10] and the theory of mutually unbiased basis [18].

Let's use some geometric group theory and combinatorics, in order to go ahead properly. As indicated in [1] (Lemma 3.6), we may introduce the symbols

$$u = XY, \;\; x = Y \;\text{ and }\; y = XYZ \tag{14}$$

and referring to the Pauli matrices X, Y, Z above and rewrite

$$\mathsf{P}_{1,2} = P = \langle u, xy, y \mid u^4 = x^2 = 1, u^2 = y^2, uy = yu, yx = xy, x^{-1}ux = u^{-1}\rangle. \tag{15}$$

The advantage of (15) is significant for several reasons: first of all, we may note a topological decomposition of P via fundamental groups of space of orbits of spheres in [1,2]. This cannot be detected if we look only at generators and relations of P. Secondly, we note that the *generalized quaternion group* is the finite 2-group (of order 2^n)

$$Q_{2^n} = \langle v, w \mid v^{2^n} = 1, v^{2^{n-2}} = w^2, w^{-1}vw = v^{-1}\rangle, \tag{16}$$

where the usual *quaternion group* Q_8 is obtained for $n = 3$, see [14]. With a bit of patience, one can find that (11) contains a copy of Q_8, see details in [4].

Let's note the relevance of the group presentations, which we have just discussed, in connection with the theory of pseudofermionic operators:

Theorem 1 (See [1], Theorem 1.2)**.** *There are two dynamical systems $\mathcal{S}$ and $\mathcal{T}$ involving pseudofermionic operators with groups of symmetries respectively P and Q_8 but with the same Hamiltonian $H_{\mathcal{S}} = H_{\mathcal{T}}$. In particular, there exist dynamical systems admitting larger groups of symmetries, whose size does not affect the dynamical aspects of the system.*

Further interesting observations can be made once the appropriate terminology and notation from [13,14] are recalled.

Notation. A subgroup H of a finite group G is said to be *maximal*, if $H \neq G$ and for any subgroup K of G such that $H \subseteq K \subseteq G$, then either $K = H$ or $K = G$. The *Frattini subgroup* of G is the intersection of all maximal subgroups of G and is denoted by $\Phi(G)$; this is a normal subgroup of G. Given two subgroups H and K of a finite group G, we say that $G = H \rtimes K$ is the *(internal) semidirect product* of H and K, if $H \cap K = 1$, H is normal in G, and $HK = G$. In this situation, we also refer to G as *split extension* of H by K, or to H as *complement* of K in G. If H and K are both cyclic groups, $G = HK$ and H is normal in G, we say that G is *metacyclic*, that is, G is an extension of H by K.

We also recall that for a finite group G, the set $Z(G) = \{x \in G \mid xy = yx,\ \forall y \in G\}$ denotes the *center* of G and it is a normal subgroup of G. For the commutator of two elements $x, y \in G$, we use the notation $[x, y] = x^{-1}x^y$, where $x^y = y^{-1}xy$ denotes the *conjugate of x by y*. The *derived subgroup* of a group G is the smallest subgroup containing all the elements of the form $[x, y]$ and is denoted by $[G, G]$. In other words, $[G, G] = \langle [x, y] \mid x, y \in G \rangle$. On the other hand, if H and K are two subgroups of G, the *derived subgroup of H and K* is $[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$, which specializes to $[G, G]$ when $H = K = G$.

Identifying abelian subgroups of large Pauli groups is, in general, a nontrivial problem. Here, we present an algebraic decomposition which enables us to do this.

Definition 1 (See [14], p. 141, §5.3, Central Product)**.** *A group G is central product of its normal subgroups H and K, if simultaneously $G = HK$, $[H, K] = 1$ and $H \cap K = Z(G)$.*

There are several variations of Definition 1, see [13] (Theorems 5.3, 7.2), or [19] (Vol. I, p. 11), or [19] (Vol. I, §4). We use the notation $G = H \circ K$, following [1,2]. We introduce a specific notation, when D_8 and Q_8 are involved in Definition 1 with $n \geq 1$:

$$E^{+}_{2^{2n+1}} = \underbrace{D_8 \circ D_8 \circ \ldots \circ D_8}_{n-\text{times}} \qquad \text{and} \qquad E^{-}_{2^{2n+1}} = \underbrace{(D_8 \circ D_8 \circ \ldots \circ D_8)}_{(n-1)-\text{times}} \circ Q_8. \tag{17}$$

When trying to generalize Theorem 1, difficulties emerge naturally, as the role of Q_8 is rather special.

Proposition 1 (See [4], Lemma 3.1)**.** *The group Q_{2^n} contains no subgroups isomorphic to $\mathbb{Z}(2) \times \mathbb{Z}(2)$, has at most one element of order 2 and can be written as the product of two normal subgroups $Q_{2^n} = HK$ with $H \cap K \neq 1$, but the condition $[H, K] = 1$ may fail.*

On the other hand, we have the following result:

Theorem 2 (See [4], Theorem 1.2)**.** *There exists a dynamical system $\mathcal{S}$ possessing a nonselfadjoint Hamiltonian H_{eff} describing the two level atom interacting with an electromagnetic field, and H_{eff} can be decomposed in pseudofermionic operators* **a** *and* **b** *which realize the generalized quaternion group Q_{2^n} for all $n \geq 3$.*

The construction of Q_{2^n} via **a** and **b** is quite technical, so we just sketch the main ideas:

Construction of pseudofermionic operators a and b as per Theorem 2

- first one finds an appropriate faithful complex representation of Q_{2^n} in $\mathrm{GL}_2(\mathbb{C})$ for all $n \geq 3$, see [4] (Lemma 3.3);
- then one observes that this representation is given by (2×2)-matrices complex nonsingular matrices which allow to write **a** and **b** (essentially our pseudofermionic operators **a** and **b** are products of matrices which arise from the representation of the previous step), see [4] (Proof of Lemma 3.3, Equation (39));
- now one can write the (effective) Hamiltonian H_{eff} (referring to a model of Maamache and others [5,6]) in terms of **a** and **b** which we have just defined, see [4] (Proof of Lemma 3.3, Equation (42)).
- finally, one checks that the CAR are satisfied for **a** and **b**, see [4] (Proof of Lemma 3.3).

We shall note that our interest in the decomposition in central products is due to the evidence from the mathematical physics that finite groups of the form $A = Q_8 \circ B$, where B is an abelian group containing at most one element of order 2, imply at the level of pseudofermionic operators an easier form of the Hamiltonian. Therefore we may better understand the dynamical properties when we detect a structural decomposition.

2. Some Independent Results on Finite Extraspecial *p*-Groups

We begin with some well known facts on finite metacyclic groups. Looking [14,19], metacyclic groups are not necessarily split extensions. Metacyclic groups were classified by Zassenhaus, see [14] (10.1.10). Of course, cyclic groups are metacyclic by definition. On the other hand, *minimal nonmetacyclic groups* are groups which are not metacyclic, but all whose proper subgroups are metacyclic. A finite group which is nonabelian but all of its proper subgroups are abelian is called *minimal nonabelian*. A finite group which is nonabelian but all of whose proper quotients are abelian is called *just nonabelian*. Finite just nonabelian, finite minimal nonabelian and finite minimal nonmetacyclic groups are described in [14,19–24].

Theorem 3. $\mathsf{P}_{1,2} = E_8^+ \circ \mathbb{Z}(4)$ *is minimal nonmetacyclic.*

We shall also mention that for a prime p and a finite p-group G of order p^m for some positive integer m, we say that G is a *modular p-group* if any two subgroups H and K of G are *permutable*, that is, if the condition $HK = KH$ is satisfied by all subgroups H and K of G. Of course, normal subgroups are always permutable in a group, hence one can specialize the notion of (finite) modular p-group, introducing the notion of (finite) *Hamiltonian p-group*, that is, a *p-group* in which each subgroup is normal. Modular groups and Hamiltonian groups have been studied for a long time by Iwasawa [19] (Vol. II, §73), but we will focus here on finite modular groups and finite Hamiltonian groups.

Definition 2 (See [19], Vol.II, Appendix 17)**.** *Denoting with Q the finite generalized quaternion 2-group, that is, for some $m \geq 2$*

$$Q = Q_{2^{m+1}} = \langle a, b \mid a^{2^m} = 1, b^2 = a^{2^{m-1}}, b^{-1}ab = a^{-1} \rangle, \tag{18}$$

we say that a finite 2-group G is a $Q^{\times}$-group if for some $r \geq 0$

$$G = Q \times \underbrace{\mathbb{Z}(2) \oplus \ldots \oplus \mathbb{Z}(2)}_{r-\text{times}} = Q \times \mathbb{Z}(2)^r. \tag{19}$$

Denoting with D the finite generalized dihedral 2-group, that is, for some $m \geq 2$

$$D = D_{2^{m+1}} = \langle a, b \mid a^{2^m} = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \tag{20}$$

we say that a finite 2-group G is a $D^{\times}$-group if for some $s \geq 0$

$$G = D \times \underbrace{\mathbb{Z}(2) \oplus \ldots \oplus \mathbb{Z}(2)}_{s-\text{times}} = D \times \mathbb{Z}(2)^s. \tag{21}$$

Of course, Definition 2 gives the usual *quaternion group* Q_8 of order 8 in (18) and the usual *dihedral group* D_8 of order 8 in (20) when $m = 2$. Note also that a (finite) Hamiltonian 2-group G should be of the form $G = Q_8 \times \mathbb{Z}(2)^r$ for some $r \geq 0$ by a result of Dedekind and Hamilton [19] (Vol. I, §1, Theorem 1.20). This structure is peculiar and emphasizes the relevance of Definition 2 in the theory of finite nonabelian 2-groups. In fact, finite

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Hamiltonian 2-groups are $Q^{\times}$-groups, according to Definition 2. We are motivated to introduce some additional notions, due to the relevance of D_8 and Q_8 in the classification of 2-groups by a result of Ward [19] (Vol. II, §56, Theorem 56.1).

Definition 3 (See [19], Vol.II, §56)**.** *A section of a finite group G is an epimorphic image of some subgroup of G. A finite 2-group G is quaternion-free if it has no sections isomorphic to Q_8, that is, if there are no epimorphic images of subgroups of G which are isomorphic to Q_8. In analogy, a finite 2-group is dihedral-free if it has no sections isomorphic to D_8.*

Leveraging on these notions, there are relevant information on the structure of subgroups and quotients of large Pauli groups, which are of the form $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ for n large enough, instead of $E^+_8 \circ \mathbb{Z}(4) = D_8 \circ \mathbb{Z}(4) = \mathsf{P}_{1,2}$ for $n = 1$.

Theorem 4. *The following conditions are satisfied:*

(i). $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ *possesses nonabelian maximal subgroups which are $Q^{\times}$-groups and nonabelian maximal subgroups which are $D^{\times}$-groups;*
(ii). *A subgroup of $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ is modular if and only if it is dihedral-free;*
(iii). *A subgroup of $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ is nonmodular quaternion-free if and only if it is a Wilkens group.*

In particular, abelian subgroups of large Pauli groups $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ are characterized to be dihedral-free, so Theorem 4 (ii) and (iii) give a complete characterization of abelian subgroups and abelian quotients of large Pauli groups. Note also that the notion of *Wilkens group* is related to Wilkens Theorem [19] (Vol. II, §79, Theorem 79.7) on the classification of nonmodular quaternion-free finite 2-groups. These groups were classified in 2002 [25] and in fact Wilkens 2-groups are exactly the subgroups H of the central product $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ in Theorem 4 (iii) where one doesn't have quotients of the form $K/L \simeq Q_8$ with L normal subgroup of H contained in K and K subgroup of H. In addition such H shouldn't be modular, which essentially means for 2-groups, neither to be elementary abelian, nor to be isomorphic to Q_8.

Remark 1. *A brief historical observation is compulsory here: William Rowan Hamilton is the same mathematician, who started to study the "Hamiltonian groups" which we have discussed in this section, but he is also the same author, who introduced the notion of "Hamiltonian for a dynamical system" which is also described along the present contribution. There are several profound reasons why William Rowan Hamilton was interested to study these two concepts. Later on Emmy Noether will develop a theory of groups of symmetries for dynamical systems along a similar line of investigation: a very rough evidence that a reader can find is that Q_8 is "highly commutative" at the level of subgroups (even if is a nonabelian group), then it is not by chance that some dynamical systems (see for instance, Theorem 1) present Hamiltonians which can be simplified due to large symmetries. A classical reference of Arnold [26] can be consulted for a systematic approach.*

3. Previous Results from the Literature

The *Heisenberg group* on the field $\mathbb{F}_p$ with p elements (p odd prime) is

$$\mathbb{H}(\mathbb{F}_p) = \left\{ \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \mid p, q, t \in \mathbb{F}_p \right\} = \{ M(p, q; t) \mid p, q, t \in \mathbb{F}_p \} \tag{22}$$

with respect to the usual matrix product in the general linear group $\mathrm{GL}_3(\mathbb{F}_p)$.

$$Z(\mathbb{H}(\mathbb{F}_p)) = \{ M(0, 0; t) \mid t \in \mathbb{F}_p \} = [\mathbb{H}(\mathbb{F}_p), \mathbb{H}(\mathbb{F}_p)] = \Phi(\mathbb{H}(\mathbb{F}_p)) \simeq \mathbb{F}_p, \tag{23}$$

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showing again that the group is nonabelian. We have also that

$$[[\mathbb{H}(\mathbb{F}_p), \mathbb{H}(\mathbb{F}_p)], \mathbb{H}(\mathbb{F}_p)] = 1 \text{ and} \tag{24}$$

$$\mathbb{H}(\mathbb{F}_p)/Z(\mathbb{H}(\mathbb{F}_p)) = \mathbb{H}(\mathbb{F}_p)/[\mathbb{H}(\mathbb{F}_p), \mathbb{H}(\mathbb{F}_p)] \simeq \mathbb{F}_p{}^2. \tag{25}$$

The condition $[[\mathbb{H}(\mathbb{F}_p), \mathbb{H}(\mathbb{F}_p)], \mathbb{H}(\mathbb{F}_p)] = 1$ means that $\mathbb{H}(\mathbb{F}_p)$ is *nilpotent of class two*, following a classical terminology, see [14] (Definitions, p.118).

The role of the Heisenberg group is important in our discussions for mostly two reasons: the first is that it is a classical example of extraspecial p-group (we will discuss details later), the second is that it is the structural analog of D_8 when we deal with 2-groups (i.e., when we must use $p = 2$). Note also that D_8 appears in one of the relevant factorizations of $P_{1,2}$ which we will investigate later on.

The quotient group $G/[G, G]$ of a finite group G is always abelian, but not necessarily of the form $G/[G, G] \simeq \mathbb{Z}(p)^r$, that is, *elementary abelian p-group of rank r* (or briefly, *elementary abelian p-group*). From (25) we have $\mathbb{H}(\mathbb{F}_p)/[\mathbb{H}(\mathbb{F}_p), \mathbb{H}(\mathbb{F}_p)]$ elementary abelian. The same applies to $D_8/[D_8, D_8]$. Definition 2 involves the notion of elementary abelian 2-group. The presence of an elementary abelian quotient occurs for $\mathsf{P}_{1,2}$ in the sense of the following result.

Lemma 1 (See [1], Lemma 3.6)**.** *The Pauli group $\mathsf{P}_{1,2}$ of order 16 with Pauli matrices X, Y, Z as in (11) admits the following presentation with 3 generators and 6 relations:*

$$\langle u, a, b \mid u^4 = a^2 = 1, u^2 = b^2, a^{-1}ua = u^{-1}, ub = bu, ab = ba \rangle, \tag{26}$$

where $u = XY$, $a = Y$, and $b = XYZ$. Moreover, $\mathsf{P}_{1,2} = AB$ with A normal subgroup isomorphic to D_8 and B cyclic normal subgroup of order four. In addition $\mathsf{P}_{1,2}$ has no elements of order eight, $Z(\mathsf{P}_{1,2})$ is cyclic of order four, $[\mathsf{P}_{1,2}, \mathsf{P}_{1,2}] \subseteq Z(\mathsf{P}_{1,2})$, $[\mathsf{P}_{1,2}, \mathsf{P}_{1,2}] = \Phi(\mathsf{P}_{1,2})$ has order two and $\mathsf{P}_{1,2}/Z(\mathsf{P}_{1,2})$ is 2-elementary abelian of rank 2.

Abelian groups may be decomposed in direct products of cyclic groups [14] (4.2.10), but Lemma 1 shows that $\mathsf{P}_{1,2}$ can be decomposed into a central product of a maximal nonabelian subgroup $\simeq D_8$ by a cyclic subgroup $\simeq \mathbb{Z}(4)$. This allows us to detect structural properties and is a classical line of research in group theory, see [14,19,22]. We recall the following definition which is well known in the literature:

Definition 4 (See [27], Definition 2.2)**.** *A group G of order $|G| = p^n$ is extraspecial, if $Z(G) = [G, G] = \Phi(G)$ has order p.*

The structure of these groups can be described in terms of Definition 1. For a group G, the *exponent of* G is the least natural number l such that $g^l = 1$ for all $g \in G$. While finite groups have always finite exponent, this is no longer true in the infinite case by considering the Prüfer group $\mathbb{Z}(p^\infty)$, see ([28], Example 1.38).

Lemma 2 (See [14], Exercise 5.3.6)**.** *If p is odd, any nonabelian p-group of order p^3 must be isomorphic either to*

$$E_1 = \langle x, y \mid x^p = y^p = 1, x^{-1}[x, y]x = y^{-1}[x, y]y = [x, y] \rangle, \tag{27}$$

which is called nonabelian p-groups of order p^3 and exponent p, or to

$$E_2 = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{1+p} \rangle, \tag{28}$$

which is called nonabelian p-groups of order p^3 and exponent p^2.

In fact one can see that E_1 has no elements of order p^2, while E_2 has.

Lemma 3 (See [14], Exercise 5.3.7 (i))**.** *Any nonabelian* 2*-group of order* 8 *either is isomorphic to D_8 or to Q_8.*

Any nonabelian p-group of order p^3 is extraspecial, nevertheless if p is even, or odd. Now we report the main classification of extraspecial p-groups.

Lemma 4 (See [14], Exercise 5.3.7 (i) and (ii))**.** *A nonabelian extraspecial group G of $|G| = p^{2n+1}$ has $Z(G)$ of order p and p-elementary abelian quotient $G/Z(G)$. Moreover, if $p = 2$, then G is the central product of D_8's or a central product of D_8's and a single Q_8. If $p > 2$, then either G has exponent p, or else it is a central product E_1's and a single E_2.*

The Pauli group $\mathsf{P}_{1,2}$ cannot be described by Lemma 4, in fact $|\mathsf{P}_{1,2}| = 16$ and both $\mathsf{P}_{1,2} \neq D_8 \circ D_8$ and $\mathsf{P}_{1,2} \neq D_8 \circ Q_8$. This means that $\mathsf{P}_{1,2}$ is not an extraspecial 2-group. An alternative argument can be used if we note that $|Z(\mathsf{P}_{1,2})| > 2$ by Lemma 1.

Remark 2. *If $p > 2$, Lemma 4 shows that G may be decomposed in the central product of finitely many factors isomorphic either to E_1 or E_2. If $p = 2$, the same is true but now the factors must be isomorphic either to D_8 or to Q_8. Because of these restrictive conditions, it is usual to talk about "the" extraspecial p-group $E_{p^{2n+1}}$ of order p^{2n+1}. In particular, if $p = 2$, there are only two options: $E^+_{2^{2n+1}}$ and $E^-_{2^{2n+1}}$ as per (17).*

In order to get decompositions for large Pauli groups, we need to introduce the notion of extraspecial p-group in a more general way, that is, up to homomorphic images.

Definition 5 (See [29], Definition 3.1)**.** *A group G of order $|G| = p^n$ is generalized extraspecial, if $[G, G] = \Phi(G)$ is of order p and contained in $Z(G)$.*

Generalized extraspecial p-groups with $Z(G) = \Phi(G)$ are extraspecial by Definition 4.

Remark 3. *One has to be very careful with Definition 5, since different authors have different meanings for the notion of generalized extraspecial p-group. Robinson [14] (Exercise 5.3.8), Kurdachenko and others [22] (Chapter 11) refer to "generalized extraspecial p-groups" as groups in Definition 5 with the additional requirement that $Z(G)$ is cyclic. This is absent for instance in [29], so the terminology is not uniform.*

Following [19], in a finite p-group G in which the order of an element $g \in G$ is denoted by $o(g)$, we may introduce the subgroups

$$\Omega_m(G) = \langle g \in G \mid o(g) \leq p^m \rangle \tag{29}$$

which is called *omega subgroup* of G. One can describe the groups in Definition 5 below.

Lemma 5. *Let G be a generalized extraspecial p-group. Then*

(i). Either $G \simeq E \times \mathbb{Z}(p)^r$, where E is extraspecial and $r \geq 0$,
(ii). Or $G \simeq (E \circ \mathbb{Z}(p^2)) \times \mathbb{Z}(p)^s$, where E is extraspecial and $s \geq 0$.

Proof. See Lemma 3.2 in [29]. □

In Lemma 5 (i) it is not difficult to check that $Z(G) \simeq \Phi(G) \times \mathbb{Z}(p)^r$, while in the other case $Z(G) \simeq \mathbb{Z}(p^2) \times \mathbb{Z}(p)^s$. In particular, if $Z(G)$ is cyclic in Lemma 5, and so in Definition 5, either G is extraspecial, one has the following description.

Lemma 6. *A finite generalized extraspecial p-group G with cyclic center $Z(G)$ satisfies the following conditions:*

(i). $[G,G] \subseteq Z(G)$ *and* $G/Z(G)$ *is elementary abelian of even rank;*
(ii). G *can be decomposed as a central product;*
(iii). $G = H/L$, *where* $H = E \times C$, C *is cyclic and* E *extraspecial. Moreover* $Z(E) = [E,E] = \langle e \rangle$ *and* $c \in \Omega_1(C)$ *imply* $L = \langle ce^{-1} \rangle$ *cyclic of exponent* p;
(iv). G *is just nonabelian.*

Proof. (i) and (ii) follow from the definitions and are shown in [14] (Exercises 5.3.8 and 5.3.9). (iii) and (iv) can be deduced by [22] (Theorem 11.2). □

Note that Lemma 6 does not guarantee that a nonabelian generalized extraspecial finite p-group has center of prime order. In fact, the moment this is true, we get extraspecial finite p-groups, since the condition $[G,G] = Z(G)$ is automatically satisfied. Therefore Definition 5 is more general than Definition 4; this becomes evident if we look at Lemma 6 (iii) and note that there are homomorphic images of direct products of the form $E \times C$ of an extraspecial finite p-groups E by a cyclic finite p-groups C.

We shall mention also another variation on the theme of extraspecial finite p-groups:

Definition 6 (See [27], Definition 2.2)**.** *A group G of order $|G| = p^n$ is almost extraspecial, if $[G,G] = \Phi(G)$ is of order p and $Z(G)$ is cyclic of order p^2.*

From the definitions, extraspecial p-groups are generalized extraspecial, but one can produce finite p-groups G with $[G,G] = \Phi(G)$ of order p contained in $Z(G) \simeq \mathbb{Z}(p)^2$, so there are generalized extraspecial p-groups which aren't almost extraspecial.

Lemma 7. *Let G be a finite 2-group and n, m integers such that $n \geq m \geq 1$.*

(i). *If G is extraspecial, then $|G| = 2^{2n+1}$ and either $G \simeq E^+_{2^{2n+1}}$, or $G \simeq E^-_{2^{2n+1}}$.*
(ii). *If G is almost extraspecial, then $|G| = 4^{n+1}$ and $G \simeq E^+_{2^{2n+1}} \circ \mathbb{Z}(4) \simeq E^-_{2^{2n+1}} \circ \mathbb{Z}(4)$.*
(iii). *If G is generalized extraspecial of order $|G| = 4^{n+1}$, then*
  (a). *Either $G \simeq E^+_{2^{2m+1}} \times \mathbb{Z}(2)^{2(n-m)+1}$,*
  (b). *Or $G \simeq E^-_{2^{2m+1}} \times \mathbb{Z}(2)^{2(n-m)+1}$,*
  (c). *Or $G \simeq (E^+_{2^{2m+1}} \circ \mathbb{Z}(4)) \times \mathbb{Z}(2)^{2(n-m)}$,*
  (d). *Or $G \simeq (E^-_{2^{2m+1}} \circ \mathbb{Z}(4)) \times \mathbb{Z}(2)^{2(n-m)}$.*

In particular, generalized extraspecial 2-groups in (c)–(d) are almost extraspecial for $m = n$.

Proof. (i). This follows from Lemma 4, Remark 2. (ii). This follows from [27] (Theorem 2.3). (iii). This follows from Lemma 5, combined with Lemma 4, Remark 2. □

An observation about central products and homomorphic images might be useful now: actually, from [1] (Lemma 3.5) and other results of [14,19,22], a central product of two groups $A \circ B$ can be identified with an appropriate homomorphic image of the direct product $A \times B$.

Remark 4. *The Pauli group $\mathsf{P}_{1,2}$ is central product of a subgroup A isomorphic to D_8 by another B isomorphic to $\mathbb{Z}(4)$, looking at Lemma 1 and Definition 1; in other words, $\mathsf{P}_{1,2} = A \circ B \simeq D_8 \circ \mathbb{Z}(4)$.*

In particular, $\mathsf{P}_{1,2}$ *is homomorphic image of* $D_8 \times \mathbb{Z}(4)$*. Moreover* $\mathsf{P}_{1,2}$ *is not extraspecial by Lemma 4, but satisfies Definitions 5 and 6 so it is both generalized extraspecial and almost extraspecial.* The fact that $D_8 \circ \mathbb{Z}(4) \simeq Q_8 \simeq \mathbb{Z}(4)$ *can be checked directly with software of computational group theory (i.e.: GAP) or look at [1,4].*

Minimal nonabelian groups were originally classified by Redei [19] (Exercise 8a, §1). Again we find an interesting behavior of $\mathsf{P}_{1,2}$.

Lemma 8 (Minimal Nonabelian and Minimal Nonmetacyclic p-Groups)**.**
Let G be a finite p-group.

(i). *G is minimal nonabelian if and only if the minimal number of generators of G is two and* $|[G,G]| = p$*. In that case* $\Phi(G) = Z(G)$*.*

(ii). *G is minimal nonabelian metacyclic if and only if G is minimal nonabelian and* $|\Omega_1(G)| \leq p^2$*, in which case either* $G \simeq Q_8$*, or* $G = \langle a, b \mid a^{p^m} = b^{p^n} = 1, b^{-1}ab = a^{1+p^{m-1}} \rangle$ *for some* $m \geq 2$ *and* $n \geq 1$*.*

(iii). *If G is minimal nonabelian nonmetacyclic, then* $G = \langle a, b \mid a^{p^m} = b^{p^n} = 1, [a,b] = c, c^p = [a,c] = [b,c] = 1 \rangle$ *for some* $m \geq n \geq 1$*, where in case* $p = 2$ *we must have* $m > 1$*. Here* $|G| = p^{m+n+1}$*,* $Z(G) = \langle a^p, b^p, c \rangle$ *and* $\Omega_1(G) = \langle a^{p^{m-1}}, b^{p^{n-1}}, c \rangle$*.*

(iv). *If G is nonabelian and A maximal abelian normal subgroup of G, then for any* $x \in G \setminus A$ *there is an* $a \in A$ *such that* $\langle x, a \rangle$ *is minimal nonabelian.*

(v). *If* $p = 2$ *and G is minimal nonmetacyclic, then one (and only one) of the following conditions is satisfied: either* $G \simeq \mathbb{Z}(2)^3$*, or* $G \simeq Q_8 \times \mathbb{Z}(2)$*, or* $G \simeq \mathsf{P}_{1,2}$*, or* $G \simeq \langle a, b, c \mid a^4 = b^4 = c^4 = [a,b] = 1, a^2 = c^2, [a,c] = b^2, [b,c] = a^2 \rangle$*.*

(vi). *The minimal nonabelian subgroups of G generate G.*

Proof. (i), (ii) and (iii) report ([30], Lemmas 3.1 and 3.2). (iv) reports [21] (Lemma 2.1). (v) reports [20] (Theorem 7.1). (vi) reports [19] (Proposition 10.28, §10, Vol. I). □

In order to detect abelian subgroups in $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$, the notions of $Q^\times$-groups and $D^\times$-groups in Definition 2 are strongly related to the presence of nonabelian maximal subgroups.

Theorem 5 (Janko-Miller Theorem on Nonabelian Maximal Subgroups of 2-Groups)**.**

(i). *Suppose that all nonabelian maximal subgroups of a finite nonabelian 2-group G are* $Q^\times$*-groups. Then either G is minimal nonabelian or G is a* $Q^\times$*-group.*

(ii). *Suppose that all nonabelian maximal subgroups of a finite nonabelian 2-group G are* $D^\times$*-groups. Then either G is a* $D^\times$*-group, or* $G \simeq D_{2^5}$*.*

Proof. See [19] (Theorem A.17.2 and Proposition A.17.4, Appendix 17, Vol. II). □

Iwasawa and Napolitani [19] (§73, Theorem 73.15, Vol. II) classified nonhamiltonian modular p-groups via appropriate products of abelian normal subgroups whose quotients are cyclic. We state Iwasawa's Theorem in case of 2-groups:

Theorem 6 (Iwasawa's Theorem for 2-Groups)**.** *A finite 2-group G is modular if and only if G is dihedral-free. A finite 2-group G is modular and quaternion-free if and only if G contains an abelian normal subgroup A with cyclic quotient G/A and there exists an element* $t \in G$ *and an integer* $s \geq 2$ *such that* $G = \langle A, t \rangle$ *and* $t^{-1}at = a^{1+2^s}$ *for all* $a \in A$*.*

Proof. See [19] (Proposition 79.6, §79, Vol. II). □

Of course, finite abelian groups are both quaternion-free and dihedral-free, and so the intuition motivates to think that finite 2-groups which are close to be abelian should be quaternion-free, or dihedral-free. This intuition may be formalized.

Theorem 7 (Ward's Theorem)**.** *Let G be a finite nonabelian 2-group. If G is quaternion-free, then G has a characteristic subgroup of index 2.*

Proof. See [19] (Theorem A.24.1, Appendix 24, Vol. II), or [19] (Theorem 56.1, §56, Vol. II). □

Finally, we end with an important result of classification for finite nonmodular 2-groups which are quaternion-free.

Theorem 8 (Wilkens Theorem)**.** *A finite 2-group G is nonmodular and quaternion-free if and only if one of the following conditions is satisfied:*

(i). $G = N \rtimes \langle x \rangle$*, where N is a maximal abelian normal subgroup of G of exponent* > 2 *and, if t is the involution in* $\langle x \rangle$*, then every element of N is inverted by t;*

(ii). $G = N\langle x \rangle$*, where N is a maximal elementary abelian normal subgroup of G and* $\langle x \rangle$ *is not normal in G;*

(iii). $G = \langle N, x, t \rangle$*, where N is an elementary abelian normal subgroup of G and t is an involution with* $[N, t] = 1$.

Proof. See [19] (Vol.II, §79, Theorem 79.7). □

The condition (i) of Theorem 8 defines a *Wilkens group of first type*. The condition (ii) of Theorem 8 defines a *Wilkens group of second type*. The condition (iii) of Theorem 8 defines a *Wilkens group of third type*. Briefly, we say that a finite 2-group G is a *Wilkens group* if G is a Wilkens group either of first, or of second, or of third type. This notion will be very useful to detect nonmodular nonabelian quaternion-free subgroups in $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$.

4. Some Arguments and Final Observations

We are going to prove Theorem 3, where the main argument follows from what we noted in Lemmas 1, 6 and 8, and Theorem 4.

Proof of Theorem 3. For $\mathsf{P}_{1,2}$ we find nonabelian subgroups from Lemma 1 so it cannot be minimal nonabelian. Lemma 8 (v) shows that $\mathsf{P}_{1,2}$ is minimal nonmetacyclic. □

Thanks to the results of the previous sections, we may also prove the next result.

Proof of Theorem 4. (i). From $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$, we note that this is neither a $Q^\times$-group, nor a $D^\times$-group, nor has order 32. Theorem 5 may be applied in the present situation: the fact that $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ is neither minimal nonabelian, nor a $Q^\times$-group, nor a $D^\times$-group, nor of order 32 implies the existence of nonabelian maximal subgroups which are $Q^\times$-groups or $D^\times$-groups. (ii). The characterization of modular 2-groups in Theorem 6 shows that any subgroup of $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ is modular if and only if it is dihedral free. (iii). Nonmodular quaternion-free 2-subgroups of $E^+_{2^{2n+1}} \circ \mathbb{Z}(4)$ are described by Theorem 8. □

We end with a few comments on applications of Theorems 3 and 4.

Remark 5. *Of course, the first area of application of Theorems 3 and 4 deals with quantum information theory and problems of mutually unbiased bases. We have already mentioned [12,16–18], where it is fundamental to classify abelian subgroups of large Pauli groups.*

Let's say something more.

Remark 6. *It was noted above that:*

(i). *Some dynamical systems involving pseudofermions are affected by the structure of central product of* (13) *in the sense of Lemma 1, but replacing the role of* D_8 *with that of* Q_8*, see Theorem 1;*

(ii). *Formalizations with pseudofermions are possible also for finite p-groups when* $p \neq 2$*, see [3,4].*

(iii). *On the basis of Theorems 1 and 2, we may consider the finite extraspecal 2-group* $(Q_8 \circ Q_8) \circ \mathbb{Z}(4) \simeq Q_8 \circ \mathsf{P}_{1,2}$ *and proceed to a realization of* Q_8 *with two pseudofermionic operators* $\mathbf{a}_1$ *and* $\mathbf{b}_1$*, along with two more pseudofermionic operators* $\mathbf{a}_2$ *and* $\mathbf{b}_2$ *for the Pauli group* $P = \mathsf{P}_{1,2}$*. Therefore one could produce in this way a dynamical systems* $\mathcal{S}$ *admitting finite extraspecial 2-groups of symmetries* $(Q_8 \circ Q_8) \circ \mathbb{Z}(4)$*, but whose Hamiltonians* $H_{\mathcal{S}}$ *is the same of the dynamical system* $\mathcal{T}$ *admitting finite extraspecial 2-groups of symmetries* $Q_8 \circ Q_8$*.*

(iv). *The physical properties of the dynamical systems which are produced as per (iii) above deserve to be investigated and (to the best of our knowledge) are not known.*

In fact both the decompositions $\mathsf{P}_{1,2} \simeq D_8 \circ \mathbb{Z}(4)$ *and* $\mathsf{P}_{1,2} \simeq Q_8 \circ \mathbb{Z}(4)$ *are valid and the last one may be formalized at the level of functional operators as per [1] (Proof of Theorem 1.2). This has the advantage to recognize conservation laws, see [1] (Remark 5.1).*

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References

1. Bagarello, F.; Bavuma, Y.; Russo, F.G. Topological decompositions of the Pauli group and their influence on dynamical systems. *Math. Phys. Anal. Geom.* **2021**, *24*, 16. [CrossRef]
2. Bagarello, F.; Bavuma, Y.; Russo, F.G. On the Pauli group on 2-qubits in dynamical systems with pseudofermions. *Forum Math.* **2024**, *36*, 585–597. [CrossRef]
3. Bavuma, Y. A short note on the topological decomposition of the central product of groups. *Trans. Comb.* **2022**, *11*, 123–129.
4. Bavuma, Y.; Russo, F.G. Dynamical systems involving pseudo-fermionic operators and generalized quaternion groups, In *Quantum Mathematics II, Contributions Based on the Presentations at the INdAM Quantum Meetings, IQM22, Milan, Italy, Spring 2022*; Correggi, M., Falconi, M., Eds.; Springer INdAM Ser. 58; Springer: Singapore, 2023; pp. 3–16.
5. Cherbal, O.; Maamache, M. Time-dependent pseudofermionic systems and coherent states. *J. Math. Phys.* **2016**, *57*, 022102. [CrossRef]
6. Cherbal, O.; Drir, M.; Maamache, M.; Trifonov, D.A. Fermionic coherent states for pseudoHermitian two-level systems. *J. Phys. A* **2007**, *40*, 1835–1844. [CrossRef]
7. Duplij, S. Polyadic sigma matrices. *J. Math. Phys.* **2024**, *65*, 083509. [CrossRef]
8. Bender, C. Making sense of non-Hermitian Hamiltonians. *Rep. Progr. Phys.* **2007**, *70*, 947–1018. [CrossRef]
9. Bagarello, F.; Bavuma, Y.; Russo, F.G. Some consequences of the Grunewald-O'Halloran conjecture for pseudoquonic operators. *Forum Math.* 2026, *in press*. [CrossRef]
10. Gottesman, D. Stabilizer Codes and Quantum Error Correction. Ph.D. Thesis, Pasadena, CA, USA, 1997.
11. Rocchetto, A.; Russo, F.G. Decomposition of Pauli groups via weak central products. *arXiv* **2020**, arXiv:1911.10158. [CrossRef]
12. Wootters, W.K. Quantum measurements and finite geometry. *Found. Phys.* **2006**, *36*,112–126. [CrossRef]
13. Gorenstein, D. *Finite Groups*; Chelsea Publishing Company: New York, NY, USA, 1980.
14. Robinson, D. *A Course in the Theory of Groups*; Springer: Berlin, Germany, 1980.
15. Bavuma, Y.; D'Angeli, D.; Donno, A.; Russo, F.G. On an infinite family of integral Cayley graphs of Pauli groups. *J. Algebra* **2024**, *659*, 148–182. [CrossRef]
16. Jorrand, P.; Planat, M. On group theory for quantum gates and quantum coherence. *J. Phys. A Math. Theor.* **2008**, *41*, 182001.

17. Zauner, G. Quantendesigns: Grundzüge Einer Nichtkommutativen Designtheorie. Ph.D. Dissertation, Universität Wien, Wien, Austria, 1999.
18. Durt, T.; Englert, B.G.; Bengtsson, I.; Zyczkowski, K. On mutually unbiased bases. *Int. J. Quantum Inf.* **2010**, *8*, 535–640. [CrossRef]
19. Berkovich, Y.; Janko, Z. *Groups of Prime Power Order*; Vol. I and Vol. II, de Gruyter Expositions in Mathematics 46 and 47; de Gruyter: Berlin, Germany, 2008.
20. Janko, Z. Finite 2-groups with exactly four cyclic subgroups of order 2^n. *J. Reine Angew. Math.* **2004**, *566*, 135–181. [CrossRef]
21. Janko, Z. On finite nonabelian 2-groups all of whose minimal nonabelian subgroups are of exponent 4. *J. Algebra* **2007**, *315*, 801–808. [CrossRef]
22. Kurdachenko, L.; Otál, J.; Subbotin, I. *Groups with Prescribed Quotient Groups and Associated Module Theory*; World Scientific: Singapore, 2002.
23. Otera, D.E.; Russo, F.G. A note on just-non-Ω groups. *Beitr. Algebra Geom.* **2011**, *52*, 75–81. [CrossRef]
24. Russo, F.G. On compact just non-Lie groups. *J. Lie Theory* **2007**, *17*, 625–632. [CrossRef]
25. Wilkens, B. On quaternion-free 2-groups. *J. Algebra* **2002**, *258*, 477–492 [CrossRef]
26. Arnold, V.I. *Mathematical Methods of Classical Mechanics*; Springer: New York, NY, USA, 1989.
27. Bouc, S.; Mazza, N. The Dade gruop of (almost) extraspecial *p*-groups. *J. Pure Appl. Algebra* **2004**, *192*, 21–51. [CrossRef]
28. Hofmann, K.H.; Morris, S. *The Structure of Compact Groups*; de Gruyter: Berlin, Germany, 2006.
29. Stancu, R. Almost all extraspecial *p*-groups are resistant. *J. Algebra* **2002**, *249*, 120–126. [CrossRef]
30. Berkovich, Y.; Janko, Z. Structure of finite *p*-groups with given subgroups. *Contemp. Math.* **2006**, *402*, 13–93.