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Article

Recurrence Formulas for Choulet Sequences

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Abstract

In this paper, we consider the so-called Choulet sequence a_n, $n = 0, 1, 2, 3, \ldots$ For any complex numbers a_0, a_1, k, l, it is defined by the formula $a_{n+1} = \sum_{j=0}^{n} a_j a_{n-j} + k(n+1) + l$ for $n = 1, 2, 3, \ldots$ We derive the recurrence formula of the form $(n+1)a_n + \sum_{j=1}^{5}(u_j + nv_j)a_{n-j} = 0$ for $n \geq 5$, where u_j, v_j, $j = 1, \ldots, 5$, are some constants that depend on a_0, a_1, k, l, but not on n. In the case when $k = 0$, we obtain a shorter recurrence formula $(n+1)a_n + \sum_{j=1}^{4}(u_j + nv_j)a_{n-j} = 0$ for $n \geq 4$. Both these formulas correspond to the recurrence formulas presented in the OEIS (The On-Line Encyclopedia of Integer Sequences) for many initial vectors $(a_0, a_1, k, l) \in \mathbb{Z}^4$. Most of them are listed as conjectures. So here we verify all of them using the same procedure by just inserting the values of a_0, a_1, k, l into the derived formulas. We also show that the Choulet sequence is a linear recurrence sequence if and only if $(k, l) = (-a_1^2, 2a_1^2 + a_1 - 2a_0 a_1)$.

Keywords: Choulet sequence; generating function; recurrence formulas

MSC: 11B37

1. Introduction

Consider the sequence of complex numbers a_n, $n = 0, 1, 2, 3, \ldots$, defined by

$$a_{n+1} = \sum_{j=0}^{n} a_j a_{n-j} + k(n+1) + l, \quad n = 1, 2, 3, \ldots, \tag{1}$$

where a_0, a_1, k, l are some fixed complex numbers. For various small $a_0, a_1, k, l \in \mathbb{Z}$ these sequences were introduced by Richard Choulet in [1]: see, for instance, the following entries: A176604–A176607, A176609–A176612, A176645, A176648, A176675, A176677, A176678, A176749–A176757, A176759, A176826, A177111, A177113, A177115, A177117, A177118, A177127–A177131, A177168–A177172.

As is usual for the entries in [1], a combinatorial, a number theoretic or another mathematical context of each such sequence is explained. In all the cases mentioned above, a recurrence formula of the form

$$(n+1)a_n + \sum_{j=1}^{d}(u_j + nv_j)a_{n-j} = 0, \tag{2}$$

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$n = d, d+1, d+2, \dots$, with some explicit u_j, v_j is provided in the respective entry for either $d = 4$ or $d = 5$ as a conjecture. Note that in the case when $a_0 = a_1 = 1$ and $k = l = 0$ the sequence (1) is the Catalan sequence $C_0 = C_1 = 1$,

$$C_{n+1} = \sum_{j=0}^{n} C_j C_{n-j}, \quad n = 1, 2, 3, \dots, \tag{3}$$

which itself is the sequence A000108 in [1]. A wide range of combinatorial interpretations of Catalan numbers is described in Stanley's monograph [2]. It seems that Catalan numbers first appeared in 18th century [3]. See also [4,5] and some very recent papers [6–9]. It is well known that

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{4}$$

and so the recurrence relation

$$(n+1)C_n + (2-4n)C_{n-1} = 0 \tag{5}$$

holds for each $n \geq 1$. On the other hand, in [10], it was shown that C_n, $n = 0, 1, 2, 3, \dots$, does not satisfy a linear recurrence sequence (the one with constant coefficients; see (10) below). Note that (5) is also of the form (2) with $d = 1$.

Very recently, in [11], Niu considered the sequence (1) with $(a_0, a_1, k, l) = (1, 1, 0, -1)$ and proved for it the recurrence formula

$$(n+1)a_n + (2-6n)a_{n-1} + (9n-13)a_{n-2} - 4a_{n-3} + (16-4n)a_{n-4} = 0$$

for each $n \geq 4$ as it was conjectured in A176677 of [1]. (This formula is of the form (2) with $d = 4$.) The proof in [11] involves some nontrivial calculations and computational verification with the Python standard library plus SymPy 1.14. The aim of our note is to prove this result for any sequence (1) (with arbitrary choice of a_0, a_1, k, l) and do it without any computer assistance. Using this, we will establish all the formulas exactly as they are conjectured in the above mentioned entries of [1].

In order to derive (2) for the sequence (1) with arbitrary $a_0, a_1, k, l \in \mathbb{C}$, we set

$$Q(z) = (1-z)^2 - 4a_0 z(1-z)^2 - 4(a_1 - a_0^2)z^2(1-z)^2 - 4kz^3 - 4(k+l)z^3(1-z) \tag{6}$$

and show the following:

Theorem 1. *There is a positive number $\delta = \delta(a_0, a_1, k, l)$ such that the generating function $G(z) = \sum_{j=0}^{\infty} a_j z^j$ of the sequence (1) is analytic in the disc $|z| < \delta$ and for each such z satisfies*

$$q_0(z)G'(z) + q_1(z)G(z) = q_2(z), \tag{7}$$

where $q_0(z) = z(1-z)Q(z)$,

$$q_1(z) = (1-2z)Q(z) - \frac{z(1-z)Q'(z)}{2}, \tag{8}$$

$$q_2(z) = \frac{Q'(z)(z-1) - 2Q(z)}{4}, \tag{9}$$

and where Q is defined in (6).

In the next section, we will prove Theorem 1. Using it, in Section 3, we will derive the formula (2) with $d = 5$ for any sequence (1) (see Theorem 3). In Section 4, assuming that

$k = 0$, we will derive the recurrence (2) with $d = 4$ (see Theorem 4). Finally, in Section 5 we will show that the Choulet sequence is not a linear recurrence sequence except for one explicitly described case. Recall that a sequence s_n, $n = 0, 1, 2, 3, \ldots$, is a *linear recurrence sequence* if for some $d \in \mathbb{N}$ and some constants $b_1, \ldots, b_d$ we have

$$s_n = b_1 s_{n-1} + \cdots + b_d s_{n-d} \tag{10}$$

for $n = d, d+1, d+2, \ldots$ In Section 5, we will show that

Theorem 2. *The sequence* (1) *is not a linear recurrence sequence except for the case when* $(k, l) = (-a_1^2, 2a_1^2 + a_1 - 2a_0 a_1)$. *Then the sequence* (1) *is* $a_0, a_1, a_1, a_1, \ldots$

Note that $a_0, a_1, a_1, a_1, \ldots$ is indeed a linear recurrence sequence because of the relation (10), where $s_n = a_n$, $n = 0, 1, 2, 3, \ldots$, holds for $d = 2$ and $(b_1, b_2) = (1, 0)$.

2. Proof of Theorem 1

We will first establish the following lemma.

Lemma 1. *For the sequence* a_n, $n = 0, 1, 2, 3, \ldots$, *defined in* (1) *there is a positive constant* $\rho = \rho(a_0, a_1, k, l)$ *such that* $|a_n| < \rho^n$ *for each* $n = 1, 2, 3, \ldots$

Proof. Take

$$\rho > \max\{2, 2(|k| + |l|), |a_0|, |a_1|, \sqrt{|a_0 a_1| + |k| + |l|}\}. \tag{11}$$

Note that (11) implies

$$|k|(n+1) + |l| \le (|k| + |l|)(n+1) < \frac{\rho(n+1)}{2} \le \rho 2^{n-1} < \rho^n \tag{12}$$

for $n \ge 1$. We claim that

$$|a_{n+1}| < C_{n+1} \rho^{n+1} \tag{13}$$

for each $n \ge 0$. Since $C_n = \frac{1}{n+1}\binom{2n}{n} < \binom{2n}{n} < 4^n$ for $n \ge 1$ (see (4)), this will complete the proof of the lemma with the constant 4ρ.

The proof of (13) is by induction on n. Since, by (11), $|a_1| < \rho = C_1 \rho$ and

$$|a_2| = |2a_0 a_1 + 2k + l| \le 2|a_0 a_1| + 2|k| + 2|l| < 2\rho^2 = C_2 \rho^2,$$

the inequality (13) holds for $n = 0$ and $n = 1$. Fix $n \ge 2$. Assume that the inequality $|a_j| < C_j \rho^j$ holds for $j = 1, 2, \ldots, n$. We will show that it then holds for $j = n+1$ as well, which will complete the proof of (13). Note that, by (1),

$$a_{n+1} = 2a_0 a_n + \sum_{j=1}^{n-1} a_j a_{n-j} + k(n+1) + l.$$

Here, $2|a_0 a_n| < 2\rho C_n \rho^n = 2C_n \rho^{n+1}$ by (11) and the induction assumption. Hence, by (3), (11), (12) and $|a_j a_{n-j}| < C_j C_{n-j} \rho^j \rho^{n-j} = C_j C_{n-j} \rho^n$, we deduce

$$\begin{aligned}
|a_{n+1}| &< 2C_n \rho^{n+1} + \rho^n \sum_{j=1}^{n-1} C_j C_{n-j} + |k|(n+1) + |l| \\
&= 2C_n \rho^{n+1} + (C_{n+1} - 2C_n)\rho^n + |k|(n+1) + |l| \\
&< 2C_n \rho^{n+1} + (C_{n+1} - 2C_n)\rho^n + \rho^n = 2C_n \rho^{n+1} + (C_{n+1} - 2C_n + 1)\rho^n.
\end{aligned}$$

Using $C_{n+1} - 2C_n \geq 1$ for $n \geq 2$, we see that this is less than $C_{n+1}\rho^{n+1}$ because

$$(C_{n+1} - 2C_n)\rho > 2(C_{n+1} - 2C_n) \geq C_{n+1} - 2C_n + 1,$$

which completes the proof of (13). $\square$

We can now prove Theorem 1. By Lemma 1, the generating function $G(z)$ of (1) is absolutely convergent in the open disc $|z| < \rho^{-1}$. In particular, the function $G(z)$ is analytic in $|z| < \rho^{-1}$. Using (1) we immediately get

$$G(z) - a_0 - a_1 z = \sum_{n=1}^{\infty} a_{n+1} z^{n+1} = \sum_{n=1}^{\infty} \sum_{j=0}^{n} a_j a_{n-j} z^{n+1} + \sum_{n=1}^{\infty} (k(n+1)+l) z^{n+1}.$$

Here,

$$\sum_{n=1}^{\infty} \sum_{j=0}^{n} a_j a_{n-j} z^{n+1} = z\Big(-a_0^2 + \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_j z^j a_{n-j} z^{n-j} \Big) = -a_0^2 z + z\Big(\sum_{j=0}^{\infty} a_j z^j \Big)^2 = zG(z)^2 - a_0^2 z$$

and

$$\sum_{n=1}^{\infty} (k(n+1)+l) z^{n+1} = kz^2 \sum_{n=1}^{\infty} n z^{n-1} + (k+l) z^2 \sum_{n=1}^{\infty} z^{n-1} = \frac{kz^2}{(1-z)^2} + \frac{(k+l)z^2}{1-z}.$$

Hence,

$$G(z) - a_0 - a_1 z = zG(z)^2 - a_0^2 z + \frac{kz^2}{(1-z)^2} + \frac{(k+l)z^2}{1-z},$$

which yields

$$zG(z)^2 - G(z) = -a_0 - (a_1 - a_0^2)z - \frac{kz^2}{(1-z)^2} - \frac{(k+l)z^2}{1-z}.$$

In view of (6), we can rewrite the last equality as

$$zG(z)^2 - G(z) = \frac{Q(z) - (1-z)^2}{4z(1-z)^2} = \frac{Q(z)}{4z(1-z)^2} - \frac{1}{4z}.$$

Thus,

$$4z^2 G(z)^2 - 4zG(z) + 1 = \frac{Q(z)}{(1-z)^2}.$$

The left-hand side of the above equality equals $(2zG(z) - 1)^2$. So multiplying both sides by $(1-z)^2$, we obtain

$$(2z(1-z)G(z) - 1 + z)^2 = Q(z). \tag{14}$$

Recall that the function $G(z)$ is analytic in the open disc $|z| < \rho^{-1}$. Take a positive number $\delta \leq \rho^{-1}$ for which the polynomial Q is not vanishing in $|z| < \delta$. (Such δ exists due to $Q(0) = 1 \neq 0$.) Taking the derivatives of both sides of (14) and dividing the resulting equality by (14), we get

$$\frac{2(2z(1-z)G'(z) + (2-4z)G(z) + 1)}{2z(1-z)G(z) - 1 + z} = \frac{Q'(z)}{Q(z)}$$

for any complex z satisfying $|z| < \delta$, and hence,

$$4z(1-z)Q(z)G'(z) + ((4-8z)Q(z) - 2z(1-z)Q'(z))G(z) = Q'(z)(z-1) - 2Q(z).$$

Dividing by 4, we complete the proof of the theorem.

3. General Case

By the definition of Q in (6) and $q_0(z) = z(1-z)Q(z)$, we find that

$$q_0(z) = z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + c_6z^6, \tag{15}$$

with

$$\begin{cases} c_2 &= -4a_0 - 3, \\ c_3 &= 4a_0^2 + 12a_0 - 4a_1 + 3, \\ c_4 &= -12a_0(a_0+1) + 12a_1 - 8k - 4l - 1, \\ c_5 &= 12a_0^2 + 4a_0 - 12a_1 + 12k + 8l, \\ c_6 &= -4a_0^2 + 4a_1 - 4k - 4l. \end{cases} \tag{16}$$

Likewise, by (6) and (8), we deduce

$$q_1(z) = 1 - (3+2a_0)z + 3(1+2a_0)z^2 - (1+6a_0-4k-2l)z^3 + 2(a_0-l)z^4.$$

Thus,

$$q_1(z) = 1 + b_1z + b_2z^2 + b_3z^3 + b_4z^4, \tag{17}$$

with

$$(b_1, b_2, b_3, b_4) = (-2a_0 - 3, 6a_0 + 3, -6a_0 - 1 + 4k + 2l, 2a_0 - 2l). \tag{18}$$

The polynomial q_2 in (9) is of degree at most 4, since so is Q. Fix $n \geq 5$ and consider the coefficient for z^n in

$$q_0(z)G'(z) = (z + c_2z^2 + \cdots + c_6z^6)\sum_{j=1}^{\infty} ja_jz^{j-1}.$$

Clearly, it is equal to

$$na_n + c_2(n-1)a_{n-1} + \cdots + c_6(n-5)a_{n-5}.$$

The coefficient for z^n in $q_1(z)G(z) = q_1(z)\sum_{j=0}^{\infty} a_jz^j$ equals

$$a_n + b_1a_{n-1} + \cdots + b_4a_{n-4}.$$

Since the coefficient for z^n, $n \geq 5$, on the right hand side of (7) vanishes, we derive that

$$(n+1)a_n + \sum_{j=1}^{4}(b_j + c_{j+1}(n-j))a_{n-j} + c_6(n-5)a_{n-5} = 0, \tag{19}$$

where $c_2, c_3, c_4, c_5, c_6, b_1, b_2, b_3, b_4$ are given in (16) and (18). Note that they depend on a_0, a_1, k, l but not on n.

In the sequence A176610 we have $(a_0, a_1, k, l) = (1, 0, 1, 1)$. So, by (18), we get $(b_1, b_2, b_3, b_4) = (-5, 9, -1, 0)$, while (16) implies

$$(c_2, c_3, c_4, c_5, c_6) = (-7, 19, -37, 36, -12).$$

Inserting these values into (19), we obtain

$$\sum_{j=0}^{5} \zeta_{n-j} a_{n-j} = 0 \tag{20}$$

for $n \geq 5$, where $(\zeta_n, \zeta_{n-1}, \zeta_{n-2}, \zeta_{n-3}, \zeta_{n-4}, \zeta_{n-5})$ is equal to

$$(n+1, 2-7n, 19n-29, 110-37n, 36(n-4), 12(5-n)),$$

which is exactly the same as conjectured for that sequence.

In the sequence A176611 we have $(a_0, a_1, k, l) = (1, 1, 1, 1)$. Now, by (18), we get $(b_1, b_2, b_3, b_4) = (-5, 9, -1, 0)$, while (16) implies

$$(c_2, c_3, c_4, c_5, c_6) = (-7, 15, -25, 24, -8).$$

Inserting these values into (19), we obtain (20), where $(\zeta_n, \zeta_{n-1}, \zeta_{n-2}, \zeta_{n-3}, \zeta_{n-4}, \zeta_{n-5})$ is equal to

$$(n+1, 2-7n, 3(5n-7), 74-25n, 24(n-4), 8(5-n)).$$

As above, this is exactly the same formula as it was conjectured for that sequence in [1].

The next sequence A176612 is $(a_0, a_1, k, l) = (1, 2, 1, 1)$. As above, we get the relation (20) with $(\zeta_n, \zeta_{n-1}, \zeta_{n-2}, \zeta_{n-3}, \zeta_{n-4}, \zeta_{n-5})$ equal to

$$(n+1, 2-7n, 11n-13, 38-13n, 12(n-4), 4(5-n)).$$

Again, this is exactly the same formula as it was conjectured for that sequence.

For the sequences A176645, where $(a_0, a_1, k, l) = (1, 4, 1, 1)$, A176648, where $(a_0, a_1, k, l) = (1, 5, 1, 1)$, and A176759, where $(a_0, a_1, k, l) = (1, 0, 1, -1)$, we get the result as claimed in OEIS. Similar sequences A177111, A177113, A177115, A177117, A177118 can be verified in exactly the same way. (Those five are claimed to be verified.) The sequence A176826, with $(a_0, a_1, k, l) = (1, 1, 1, -1)$, gives (20) with

$$(\zeta_n, \zeta_{n-1}, \zeta_{n-2}, \zeta_{n-3}, \zeta_{n-4}, \zeta_{n-5}) = (n+1, 2-7n, 3(5n-7), 46-17n, 4(2n-7), 0).$$

So, the recurrence is

$$(n+1)a_n + (2-7n)a_{n-1} + 3(5n-7)a_{n-2} + (46-17n)a_{n-3} + 4(2n-7)a_{n-4} = 0$$

as claimed in A176826 of [1]. Observe that it does not have the term with a_{n-5}, since

$$c_6 = -4a_0^2 + 4a_1 - 4k - 4l = 0,$$

and that it holds for $n \geq 4$, because the polynomial q_2 in (9) is cubic.

Using (16) and (18), we can write (19) in the following equivalent form:

Theorem 3. *For the sequence* (1) *we have*

$$(n+1)a_n + \zeta_{n-1}a_{n-1} + \zeta_{n-2}a_{n-2} + \zeta_{n-3}a_{n-3} + \zeta_{n-4}a_{n-4} + \zeta_{n-5}a_{n-5} = 0 \tag{21}$$

for $n = 5, 6, 7, \ldots$, *where*

$$\begin{cases} \zeta_{n-1} &= -(4a_0+3)n + 2a_0, \\ \zeta_{n-2} &= (4a_0^2 + 12a_0 - 4a_1 + 3)n - (8a_0^2 + 18a_0 - 8a_1 + 3), \\ \zeta_{n-3} &= (-12a_0^2 - 12a_0 + 12a_1 - 8k - 4l - 1)n + (36a_0^2 + 30a_0 - 36a_1 + 28k + 14l + 2), \\ \zeta_{n-4} &= (12a_0^2 + 4a_0 - 12a_1 + 12k + 8l)n - (48a_0^2 + 14a_0 - 48a_1 + 48k + 34l), \\ \zeta_{n-5} &= -4(a_0^2 - a_1 + k + l)(n-5). \end{cases}$$

4. The Case $k = 0$

In the case $k = 0$ we have $Q(1) = 0$, so $Q(z)$ is divisible by $1 - z$, namely,

$$F(z) = \frac{Q(z)}{1-z} = (1-z) - 4z(1-z)a_0 - 4z^2(1-z)(a_1 - a_0^2) - 4lz^3 \in \mathbb{C}[z]. \tag{22}$$

Since $k = 0$, each of the polynomials q_0, q_1, q_2 in Theorem 1 is divisible by $1 - z$. Therefore, we will consider the polynomials $Q_j(z) = q_j(z)/(1-z)$, $j = 0, 1, 2$, for which (7) holds with polynomials Q_j of smaller degree.

In particular, using (6) with $k = 0$, we find that

$$Q_0(z) = \frac{q_0(z)}{1-z} = zQ(z) = z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5,$$

with

$$(c_2, c_3, c_4, c_5) = (-4a_0 - 2, 4a_0^2 + 8a_0 - 4a_1 + 1, -8a_0^2 - 4a_0 + 8a_1 - 4l, 4a_0^2 - 4a_1 + 4l).$$

Likewise, from (8) and (22), we get

$$Q_1(z) = \frac{q_1(z)}{1-z} = \frac{(1-2z)Q(z)}{1-z} - \frac{zQ'(z)}{2} = (1-2z)F(z) - \frac{zQ'(z)}{2}.$$

Thus,

$$Q_1(z) = 1 + b_1z + b_2z^2 + b_3z^3,$$

with

$$(b_1, b_2, b_3) = (-2a_0 - 2, 4a_0 + 1, 2l - 2a_0). \tag{23}$$

This time, by (9) and (22),

$$Q_2(z) = \frac{q_2(z)}{1-z} = -\frac{2F(z) + Q'(z)}{4}$$

is of degree at most 3. In fact,

$$Q_2(z) = a_0 + 2(a_1 - a_0^2 - a_0)z + (4a_0^2 + a_0 - 4a_1 + 3l)z^2 + 2(a_1 - a_0^2 - l)z^3. \tag{24}$$

So, from

$$Q_0(z)G'(z) + Q_1(z)G(z) = Q_2(z), \tag{25}$$

it follows that

$$(n+1)a_n + \sum_{j=1}^{3}(b_j + c_{j+1}(n-j))a_{n-j} + c_5(n-4)a_{n-4} = 0$$

for each $n \geq 4$. Inserting the values of c_j as above and b_j from (23), we arrive at the following recurrence relation:

Theorem 4. *If a_n, $n = 0, 1, 2, 3, \dots$ is the sequence (1) with $k = 0$, then*

$$\begin{aligned}(n+1)a_n &+ (-2(2a_0+1)n + 2a_0)a_{n-1} \\ &+ ((4a_0^2 + 8a_0 - 4a_1 + 1)n - (8a_0^2 + 12a_0 - 8a_1 + 1))a_{n-2} \\ &+ ((-8a_0^2 - 4a_0 + 8a_1 - 4l)n + (24a_0^2 + 10a_0 - 24a_1 + 14l))a_{n-3} \\ &+ 4(a_0^2 - a_1 + l)(n-4)a_{n-4} = 0\end{aligned}$$

for each $n \geq 4$.

Note that the recurrence relation of Theorem 4 proves the corresponding recurrence formulas in A176604–A176607, A176609, A176675, A176677 (the one proved in [11]), A176678, A176749–A176757, A177127–A177131, A177168–A177172.

We remark that in the case of A176675, with $(a_0, a_1, k, l) = (1, 0, 0, -1)$, Theorem 4 implies

$$(n+1)a_n + (2 - 6n)a_{n-1} + (13n - 21)a_{n-2} + 4(5 - 2n)a_{n-3} = 0 \tag{26}$$

for $n \geq 4$. Here, the last coefficient for a_{n-4} in Theorem 4 is $4(a_0^2 - a_1 + l)(n-4) = 0$. The recurrence (26) also holds for $n = 3$, because the polynomial $Q_2(z) = 1 - 4z + 2z^2$ in (25) (see (24)) is quadratic.

Moreover, if we consider the sequence (1) with $(a_0, a_1, k, l) = (l, l(l+1), 0, l)$, then Theorem 4 yields the recurrence relation

$$(n+1)a_n + 2(l - (2l+1)n)a_{n-1} + (4l+1)(n-1)a_{n-2} = 0$$

for each $n \geq 4$. This time, by (24), we have $Q_2(z) = l$, so the above relation holds for $n = 2$ and $n = 3$ as well.

5. Proof of Theorem 2

From (14), we deduce that $2z(1-z)G(z) - 1 + z$ is equal to $\pm\sqrt{Q(z)}$. By (6), we have $Q(0) = 1$, while $2z(1-z)G(z) - 1 + z$ at $z = 0$ is -1. So, for some $\delta_0 > 0$, in the open disc $|z| < \delta_0$ we must have

$$2z(1-z)G(z) - 1 + z = -\sqrt{Q(z)}.$$

This implies the next representation for the generating function

$$G(z) = \frac{1 - z - \sqrt{Q(z)}}{2z(1-z)} \tag{27}$$

for each $z \neq 0, 1$ satisfying $|z| < \delta_0$.

We claim the following:

Lemma 2. *The polynomial Q defined in (6) is the square of a polynomial if and only if $k = -a_1^2$ and $l = 2a_1^2 + a_1 - 2a_0a_1$.*

Proof. By (6), the coefficients of Q for $1, z, z^2, z^3, z^4$ are

$$1, \quad -2(2a_0+1), \quad 1 + 8a_0 + 4a_0^2 - 4a_1, \quad -4(a_0 + 2a_0^2 - 2a_1 + 2k + l), \quad 4(a_0^2 - a_1 + k + l),$$

respectively.

Assume that Q is the square of a polynomial, say R. From $Q(z) = R(z)^2$ and $Q(0) = 1$, we find that $R(0) = \pm 1$. Replacing R by $-R$, if necessary, we can assume that $R(0) = 1$. Next, by considering the coefficients for z, we immediately get $R(z) = 1 - (2a_0 + 1)z + \tau z^2$, with some constant τ. Considering the coefficients for z^2 in

$$Q(z) = (1 - (2a_0 + 1)z + \tau z^2)^2,$$

we deduce

$$2\tau + (2a_0 + 1)^2 = 1 + 8a_0 + 4a_0^2 - 4a_1,$$

so $\tau = 2(a_0 - a_1)$. Using this expression for τ and considering the coefficients for z^4, we find that

$$\tau^2 = 4a_0^2 + 4a_1^2 - 8a_0a_1 = 4a_0^2 - 4a_1 + 4k + 4l,$$

which implies

$$k + l = a_1^2 + a_1 - 2a_0a_1. \tag{28}$$

The coefficients for z^3 show that

$$-2\tau(2a_0 + 1) = -4(a_0 - a_1)(2a_0 + 1) = -4(a_0 + 2a_0^2 - 2a_1 + 2k + l).$$

This is equivalent to

$$2a_0^2 - 2a_0a_1 + a_0 - a_1 = a_0 + 2a_0^2 - 2a_1 + 2k + l,$$

and hence $2k + l = a_1 - 2a_0a_1$. Therefore, combining this equality with (28), we conclude that the only possibility is $k = -a_1^2$ and $l = 2a_1^2 + a_1 - 2a_0a_1$.

Conversely, with those values of k, l, the identity

$$Q(z) = (1 - (2a_0 + 1)z + 2(a_0 - a_1)z^2)^2 \tag{29}$$

holds. This concludes the proof of the lemma. □

We can now finish the proof of Theorem 2. As in [10] (see also Chapter 4 in [12]), we note that a_n, $n = 0, 1, 2, 3, \dots$, is a linear recurrence sequence only if its generating function $G(z) = \sum_{j=0}^{\infty} a_j z^j$ is a rational function, namely, it satisfies $G(z) = U(z)/V(z)$, where U, V are some polynomials. By (27), this is the case only if $\sqrt{Q(z)}$ is a rational function. The latter happens only if the polynomial Q is the square of a polynomial. By Lemma 2, this is never the case, unless $(k, l) = (-a_1^2, 2a_1^2 + a_1 - 2a_0a_1)$. But in that special case, using (29), we derive that

$$\begin{aligned} G(z) &= \frac{1 - z - 1 + (2a_0 + 1)z - 2(a_0 - a_1)z^2}{2z(1 - z)} = \frac{a_0 - a_0z + a_1z}{1 - z} \\ &= a_0 + \frac{a_1z}{1 - z} = a_0 + a_1z + a_1z^2 + a_1z^3 + \dots . \end{aligned}$$

Therefore, the sequence (1) is indeed $a_0, a_1, a_1, a_1, \dots$, as claimed.

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