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AROMATO INVARIANTAI GRIMUS NEUFELD MODELYJE

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FLAVOUR INVARIANTS IN THE GRIMUS-NEUFELD MODEL

MASTER’S THESIS

Theoretical Physics and Astrophysics study programme

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1 Introduction

Since the initial construction and success of the Standard Model (SM), particle physicists have theorized numerous new ways to expand on it. Despite its experimental predictions, the SM leaves several unanswered questions, suggesting incompleteness of the theory [1]. Some of these concern the observable flavour characteristics. The flavour in this context refers to the distinction between different particle generations. The various particles in the Standard model (or its theoretical expansions) will have a number equivalent particles, possessing different masses that correspond to the different flavour states of the field. The flavour study of a given model might focus on these masses, inter-flavour mixing and the emergent sources of charge-parity (CP) violation. The later refers to the combination of two transformations: charge conjugations (C) and parity (P), which, put together, are equivalent to the transformation between particles and antiparticles.

One way of expanding the SM is positing an increased amount of flavours for the fields that are already in the SM. In this work, we focus on such an extension called the Grimus-Neufeld model (GNM) [2]. Our choice of the model is motivated by it being the simplest case that includes two common methods of expanding the SM: the seesaw mechanism with Majorana neutrinos and additional Higgs doublets. The former assumes existence of some number of Majorana neutrinos with heavy masses. These couple with the SM neutrinos through Yukawa interactions, which in turn acquire masses that are inversely proportional to the heavy Majorana mass [3]. At the same time, increasing the number of Higgs doublets gives a significantly more complicated scalar potential and increases the number of possible Yukawa interactions. A case of such an expansion used in the GNM is the Two-Higgs-Doublet Model (2HDM) [4].

The flavour characteristic of any theory are described by the parameters in the couplings in a Lagrangian. These can often be absorbed by our ability to freely redefine field flavour basis via unitary transformations [5]. In certain cases, the physical meaning behind theory might suggest a seemingly natural choice of field definitions. For example, CP violation in SM quark sector is often described by formulating the theory within the quark mass basis as it conveniently reflects our physical interpretations [6].

If we want to move beyond the Standard Model, the lack of physical observations might prevent us from finding such a natural choice of basis. Thus, to be certain of our predictions (for example, an unambiguous existence of CP violation) the theory might need to be formulated in terms of its basis invariants [7]. The so called basis invariants emerge from the theory as particular combinations (structures) of the freely transforming elements that are found in the Lagrangian, be they fields or spurions (couplings transforming as fields). The composition, inter-dependencies and the overall set-structure of these invariants is itself a wide-ranging subject of the invariant theory in the field of mathematics.

One of the earlier uses of basis invariant compositions, in the description of physical phenomena, was the original derivation of Jarlskog (J-)invariant in the three generational quark model [6]. For a more extensive picture of a model, the entire set of linearly independent invariant quantities could be found. Such strategies are taken in [8] to discuss the phase invariants in the three generational seesaw model and in [5] to discuss the construction of basis independent J-invariants in the quark Yukawa sector.

For more complicated models the task of finding all the independent basis invariants is itself a non-trivial one. A systematic approach to this issue employs the help of Hilbert series (HS) [9], the plethystic exponential (PE) [10] and the plethystic logarithm (PL) [11]. The Hilbert series is a useful tool originating in the theory of invariants that describes the entire structure of basis invariants by a single polynomial expression. As long as we know the basic building blocks of a model, as well as the properties of their basis transformations we can construct HS for its invariants simply by using a PE-based generating function. Furthermore, the application of the PL helps us analyze the generated set of basis invariants by describing the main elements of the basis invariant set, as well as the relationships between them.

In recent years these methods were widely applied a variety of physical models. Most relevant for us are the applications in flavour physics. One of the earlier ones can be found in [9], where manually (without the PE) constructed Hilbert series were applied to examine the basis invariants in the SM quark sector and various seesaw extensions of the lepton Yukawa sector. Plethystics-based HS generation was later used to construct the SM quark sector invariant ring in [12]. Similarly, the leptonic Yukawa sector was examined in [13], by generating the HS for 1- and 2-Majorana neutrino seesaw models (the later being a minimal extension of the SM, consistent with the physically observable phenomena). Furthermore, especially relevant for us is the application of the HS and the plethystic methods in describing the scalar potential of 2HDM in terms of invariants by [7] and [14] as well as a more general application to multiple-Higgs doublet models in [15]. Since both 1-Majorana seesaw, as well as 2HDM models constitute the parts of the GNM, reproducing the results in these papers is one of our subgoals.

We apply the *DECO* package [16], developed to deal with the EFT HS calculations in [17] and [18] to describe some of the more complicated invariant structures. In fact, one of the biggest difficulties when dealing with HS generation are the taxing computations. The more complex models tend to yield extreme intermediary expressions when using the pletystics-based approach. Often, these expressions prove to be not handled particularly well by tools such as *Mathematica*. A good description of these difficulties can be found in [19].

In this paper, we seek to initiate the study of the Grimus-Neufeld model in terms of flavour invariants by finding and classifying the set of basis invariant constructions in the Yukawa and the scalar sectors of the GNM. We use the plethystic exponential to generate the Hilbert series for a number of subsections of the GNM as well as the full theory. Furthermore, for each of these we use the plethystic logarithm to analyze the complete sets of flavour invariants in terms of their generator basis. We take a gradual approach, examining the model in terms of its parts, verifying some of the results from previous research on the way. We also investigate the mixed invariants - ones containing inputs from both: Yukawa and Higgs sector. As far as we are aware, these mixed parts of the invariant structure were not described in such context previously. We see our work as a potential stepping stone for a broader examinations of the flavour invariants in the GNM as well as other theories, combining the seesaw and the multiple-Higgs doublet models.

2 The ring structure of basis invariants

We begin by explaining the concept of basis invariants in a more mathematical sense as well as what we mean by the word "structure" in this context. For both of these purposes we mostly refer to the book on invariant theory by Mukai [20]. There, an interested reader might also find a more extensive overview of the subject.

2.1 The transforming space

First, we describe some preliminary ideas. We start by defining a n-dimensional, complex vector space V. For now, we stay in the realm of mathematical abstraction. However, not to lose the attention of physicists, we mention that this vector space will later correspond to the flavour spaces of various elementary particles. Any element in V can be expressed as a certain sum of its basis elements: $v_1, ..., v_n$:

$$\text{Any } u \in V : \quad u = \sum_i c_i v_i, \text{ where } c_i \in \mathbb{C} \tag{1}$$

These basis elements are of the form:

$$v_1 = q_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad v_2 = q_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad v_3 = q_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \quad \ldots \tag{2}$$

Here, we separated complex arbitrary variables q_i, from the orthogonal basis elements, which we will mark as e_i:

$$v_i = q_i e_i \tag{3}$$

Now, we can classify all the vectors in the space V in terms of the elements q_i, by defining a map p:

$$\begin{aligned} p : \quad V &\twoheadrightarrow Q^1 \\ e_i &\mapsto 1 \end{aligned} \tag{4}$$

Here, we define Q^1 as a subset of first order elements in a complex polynomial ring Q, generated by the variables q_i:

$$Q = \mathbb{C}[q_1, \ldots, q_n]. \tag{5}$$

For some physical context, the polynomial ring Q might correspond to the Lagrangian parameter space. An arbitrary vector $u \in V$, is mapped to it as:

$$u = \sum_i c_i v_i \mapsto \sum_i c_i q_i \in Q \tag{6}$$

With the use of tensor multiplication, we might also construct higher order spaces:

$$V^k = \overbrace{V \otimes V \otimes \cdots}^{k} \tag{7}$$

The basis elements of V^k, which we will mark as $v_j^{(k)}$ will be the tensor products of k original elements $q_i e_i$:

$$v_j^{(k)} = q_j^{(k)} e_j^{(k)} \in \{\bigotimes_i (q_i e_i)^{\otimes l_i} \mid \sum_i l_i = k,\ i = 1, ..., n\}, \text{ with } j = 1, \dots n^k. \tag{8}$$

Using a similar procedure as in eq.(4), we can define a mapping of V^k to a k-order subring $Q^k \subset Q$, where:

$$Q^k \subset \mathbb{C}[q_1^{(k)}, \dots, q_{n^k}^{(k)}] \subset Q \tag{9}$$

For example, if V is a 2 dimensional vector space, generated by basis elements v_1 and v_2, we can construct a second order space $V^2 = V \otimes V$, with the basis elements:

$$v_j^{(2)} = q_j^{(2)} e_j^{(2)} \in \{q_1^2(e_1^{\otimes 2}), q_1 q_2(e_1 \otimes e_2), q_2 q_1(e_2 \otimes e_1), q_2^2(e_2^{\otimes 2})\}, \text{ with } j = 1, 2, 3, 4. \tag{10}$$

Which we can map on a polynomial subring:

$$Q^2 \subset \mathbb{C}[q_1^2, q_1 q_2, q_2^2] \subset Q \tag{11}$$

All vectors in each of these spaces can be written as algebraic combinations (additions, tensor products and multiplications by a complex constant) of the basis elements. Hence, we can define the set of vectors in a unified space:

$$\bar{V} = \bigcup_{k=0}^{\infty} V^k \tag{12}$$

as a set of all additive and/or multiplicative combinations of v_i. In other words, the unified space $\bar{V}$ is a complex ring, generated by the basis elements of V:

$$\bar{V} = \mathbb{C}[v_1, ..., v_n] \tag{13}$$

Applying p from eq.(4) surjectively maps it to the ring Q, meaning that every element in Q classifies at least one element in $\bar{V}$.

Now, a particular theory might allow us to transform the basis elements of V in accordance to the symmetries of some group G. More specifically, we might act on the basis elements v_i with the group representations $\rho(G)$, where:

$$\rho(G) : G \underset{\sim}{\rightarrow} GL(n) \tag{14}$$

The action of the representation of a group G on the vector space V is a mapping:

$$\begin{aligned} \rho(G)V: \quad (\rho(G), V) &\to V \\ v_i &\mapsto \rho(g)v_i \end{aligned} \tag{15}$$

We say that the elements of the space V transform as n-dimensional representation of the group G. The higher order spaces V^k are also transformed by this group action, such that:

$$\begin{aligned} \rho(G)V^k: \quad (\rho(G), V^k) &\to V^k \\ v_i^{(k)} &\mapsto v_i^{(k)} \prod_{i=1}^{k} \rho(g). \end{aligned} \tag{16}$$

At the same time, such an action will impact the elements of a polynomial ring Q:

$$\begin{aligned} \rho(G)Q: \quad (\rho(G), Q) &\to Q \\ q_i &\mapsto (\rho(g)\vec{q})_i \end{aligned} \tag{17}$$

Where, $\vec{q}$ is defined as:

$$\vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \tag{18}$$

Again, we can use an example with 2-dimensional space. Lets take the group action of $\mathbb{Z}_2$, containing the elements $\{e, x\}$, such that:

$$ee = xx = e, \quad ex = xe = x. \tag{19}$$

represented by the (2×2) matrices:

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{20}$$

The action of the representation of $\mathbb{Z}_2$ on the vector space V is defined by:

$$\rho(e)v_i = v_i, \quad \rho(x)v_1 = q_1(\rho(x)e_1) = q_1e_2, \quad \rho(x)v_2 = q_2e_1. \tag{21}$$

In this case, the action of the group element x impacts Q by exchanging the variables $q_1 \leftrightarrow q_2$.

2.2 The basis invariants

A set of g-invariants is the set of elements in the ring Q that remain unchanged by a basis transformation $\rho(g)$:

$$Q^g = \{I^g \in Q \mid \rho(g)I^g = I^g, \text{for } g \in G\}. \tag{22}$$

Basis invariance is a more constricted concept of g-invariance, where we might define a basis invariant I as an element of Q that remains unchanged under basis transformations for all g. Again, the set of basis invariants is the set of all elements $I \in Q$, such that they remain unchanged under any allowed basis transformation of V:

$$Q^G = \{I \in Q \mid \rho(g)I = I, \text{for all } g \in G\}. \tag{23}$$

Alternatively, we can define it as an intersection of all possible g-invariants:

$$Q^G = \bigcap_{g\in G} Q^g. \tag{24}$$

Furthermore, all algebraic combinations of basis invariants correspond to another basis invariant. Hence, the set of all basis invariants is a subring (or a subspace) of all elements of Q:

$$Q^G \subset \mathbb{C}[q_1, ..., q_n]. \tag{25}$$

Now we define some important ideas that are used when describing the structure of invariant rings. Two invariants are said to be algebraically dependent if there is a non-trivial polynomial relationship between them, where [7]:

$$P(I_a, I_b, ...) = 0. \tag{26}$$

Here $P(...)$ is a polynomial. Linear dependence is an equivalent notion, except with the polynomial relationship (eq. (26)) being exclusively linear (of polynomial order 1). The algebraic relationship between invariants (such as in eq. (26)) are called syzygies.

Lastly, the minimal set of invariants that, through algebraic combinations, can define all other invariants is called the generating set and its elements - generators. These correspond to an important notion of ring theory with the same name. Importantly, the generating invariants do not have to be algebraically independent (the reason for this is that the ring structure does not necessarily include the operation of division).

Extending our previous example, we can define the invariants of the 2-dimensional space V that transforms under the representation of $\mathbb{Z}_2$. These are the set of polynomials in Q that remain unchanged under $q_1 \leftrightarrow q_2$. Hence the $\mathbb{Z}_2$-invariant ring can be defined thusly:

$$Q^{\mathbb{Z}_2} = \mathbb{C}[q_1 + q_2, q_1q_2] \tag{27}$$

with, $q_1 + q_2$ and q_1q_2 being its generating invariants. Since we cannot construct any polynomial relationships between these, there are no syzygies in this example and thus both generators are algebraically independent.

To answer the second question, posed at the beginning, the word "structure" refers precisely to the constitution of the ring of invariants. Before moving on we have to discuss the main aspects of what we wish to understand about it. First we specify that we want to categorize the ring of invariants only up to a linear dependency. This means that any invariant that can be expressed as a linear function of another invariant is considered as being the same invariant:

$$I \equiv a \cdot I + b, \text{ for any } a, b \in \mathbb{C}. \tag{28}$$

When describing the invariant rings it is important to find the set of generating invariants and answer which of them are algebraically independent. In a more physical interpretation, these correspond to the parameters of the model. Finally, when describing rings of basis invariants it is also useful finding and categorize the syzygies between these generating invariants. The methods that we use are the subject of the rest of this section.

2.3 Hilbert-Poincaré series

In the study of invariants, one often encounters a mathematical tool called the Hilbert-Poincaré series (also Weyl or Molien series). To save some space, we will refer to it simply as the Hilbert series (HS). It is defined as an element of the invariant ring $\mathbb{C}[q_1, ..., q_n]^G$ that contains every other element (classified up to linear dependence) within it. In other words, it is a polynomial that we obtain by adding every single monomial term in Q^G. Thus, we define the multi-graded Hilbert series as:

$$\mathfrak{H}(q_1, ..., q_n) = \sum_{k=0}^{\infty} f_k(q_1, ..., q_n), \tag{29}$$

where f_k are the sums of all homogeneous invariant polynomials of its variables of degree k. We can set all variables to be equal (up to a degree):

$$\{q_1, .., q_n\} \rightarrow \{q^{w_1}, ..., q^{w_n}\}, \tag{30}$$

where w_i are the arbitrarily chosen weights of each element. This way we define the mono-graded Hilbert series as:

$$\mathfrak{H}(q) = \sum_{k=0}^{\infty} |Q^G \cap Q^k| q^k. \tag{31}$$

It is a polynomial in the elements of subspace $Q_{\vec{q}\rightarrow q}$, where the constants denote the number of invariant elements in the subspace Q^k: $|Q^G \cap Q^k|$. Note that HS is an infinite polynomial function. However, if the ring of invariants is finitely generated, we can write down the HS in terms of its generating invariants:

$$\mathfrak{H}(q_1, ..., q_n) = \prod_{l}^{m} (1 + I_l^0(q_1, ..., q_n) + I_l^0(q_1, ..., q_n)^2 + ...). \tag{32}$$

Here, I_l^0 are these generators, written as specific monomials in basis elements q_i. It should be clear that eq. (32) indeed includes all of their possible combinations. Then, using Taylor expansion:

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + ..., \tag{33}$$

we can rewrite eq. (32) in a more compact manner:

$$\mathfrak{H}(q_1, ..., q_n) = \frac{1}{\prod_l (1 - I_l^0(q_1, ..., q_n))}. \tag{34}$$

Rewriting the generating invariants as monomials, we get a general case of finitely generated HS of the form:

$$\mathfrak{H}(q_1, ..., q_n) = \frac{1}{\prod_j^m (1 - \prod_{i=1}^n q_i^{l_{ij}})^{d_j}}. \tag{35}$$

Here, the numbers $l_{ij} \in \mathbb{Z}$ describe the composition of the generating invariants, while d_j describes the number of generators of the specific composition. Note that the latter does not need to be positive

[11]. In fact, it being negative implies that it corresponds to an algebraically dependent generator. Hence, by putting some factors of $(1 - \dots)$ in the numerator, we get the standard form for HS:

$$\mathfrak{H}(q_1, \dots, q_n) = \frac{F(q_1, \dots, q_n)}{\prod_j^p (1 - I_j^0(q_1, \dots, q_n))}. \tag{36}$$

In this form, the generators in the denominator I_j^0 are all algebraically independent and thus correspond to the (physical) parameters of the model. These, are called the primary generating invariants. We denote their number as p. The numerator is a polynomial that includes the algebraically dependent generators. These are called the secondary generators. Writing HS in this form automatically accounts for terms that are made redundant by syzygies among the generating invariants [11]. Rewriting this in mono-graded form we get:

$$\mathfrak{H}(q) = \frac{F(q)}{\prod_j^p (1 - q^{l_j})}. \tag{37}$$

The interpretation is the same, except we can also see some additional characteristics [9]. First, the polynomial $F(q)$ is palindromic:

$$F(q) = q^{l_F} F\left(\frac{1}{q}\right), \tag{38}$$

where l_F is the degree of $F(q)$. Second, the condition:

$$p \leq \sum_j^p l_j - l_F \leq \dim V, \tag{39}$$

has to be satisfied [9]. These characteristics can be used to check the accuracy of a constructed HS.

Knowing the HS in the form eq. (37) and expanding it lets us easily obtain any invariant in a particular model and automatically account for relations between them. Furthermore, even after abstracting certain information (by turning to the mono-graded description) we can use HS to describe certain aspects of the invariant ring (for example, counting the linearly independent invariants of any order).

However, calculating the Hilbert series for a specific model can be rather cumbersome. A case in point, taking what we might call the "hands-on" approach (by manually constructing all the generators and checking for their dependencies) was found to be untenable for more complex theories. Such was the conclusion reached by [9], when they attempted to count the flavour invariants of 3-Majorana neutrino seesaw model. Therefore, before making any use of HS we need a systematic way to derive it in the form presented in eq. (37). The two following subsections outline precisely such a method.

2.4 Molien theorem

First, we discuss the case with finite groups, intending to later expand the method to infinite Lie groups. Again, in this section we base most of our discussion on [20] as well as [21]. We define the averaging operator for any element $q \in Q^k$:

$$E(q) = \frac{1}{|G|} \sum_{g \in G} \rho(g) q. \tag{40}$$

Here $|G|$ is the magnitude of the group. An important quality of this operator can be shown by acting on it with any arbitrary element of the representation of G:

$$\rho(g_2) E(q) = \frac{1}{|G|} \sum_{g_1 \in G} \rho(g_2) \rho(g_1) q = \frac{1}{|G|} \sum_{g \in G} \rho(g) q. \tag{41}$$

Since it remains unchanged, it is clear, that an element acted with this averaging operator becomes part of the invariant subspace $E(q) \in Q_k^G$. In other words, the averaging operator E projects the entire space Q^k to the subspace of invariants Q_k^G. Therefore it also satisfies the characteristics of a projection operator:

$$\begin{aligned} E(E(q)) &= \frac{1}{|G|} \sum_{g_2 \in G} \rho(g_2) \frac{1}{|G|} \sum_{g_1 \in G} \rho(g_1) q = \frac{1}{|G|^2} \sum_{g_2 \in G} \sum_{g_1 \in G} \rho(g_2 g_1) q \\ &= \frac{1}{|G|^2} \sum_{g_2 \in G} \sum_{g \in G} \rho(g) q = \frac{|G|}{|G|^2} \sum_{g \in G} \rho(g) q = \frac{1}{|G|} \sum_{g \in G} \rho(g) q = E(q). \end{aligned} \tag{42}$$

Here, between the second and the third lines we applied an automorphism $G \to G : g_1 \to g_2 g_1 = g$. This property guarantees that the eigenvalues of operator E is either 1 or 0:

$$E^2 q = \lambda^2 q = \lambda q = E q : \quad \lambda = 0, 1. \tag{43}$$

The operator E decomposes the space Q^k into a direct sum of its kernel and image:

$$Q^k \cong \ker E \oplus \operatorname{im} E = \ker E \oplus Q_k^G \tag{44}$$

The eigenvalues that are equal to 1 are all associated with the image of the operator ($\operatorname{im} E$) while the ones, equal to 0 are associated with $\ker E$ [21]. Hence, the sum of all eigenvalues of E are equal to the dimension of the invariant subspace Q_k^G:

$$\operatorname{Tr} E = \operatorname{Tr} \left(\frac{1}{|G|} \sum_{g \in G} \rho(g) \right) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} \rho(g) = \dim Q_k^G. \tag{45}$$

The trace over the representation of a group:

$$\operatorname{Tr} \rho(g) \equiv \chi[\rho(g)], \tag{46}$$

is called the character of representation. As a result, we can express the Hilbert series via the formula:

$$\mathfrak{H} = \sum_{k=0} \frac{1}{|G|} \sum_{g \in G} \chi[\rho(g)^k] q^k, \text{ where } q \in Q. \tag{47}$$

Within the context of this subsection, it should be clear that the expressions in eq. (31) and eq. (47) are equivalent. Furthermore, we can use the following identity [21]:

$$\sum_{k=0} \chi[\rho(g)^k] q^k = \frac{1}{\det(\mathrm{id} - q\rho(g))}, \tag{48}$$

to finally derive the so-called Molien theorem:

$$\mathfrak{H}(q) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathrm{id} - q\rho(g))}. \tag{49}$$

This expression is an important stepping stone. It allows us to systematically generate the invariants comprising HS just by knowing the properties of the group G. However, in this form, it is still not very applicable to our case. The main problem comes from the fact that it can only be used with finite groups. In the next subsection we show how this problem can be addressed and derive the final form of HS generating function that is used in this paper.

2.4.1 The plethystic exponential and Hilbert series generating function

Applying the Molien formula in eq. (49) to infinite Lie groups amounts to to taking a continuous integral over the parameter space of G instead of a discrete sum of the elements of G:

$$\frac{1}{|G|} \sum_{g \in G} \to \oint dg. \tag{50}$$

However, before discussing the details of that, we should take care of the determinant in eq. (49), since continuously integrating it could prove to be problematic. For this purpose, we can use the relation:

$$\log \det A = \operatorname{Tr} \log A. \tag{51}$$

It allows us to perform the following change to eq. (49):

$$\mathfrak{H}(q, G) = \frac{1}{|G|} \sum_{g \in G} \exp\left(-\operatorname{Tr} \log(1 - q\rho(g)) \right). \tag{52}$$

Now we can Taylor expand the logarithm:

$$\log(1 - A) = -\sum_{r=1}^{\infty} \frac{A^r}{r}, \quad \log(1 + A) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{A^r}{r}, \tag{53}$$

and arrive at:

$$\mathfrak{H}(q, G) = \frac{1}{|G|} \sum_{g \in G} \exp\left(\operatorname{Tr} \sum_{r=1}^{\infty} \frac{(q\rho(g))^r}{r} \right) = \frac{1}{|G|} \sum_{g \in G} \exp\left(\sum_{r=1}^{\infty} \frac{q^r}{r} \operatorname{Tr} \rho(g)^r \right). \tag{54}$$

Again, the trace over the representation of a group is the character $\chi[\rho(g)]$. The exponential in eq. (54) is called the plethystic exponential:

$$\mathrm{PE}\big[q, \rho(g)\big] = \exp\left(\sum_{r=1}^{\infty} \frac{q^r}{r} \chi[\rho(g)^r]\right). \tag{55}$$

Now, the character of representation is invariant to the action of conjugation by the group elements:

$$\chi[\rho(g)] \to \chi[\rho(hgh^{-1})] = \chi[\rho(h)\rho(g)\rho(h^{-1})] = \chi[\rho(g)\rho(h^{-1})\rho(h)] = \chi[\rho(g)]. \tag{56}$$

Hence, instead of working with the entire group, we can instead represent the group actions in eq. (55) with actions of its maximal torus $T \subset G$. This, being a subgroup, where every element corresponds to a unique conjugacy class. The representations of these are sets of diagonal matrices. For example, the $SU(2)$ maximal torus is in the dimension-2 representation is the set of (2×2) matrices, taking the form:

$$\rho(t(x) \in T) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \tag{57}$$

where x is a complex parameter spanning the values of the unitary circle. Taking its trace gives a polynomial:

$$\chi_2^{SU(2)}(x) = x + \frac{1}{x}. \tag{58}$$

Hence, we reduce the problem of integrating all elements in an infinite group to one of taking the integral through a unitary circle, in terms of parameters x_j (the index indices that the different groups might be represented by different number of parameters). The expression of the PE is therefore:

$$\mathrm{PE}(q, \chi_{\mathrm{R}}^G) = \exp\left(\sum_{r=1}^{\infty} \frac{q^r \chi_{\mathrm{R}}^G(x_j^r)}{r}\right) \tag{59}$$

However, the conjugacy classes of a given group are likely to have different sizes. Taking this into account, we add a weight function to the integral, called the Haar measure:

$$\lim_{|G|\to\infty} \frac{w_t}{|G|} \equiv d\mu(x_j) \tag{60}$$

Finally, taking an integral over x_j, we get the Hilbert series generating function for continuous Lie groups:

$$\mathfrak{H}(q, \chi_{\mathrm{R}}^G) = \oint_{|x_j|=1} d\mu(x_j)\ \mathrm{PE}(q, \chi_{\mathrm{R}}^G). \tag{61}$$

We list the expressions for the characters and the Haar measures of some relevant groups and their representations in the Appendix.A

2.5 Plethystic logarithm and the algebraic dependence of invariants

Another useful tool in the study of the rings of invariants is the plathystic logarithm (PL). It is an inverse function of the plethystic exponential ($\text{PL} = \text{PE}^{-1}$). It is defined as follows [11]:

$$\text{PL}\big(\mathfrak{H}(q)\big) = \sum_r \frac{\mu(r)}{r} \log\big(\mathfrak{H}(q^r)\big), \tag{62}$$

where $\mu(r)$ is the Möbius function:

$$\mu(r) = \begin{cases} 0, & r \text{ has at least one repeated prime factors} \\ 1, & r = 1 \\ (-1)^n, & r \text{ is a product of } n \text{ distinct primes} \end{cases}. \tag{63}$$

The input of PL is the Hilbert series, and the output is a series of the form:

$$\text{PL}(\mathfrak{H}(q_1, ..., q_n)) = \sum_j d_j \left(\prod_{i=1}^{n} q_i^{l_{ij}} \right), \tag{64}$$

where the product in the brackets represents an invariant composition and the coefficient d_j is the same as the powers in eq. (35). A positive d_j contributes a factor in the denominator of HS, while a negative contributes a factor in the numerator. However, the numerator of HS consists of secondary invariants that are algebraically dependent on the ones in the denominator via syzygies. Hence, the usual interpretation of the results of the PL calculations is as follows: the positive coefficients show how many basis elements of a certain composition there are in the V^G, while the negative coefficients correspond to the number of syzygies of the respective compositions [11]. This reading is entirely applicable when the series expansion of the PL is a finite polynomial. A mono-graded form of the PLs are mainly used to count the total number generating invariants, as well as the syzygies between them. The multi-graded version clarifies the composition of the said invariants and syzygies. Also, finding the PL might allow reconstructing HS in their rational form (by using eq. (35)). However, in many cases, the PLs yield infinite polynomials. It does not mean that the ring of invariants has an infinite basis, only that the numerator in the rational form of HS eq. (36) cannot be finitely factored into polynomials of the form $(1 - q^j)^{d_j}$. The interpretation of infinite mono-graded PL polynomials is slightly different:

1. The lower order d_j coefficients in eq. (64) of a non-terminating mono-graded PL show the difference between the number of generating invariants and the number of syzygies of the corresponding degree. In terminating PLs, the generator and syzygy terms appear at different orders and thus their number is identical to the difference between the two. At non-terminating PLs, these might overlap, thus complicating the counting procedure.

2. Furthermore, at higher orders, the infinite PL expansion might contain additional sets of positive terms. These, however, should not be interpreted as additional generators, but instead as representing the higher order syzygies between syzygies [22]. Then again, even higher order

negative terms represent the relationships between these, and so on *ad infinitum*.

An issue caused by such behaviour of the infinite PL series is that we cannot naively use the mono-graded PL coefficients to count the number of ring generators. Instead, we should look at the multi-graded version, where the generators and syzygies of the same order are separated by different compositions. Then, the total number of generators and syzygies can be found by separating the positive and negative terms and afterwards applying the mono-grading of the series. This way, the generators and the syzygies of the same degree do not cancel out. This method does not solve this issue completely, as in some cases the grading structure of a syzygy corresponds to the grading structure of a generator, thus retaining the said canceling behaviour. Regardless, we can still use it in many cases to arrive at a more-correct version of the PL. Another, more complicated issue is the difficulty of separating the positive terms, representing the generators, and the positive terms representing the higher order positive terms, representing the syzygies between syzygies. To do this, we follow the conjecture, proposed by [13] and discussed in [22]. It posits the separation of the two categories of positive terms by "pure negative" orders (the multi-graded PL orders that contain only negative terms).

3 The Grimus-Neufeld model

The Grimus-Neufeld model (GNM) combines two common methods of supplementing the SM: a single Majorana neutrino and the two-Higgs doublet model (2HDM). This can be used to extend the formulation of the theory to include the mass generation for two SM neutrinos: one via the seesaw mechanism and another via radiative one-loop corrections (due to interactions with the second Higgs doublet [2]). The resulting theory has two physical light neutrino masses, which are consistent with the observations of neutrino mass hierarchies [23]. Furthermore, the theory provides potential new sources for CP violation (in both 2HDM potential and the Yukawa sector) [24]. We dedicate this section of our paper to specify the various aspects of the GNM.

3.1 Two Higgs doublet model

The general 2HDM potential is [14]:

$$V_{2HDM} = \phi_a^\dagger (m^2)^a_b \phi^b + \phi_a^\dagger \phi_b^\dagger Z^{ab}_{cd} \phi^c \phi^d; \quad a, b, c, d = 1, 2. \tag{65}$$

The coupling parameters here are not all independent:

$$(m^2)^a_b = \left((m^2)^b_a\right)^*; \quad Z^{ab}_{cd} = \left(Z^{cd}_{ab}\right)^*; \quad Z^{ab}_{cd} = Z^{ba}_{dc} \tag{66}$$

To account for these interrelations between the couplings, we can rewrite the potential as [14]:

$$\begin{aligned} V_{2HDM} = \quad & m_{11}^2 \phi_1 \phi_1^\dagger + m_{22}^2 \phi_2 \phi_2^\dagger + m_{12}^2 \phi_1 \phi_2^\dagger + m_{21}^2 \phi_2 \phi_1^\dagger \\ & + \lambda_1 (\phi_1 \phi_1^\dagger)^2 + \lambda_2 (\phi_2 \phi_2^\dagger)^2 \\ & + \lambda_3 (\phi_1 \phi_1^\dagger)(\phi_2 \phi_2^\dagger) + \lambda_4 (\phi_1 \phi_2^\dagger)(\phi_2 \phi_1^\dagger) \\ & + \left[\frac{1}{2}\lambda_5 (\phi_1 \phi_2^\dagger)^2 + \left(\lambda_6 (\phi_1 \phi_1^\dagger) + \lambda_7 (\phi_2 \phi_2^\dagger)\right)(\phi_1 \phi_2^\dagger) + h.c.\right] \end{aligned} \tag{67}$$

The square brackets distinguish CP-violating parameters.

The description of flavour invariants in 2HDM was done by [7] and [14] using the same methodology of HS generation. However, their analyses were confined exclusively to the scalar potential and did not account for any effects from the Yukawa sector. As a sub-goal of this paper, we hope to contribute to this part of the analysis of 2HDM flavour invariants.

3.2 Seesaw mechanism

The seesaw mechanism is one of the most commonly used mechanisms to explain the presence of light neutrino masses in experimental observations [23]. A brief description of the seesaw mechanism can be found in [3]. In short, the mechanism assumes the existence of some number of heavy Majorana neutrinos that couple to the SM neutrinos through the Yukawa interaction terms.

The Lagrangian of n-flavor lepton sector is therefore [2]:

$$\mathcal{L} = \mathcal{K} - (Y_E^a)^{ij}(L_i^\dagger \phi_a)E_j - (Y_N^a)^{ij}(L_i^\dagger i\sigma^2 \phi_a^*)N_j - \frac{1}{2}M^{ij}N_i^T N_j + h.c. \tag{68}$$

Here $\mathcal{K}$ is the kinetic part of the Lagrangian, while Y and M are the couplings between the fields, the latter being the Majorana masses of the heavy neutrinos. The fields of the model include: standard left-handed lepton $SU(2)_L$ doublet:

$$L_i = \begin{pmatrix} e_i \\ \nu_i \end{pmatrix}, \tag{69}$$

with flavour indices $i, j \leq n$, Higgs doublets:

$$\phi_a = \begin{pmatrix} \phi_a^+ \\ \phi_a^0 \end{pmatrix} = \begin{pmatrix} \phi_a^+ \\ \frac{1}{\sqrt{2}}(v_a + H_a + iA_a) \end{pmatrix}, \tag{70}$$

where a denotes the Higgs field index in multiple-Higgs field models, as well as $SU(2)$ singlet electron and neutrino fields: E and N, respectively.

At low energies ($\mathcal{K}_N << M$), the kinetic term of a Majorana neutrino field can be neglected due to its high mass:

$$\mathcal{L}_N = -Y_N^{ij}(L_i^\dagger i\sigma^2 \phi^*)N_j + \frac{1}{2}M^{ij}N_i^T N_j + h.c. \tag{71}$$

Doing so, results in an effective Yukawa Lagrangian:

$$\mathcal{L} = -Y_E^{ij}(L_i^\dagger i\sigma^2 \phi)E_j - \frac{1}{2}(L_i^\dagger i\sigma^2 \phi^*)(C_5)^{ij}(\phi^* i\sigma^2 L_j^\dagger) + h.c. \tag{72}$$

Here $C_5 = \frac{Y_N Y_N^T}{M}$ is an effective order 5 (dimension $[mass^{-1}]$) coupling that is the source of SM neutrino masses $m_\nu = \frac{Y_N^2 v^2}{M}$. This being inversely proportional to the heavy Majorana mass ensures that SM neutrinos are much lighter than their counterparts - hence the name "seesaw".

In the GNM, only one Majorana neutrino is added, and thus the seesaw mechanism provides mass to only one SM neutrino. This, however, is contradictory to the observed neutrino oscillations, since they require the existence of at least two SM neutrino masses. This problem is solved in the GNM by combining the two previously described extensions. This, however, requires one-loop corrections to the theory [2].

3.3 The flavour transformations in GNM

From eq (65) and eq. (71) the GNM lepton sector has 4 fields: a left handed fermion doublet L, a right handed charged fermion singlet E, a Majorana fermion singlet N, and a two-flavour scalar Higgs doublet ϕ. Each of these fields possesses flavour transformations. We list these in Table.1 We might use these field transformations to describe the flavour characteristics of the couplings in the model Table.(2). It should also be kept in mind that the full description includes the hermitian conjugates of these couplings, which transform in opposite ways. We can simplify the upcoming calculations by making certain changes to how we describe the GNM flavour space. Lets start with

Field	Flavour symmetries	Flavour transformations
L_i	$U(3)$	$L_i \to L_j \mathcal{U}^j_i; \quad i,j = 1,2,3$
E_i	$U(3)$	$E_{\bar{i}} \to E_{\bar{j}} \mathcal{U}'^{\bar{j}}_{\bar{i}}; \quad \bar{i},\bar{j} = 1,2,3$
N	$U(1)$	$N \to N\eta; \quad \eta = e^{i\theta_N}$
ϕ_a	$U(2)$	$\phi_a \to \phi_b \mathcal{R}^b_a; \quad a,b = 1,2$

1 table The GNM fields and their flavour symmetries.

Coupling	Flavour basis transformations	Representations in $U(3)_E \times U(3)_L \times U(2)_\phi \times U(1)_N$
Y_E	$(\mathbf{Y_E})^{\mathbf{a}}_{\mathbf{i\bar{i}}} \to (\mathcal{U}^\dagger)^j_i \mathcal{R}^a_b (\mathbf{Y_E})^{\mathbf{b}}_{\mathbf{j\bar{j}}} \mathcal{U}'^{\bar{j}}_{\bar{i}}$	$(3, \bar{3}, 2, 0)$
Y_N	$(\mathbf{Y_N})^{\mathbf{a}}_{\mathbf{i}} \to (\mathcal{U}^\dagger)^j_i (\mathcal{R}^*)^a_b (\mathbf{Y_N})^{\mathbf{b}}_{\mathbf{j}} \eta$	$(0, \bar{3}, \bar{2}, 1)$
M	$\mathbf{M} \to \eta \mathbf{M} \eta$	$(0, 0, 0, 2)$
m^2	$(\mathbf{m^2})^{\mathbf{a}}_{\mathbf{b}} \to (\mathcal{R}^\dagger)^a_c (\mathbf{m^2})^{\mathbf{c}}_{\mathbf{d}} \mathcal{R}^d_b$	$(0, 0, \bar{2} \otimes 2, 0)$
Z	$Z^{ab}_{cd} \to (\mathcal{R}^\dagger)^a_e (\mathcal{R}^\dagger)^b_f \mathbf{Z}^{\mathbf{ef}}_{\mathbf{gh}} \mathcal{R}^g_c \mathcal{R}^h_d$	$(0, 0, \bar{2} \otimes \bar{2} \otimes 2 \otimes 2, 0)$

2 table The GNM couplings and their transformations. The first three numbers in the brackets show the dimension of representations of the corresponding groups (with bar denoting anti-fundamental representation), while the last number shows the charge of the $U(1)_N$ group representation. When using mono-graded description of the invariant ring, we prescribe the weights by the measure of these couplings (meaning that in this table, all couplings have $w = 1$ as written in eq. (**??**).

the simplest coupling M. It is essentially just a scalar. In any invariant construction, it contributes only a scalar factor. However, it is not invariant to the unitary $U(1)_M$ rephasings. This can be solved simply by multiplying it by its hermitian (complex) conjugate, thus giving us an invariant:

$$M_0^2 = MM^* \to M(\eta\eta^*)^2 M^* = MM^* = M_0^2 \tag{73}$$

This reduces the amount of calculations needed. Furthermore, dealing with $SU(n)$ groups is somewhat more convenient than $U(n)$, since these have one fewer integration parameter. Thus, we might reduce the unitary group symmetries of couplings into direct products of special unitary groups and $U(1)$, by using the isomorphism relation [25]:

$$U(n) \cong \big(SU(n) \times U(1)\big)/\mathbb{Z}_n. \tag{74}$$

Hence, we can change the description of the theory to:

$$SU(3)_L \times SU(2)_\phi \times U(1)_M \times U(1)_L \times U(1)_\phi. \tag{75}$$

As for the invariants themselves, we specify the notation as:

$$I^{(i)}_{a,b,c,d,e} \equiv (Y_E \otimes Y_E^\dagger)^{\otimes a} \otimes (Y_N \otimes Y_N^\dagger)^{\otimes b} \otimes (m^2_{(3)})^{\otimes c} \otimes Z^{\otimes d}_{(3)} \otimes Z^{\otimes e}_{(5)} \tag{76}$$

where i accounts for distinct invariants of equivalent coupling composition. The actual invariants are found by taking separate traces over the indices of each group in these compositions. That way, the transformations in Table. 2 cancel out, giving the flavour invariants.

3.3.1 The GNM flavour space building blocks - lepton sector

The above description is sufficient to start calculating HS for the GNM flavour space. However, for further analysis, it is useful to redefine the flavour space in terms of basis building blocks that provide more information about the underlying structure of the basis invariants composed of them. We mostly do this by constructing adjoint representations out of the couplings in the previous section. First, we see that among all the described elements, only Y_E contains indices of $U(3)_E$ symmetry. This means that any invariant constructed out of it includes a contraction of Y_E with $Y_E^\dagger$ through the barred indices:

$$\mathcal{Y}_E = \langle Y_E Y_E^\dagger \rangle_{U(3)_E} = (Y_E)_{\bar{i}} (Y_E^\dagger)^{\bar{i}} \xrightarrow{U(3)_E} (Y_E)_{\bar{j}} \mathcal{U}'^{\bar{j}}_{\bar{i}} (\mathcal{U}'^\dagger)^{\bar{i}}_{\bar{k}} (Y_E^\dagger)^{\bar{k}} = \delta^{\bar{j}}_{\bar{k}} (Y_E)_{\bar{j}} (Y_E^\dagger)^{\bar{k}} = \mathcal{Y}_E. \tag{77}$$

The $\langle \ldots \rangle_G$ notation here means symmetric contraction over the indices of G. While it usually corresponds to a trace over the group indices, it can also mean an antisymmetric contraction with the Levi-Civita symbol. In those cases, we will add clarifications. The contracted coupling $\mathcal{Y}_E$ is only a $U(3)_E$ invariant. It still transforms with uncontracted indices of $U(3)_L$ and $U(2)_\phi$:

$$[\mathcal{Y}_E]^{ia}_{jb} \rightarrow (\mathcal{U}^\dagger)^i_k \mathcal{R}^a_c [\mathcal{Y}_E]^{kc}_{ld} (\mathcal{R}^\dagger)^d_b \mathcal{U}^l_j. \tag{78}$$

In other terms, the coupling $\mathcal{Y}_E$ is a representation $(0, \bar{3} \otimes 3, 2 \otimes \bar{2}, 0)$ of the flavour groups.

Next, we can decompose the couplings into irreducible representations of the respective groups. We can use the known relations of Clebsch-Gordan decompositions that we might find in [15]. The ones we need for our purpose are:

$$\begin{aligned} SU(2): &\quad 2 \otimes 2 = 1 \oplus 3, \\ SU(3): &\quad 3 \otimes \bar{3} = 1 \oplus 8. \end{aligned} \tag{79}$$

These can be done for more complex cases by using the method of Young tableaux [26]. Using these, we can start decomposing the couplings. The contracted electron Yukawa coupling decomposes in the following way:

$$\begin{aligned} \mathcal{Y}_E &\equiv (3_{SU(3)_L} \otimes \bar{3}_{SU(3)_L}) \otimes (2_{SU(2)_\phi} \otimes 2_{SU(2)_\phi}) \\ &\equiv (1_{SU(3)_L} \oplus 8_{SU(3)_L}) \otimes (1_{SU(2)_\phi} \oplus 3_{SU(2)_\phi}) \\ &\equiv (1_{SU(3)_L} \otimes 1_{SU(2)_\phi}) \oplus (1_{SU(3)_L} \otimes 3_{SU(2)_\phi}) \\ &\oplus (8_{SU(3)_L} \otimes 1_{SU(2)_\phi}) \oplus (8_{SU(3)_L} \otimes 3_{SU(2)_\phi}) \\ &= \mathcal{Y}_{(1,1)} \oplus \mathcal{Y}_{(8,1)} \oplus \mathcal{Y}_{(1,3)} \oplus \mathcal{Y}_{(8,3)} \end{aligned} \tag{80}$$

The double-singlet $\mathcal{Y}_{(1,1)}$ element is already an invariant. The second and third elements are an octuplet-singlet $\mathcal{Y}_{(8,1)}$ and a singlet-triplet $\mathcal{Y}_{(1,3)}$. These correspond to the adjoint representations of $SU(3)_L$ and $SU(2)_\phi$ respectively. The last one is an octet-triplet $\mathcal{Y}_{(8,3)}$ that transforms in the adjoint representations of both groups. We extend this description to the neutrino Yukawa couplings, defining the building blocks:

$$\begin{aligned} \bar{\mathcal{Y}}_N^{U(2)} &= \langle Y_N Y_N^\dagger \rangle_{U(2)_\phi}, \\ \bar{\mathcal{Y}}_N^{U(3)} &= \langle Y_N Y_N^\dagger \rangle_{U(3)_L}. \end{aligned} \tag{81}$$

Using the decompositions in eq. (79), we can define the basis elements:

$$\begin{aligned} \bar{\mathcal{Y}}_N^{U(2)} &= \bar{\mathcal{Y}}_{(1)}^{U(2)} \oplus \bar{\mathcal{Y}}_{(8)}, \\ \bar{\mathcal{Y}}_N^{U(3)} &= \bar{\mathcal{Y}}_{(1)}^{U(3)} \oplus \bar{\mathcal{Y}}_{(3)}. \end{aligned} \tag{82}$$

Where:

$$\bar{\mathcal{Y}}_{(1)}^{U(2)} = \bar{\mathcal{Y}}_{(1)}^{U(3)} = \bar{\mathcal{Y}}_{(1)}. \tag{83}$$

We can also use the doubly-adjoint representation $\bar{\mathcal{Y}}_{(8,3)}$. There is, however, a significant caveat here. Using these building blocks in calculations of Hilbert series might lead to the overcounting of linearly independent invariants. That is because these turn out to not be entirely independent of each other when one takes into account the cyclicity of the trace. Two main problems might arise:

1. Overcounting due to invariants being cycled into each other. An example of this can be seen in eq. (81), where we can use the cyclic property of the trace to see that their quadratic combinations are equivalent under a trace:

$$\langle \bar{\mathcal{Y}}_3^2 \rangle \equiv \langle \langle Y_N Y_N^\dagger \rangle^2_{SU(3)_L} \rangle_{SU(2)_\phi} = \langle \langle Y_N Y_N^\dagger \rangle^2_{SU(2)_\phi} \rangle_{SU(3)_L} \equiv \langle \bar{\mathcal{Y}}_8^2 \rangle$$

2. Overcounting of linearly independent invariants of certain degrees due to a hidden dependence on lower order invariants. An example of such a problem could be seen in the cubic invariant:

$$\langle \bar{\mathcal{Y}}_8^3 \rangle \equiv \langle \bar{\mathcal{Y}}_3^3 \rangle.$$

   Here, we see that the trace of the (3×3) matrix raised to the third power (which normally would be an independent invariant) can be rotated into the trace of the (2×2) matrix cubed. By applying Cayley-Hamilton's theorem ([9]), we can see that this invariant decomposes into combinations of lower order invariants. This second point differs from the first one because it arises due to the groups $SU(2)_\phi$ and $SU(3)_L$ having different dimensions.

These relations are not automatically accounted for by the HS generating function, therefore, we are not able to use these building blocks when calculating the HS for the GNM Yukawa sector. However, these can be useful when showing the inner structures of the generating invariants. For the generating function, we use the fundamental representations.

3.3.2 The GNM flavour space building blocks - Higgs sector

The Higgs potential coupling m^2 decomposes more simply:

$$m^2 \equiv 2_{SU(2)_\phi} \otimes 2_{SU(2)_\phi} = 1_{SU(2)_\phi} \oplus 3_{SU(2)_\phi} \equiv m^2_{(1)} \oplus m^2_{(3)}. \tag{84}$$

Again, we get a singlet $m^2_{(1)}$, which we can exclude from integration, and a triplet $m^2_{(3)}$, transforming in the adjoint representation. The last coupling is Z, transforming as $(2 \otimes 2 \otimes 2 \otimes 2)$ in $SU(2)_\phi$. The decomposition into irreducible representations was done in [14]. In this case, there were additional caveats to the reduction that needed to be accounted for. The result of the Young tableaux decomposition is:

$$2_{SU(2)_\phi} \otimes 2_{SU(2)_\phi} \otimes 2_{SU(2)_\phi} \otimes 2_{SU(2)_\phi} = 1_{SU(2)_\phi} \oplus 1_{SU(2)_\phi} \oplus 3_{SU(2)_\phi} \oplus 3_{SU(2)_\phi} \oplus 3_{SU(2)_\phi} \oplus 5_{SU(2)_\phi} \quad (85)$$

However, the added complication here is that out of the three triplet representations of Z, one vanishes while the remaining two are dependent on each other. This is the result of the relations in eq. (66). For more details on this, we again refer to [14]. The decomposition is therefore:

$$Z = Z_{(1)} \oplus Z_{(\bar{1})} \oplus Z_{(3)} \oplus Z_{(5)}. \quad (86)$$

Two distinct singlets $Z_{(1)}$ and $Z_{(\bar{1})}$ are (as discussed) already invariant. The triplet $Z_{(3)}$ transforms in the adjoint representation of $SU(2)_\phi$. The quintuplet $Z_{(5)}$ transforms in the 5-representation of $SU(2)_\phi$. In the case of the Higgs potential, the problems of overcounting that arise in the lepton sector are not present. That is because we have only one symmetry group. Therefore, we can use these building blocks when calculating the HS for GNM.

3.3.3 Explicit decomposition of the couplings

Putting everything together, the resultant description of GNM flavour characteristics is presented in Table.(3) Defining the basis elements for our flavour space is enough to generate the

The element of the flavour space	Representations in $SU(3)_L \times SU(2)_\phi$	Prescribed weights in the mono-graded HS
$\mathcal{Y}_{(8,1)}$	$(8,1)$	2
$\mathcal{Y}_{(1,3)}$	$(1,3)$	2
$\mathcal{Y}_{(8,3)}$	$(8,3)$	2
$\bar{\mathcal{Y}}_{(3)}$	$(1,3)$	2
$\bar{\mathcal{Y}}_{(8)}$	$(8,1)$	2
$m^2_{(3)}$	$(1,3)$	1
$Z_{(3)}$	$(1,3)$	1
$Z_{(5)}$	$(1,5)$	1
$M^2_{(0)}$	invariant	2
$\mathcal{Y}_{(1,1)}$	invariant	2
$\bar{\mathcal{Y}}_{(1)}$	invariant	2
$m^2_{(1)}$	invariant	1
$Z_{(1)}$	invariant	1
$Z_{(\bar{1})}$	invariant	1

3 table The elements of GNM flavour space and their transformation characteristics. The last six elements are completely invariant to the possible flavour transformations.

complete ring of invariants. However, any further analysis might require knowledge of how these connect to the couplings in the Lagrangian. This can be done by writing the decomposition into irreducible representations explicitly. We start by raising all of the indices in eq. (78):

$$[\langle (Y_E)^{ic}(Y_E^{\dagger})_{la}\rangle_{SU(3)_E}]\epsilon_{bc}\epsilon^{jkl} = [\mathcal{Y}_E]^{ic}_{la}\epsilon_{bc}\epsilon^{jkl} = [\mathcal{Y}_E]^{ijk}_{ab} \tag{87}$$

Note that j and k are antisymmetric. Hence, we can construct the appropriate Young tableaux for these elements:

$$[\,\mathcal{Y}_E]^{ijk}_{ab} = \left(\begin{array}{|c|}\hline i \\ \hline \end{array} \otimes \begin{array}{|c|}\hline j \\ \hline k \\ \hline \end{array}\right)_{SU(3)} \otimes \left(\begin{array}{|c|}\hline a \\ \hline \end{array} \otimes \begin{array}{|c|}\hline b \\ \hline \end{array}\right)_{SU(2)}$$

Decomposing into irreps yields the following for $SU(2)$ indices:

$$\begin{array}{|c|}\hline a \\ \hline \end{array} \otimes \begin{array}{|c|}\hline b \\ \hline \end{array} \to \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}_{(1)} \oplus \begin{array}{|c|c|}\hline a & b \\ \hline \end{array}_{(3)}$$

and for $SU(3)$:

$$\begin{array}{|c|}\hline i \\ \hline \end{array} \otimes \begin{array}{|c|}\hline j \\ \hline k \\ \hline \end{array} \to \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array}_{(1)} \oplus \begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}_{(8)}$$

For more information about Young tableaux as well as decompositions into irreps, we suggest [26] and [27]. From these, we see that in order to get the singlet elements $\mathcal{Y}_E$, we need to anti-symmetrize the indices of $[\mathcal{Y}_E]^{ijk}_{ab}$ in each group:

$$\begin{aligned}
\mathcal{Y}_{(1,1)} =& \frac{1}{2\times 6}\Big([\mathcal{Y}_E]^{ijk}_{ab} + [\mathcal{Y}_E]^{jki}_{ab} + [\mathcal{Y}_E]^{kij}_{ab} - [\mathcal{Y}_E]^{ikj}_{ab} - [\mathcal{Y}_E]^{jik}_{ab} - [\mathcal{Y}_E]^{kji}_{ab} \\
&- [\mathcal{Y}_E]^{ijk}_{ba} - [\mathcal{Y}_E]^{jki}_{ba} - [\mathcal{Y}_E]^{kij}_{ba} + [\mathcal{Y}_E]^{ikj}_{ba} + [\mathcal{Y}_E]^{jik}_{ba} + [\mathcal{Y}_E]^{kji}_{ba}\Big) \\
=& \frac{1}{6}\Big([\mathcal{Y}_E]^{ijk}_{ab} + [\mathcal{Y}_E]^{jki}_{ab} + [\mathcal{Y}_E]^{kij}_{ab} + [\mathcal{Y}_E]^{ikj}_{ba} + [\mathcal{Y}_E]^{jik}_{ba} + [\mathcal{Y}_E]^{kji}_{ba}\Big) \\
=& \frac{1}{6}\Big([\mathcal{Y}_E]^{i'j'k'}_{a'b'}\epsilon_{i'j'k'}\epsilon^{a'b'}\Big)\epsilon^{ijk}\epsilon_{ab}.
\end{aligned} \tag{88}$$

Here, to get the second equality, we used the anti-symmetry of the first and second $SU(3)$ indices, while the last expression is obtained by using the known relationships between the Dirac delta and Levi-Civita symbols:

$$\begin{aligned}
\epsilon^{a'b'}\epsilon_{ab} =& \delta^{a'}_a\delta^{b'}_b - \delta^{b'}_a\delta^{a'}_b \\
\epsilon^{i'j'k'}\epsilon_{ijk} =& \delta^{i'}_i(\delta^{j'}_j\delta^{k'}_k - \delta^{k'}_j\delta^{j'}_k) + \delta^{j'}_i(\delta^{k'}_j\delta^{i'}_k - \delta^{i'}_j\delta^{k'}_k) + \delta^{k'}_i(\delta^{i'}_j\delta^{j'}_k - \delta^{j'}_j\delta^{i'}_k)
\end{aligned} \tag{89}$$

To find the singlet-triplet decomposed coupling, we have to symmetrize $SU(2)$ and anti-symmetrize $SU(3)$ indices:

$$\mathcal{Y}_{(1,3)} = \frac{1}{6}\Big([\mathcal{Y}_E]^{ijk}_{ab} + [\mathcal{Y}_E]^{jki}_{ab} + [\mathcal{Y}_E]^{kij}_{ab} - [\mathcal{Y}_E]^{ikj}_{ba} - [\mathcal{Y}_E]^{jik}_{ba} - [\mathcal{Y}_E]^{kji}_{ba}\Big). \tag{90}$$

Finding the octupet-singlet element requires symmetrizing i and j, while anti-symmetrizing i, k and j, k, as well as anti-symmetrizing $SU(2)$ indices.

$$\mathcal{Y}_{(8,1)} = \frac{1}{6}\Big([\mathcal{Y}_E]^{ijk}_{ab} - [\mathcal{Y}_E]^{jki}_{ab} + [\mathcal{Y}_E]^{ikj}_{ba} - [\mathcal{Y}_E]^{jik}_{ba}\Big). \tag{91}$$

Finally, we can construct the octuplet-singlet element by symmetrizing all indices:

$$\mathcal{Y}_{(8,3)} = \frac{1}{6}\Big([\mathcal{Y}_E]^{ijk}_{ab} - [\mathcal{Y}_E]^{jki}_{ab} - [\mathcal{Y}_E]^{ikj}_{ba} + [\mathcal{Y}_E]^{jik}_{ba}\Big). \tag{92}$$

Equivalently, we perform the explicit decompositions of the neutrino couplings, with:

$$[(Y_N)^{ic}(Y_N^\dagger)_{la}]\epsilon_{bc}\epsilon^{jkl} = [\mathcal{Y}_N]^{ic}_{la}\epsilon_{bc}\epsilon^{jkl} = [\mathcal{Y}_N]^{ijk}_{ab} \tag{93}$$

First, we present the singlet:

$$\bar{\mathcal{Y}}_{(1)} = \frac{1}{6}\Big([\mathcal{Y}_N]^{i'j'k'}_{a'b'}\epsilon_{i'j'k'}\epsilon^{a'b'}\Big)\epsilon^{ijk}\epsilon_{ab}. \tag{94}$$

The adjoint terms are:

$$\begin{aligned}\bar{\mathcal{Y}}_{(3)} &= \frac{1}{6}\Big([\mathcal{Y}_N]^{ijk}_{ab} - [\mathcal{Y}_N]^{ikj}_{ba}\Big)\epsilon_{i'j'k'}\epsilon^{ijk},\\ \bar{\mathcal{Y}}_{(8)} &= \frac{1}{6}\Big([\mathcal{Y}_N]^{ijk}_{ab} + [\mathcal{Y}_N]^{jki}_{ba}\Big)\epsilon^{a'b'}\epsilon_{ab}.\end{aligned} \tag{95}$$

Summarizing the Yukawa sector, in both electron and neutrino cases, we have four kinds of building blocks:

1. Singlet-singlet $\mathcal{Y}_{(1)}, \bar{\mathcal{Y}}_{(1)}$, both being just traces of quadratic combinations of their respective Yukawa couplings.

2. Octuplet-singlet $\mathcal{Y}_{(8)}, \bar{\mathcal{Y}}_{(8)}$. These describe the $SU(3)_L$ characteristics of the Yukawa sector, transforming in adjoint representations. These correspond to traces over $SU(2)_\phi$ indices.

3. Singlet-triplet $\mathcal{Y}_{(3)}, \bar{\mathcal{Y}}_{(3)}$ - adjoint in $SU(2)_\phi$, traces over $SU(3)_L$.

4. Octuplet-triplet $\mathcal{Y}_{(8,3)}, \bar{\mathcal{Y}}_{(8,3)}$: adjoint in both $SU(3)_L$ and $SU(2)_\phi$.

The explicit decompositions of Higgs couplings can be found in [7]. We list them here as well.

The singlet elements are:

$$
\begin{aligned}
m^2_{(1)} =& -\frac{1}{2}(m^2)^{a'}_{a'}\epsilon_{ab},\\
Z_{(1)} =& \frac{1}{2}(Z^{a'b'}_{c'd'}\epsilon_{a'b'}\epsilon^{c'd'})\epsilon^{ab}\epsilon_{cd},\\
Z_{(\bar{1})} =& \frac{1}{2}(Z^{a'b'}_{a'b'} + Z^{a'b'}_{b'a'})\epsilon^{ab}\epsilon_{cd}.
\end{aligned}
\tag{96}
$$

The expressions for the $SU(2)$ triplets and the quintuplet building blocks can be found in [7].

3.4 The CP characteristics of the flavour space

One might use the invariant description of a given model to examine its CP violating properties. A lack of CP-odd generating invariants (invariants transforming anti-symmetrically under discrete CP transforms) in a model is sufficient to say that it is not CP-violating. While vice-versa is not necessarily true, identifying the CP-odd parts of an invariant subring is a useful step in examining the emergence of CP-violating phenomena in the said model. Therefore, we also specify how the basis elements of the flavour space change under CP transformations.

Lets start by defining the general CP transformation:

$$
\begin{aligned}
&L_i \xrightarrow{CP} (L^*)^j \mathcal{U}^i_j,\\
&(L^\dagger)^i \xrightarrow{CP} (L^T)_j (\mathcal{U}^\dagger)^j_i,\\
&\phi_a \xrightarrow{CP} (\phi^*)^b \mathcal{R}^a_b,\\
&(\phi^\dagger)^a \xrightarrow{CP} (\phi^T)_b (\mathcal{R}^\dagger)^b_a,\\
&N \xrightarrow{CP} N^*\eta,\\
&N^\dagger \xrightarrow{CP} N^T\eta^*.
\end{aligned}
\tag{97}
$$

For the sake of completeness, we should mention that we only specified transformations in the flavour space. In actuality, the CP also contributes a transformation in the Lorentz as well as gauge spaces. However, we suppress these, as they don't impact the subject of this paper [7]. We can also suppress the transformation of the right handed electron field E because, in our description of the model, these transformations are already traced over. Alternatively, looking at the Lagrangians in eq. (71) and eq. (65), we can see that the same outlined CP transformations can be described by the following transformations of the couplings:

$$
\begin{aligned}
&(Y_E)^i_a \xrightarrow{CP} (Y_E^\dagger)^b_j (\mathcal{R}^\dagger)^a_b \mathcal{U}^j_i = (Y_E^*)^j_b (\mathcal{R}^*)^b_a (\mathcal{U}^T)^i_j, \\
&(Y_N)^i_a \xrightarrow{CP} (Y_N^\dagger)^b_j (\mathcal{R}^T)^a_b \mathcal{U}^j_i \eta^* = (Y_N^*)^j_b (\mathcal{R}^\dagger)^b_a (\mathcal{U}^T)^i_j \eta^*, \\
&M \xrightarrow{CP} M^* (\eta^*)^2, \\
&(m^2)^a_b \xrightarrow{CP} (m^2)^d_c (\mathcal{R}^\dagger)^b_d (\mathcal{R})^c_a = ((m^2)^T)^c_d (\mathcal{R}^*)^d_b (\mathcal{R}^T)^a_c, \\
&Z^{ab}_{cd} \xrightarrow{CP} Z^{gh}_{ef} (\mathcal{R}^\dagger)^c_g (\mathcal{R}^\dagger)^d_h (\mathcal{R})^e_a (\mathcal{R})^f_b = (Z^T)^{ef}_{gh} (\mathcal{R}^*)^g_c (\mathcal{R}^*)^h_d (\mathcal{R}^T)^a_e (\mathcal{R}^T)^b_f .
\end{aligned}
\tag{98}
$$

As well as their hermitian-conjugate counterparts (without m^2 and Z, since they are hermitian, as demonstrated in eq (66)):

$$
\begin{aligned}
&(Y_E^\dagger)^a_i \xrightarrow{CP} (Y_E)^j_b (\mathcal{R})^b_a (\mathcal{U}^\dagger)^i_j = (Y_E^T)^b_j (\mathcal{R}^T)^a_b (\mathcal{U}^*)^j_i, \\
&(Y_N^\dagger)^a_i \xrightarrow{CP} (Y_N)^j_b (\mathcal{R}^*)^b_a (\mathcal{U}^\dagger)^i_j \eta = (Y_N^T)^b_j (\mathcal{R}^\dagger)^a_b (\mathcal{U}^*)^j_i \eta, \\
&M^* \xrightarrow{CP} M\eta^2.
\end{aligned}
\tag{99}
$$

4 The flavour invariant rings of the GNM

Finally, we can start calculating the Hilbert series for the flavour invariants of the GNM scalar and lepton sectors. First, we need to write down the complete generating function for lepton Yukawa and Higgs sectors. As outlined previously, the main pieces of information that come into it, besides the couplings, are the characters of their corresponding representations, as well as the Haar measures of each group. We present these in Appendix.A. The generating function for the GNM flavour-invariant HS is:

$$
\begin{aligned}
\mathfrak{H}_{GNM} = \oint & d\mu_{SU(3)_L} d\mu_{SU(2)_\phi} \left(\prod d\mu_{U(1)} \right) \\
& \mathrm{PE}(Y_N, \chi_{\bar{3}}^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_1^{U(1)_M} \chi_{-1}^{U(1)_\phi} \chi_1^{U(1)_\phi}) \\
& \mathrm{PE}(Y_N^\dagger, \chi_3^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_{-1}^{U(1)_M} \chi_1^{U(1)_\phi} \chi_{-1}^{U(1)_\phi}) \\
& \mathrm{PE}(Y_E, \chi_3^{SU(3)_L} \chi_{\bar{3}}^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_1^{U(1)_E} \chi_{-1}^{U(1)_L} \chi_1^{U(1)_\phi}) \\
& \mathrm{PE}(Y_E^\dagger, \chi_{\bar{3}}^{SU(3)_L} \chi_3^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_{-1}^{U(1)_E} \chi_1^{U(1)_L} \chi_{-1}^{U(1)_\phi}). \\
& \mathrm{PE}(m^2_{(3)}, \chi_3^{SU(2)_\phi})\ \mathrm{PE}(Z_{(3)}, \chi_3^{SU(2)_\phi})\ \mathrm{PE}(Z_{(5)}, \chi_5^{SU(2)_\phi}).
\end{aligned}
\tag{100}
$$

This is not the complete series, since we exclude the singlet terms from the calculations. The expression for the plethystic exponential is again provided in eq. (59). Although even without all the details, we can see that the integrand is fairly complex. The integration here is essentially done by the repeated use of the Cauchy residue theorem, which states that for a complex valued function

$\phi(z)$ with poles c inside the area encircled by a contour C:

$$\oint_{z\in C}\phi(z)dz = 2\pi i \sum_{c\in[\text{Area insde } C]} Res\Big(\phi(z), c\Big), \tag{101}$$

with residues given by the general formula:

$$Res\Big(\phi(z), c\Big) = \frac{1}{(p-1)!}\lim_{z\to c}\frac{d^{p-1}}{dz^{p-1}}\Big[(z-c)^p\phi(z)\Big]. \tag{102}$$

Here p is the degree of the pole c. We use the PE for $m^2_{(3)}$ as an example to show that the integral in eq. (135) can be calculated this way:

$$\begin{aligned}
\mathrm{PE}(m^2_{(3)}, \chi_3^{SU(2)_\phi}) &= \exp\left(\sum_{r=1}^{\infty}\frac{(m^2_{(3)})^r}{r}\chi_3^{SU(2)_\phi}(w^r)\right)\\
&= \exp\left(\sum_{r=1}^{\infty}\frac{(m^2_{(3)})^r}{r}(w^{2r}+1+\frac{1}{w^{2r}})\right)\\
&= \exp\left(\sum_{r=1}^{\infty}\frac{(m^2_{(3)}w^2)^r}{r}+\sum_{r=1}^{\infty}\frac{(m^2_{(3)})^r}{r}+\sum_{r=1}^{\infty}\frac{(\frac{m^2_{(3)}}{w^2})^r}{r}\right)
\end{aligned} \tag{103}$$

Now we can undo the expansions in eq. (53), to get:

$$\begin{aligned}
\mathrm{PE}(m^2_{(3)}, \chi_3^{SU(2)_\phi}) &= \exp\left(\log(1-m^2_{(3)}w^2)+\log(1-m^2_{(3)})+\log(1-\frac{m^2_{(3)}}{w^2})\right)\\
&= \frac{1}{(1-m^2_{(3)}w^2)(1-m^2_{(3)})(1-\frac{m^2_{(3)}}{w^2})}.
\end{aligned} \tag{104}$$

The result is a function of w, with 4 poles at:

$$w_{1,2} = \pm\sqrt{m^2_{(3)}}, \quad w_{3,4} = \pm\frac{1}{\sqrt{m^2_{(3)}}}. \tag{105}$$

During the contour integration, only 2 of the poles contribute to the integration, since, depending on the coupling, either the poles 1 and 2, or the poles 3 and 4 can be inside the contour. Following this example, all the PEs in the integrand can be written this way, making the integration a matter of calculating and adding up residues. We will not burden the reader with all the details, however, it should be clear from the size of the generating function that the calculation is quite demanding. The integration on its own is not a problem here. The calculation of all the residues can be done quickly with the regular tools provided by *Mathematica*. However, a massive array of rationals is not very useful as a representation of HS. If we wish to find HS in the form described by eq. (37), we need to add all of them up. This, however, is beyond the standard tools at ones disposal and, in some cases (as suggested by [19]), possibly beyond the capabilities of current technology. Needless to say, we were not able to complete these computations. A detailed description of such kinds of calculations, as well as some ways to work around this issue, can be found in [19]. Other papers, such as [15], completed such integrations by using completely different methods from the start (without using the residue theorem) [25]. We, however, used a slightly simplified method. Instead of obtaining the entire HS in their rational form, we calculated the Taylor expanded version up to a certain degree. Essentially, this is an approximation of HS that, depending on the degree of the approximation, might cut off some of the important information about the entire structure of the invariant space. However, since we know that our ring of invariants is finitely generated, we can also expect that after a certain degree, this approximation will include all the relevant information. To do so, we used the package called *DECO* from [18], which can be run on *FORM*. The said package was created for the purpose of generating gauge invariant operators for effective field theories (EFTs). The entire collection of these essentially amounts to HS. The main differences are that for EFTs, the elements of a symmetric space include fermionic fields and invariants containing derivatives. Furthermore, the HS would need to account for the equations of motion (EOM) and integration by parts (IBP) redundancies. Luckily, we can disregard this added complexity since it does not exist in the context of flavour. We set the programme to treat the couplings as scalar fields as well as nullify the terms that include derivatives.

4.1 The HS for SM-like Quark model

We start by verifying the method with SM-based theory. That is the quark Yukawa sector, with one Higgs flavour. The flavour symmetries here are described by the group $SU(3)_Q \times SU(3)_U \times SU(3)_D$. The Q here refers to the left handed quark flavour space, while U and D are the right handed quark flavour spaces. As transforming flavour basis elements, we use two quark Yukawa couplings Y_U and Y_D, both acting similarly to the electron coupling in the lepton sector. We also contract the right-handed flavour transformations (indexed as U and D) by taking traces over them, as we did with the electron couplings in eq. (87). Thus, we can use the elements:

$$\begin{aligned} \mathfrak{U} =& \langle Y_D Y_D^\dagger \rangle_{U(3)_U} \\ \mathfrak{D} =& \langle Y_U Y_U^\dagger \rangle_{U(3)_D} \end{aligned} \tag{106}$$

as well as using their decompositions into irreps: the $SU(3)_Q$ singlets $\mathfrak{U}_1$, $\mathfrak{D}_1$ and octuplets $\mathfrak{U}_8$, $\mathfrak{D}_6$. The HS constructed out of these does not result in overcounting since: (1) no elements in eq. (106) can be rotated into each other, and (2) all groups in the theory have equal dimensions. Thus, the generating function for the SM quark sector flavour invariant Hilbert series is:

$$\mathfrak{H}_{Quark} = \oint d\mu_{SU(3)_Q} \, \mathrm{PE}(\mathfrak{U}_{(8)}, \chi_8^{SU(3)_L}) \, \mathrm{PE}(\mathfrak{D}_{(8)}, \chi_8^{SU(3)_L}). \tag{107}$$

Calculating the Taylor expanded version with *DECO*, we find the following result for mono-graded HS:

$$\begin{aligned} \mathfrak{H}_{Quark}(t) = & \, 1 + 3t^4 + 4t^6 + 7t^8 + 12t^{10} + 24t^{12} + 28t^{14} \\ & + 55t^{16} + 76t^{18} + 111t^{20} + 160t^{22} + 238t^{24} + O(t^{26}) \end{aligned} \tag{108}$$

and the mono-graded PL yields a terminating series:

$$\mathrm{PL}_{Quark}(t) = 3t^4 + 4t^6 + t^8 + t^{12} - t^{24}. \tag{109}$$

Since the model is comparatively simple, we can also calculate the rational expression of HS by performing the integrations as described in the previous section or, alternatively, by reconstructing from the PL, by applying eq. (35). The results are:

$$\begin{aligned} \mathfrak{H}_{Quark}(t) &= \frac{1 - t^{24}}{(1 - t^4)^3(1 - t^6)^4(1 - t^8)(1 - t^{12})} \\ &= \frac{1 + t^{12}}{(1 - t^4)^3(1 - t^6)^4(1 - t^8)}. \end{aligned} \tag{110}$$

As we can see from the rational form of HS, as well as the PL, the total number of generating invariants is 11 (with two added singlets of order t^2). A detailed analysis of this particular ring of invariants has been conducted in [8] and [28]. All of the generating invariants correspond to particular parameters of the model. These are the following:

1. 3 up-quark masses: $I_{2,0} = \mathfrak{U}_1$, $I_{4,0} = \mathfrak{U}_8^2$, $I_{6,0} = \mathfrak{U}_8^3$;
2. 3 down-quark masses: $I_{0,2} = \mathfrak{D}_1$, $I_{0,4} = \mathfrak{D}_8^2$, $I_{0,6} = \mathfrak{D}_8^3$;
3. 3 mass mixing angles: $I_{2,2} = \mathfrak{U}_8\mathfrak{D}_8$, $I_{4,2} = \mathfrak{U}_8^2\mathfrak{D}_8$, $I_{2,4} = \mathfrak{U}_8\mathfrak{D}_8^2$;
4. 1 complex phase: $I_{4,4} = \mathfrak{U}_8^2\mathfrak{D}_8^2$

There is one more generating invariant, however, it is not entirely independent from the rest. That is the CP-violating order-6 invariant:

$$I_{6,6} = \langle \mathfrak{U}_8^2\mathfrak{D}_8\mathfrak{U}_8\mathfrak{D}_8^2 \rangle \tag{111}$$

To express the CP violating aspect of this invariant, we can use its linear combinations with its CP-transformed version and construct two invariants:

$$\begin{aligned} I_{6,6}^+ =& \langle \mathfrak{U}_8^2\mathfrak{D}_8\mathfrak{U}_8\mathfrak{D}_8^2 \rangle + \langle \mathfrak{D}_8^2\mathfrak{U}_8\mathfrak{D}_8\mathfrak{U}_8^2 \rangle \\ I_{6,6}^- =& \langle \mathfrak{U}_8^2\mathfrak{D}_8\mathfrak{U}_8\mathfrak{D}_8^2 \rangle - \langle \mathfrak{D}_8^2\mathfrak{U}_8\mathfrak{D}_8\mathfrak{U}_8^2 \rangle \\ \equiv& \langle \left[\mathfrak{U}, \mathfrak{D}\right]^3 \rangle = \det[\mathfrak{U}_8, \mathfrak{D}_8] \end{aligned} \tag{112}$$

The CP-symmetrised invariant $I_{6,6}^+$ can be expressed as a linear combination of lower order invariants, as demonstrated in [9]. The CP-antisymmetrized invariant $I_{6,6}^-$ is linearly independent. It corresponds to the Standard Model Jarlskog invariant. The connection between the expression in eq. (112) and the more standard definition of the SM Jarlskog invariant, written in the quark mass basis [6]:

$$\det[\mathfrak{U}_8, \mathfrak{D}_8] \equiv -2(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)(m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)J. \tag{113}$$

can found, by following [29]. It is, in fact, only a secondary invariant among the generating set. That is because of higher order syzygies that we see by the term $-t^{24}$ in the PL eq (109). This particular syzygy is also presented in [8]. The algebraic non-independence of the CP-violating generator reflects the fact that the Jarlskog invariant is essentially responsible for the direction of the CP violating phase, while the size is entirely dependent on CP-even invariants of lower orders [9]. Another way to look at it is that the primary generators describe the physical parameters of the model, with certain degeneracies accounted by the secondary generators.

4.2 Hilbert series for the leptonic sector

We move to discuss the invariants of the leptonic sector. As we have seen, the complexity of the SM quark sector was fairly manageable. However, as we will show shortly, a simple addition of a $SU(2)$ flavour symmetry to the model complicates things significantly. That is why we are going to approach the complete GNM in a slightly more bottom-up manner. We start by examining the component parts of the model, adding them up to hopefully arrive at a more complete picture of the invariant ring.

4.2.1 HS for the 1 Majorana neutrino seesaw model

First, we focus on a simple sub-theory of the GNM to see if the results of this method correspond to previous work. We generate its ring of invariants by ignoring the elements added by the 2HDM in the Table.3. Therefore, the couplings we use for this are: $\mathcal{Y}_{(1)}, \mathcal{Y}_{(8)}, Y_N, Y_N^\dagger, M^2_{(0)}, m^2_{(1)}, Z_{(1)}$. The first two couplings (of which one is invariant) correspond to $\mathcal{Y}_{(1,1)}, \mathcal{Y}_{(8,1)}$. Also, since the Higgs field adds only $U(1)_\phi$ symmetry, the Higgs couplings $m^2_{(1)}, Z_{(1)}$ are both already invariants, as well as $M^2_{(0)}$. The generating function is therefore:

$$
\begin{aligned}
\mathfrak{H}_M = \oint & d\mu_{SU(3)_L} d\mu_{U(1)_M} d\mu_{U(1)_L} d\mu_{U(1)_\phi} \\
& \mathrm{PE}(Y_N, \chi_{\bar{3}}^{SU(3)_L} \chi_1^{U(1)_M} \chi_{-1}^{U(1)_\phi} \chi_1^{U(1)_\phi})\ \mathrm{PE}(Y_N^\dagger, \chi_3^{SU(3)_L} \chi_{-1}^{U(1)_M} \chi_1^{U(1)_\phi} \chi_{-1}^{U(1)_\phi}) \\
& \mathrm{PE}(\mathcal{Y}_{(8)}, \chi_8^{SU(3)_L}).
\end{aligned}
\tag{114}
$$

Here, the singlet couplings are excluded. Calculating the mono-graded HS (up to degree 16) gives the result:

$$
\begin{aligned}
\mathfrak{H}_M(t) = 1 &+ 3t^2 + 2t^3 + 7t^4 + 6t^5 + 16t^6 + 14t^7 + 31t^8 + 30t^9 + 55t^{10} \\
&+ 56t^{11} + 94t^{12} + 96t^{13} + 151t^{14} + 158t^{15} + 232t^{16} + O(t^{17}).
\end{aligned}
\tag{115}
$$

Using eq. (62) to calculate the plethystic logarithm, we obtain a terminating series:

$$
\mathrm{PL}_M(t) = t^2 + 2t^4 + 2t^6. \tag{116}
$$

In full description, we should account for the exclusion of 4 singlets from the calculations. The lack of terms with minus signs on the right-hand side suggests that there are no syzygies in this theory.

Due to the simplicity of these expressions, we can use the information provided by the PL (and the connection between eq. (64) and eq. (35)) to reconstruct the rational form of HS:

$$\mathfrak{H}_M(t) = \frac{1}{(1-t^2)(1-t^4)^2(1-t^6)^2}. \tag{117}$$

Taylor-expanding this indeed gives eq. (115). The composition of these invariants becomes apparent in multi-graded form. The multi-graded PL is:

$$\mathrm{PL}_M = \bar{\mathcal{Y}}_{(1)} + \mathcal{Y}^2_{(8)} + \bar{\mathcal{Y}}_{(8)}\mathcal{Y}_{(8)} + \mathcal{Y}^3_{(8)} + \bar{\mathcal{Y}}_{(8)}\mathcal{Y}^2_{(8)}. \tag{118}$$

Note that we use the definitions provided in eq. (95) to describe the contributions of the neutrino Yukawa coupling. We should clarify that, while not explicitly written, the actual invariant quantities are traces over the indices of these combinations. The multi-graded HS for 1-Majorana neutrino seesaw model is (with singlet building blocks included):

$$\begin{aligned}\mathfrak{H}_M =& \frac{1}{(1-m^2_{(1)})(1-Z_{(1)})(1-\bar{\mathcal{Y}}_{(1)})(1-\mathcal{Y}_{(1)})(1-\mathcal{Y}^2_{(3)})} \\ & \frac{1}{(1-M^2_{(0)})(1-\bar{\mathcal{Y}}_{(8)}\mathcal{Y}_{(8)})(1-\mathcal{Y}^3_{(8)})(1-\bar{\mathcal{Y}}_{(8)}\mathcal{Y}^2_{(8)})}.\end{aligned} \tag{119}$$

Counting the number of factors in the denominator, we find that there are $p = 9$ primary generating invariants, all corresponding to the parameters of the model. This is consistent with the analysis of seesaw models in [9], as well as the HS generated in [13]. Using their formulas for counting the parameters in the seesaw model, we see that for $n_L = 3, n_N = 1$ theory, we should expect $p_{\text{lepton}} = 7$ parameters in the lepton sector (which correspond to our invariants):

- $n_L = 3$ electron masses ($\mathcal{Y}_{(1)}, \mathcal{Y}^2_{(8)}, \mathcal{Y}^3_{(8)}$),
- $n_N = 1$ SM neutrino mass ($\bar{\mathcal{Y}}_{(1)}$),
- $n_N = 1$ heavy Majorana neutrino mass ($M^2_{(0)}$),
- $n_N n_L - n_N = 2$ mixing angles ($\bar{\mathcal{Y}}_{(8)}\mathcal{Y}_{(8)}, \bar{\mathcal{Y}}_{(8)}\mathcal{Y}^2_{(8)}$),
- $n_N n_L - n_L = 0$ CP violating phases.

The SM Higgs subspace includes $p_{\text{Higgs}} = 2$ more parameters (again, corresponding to $m^2_{(1)}, Z_{(1)}$), which adds up to a total of $p = 9$ parameters across both sectors.

4.2.2 The HS for the GNM lepton invariants

Continuing with our modular strategy, we add Higgs $SU(2)_\phi$ symmetry to the seesaw theory covered above. Lets start with the neutrino Yukawa coupling, Y_N, transforming as a fundamental representation in both $SU(3)_L$ and $SU(2)_\phi$ symmetries.. The generating function for HS is:

$$
\begin{aligned}
\mathfrak{H}_{Y_N} = \oint & d\mu_{SU(3)_L} d\mu_{SU(2)_\phi} d\mu_{U(1)_M} d\mu_{U(1)_L} d\mu_{U(1)_\phi} \\
& \mathrm{PE}(Y_N, \chi_{\bar{3}}^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_1^{U(1)_M} \chi_{-1}^{U(1)_\phi} \chi_1^{U(1)_\phi}) \\
& \mathrm{PE}(Y_N^\dagger, \chi_3^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_{-1}^{U(1)_M} \chi_1^{U(1)_\phi} \chi_{-1}^{U(1)_\phi}).
\end{aligned}
\tag{120}
$$

We can perform the integration to find the rational expression for HS:

$$
\mathfrak{H}_{Y_N} = \frac{1}{\left(1 - Y_N Y_N^\dagger\right)\left(1 - (Y_N Y_N^\dagger)^2\right)} = \frac{1}{\left(1 - \bar{\mathcal{Y}}_{(1)}\right)\left(1 - \bar{\mathcal{Y}}^2_{(3 \text{ or } 8)}\right)}
\tag{121}
$$

Interestingly, we find that the addition of $SU(2)$ transformations to the leptonic seesaw model increases the number of "pure" Y_N invariants from one to two. Following the example of the SM quark sector, we can reason that these "pure" invariants correspond to two neutrino masses present in the GNM extension. In standard descriptions, the second neutrino mass emerges as a consequence of one-loop interactions between neutrino fields and the two-flavour Higgs field [2]. The actual value of the second neutrino mass contains the contribution of additional 2HDM potential terms. This, however, suggests that the mere introduction of $SU(2)$ symmetry to the leptonic sector provides a necessary condition to generate an additional neutrino mass in the GNM (this being the non-zero value of the quartic generator). This should not be particularly surprising, as such conclusion is already reflected in another popular minimal extension of the leptonic sector - the two-Majorana neutrino model [13]. Writing the Y_N invariants in terms of adjoint building blocks, we get:

$$
\begin{aligned}
\mathcal{I}_{02000} &= \langle \bar{\mathcal{Y}}_{(1)} \rangle \\
\mathcal{I}_{04000} &= \langle \bar{\mathcal{Y}}^2_{(3)} \rangle = \langle \bar{\mathcal{Y}}^2_{(8)} \rangle
\end{aligned}
\tag{122}
$$

Using DECO, we can also verify the consistency of the results by generating the Taylor-expanded version, as well as the PL:

$$
\begin{aligned}
&\mathfrak{H}_{Y_N}(t) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 4t^{12} + 4t^{14} + 6t^{16} + 6t^{18}, \\
&\mathrm{PL}_{Y_N}(t) = t^2 + 2t^4.
\end{aligned}
\tag{123}
$$

4.2.3 Y_E invariant subring

Next, we cover the "pure" invariants of Y_E. However, while previously we were able to obtain a quite simple exact expression for HS, here the ring structure is far more complicated. That is because we have to account for the entire $SU(3)_E$ symmetry, which is not present anywhere else. We also cannot use the contracted version of the invariant building blocks from eq. (82) to construct the generating function, as the cyclic relationships between them lead to significant overcounting.

The generating function is:

$$
\begin{aligned}
\mathfrak{H}_{Y_E} = \oint & d\mu_{SU(3)_E} d\mu_{SU(3)_L} d\mu_{SU(2)_\phi} d\mu_{U(1)_E} d\mu_{U(1)_L} d\mu_{U(1)_\phi} \\
& \mathrm{PE}(Y_E, \chi_3^{SU(3)_L} \chi_{\bar{3}}^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_1^{U(1)_E} \chi_{-1}^{U(1)_L} \chi_1^{U(1)_\phi}) \\
& \mathrm{PE}(Y_E^\dagger, \chi_{\bar{3}}^{SU(3)_L} \chi_3^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_{-1}^{U(1)_E} \chi_1^{U(1)_L} \chi_{-1}^{U(1)_\phi}).
\end{aligned} \tag{124}
$$

Due to the complexity of these HS, we use the approximate approach with DECO to obtain the mono-graded HS:

$$
\begin{aligned}
\mathfrak{H}_{Y_E}(t) = 1 &+ t^2 + 4t^4 + 8t^6 + 21t^8 + 42t^{10} + 111t^{12} + 229t^{14} + 534t^{16} \\
&+ 1125t^{18} + 2405t^{20} + 4850t^{22} + 9767t^{24} + 18778t^{26} + 35682t^{28} + O(t^{30}).
\end{aligned} \tag{125}
$$

and the mono-graded PL:

$$
\begin{aligned}
\mathrm{PL}_{Y_E}(t) =& t^2 + 3t^4 + 4t^6 + 7t^8 + 9t^{10} + 28t^{12} + 39t^{14} + 70t^{16} \\
&+ 101t^{18} + 92t^{20} - 2t^{22} - 379t^{24} - 1212t^{26} - 2956t^{28} + O(t^{30}).
\end{aligned} \tag{126}
$$

Here we encounter a non-terminating PL series. As discussed in the section 2.5, the coefficients of non-terminating PL show the difference between generating and syzygy terms at the corresponding order. Hence, to arrive at a trustworthy count of the generators the PL should be corrected. Unfortunately, in this case using the multi-graded PL does not help. That is because the multi-graded composition of this series is not particularly more complicated than the mono-graded one, essentially consisting invariants and syzygies of the form:

$$
\begin{aligned}
I_{n,n}^{Y_E} &\equiv (Y_E)^{\otimes n} \otimes (Y_E^\dagger)^{\otimes n} \\
S_{n,n}^{Y_E} &\equiv f\left[(Y_E)^{\otimes n} \otimes (Y_E^\dagger)^{\otimes n}\right]
\end{aligned} \tag{127}
$$

The actual complicated structure of these is hidden in the multiplicity of ways of contracting the indices of these building blocks. Taking the naive approach to the counting, we can estimate that the subring of a pure electron Yukawa coupling invariants is generated by the total of 354 invariants - a significant increase from the previous models. Not all of them are independent of each other due to the higher order syzygies, accounted for by the negative-sign terms in the PL. These numbers are again a mere estimation and should be corrected by extensive and detailed study of this particular ring or, possibly, other approaches of which, so far, we are unaware.

4.2.4 The complete GNM leptonic sector

Finally, we use the combination of eq (120) and eq. (124) to find the combined HS for the GNM leptonic Yukawa sector:

$$\begin{aligned}\mathfrak{H}_{lepton} = \oint d\mu_{SU(3)_E} d\mu_{SU(3)_L} d\mu_{SU(2)_\phi} d\mu_{U(1)_M} d\mu_{U(1)_L} d\mu_{U(1)_\phi} \\ \mathrm{PE}(Y_N, \chi_{\bar{3}}^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_1^{U(1)_M} \chi_{-1}^{U(1)_\phi} \chi_1^{U(1)_\phi}) \\ \mathrm{PE}(Y_N^\dagger, \chi_3^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_{-1}^{U(1)_M} \chi_1^{U(1)_\phi} \chi_{-1}^{U(1)_\phi}) \\ \mathrm{PE}(Y_E, \chi_3^{SU(3)_L} \chi_{\bar{3}}^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_1^{U(1)_E} \chi_{-1}^{U(1)_L} \chi_1^{U(1)_\phi}) \\ \mathrm{PE}(Y_E^\dagger, \chi_{\bar{3}}^{SU(3)_L} \chi_3^{SU(3)_L} \chi_2^{SU(2)_\phi} \chi_{-1}^{U(1)_E} \chi_1^{U(1)_L} \chi_{-1}^{U(1)_\phi}).\end{aligned} \tag{128}$$

The resulting HS is:

$$\begin{aligned}\mathfrak{H}_{lepton}(t) = 1 + 2t^2 + 10t^4 + 34t^6 + 142t^8 + 534t^{10} + 2091t^{12} \\ + 7598t^{14} + 26857t^{16} + 89880t^{18} + O(t^{20}).\end{aligned} \tag{129}$$

And the PL:

$$\begin{aligned}\mathrm{PL}_{lepton}(t) =2t^2 + 7t^4 + 16t^6 + 56t^8 + 172t^{10} + 553t^{12} \\ + 1402t^{14} + 2803t^{16} + 2222t^{18} + O(t^{20}).\end{aligned} \tag{130}$$

It is apparent that the combination of electron and neutrino couplings makes the number of generators "explode". This makes the detailed analysis of the resulting ring very difficult. Still, we can attempt to deal with this complexity by taking advantage of the sectional strategy we have been using so far. First, we make an assumption that the entire leptonic HS is composed of three non-intersecting categories of generators: pure neutrino, pure electron, and mixed. Furthermore, using the definition of the HS provided in eq. (35), we can reason that the HS of invariant subrings generated by each category can be factored as follows:

$$\mathfrak{H}_{lepton} = \mathfrak{H}_{Y_N} \cdot \mathfrak{H}_{Y_E} \cdot \mathfrak{H}_{mix}. \tag{131}$$

Since we have already found $\mathfrak{H}_{Y_N}$, $\mathfrak{H}_{Y_E}$, and $\mathfrak{H}_{lepton}$, we can separate the mixed part of the leptonic HS by simple division:

$$\mathfrak{H}_{mix} = \frac{\mathfrak{H}_{lepton}}{\mathfrak{H}_{Y_N} \cdot \mathfrak{H}_{Y_E}}. \tag{132}$$

The resultant HS from mixed leptonic invariants is:

$$\mathfrak{H}_{mix}(t) = 1 + 3t^4 + 12t^6 + 55t^8 + 199t^{10} + 760t^{12} + 2512t^{14} + 8032t^{16} + O(t^{18}) \tag{133}$$

Finding its plethystic logarithm yields:

$$\mathrm{PL}_{mix}(t) = 3t^4 + 12t^6 + 49t^8 + 163t^{10} + 525t^{12} + 1363t^{14} + 2733t^{16} + O(t^{18}) \tag{134}$$

As we can see, most of the invariants in the leptonic sector are mixed electron-neutrino coupling invariants.

4.3 The HS for the 2HDM

The invariant ring of the 2HDM was already extensively covered in [14], [7], and [15]. Therefore, we refer to these sources for a more detailed analysis. Here, we mostly provide our own verifications of the results found in these papers. The generating function for 2HDM flavour invariant HS can be constructed out of 2HDM building blocks, provided in Table.3:

$$\mathfrak{H}_{HP} = \oint \mu_{SU(2)_\phi}\, \mathrm{PE}(m^2_{(3)}, \chi_3^{SU(2)_\phi})\, \mathrm{PE}(Z_{(3)}, \chi_3^{SU(2)_\phi})\, \mathrm{PE}(Z_{(5)}, \chi_5^{SU(2)_\phi}). \tag{135}$$

Using this, we can calculate the rational form of multi-graded HS:

$$\mathfrak{H}_{2HDM} = \frac{N_{2HDM}(m^2_{(3)}, Z_{(3)}, Z_{(5)})}{D_{2HDM}(m^2_{(3)}, Z_{(3)}, Z_{(5)})}, \tag{136}$$

where the denominator is given by:

$$\begin{aligned} D_{2HDM} = &\left(1-(m^2_{(3)})^2\right)\left(1-m^2_{(3)}Z_{(3)}\right)\left(1-Z^2_{(3)}\right)\left(1-(m^2_{(3)})^2Z_{(5)}\right) \\ &\left(1-Z^2_{(3)}Z_{(5)}\right)\left(1-Z^2_{(5)}\right)\left(1-(m^2_{(3)})^2Z^2_{(5)}\right)\left(1-Z^2_{(3)}Z^2_{(5)}\right)\left(1-Z^3_{(5)}\right). \end{aligned} \tag{137}$$

And the numerator can be found in [7]. Calculating the mono-graded version gives:

$$\mathfrak{H}_{HP}(t) = \frac{1+t^3+4t^4+2t^5+4t^6+t^7+t^{10}}{(1-t^2)^4(1-t^3)^3(1-t^4)}. \tag{138}$$

Using *DECO*, we calculate the approximation of HS:

$$\begin{aligned} \mathfrak{H}_{2HDM}(t) = &1 + 4t^2 + 4t^3 + 15t^4 + 18t^5 + 53t^6 + 65t^7 + 148t^8 + 198t^9 + 371t^{10} \\ &+ 499t^{11} + 853t^{12} + 1129t^{13} + 1786t^{14} + 2354t^{15} + 3498t^{16} + O(t^{17}). \end{aligned} \tag{139}$$

As well as the plethystic logarithm:

$$\begin{aligned} \mathrm{PL}_{2HDM}(t) = &4t^2 + 4t^3 + 5t^4 + 2t^5 + 3t^6 - 3t^7 - 12t^8 - 12t^9 - 15t^{10} \\ &+ 6t^{11} + 28t^{12} + 60t^{13} + 75t^{14} + 8t^{15} - 105t^{16} + O(t^{17}). \end{aligned} \tag{140}$$

In this case, the PL again does not terminate, meaning that the numerator in the rational form of HS eq. (138) cannot be expressed as a product of a finite number of factors. Furthermore, we can see that there is a slight mismatch between the number of primary generators in multi- and mono-graded

HS (eq. (137) and eq. (138)), with former showing 4 primary invariants at order-3, while the latter showing 3. This is because of the overlapping between syzygies and generators at order-6, which in mono-graded series causes the canceling between a generator and a syzygy thus resulting in a miscount. To correct this, we again can use the multi-graded PL series. By separating the negative and positive terms, we arrive at the corrected PL expression:

$$\mathrm{PL}_{2HDM}(t) = 4t^2 + 4t^3 + 5t^4 + 2t^5 + (4-1)t^6 - 3t^7 + O(t^8). \tag{141}$$

Now we see that the number of mono-graded generators is 19, which corresponds to the number of generators in the multi-graded expression. Checking these results against those in [14] and [7], we see that they are indeed consistent.

4.4 The HS for the GNM

Now that we described the component parts, we follow our strategy and calculate the HS for the invariants in the full GNM (both Higgs and the lepton Yukawa sector). This time we include all the couplings and symmetries in the Table(3). The generating function is already provided in eq. (135). Even before obtaining the result, we note that the calculations took significantly longer than in previous examples (even in the mono-graded form). We again approximate HS up to the 14th power and obtain the following expression:

$$\begin{aligned}\mathfrak{H}_{GNM}(t) = 1 &+ 6t^2 + 8t^3 + 39t^4 + 77t^5 + 262t^6 + 591t^7 + 1670t^8 + 3956t^9 \\ &+ 10123t^{10} + 23914t^{11} + 57799t^{12} + 133683t^{13} + 309678t^{14} + O(t^{15}).\end{aligned} \tag{142}$$

We immediately see that the number of couplings grows much faster than in previous examples. This explains the length and complexity of this calculation. The rapid rise in the number of terms with increasing degree implies a very large number of generating invariants. However, we can still try finding the generating basis for this model. For that purpose, we calculate the PL and obtain the result:

$$\begin{aligned}\mathrm{PL}_{GNM}(t) = &6t^2 + 8t^3 + 18t^4 + 29t^5 + 62t^6 + 105t^7 + 175t^8 + 267t^9 \\ &+ 298t^{10} + 123t^{11} - 743t^{12} - 3256t^{13} - 9347t^{14} + O(t^{15}).\end{aligned} \tag{143}$$

We note that the expression is not terminating, thus preventing us from reconstructing the rational form of HS. We can estimate that there are 6 generating invariants of degree 2, 8 of degree 3, etc., adding up to a total of 1091 generating invariants of degree no higher than 12. Furthermore, we can see that at degree 12, we have 743 syzygies, giving algebraic dependencies between the generating invariants. We also clarify that this is applicable only for the mono-graded HS. The multi-graded version is likely to have slightly different characteristics. Furthermore, without the rational form of HS, we are also unable to discern which generating invariants are primary (corresponding to the independent parameters) and which are secondary (dependent). Needless to say, analyzing the multi-graded version is going to be very difficult, and useful information about the overall structure of this ring is going to be obscured by the sheer number of terms. While the mono-graded version is not the most descriptive, we can still read off some useful information. We can apply a similar partitioning of the invariant ring, as we did with the leptonic sector invariants, to reconstruct the subring of the mixed basis invariants. This time we assume that the HS is comprised of three separate parts: the purely leptonic invariants, the purely Higgs invariants, and the mixed invariants (including couplings from both sectors). We might separate the overall HS into these sections as follows:

$$\mathfrak{H} = \mathfrak{H}_{lepton} \cdot \mathfrak{H}_{2HDM} \cdot \mathfrak{H}_{mix} \tag{144}$$

The 2HDM and the leptonic part we already know in eq. (138) and eq. (129). Using these, we can find the mixed part of the invariant ring by the following calculation:

$$\mathfrak{H}_{mix} = \frac{\mathfrak{H}}{\mathfrak{H}_{lepton} \cdot \mathfrak{H}_{2HDM}}. \tag{145}$$

While the polynomials here do not divide nicely, we remind you that at this point we only care about the approximations (of degree 16). Hence, we can just Taylor expand the expression and obtain the HS for the mixed invariants:

$$\begin{aligned}\mathfrak{H}_{mix}(t) =& 1 + 4t^3 + 6t^4 + 27t^5 + 53t^6 + 132t^7 + 260t^8 + 633t^9 \\ &+ 1269t^{10} + 2804t^{11} + 5611t^{12} + 11414t^{13} + 22009t^{14} + O\left(t^{15}\right).\end{aligned} \tag{146}$$

We can put this into the formula for PL and find the mixed basis:

$$\begin{aligned}\mathrm{PL}_{mix}(t) =& 4t^3 + 6t^4 + 27t^5 + 43t^6 + 108t^7 + 131t^8 + 279t^9 \\ &+ 141t^{10} + 117t^{11} - 1322t^{12} - 3316t^{13} - 10824t^{14} + O(t^{15}).\end{aligned} \tag{147}$$

Using the calculated PLs allows us to make the deductions that the ring of the GNM flavour invariants has:

- 3 generating order-1 invariants (0 leptonic, 3 Higgs, 0 mixed);
- 6 order-2 generating invariants (2 leptonic, 4 Higgs, 0 mixed, 1 Majorana mass);
- 8 order-3 generating invariants (0 leptonic, 4 Higgs, 4 mixed);
- 18 order-4 generating invariants (7 leptonic, 5 Higgs, 6 mixed);
- 29 order-5 generating invariants (0 leptonic, 2 Higgs, 27 mixed);
- 62 order-6 generating invariants (16 leptonic, 3 Higgs, 43 mixed);
- 105 order-7 generating invariants (0 leptonic, −3 Higgs, 108 mixed);
- 175 order-8 generating invariants (56 leptonic, −12 Higgs, 131 mixed);

- 267 order-9 generating invariants (0 leptonic, −12 Higgs, 279 mixed);
- 298 order-10 generating invariants (172 leptonic, −15 Higgs, 141 mixed);
- 123 order-11 generating invariants (0 leptonic, 6 Higgs, 117 mixed);
- −743 order-12 generating invariants (553 leptonic, 28 Higgs, −1322 mixed);

We might continue such analysis to higher orders. Taking a higher order term such that the effects of syzygies start to contribute (for example, at order-8), we can clarify the interpretation of positive and negative sign coefficients: The coefficient number 175 at order-8 corresponds to 56 leptonic and 131 mixed generating invariants, as well as 12 syzygies between Higgs sector invariants. Therefore, we see that the coefficients in a non-terminating PL are equal to the difference between the number of generating invariants and syzygies at the respective order. The numbers in this case do indeed add up: $56 + 0 + 131 - 12 = 175$. As we can see, all numbers are consistent.

4.4.1 Y_N - 2HDM mixing

While the invariant characteristics of isolated component parts of the GNM (seesaw and 2HDM) were extensively examined in previous works, here we are interested in the emergent results of adding these theories together. Therefore, in this section, we focus on one of its mixed sections - specifically the mixing of the neutrino Yukawa coupling and the 2HDM potential. These invariants, being a unique consequence of GN-type models due to their composition, might prove to be relevant in certain leptogenesis models (especially at lower loop approximations). For this subring, we can calculate the closed form of HS by including only 2HDM building blocks: $m^2_{(3)}, Z_{(3)}, Z_{(5)}$ and the adjointly transforming the Yukawa building block $\bar{\mathcal{Y}}_{(3)}$ from Table. (3). The mono-graded result is:

$$\mathfrak{H}_{Y_N+2HDM}(t) = \frac{N_{Y_N+2HDM}(t)}{\left(1-t^2\right)^4 \left(1-t^3\right)^4 \left(1-t^4\right)^2 \left(1-t^5\right)}. \tag{148}$$

with the palindromic numerator term:

$$\begin{aligned} N_{Y_N+2HDM}(t) =& 1 + 2t^3 + 7t^4 + 8t^5 + 13t^6 + 10t^7 + 10t^8 \\ &+ 10t^9 + 13t^{10} + 8t^{11} + 7t^{12} + 2t^{13} + t^{16}. \end{aligned} \tag{149}$$

Calculating the PL gives:

$$\begin{aligned} \mathrm{PL}_{Y_N+2HDM}(t) =& 5t^2 + 6t^3 + 9t^4 + 9t^5 + 10t^6 \\ &- 4t^7 - 34t^8 - 70t^9 - 106t^{10} - 56t^{11} \\ &+ 131t^{12} + 564t^{13} + 1098t^{14} + O(t^{15}). \end{aligned} \tag{150}$$

The PL series is again infinite, therefore we have to apply a more careful interpretation as described in section 2.5. The positive-sign terms in the PL correspond to the difference between the generating invariants of the ring and the syzygies of that order. Hence, before making any claims about the number of these we need to separate the positive and negative sign terms from the multi-graded PL. The result is:

$$\begin{aligned}\mathrm{PL}_{Y_N+2HDM}(t) =&4t^2 + 6t^3 + 9t^4 + 9t^5 + (11-1)t^6 \\ &+ (5-9)t^7 + (2-36)t^8 + (1-71)t^9 + (6-112)t^{10} + (63-119)t^{11} \\ &+ (240-109)t^{12} + \mathcal{O}(t^{13}).\end{aligned} \tag{151}$$

This solves the problem of term cancellation, however, we still lack a way to distinguish between the positive terms representing generators and the positive terms, representing the syzygies among syzygies, as the PL in eq. (151) has no purely negative orders. This can be accounted for by changing the mono-grading weight of the Yukawa building block $\bar{\mathcal{Y}}_{(3)}$ from previously used $w_{\bar{\mathcal{Y}}_{(3)}} = 2$ (reflecting the order of Lagrangian couplings) to $w_{\bar{\mathcal{Y}}_{(3)}} = 1$. Now we can find the corrected PL:

$$\begin{aligned}\mathrm{PL}_{Y_N+2HDM}(t) =&7t^2 + 8t^3 + 14t^4 + 8t^5 + (10-16)t^6 \\ &- 48t^7 - 123t^8 + (33-137)t^9 + (246-146)t^{10} + (783-71)t^{11} \\ &+ (1641-116)t^{12} + \mathcal{O}(t^{13}),\end{aligned} \tag{152}$$

with clear absence of positive terms at orders 7 and 8. Hence, using the conjecture, outlined in section 2.5 and [22], we posit that the generators are represented by the positive terms up to and including the order-6 in eq. (152), or, respectively, order-9 according to the original grading in eq. (151). Hence, the total number of generating invariants is 47. This includes 19 purely-Higgs-potential generators from [7] and 1 purely-Yukawa generator from eq. (121).

The HS for the multi-graded ring is:

$$\mathfrak{H}_{Y_N+2HDM} = \frac{N_{Y_N+2HDM}(\bar{\mathcal{Y}}_{(3)}, m^2_{(3)}, Z_{(3)}, Z_{(5)})}{D_{Y_N+2HDM}(\bar{\mathcal{Y}}_{(3)}, m^2_{(3)}, Z_{(3)}, Z_{(5)})}, \tag{153}$$

with the denominator being:

$$\begin{aligned}D_{Y_N+2HDM} =& \left(1-\bar{\mathcal{Y}}^2_{(3)}\right)\left(1-(m^2_{(3)})^2\right)\left(1-m^2_{(3)}Z_{(3)}\right)\left(1-Z^2_{(3)}\right)\left(1-(m^2_{(3)})^2Z_{(5)}\right) \\ &\left(1-Z^2_{(3)}Z_{(5)}\right)\left(1-Z^2_{(5)}\right)\left(1-(m^2_{(3)})^2Z^2_{(5)}\right)\left(1-Z^2_{(3)}Z^2_{(5)}\right)\left(1-Z^3_{(5)}\right) \\ &\left(1-\bar{\mathcal{Y}}^2_{(3)}Z^2_{(5)}\right)\left(1-m^2_{(3)}\bar{\mathcal{Y}}_{(3)}\right)(1-\bar{\mathcal{Y}}_{(3)}Z_{(3)})\left(1-\bar{\mathcal{Y}}^2_{(3)}Z_{(5)}\right).\end{aligned} \tag{154}$$

Since the multi-graded numerator is quite extensive, we provided it in the accompanying file [30]. We see that by setting the parameters of the 2HDM potential to zero, we get the HS from the GNM Yukawa neutrino coupling Y_N, from eq. (121). Alternatively, by disregarding $\bar{\mathcal{Y}}_{(3)}$, we get the HS for the 2HDM in eq. (138). The main additions of combining the two are the mixed invariants. The primary (algebraically independent) ones are found in the denominator of the multi-graded HS (that is the last line in eq. (154)). These are:

$$\mathcal{I}_{0,2,1,0,0} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} \rangle, \qquad \mathcal{I}_{0,2,0,1,0} = \langle \bar{\mathcal{Y}}_{(3)} Z_{(3)} \rangle,$$
$$\mathcal{I}_{0,4,0,0,1} = \langle \bar{\mathcal{Y}}^2_{(3)} Z_{(5)} \rangle, \qquad \mathcal{I}_{0,4,0,0,2} = \langle \bar{\mathcal{Y}}^2_{(3)} Z^2_{(5)} \rangle. \tag{155}$$

These correspond to independent physical parameters describing neutrino-Higgs interactions. This mixed invariant ring also adds a significant number of secondary generating invariants. More specifically, using the analysis of the corrected mono-graded PL in eq. (151), the number of secondary mixed generators is $47 - 19 - 1 - 4 = 23$. The multi-graded PL up to the lowest order negative terms allows us to find their composition quite easily. We provide the expression for the PL in [30]. The newly added mixed secondary invariants are:

1. 3 generating invariants at order-4:

$$\mathcal{I}_{0,2,0,1,1} = \langle \bar{\mathcal{Y}}_{(3)} Z_{(3)} Z_{(5)} \rangle, \quad \mathcal{I}_{0,2,1,1,0} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(3)} \rangle, \quad \mathcal{I}_{0,2,1,0,1} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(5)} \rangle. \tag{156}$$

2. 6 generators at order-5:

$$\mathcal{I}_{0,2,0,1,2} = \langle \bar{\mathcal{Y}}_{(3)} Z_{(3)} Z^2_{(5)} \rangle, \qquad \mathcal{I}_{0,2,1,0,2} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z^2_{(5)} \rangle,$$
$$\mathcal{I}_{0,2,0,2,1} = \langle \bar{\mathcal{Y}}_{(3)} Z^2_{(3)} Z_{(5)} \rangle, \qquad \mathcal{I}^{(1)}_{0,2,1,1,1} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(3)} Z_{(5)} \rangle, \tag{157}$$
$$\mathcal{I}^{(2)}_{0,2,1,1,1} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(3)} Z_{(5)} \rangle, \qquad \mathcal{I}_{0,2,2,0,1} = \langle \bar{\mathcal{Y}}_{(3)} (m^2_{(3)})^2 Z_{(5)} \rangle.$$

3. 6 at order-6:

$$\mathcal{I}_{0,2,0,2,2} = \langle \bar{\mathcal{Y}}_{(3)} Z^2_{(3)} Z^2_{(5)} \rangle, \qquad \mathcal{I}^{(1)}_{0,2,1,1,2} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(3)} Z^2_{(5)} \rangle,$$
$$\mathcal{I}^{(2)}_{0,2,1,1,2} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(3)} Z^2_{(5)} \rangle, \qquad \mathcal{I}_{0,2,2,0,2} = \langle \bar{\mathcal{Y}}_{(3)} (m^2_{(3)})^2 Z^2_{(5)} \rangle, \tag{158}$$
$$\mathcal{I}_{0,4,0,1,1} = \langle \bar{\mathcal{Y}}^2_{(3)} Z_{(3)} Z_{(5)} \rangle, \qquad \mathcal{I}_{0,4,1,0,1} = \langle \bar{\mathcal{Y}}^2_{(3)} m^2_{(3)} Z_{(5)} \rangle.$$

4. 5 at order-7:

$$\mathcal{I}_{0,2,0,2,3} = \langle \bar{\mathcal{Y}}_{(3)} Z^2_{(3)} Z^3_{(5)} \rangle, \qquad \mathcal{I}_{0,2,1,1,3} = \langle \bar{\mathcal{Y}}_{(3)} m^2_{(3)} Z_{(3)} Z^3_{(5)} \rangle,$$
$$\mathcal{I}_{0,2,2,0,3} = \langle \bar{\mathcal{Y}}_{(3)} (m^2_{(3)})^2 Z^3_{(5)} \rangle, \qquad \mathcal{I}_{0,4,0,1,2} = \langle \bar{\mathcal{Y}}^2_{(3)} Z_{(3)} Z^2_{(5)} \rangle, \tag{159}$$
$$\mathcal{I}_{0,4,1,0,2} = \langle \bar{\mathcal{Y}}^2_{(3)} m^2_{(3)} Z^2_{(5)} \rangle.$$

5. 2 at order-8:

$$\mathcal{I}_{0,4,0,1,3} = \langle \bar{\mathcal{Y}}^2_{(3)} Z_{(3)} Z^3_{(5)} \rangle, \qquad \mathcal{I}_{0,4,1,0,3} = \langle \bar{\mathcal{Y}}^2_{(3)} m^2_{(3)} Z^3_{(5)} \rangle. \tag{160}$$

6. 1 at order-9:

$$\mathcal{I}_{0,6,0,0,3} = \langle \bar{\mathcal{Y}}^3_{(3)} Z^3_{(5)} \rangle. \tag{161}$$

While secondary invariants are not independent, they tend to play an important role in describing CP violation in a given model [9]. Using CP transformations, discussed previously, we can identify the CP-odd characteristics of this invariant subring:

$$\bar{\mathcal{Y}}_{ab} = [\langle (Y_E)_a (Y_E^\dagger)^c \rangle_{SU(3)_L}] \epsilon_{bc} \xrightarrow{CP} [\langle (Y_N^*)^i_a (Y_N^T)^c_l \rangle_{SU(3)_L}] \epsilon_{bc} = -\bar{\mathcal{Y}}_{ab}. \tag{162}$$

Meaning that under CP [7]:

$$\begin{aligned} &\bar{\mathcal{Y}}_{(3)} \xrightarrow{CP} -\bar{\mathcal{Y}}_{(3)}, \\ &m^2_{(3)} \xrightarrow{CP} -m^2_{(3)}, \\ &Z_{(3)} \xrightarrow{CP} -Z_{(3)}, \\ &Z_{(5)} \xrightarrow{CP} Z_{(5)}. \end{aligned} \tag{163}$$

We find that the rules for identifying CP invariants with the added adjoint Yukawa element are equivalent to the ones found in [7]. Specifically, we see that CP odd invariants in this ring are the ones constructed from an odd number of adjoint building blocks ($m^2_{(3)}$, $Z_{(3)}$, and in our case also $\bar{\mathcal{Y}}_{(3)}$). With this in mind, we can scan the generating set to find that:

- None of the primary generating invariants are CP-odd.
- The secondary set described above includes 18 CP-odd secondary invariants, with:
  - 4 at order-5: $\mathcal{I}_{0,2,0,2,1}$, $\mathcal{I}^{(1)}_{0,2,1,1,1}$, $\mathcal{I}^{(2)}_{0,2,1,1,1}$, $\mathcal{I}_{0,2,2,0,1}$.
  - 6 at order-6: $\mathcal{I}_{0,2,0,2,2}$, $\mathcal{I}^{(1)}_{0,2,1,1,2}$, $\mathcal{I}^{(2)}_{0,2,1,1,2}$, $\mathcal{I}_{0,2,2,0,2}$, $\mathcal{I}_{0,4,0,1,1}$, $\mathcal{I}_{0,4,1,0,1}$.
  - 5 at order-7: $\mathcal{I}_{0,2,0,2,3}$, $\mathcal{I}_{0,2,1,1,3}$, $\mathcal{I}_{0,2,2,0,3}$ $\mathcal{I}_{0,4,0,1,2}$, $\mathcal{I}_{0,4,1,0,2}$.
  - 2 at order-8: $\mathcal{I}_{0,4,0,1,3}$, $\mathcal{I}_{0,4,1,0,3}$.
  - 1 at order-9: $\mathcal{I}_{0,6,0,0,3}$.

The vanishing of these 18 CP-violating invariants would strongly suggest CP non-violation in 2HDM-Majorana neutrino mixing. However, the actual number of such conditions might be lower due to higher order syzygies between these invariants, as was done for the "purely" 2HDM ring in [7]. Finding these could be the subject of further study. We also want to draw attention to the order-9 CP-odd invariant $\mathcal{I}_{0,6,0,0,3}$ as its CP-violating properties are completely independent of the CP-oddness of the 2HDM sector. This can be seen by it being the only CP-odd invariant that does not contain any of the adjointly transforming 2HDM potential building blocks that are otherwise responsible for the CP-oddness in the 2HDM model.

5 Summary and conclusions

We described the flavour space of the Grimus-Neufeld model, its lepton and Higgs sectors in terms of global unitary field and coupling transformations (Table.1 and Table.2). Furthermore, we defined the building blocks (Table. 3) akin to those in references [13] and [7] however, we found that in a model with multiple flavour symmetries, using these to generate the HS could lead to overcounting of flavour invariant constructions. These redundancies appear due to the cyclicity of the invariant index contractions. The inclusion of both $SU(3)$ and $SU(2)$ flavour symmetries in the GNM model runs into this exact issue, as described in section 3.3.1, prompting us to use the fundamental representations of the flavour space in most of our calculations.

We verified the previous results of HS and the PL expressions for the SM eq. (110), the 1-Majorana neutrino seesaw model eq. (119) and 2HDM eq. (138) (from the [12], [13] and [7] respectively). All calculations yielded the equivalent results, which supported our use of the method.

We also calculated the mono-graded HS and PL in their Taylor expanded form (using *DECO* [18]) for the GNM and its component parts in the leptonic Yukawa sector. Due to the additional Higgs flavour symmetry the latter, unlike the SM quark sector yielded significantly more generators as well as the non-terminating PL. For the neutrino Yukawa coupling Y_N we found that its subring in the GNM contains two independent generating invariants, leading us to conclude that the non-vanishing of the quartic invariant is a necessary Yukawa sector condition for the emergence of the second light neutrino mass.

We generated the mono-graded HS and PL for combined Yukawa and Higgs sector. These are presented in eq. (142) and eq. (143). Using the previously gained knowledge of its component parts, we categorized the invariant ring of the GNM into purely leptonic, Higgs and mixed invariant subrings (eq. (144)) and were able to factor out the mixed part. We computed the multi-graded expressions of the PL, showing the generating basis for the invariant ring. We listed the PL terms up to the order-12 according to these categories in the accompanying file [30].

Focusing on a specific mixed invariant subring - the 2HDM-Y_N mixed invariants, we were able to generate the rational HS expression. This allowed us to identify the primary generating set of invariants eq. (154). Using the analysis of the multi-graded PL, we were able to distinguish the generating basis of secondary invariants (using the conjecture from [13]). Comparing with the pure 2HDM invariant subring from [14], we found that the introduction of scalar-Yukawa mixing results in 27 additional mixed generators, with 4 of them being primary and 23 secondary (see section 4.4.1). Among the latter, we identified 18 of them as CP-odd, with particularly the invariant of composition $I_{0,6,0,0,3}$ being independent of the CP-violating properties of the 2HDM potential. We see a more extensive exploration of the compositions and interrelationships between these invariants, as well as the invariants in other mixed parts of model as a possible avenue for a further inquiry into the invariant ring of the GNM and its CP violating properties.

A Haar measures and characters

As outlined previously, the main pieces of information that come into it, besides the couplings, are the characters of their corresponding representations, as well as the Haar measures of each group. We can list the latter by referring to [31]:

$$
\begin{aligned}
d\mu_{U(1)}(x) =& \frac{1}{2\pi i}\frac{dx}{x}, \\
d\mu_{SU(2)}(x) =& \frac{1}{2\pi i}\frac{dx}{x}\left(1-x^2\right), \\
d\mu_{SU(3)}(x_1,x_2) =& \frac{1}{2(2\pi i)^2}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\left(1-x_1x_2\right)\left(1-\frac{x_1^2}{x_2}\right)\left(1-\frac{x_2^2}{x_1}\right),.
\end{aligned}
\tag{164}
$$

The characters we use are [10], [7]:

$$
\begin{aligned}
\text{A singlet:}\quad & \chi_1 = 1, \\
U(1)\text{ with charge }Q\text{:}\quad & \chi_Q^{U(1)}(x) = x^Q, \\
SU(2)\text{ fundamental:}\quad & \chi_2^{SU(2)}(x) = x+\frac{1}{x}, \\
\text{adjoint:}\quad & \chi_3^{SU(2)}(x) = x^2+1+\frac{1}{x^2}, \\
\text{quintuplet:}\quad & \chi_5^{SU(2)}(x) = x^4+x^2+1+\frac{1}{x^2}+\frac{1}{x^4}, \\
SU(3)\text{ fundamental:}\quad & \chi_3^{SU(3)}(x_1,x_2) = x_1+\frac{x_2}{x_1}+\frac{1}{x_2}, \\
\text{anti-fundamental:}\quad & \chi_{\bar{3}}^{SU(3)}(x_1,x_2) = x_2+\frac{x_1}{x_2}+\frac{1}{x_1}, \\
\text{adjoint:}\quad & \chi_8^{SU(3)}(x_1,x_2) = x_1x_2+\frac{x_2^2}{x_1}+\frac{x_1^2}{x_2}+2+\frac{x_2}{x_1^2}+\frac{x_1}{x_2^2}+\frac{1}{x_1x_2}.
\end{aligned}
\tag{165}
$$

The character of a product of two groups is a product of their characters:

$$
\chi^{G\times H}_{\mathrm{R}_G\otimes\mathrm{R}_H}(x_j,x'_k) = \chi^G_{\mathrm{R}_G}(x_j)\chi^H_{\mathrm{R}_H}(x'_k)
\tag{166}
$$

References

[1] B.A. Dobrescu. "Beyond the standard model". In: *2011 European School of High-Energy Physics, ESHEP 2011 - Proceedings* (Apr. 2014), pp. 119–149.

[2] W. Grimus and H. Neufeld. "Radiative neutrino masses in an SU(2) × U(1) model". In: *Nuclear Physics B* 325.1 (1989), pp. 18–32. ISSN: 0550-3213.

[3] Goran Senjanović. "Neutrino mass: From LHC to grand unification". In: *Rivista del Nuovo Cimento* 34 (Jan. 2011), pp. 1–68. DOI: `10.1393/ncr/i2011-10061-8`.

[4] Marcel Krause. "On the Renormalization of the Two-Higgs-Doublet Model". MA thesis. KIT, Karlsruhe, TP, 2016.

[5] F. J. Botella and João P. Silva. "Jarlskog-like invariants for theories with scalars and fermions". In: *Physical Review D* 51.7 (Apr. 1995), pp. 3870–3875. ISSN: 0556-2821. DOI: `10.1103/physrevd.51.3870`. URL: `http://dx.doi.org/10.1103/PhysRevD.51.3870`.

[6] C. Jarlskog. "Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation". In: *Phys. Rev. Lett.* 55 (10 Sept. 1985), pp. 1039–1042. DOI: `10.1103/PhysRevLett.55.1039`. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.55.1039`.

[7] Andreas Trautner. "Systematic construction of basis invariants in the 2HDM". In: *Journal of High Energy Physics* 2019.5 (May 2019). ISSN: 1029-8479. DOI: `10.1007/jhep05(2019)208`. URL: `http://dx.doi.org/10.1007/JHEP05(2019)208`.

[8] Elizabeth Jenkins and Aneesh V. Manohar. "Rephasing invariants of quark and lepton mixing matrices". In: *Nuclear Physics B* 792.1–2 (Mar. 2008), pp. 187–205. ISSN: 0550-3213. DOI: `10.1016/j.nuclphysb.2007.09.031`. URL: `http://dx.doi.org/10.1016/j.nuclphysb.2007.09.031`.

[9] Elizabeth E Jenkins and Aneesh V Manohar. "Algebraic structure of lepton and quark flavor invariants andCPviolation". In: *Journal of High Energy Physics* 2009.10 (Oct. 2009), pp. 094–094. ISSN: 1029-8479. DOI: `10.1088/1126-6708/2009/10/094`. URL: `http://dx.doi.org/10.1088/1126-6708/2009/10/094`.

[10] Landon Lehman and Adam Martin. "Hilbert series for constructing Lagrangians: Expanding the phenomenologist's toolbox". In: *Physical Review D* 91.10 (May 2015). ISSN: 1550-2368. DOI: `10.1103/physrevd.91.105014`. URL: `http://dx.doi.org/10.1103/PhysRevD.91.105014`.

[11] Sergio Benvenuti et al. "Counting BPS operators in gauge theories: quivers, syzygies and plethystics". In: *Journal of High Energy Physics* 2007.11 (Nov. 2007), pp. 050–050. ISSN: 1029-8479. DOI: `10.1088/1126-6708/2007/11/050`. URL: `http://dx.doi.org/10.1088/1126-6708/2007/11/050`.

[12] Amihay Hanany et al. "Hilbert series for flavor invariants of the Standard Model". In: *Journal of High Energy Physics* 2011.3 (Mar. 2011). ISSN: 1029-8479. DOI: 10.1007/jhep03(2011)096. URL: http://dx.doi.org/10.1007/JHEP03(2011)096.

[13] Bingrong Yu and Shun Zhou. "Hilbert series for leptonic flavor invariants in the minimal seesaw model". In: *Journal of High Energy Physics* 2021.10 (Oct. 2021). ISSN: 1029-8479. DOI: 10.1007/jhep10(2021)017. URL: http://dx.doi.org/10.1007/JHEP10(2021)017.

[14] Miguel P. Bento et al. "A fully basis invariant symmetry map of the 2HDM". In: *Journal of High Energy Physics* 2021.2 (Feb. 2021). ISSN: 1029-8479. DOI: 10.1007/jhep02(2021)220. URL: http://dx.doi.org/10.1007/JHEP02(2021)220.

[15] M. P. Bento. "The invariant space of multi-Higgs doublet models". In: *Journal of High Energy Physics* 2021.5 (May 2021). ISSN: 1029-8479. DOI: 10.1007/jhep05(2021)146. URL: http://dx.doi.org/10.1007/JHEP05(2021)146.

[16] Simon Calò, Coenraad Marinissen, and Rudi Rahn. *DECO*. https://github.com/cbmarini/DECO.git. 2023.

[17] Coenraad B. Marinissen, Rudi Rahn, and Wouter J. Waalewijn. "..., 83106786, 114382724, 1509048322, 2343463290, 27410087742, ... efficient Hilbert series for effective theories". In: *Physics Letters B* 808 (Sept. 2020), p. 135632. ISSN: 0370-2693. DOI: 10.1016/j.physletb.2020.135632. URL: http://dx.doi.org/10.1016/j.physletb.2020.135632.

[18] Simon Calò, Coenraad Marinissen, and Rudi Rahn. "Discrete symmetries and efficient counting of operators". In: *Journal of High Energy Physics* 2023.5 (May 2023). ISSN: 1029-8479. DOI: 10.1007/jhep05(2023)215. URL: http://dx.doi.org/10.1007/JHEP05(2023)215.

[19] Yan Xiao, Yang-Hui He, and Cyril Matti. "Standard model plethystics". In: *Physical Review D* 100.7 (Oct. 2019). ISSN: 2470-0029. DOI: 10.1103/physrevd.100.076001. URL: http://dx.doi.org/10.1103/PhysRevD.100.076001.

[20] Shigeru Mukai. *An Introduction to Invariants and Moduli*. Ed. by W. M.Translator Oxbury. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2003.

[21] Holger Schellwat. *A Gentle Introduction to a Beautiful Theorem of Molien*. 2017. arXiv: 1701.04692 [math.GM]. URL: https://arxiv.org/abs/1701.04692.

[22] A. V. Bednyakov. *On the scalar sector of 2HDM: ring of basis invariants, syzygies, and six-loop renormalization-group equations*. 2025. arXiv: 2501.14087 [hep-ph]. URL: https://arxiv.org/abs/2501.14087.

[23] R. L. Workman et al. "Review of Particle Physics". In: *PTEP* 2022 (2022), p. 083C01. DOI: 10.1093/ptep/ptac097.

[24] Simonas Draukšas. "Renormalization of the Grimus-Neufeld model". PhD thesis. Vilnius U., June 2024. DOI: 10.15388/vu.thesis.614.

[25] M. A. Aguilar and M. Socolovsky. *The Universal Covering Group of U(n) and Projective Representations*. 1999. arXiv: math-ph/9911028 [math-ph]. URL: https://arxiv.org/abs/math-ph/9911028.

[26] Howard Georgi. *Lie algebras in particle physics*. 2nd ed. Vol. 54. Reading, MA: Perseus Books, 1999.

[27] Stefan Keppeler. "Birdtracks for SU(N)". In: *SciPost Physics Lecture Notes* (Sept. 2018). DOI: 10.21468/scipostphyslectnotes.3. URL: http://dx.doi.org/10.21468/SciPostPhysLectNotes.3.

[28] Miguel P. Bento, Joao P. Silva, and Andreas Trautner. *The Basis Invariant Flavor Puzzle*. 2023. arXiv: 2308.00019 [hep-ph]. URL: https://arxiv.org/abs/2308.00019.

[29] Tatsuo Suzuki. "Some formulas for invariant phases of unitary matrices by Jarlskog". In: *Journal of Mathematical Physics* 50.12 (Dec. 2009). ISSN: 1089-7658. DOI: 10.1063/1.3272544. URL: http://dx.doi.org/10.1063/1.3272544.

[30] Jokūbas Mišeikis Vytautas Dūdėnas. *HS*. https://github.com/vytautasdudenas/HS.git. 2026.

[31] Amihay Hanany, Noppadol Mekareeya, and Giuseppe Torri. "The Hilbert series of adjoint SQCD". In: *Nuclear Physics B* 825.1–2 (Jan. 2010), pp. 52–97. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2009.09.016. URL: http://dx.doi.org/10.1016/j.nuclphysb.2009.09.016.

AROMATO BAZĖS INVARIANTAI GRIMUS-NEUFELD MODELYJE

Jokūbas Mišeikis

Santrauka

Šiame darbe mes analizavome Grimus-Neufeld modelio (GNM) aromato (angl. *flavour*) bazės invariantų žiedinę struktūrą. Pagrindinis darbo tikslas - pradėti GNM leptonų ir Higso sektorių generuojančių aromato invariantų klasifikaciją, pasinaudojant Hilberto serija (HS) ir pletistiniu formalizmu. Hilberto serija - tai serija turinti visas invariantiškas, tiesiškai nepriklausomas modelio parametrų kombinacijas. Žinant HS išraišką galima sužinoti bazės invariantų žiedo generatorių skaičių, kompoziciją ir tarpusavio sąryšius. Atitinkamo modelio HS galima surasti, pasinaudojant pletistine eksponente PE. Žinant HS, generatorių bazę ir jų tarpusavio sąryšius surasti galima su pletisinio logaritmo PL serija.

Aromato būsenos - tai skirtingos modelio laukų masės tikrinės būsenos. Standartiniame modelyje (SM) yra įtraukiami 3-jų aromatų kairiojo-chirališkumo leptoniniai laukai, 3-jų aromatų dešiniojo-chirališkumo elektroninis laukas ir 1 aromato Higso laukas. GNM tai SM plėtinys, pridedantis papildoma Higso lauko aromatą (t.y. naudojamas 2 Higso dubletų modelis (2HDM)) bei 1 papildomą Majorana neutriną (svyruoklės modelis - *seesaw model*).

Pradžiai, naudodamiesi GNM Yukawa ir skaliarinio potencialo Lagranžo tankio funkcijomis, apsibrėžėme modelio laukų aromato erdvę ir jos transformacijas. Aromato bazė GNM leptonų ir Higso sektoriuose yra keičiama, transformuojant laukų sąveikos parametrus atitinkamomis bendros aromato grupės: $U(3)_E \times U(3)_L \times U(2)_\phi \times U(1)_N$, reprezentacijomis. Šia informaciją panaudojome, konstruodami HS generuojančias funkcijas, įvairių modelių aromato invariantų žiedams.

Pavyko reprodukuoti ir patvirtinti ankstesnių straipsnių rezultatus, kuriuose buvo sugeneruotos HS: SM kvarkų sektoriaus, 1-Majorana neutrino *seesaw* modelio ir 2HDM aromatų bazės invariantams.

Pasinaudoja HS generuojančia funkcija ir programiniu paketu *DECO*, apskaičiavome HS GNM leptonų Yukawa sektoriui ir bendrai teorijai kombinuojančiai skaliarų ir leptonų Yukawa sektorius, iki 14 eilės.

Analizavome mišrių invariantų aibe, kombinuojančią Majorana neutrino Yukawa sąveikos konstantą Y_N ir 2HDM potencialo parametrus. Sugeneravome pilną HS bei PL. Naudojantis šiomis serijomis identifikavome 4 nepriklausomus bei 23 priklausomus generuojančius mišrius invariantus. Tarp pastarųjų, 18 kategorizavome kaip potencialiai laužančius krūvio jungtinumo-lyginumo simetriją (angl. *charge conjugation - parity* CP).