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EFFECTS OF TIME-DELAYED INTERACTIONS ON ORDERING TIMES OF ELEMENTARY SPIN SYSTEMS

Master’s final thesis

Photonic and nanotechnology

Study program

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Contents

1. Introduction

The study of complex systems has become an important research field connecting physics, mathematics, biology, economics, computer science, and social sciences. Many real-world systems consist of large numbers of interacting elements whose collective behavior cannot be understood only by analyzing individual components. Instead, local interactions between system elements may generate macroscopic phenomena such as self-organization, synchronization, pattern formation, or collective decision-making[1,2]. Similar collective effects appear in systems of very different physical nature, including transportation networks, ecosystems, neural systems, financial markets and human social structures[2].

The application of statistical physics methods to social systems led to the development of sociophysics and opinion dynamics research. There, individuals are represented as interacting agents whose states evolve according to stochastic interaction rules[3]. These models became useful for investigating social influence, information spreading, collective behavior, political polarization and consensus formation processes[3,4]. Because many social interactions involve large populations and probabilistic behavior, statistical physics provides a natural framework for studying the emergence of macroscopic social phenomena from microscopic interaction mechanisms.

Among the many models proposed in opinion dynamics research, the voter model became one of the simplest and most widely studied frameworks for analyzing consensus formation in interacting systems[5]. In the classical voter model, agents possess one of two opinions and randomly copy the opinion of neighboring agents. Despite its simplicity, the model exhibits nontrivial stochastic dynamics and became an important reference system for investigating nonequilibrium ordering processes[5,6].

Later studies introduced various modifications of voter-type dynamics to describe more realistic interaction mechanisms. In many real systems, interactions are not instantaneous and individuals often require time to process information, reconsider previously adopted opinions or temporarily stop participating in interactions. Such effects may significantly influence collective ordering behavior and alter consensus formation dynamics[7]. Delayed interactions and internal latent states therefore became important extensions of classical voter dynamics.

Another important direction involves collective information feedback mechanisms. Instead of interacting only through direct local imitation, agents may also respond to globally published information such as opinion surveys, popularity indicators or delayed information propagation. Polling-based interaction mechanisms introduce additional nonequilibrium effects because the system evolution may depend on outdated or delayed information about the collective state itself [8].

The goal of this thesis is to investigate how delayed interactions[9] and latency[10] mechanisms influence ordering processes in voter-type systems under assumption that all particles (agents) can interact with each other. The analysis includes the classical voter model, the Lambiotte et al. latent voter model[10] and its reversed modification, as well as polling-based models with delayed information updates[9]. Particular attention is devoted to ordering times, exit probabilities, and the influence of delayed interactions on collective dynamics.

Tasks:

1) Study the dependence of the mean ordering time and exit-state probability on the initial conditions in the voter model. Analyze how model parameters influence these characteristics.

2) Study the dependence of the mean ordering time and exit-state probability on the initial conditions in the latent voter model (as proposed in Ref. 10). Analyze how model parameters influence these characteristics.

3) Modify the latent voter model by changing the order of "state freezing" and "state change" steps. Investigate whether this modification leads to fundamental changes in the phenomenology of ordering times in the model.

4) Cast the polled and poll-delayed voter models (as proposed in Ref. 9) into the numerical framework developed in earlier tasks, and to determine whether these models exhibit fundamentally different ordering-time phenomenology.

2. Literature overview

2.1 Complex systems and opinion dynamics

Statistical physics provides one of the main theoretical frameworks for studying opinion behavior. Spin systems are widely used as simplified models for investigating ordering processes and nonequilibrium dynamics. Originally, spin models were developed to describe magnetic interactions in condensed physics. Later, similar approaches became useful in many other scientific fields because binary-state systems can also represent competing decisions, alternative social opinions, or different behavioral states[3]. The connection between statistical physics and social systems motivated the development of opinion dynamics models[3]. In these models, individuals are represented as interacting agents (or typically used physics terminology – particles) whose opinions evolve according to simple stochastic (i.e., involving randomness) interaction rules Such approaches are commonly used to study social influence, information spreading, collective decision-making, political polarization, epidemic spreading, and rumor propagation in complex social networks[3].

Among the many models proposed in opinion dynamics research, the voter model is the most widely studied, as it is simple and easily mathematically tractable. In the classical voter model, agents (particles) possess binary opinions (states) and randomly adopt the opinion of selected neighboring agents. Despite the simplicity of the interaction mechanism, the model exhibits stochastic dynamics which allow to explore consensus formation and nonequilibrium ordering processes[5] in various theoretical and real-world scenarios.

Because of its simplicity and flexibility, the voter model later became a basis for numerous generalized models incorporating additional mechanisms such as memory effects, delayed interactions, or external influences[7]. These modifications are important because interactions in real social systems are rarely instantaneous. Individuals often require time to process information, reconsider previously adopted opinions or react to external events. Similarly, information propagation in social and communication networks may involve delays caused by limited communication speed, asynchronous interactions or temporary inactivity of individuals. Such effects may significantly influence collective dynamics by modifying ordering times, stabilizing metastable states, or generating oscillatory behavior[3,7,11].

2.2 Classical voter model

The classical voter model was originally introduced in probability theory as an interacting system describing the spread of competing states among neighboring agents[12,13]. Later, it became widely used in statistical physics and sociophysics because simple local interaction rules are capable of producing nontrivial collective behavior[3].

In the classical voter model, each agent (particle) is assigned a binary opinion (state) representing one of two possible opinions. These states are commonly denoted by spin-like variables

$$s_i \in \{-1,+1\}, \tag{1}$$

where s_i represents the opinion of agent i. In some formulations, binary variables 0 and 1 are used instead of spin notation, but these are usually just labels used to denote particular states. The system evolves through stochastic interactions between neighboring agents.

Interactions are based on a simple imitation mechanism. During each elementary update step, a random agent i selected together with one of its neighboring agents j. The selected agent then adopts the opinion of the neighbor,

$$s_i \rightarrow s_j. \tag{2}$$

As a result, the probability of changing state depends entirely on the local neighborhood configuration. Unlike equilibrium spin systems such as the Ising model, the voter model does not contain an energy function or temperature parameter. The dynamics are therefore driven purely by stochastic imitation rather than by energy minimization[3].

The model may be studied on different interaction topologies (or graphs), including one-dimensional chains, two-dimensional lattices, complex networks, or fully connected systems. In the fully connected case, often referred to as the mean-field limit, every agent may interact with every other agent in the system.

In the previous notation, the opinion state was represented by spin variables $s_i \in \{-1,+1\}$. However, when studying exit probabilities and opinion fractions, it is more convenient to use an equivalent binary representation $s_i \in \{0,1\}$, where $s_i = 1$ denotes agents holding opinion 1, and $s_i = 0$ denotes agents holding the opposite opinion. In this notation, the fraction of agents holding opinion 1 is written as

$$x(t)=\frac{1}{N}\sum_{i=1}^{N} s_i, \tag{3}$$

where N denotes the total number of agents and $x(t)$ represents the fraction of agents in opinion state 1 at time t.

The classical voter model can be described with algorithm 1.

Algorithm 1. Classical voter model

1. Set the number of agents N, initial opinion fraction x_o, and copying probability p.
2. Initialize each agent with an opinion state.
3. Select a random agent i.
4. Select a random interaction partner j.
5. With probability p, agent i copies the opinion of agent j:

$$s_i \rightarrow s_j. \tag{4}$$

6. Update the global opinion fraction $x(t)$.
7. Repeat Steps 3–6 until all agents hold the same opinion.

The stochastic evolution of the system eventually leads toward one of two fully ordered states. For the classical voter model these states are absorbing, as no further changes are possible. These states also imply that consensus was reached. For any model, fully ordered states will satisfy $x = 0$ or $x = 1$.

The ordering time T measures how long the system requires to reach a fully ordered state. It represents the number of update steps required for the system to reach consensus. Minimum ordering time defined by Eq. (5).

$$T = \min\{t \geq 0 : x(t) \in \{0,1\} | x(0) = x_0\}, \tag{5}$$

Where T – ordering time; t – current time (steps); x – fraction of the state; x_o – initial fraction of the state. The minimum ordering time $T = 0$ corresponds to an initially ordered configuration. In the classical voter model, interactions are symmetric, leading to conservation of the average magnetization[3,12].

$$P_E = P(x(T) = 1), \tag{6}$$

Consequently, the probability of reaching consensus state $x = 1$ equals the initial fraction x_o.

$$P_E = E(x(t)) = x_0, \tag{7}$$

In finite systems, stochastic fluctuations eventually drive the dynamics toward one of the absorbing consensus states. The corresponding ordering time depends strongly on system size and interaction dynamics[14].

2.3 Fully connected graph assumption

In interacting particle systems, the dynamics of the model strongly depend on the underlying interaction topology[15]. In lattice-based systems, each particle interacts only with a limited number of neighboring particles, as determined by the geometry of the lattice. Such local interactions may lead to spatial correlations and domain formation during the evolution of the system[16].

A commonly used simplification in statistical physics is mean-field approximation. In this approach, spatial structure and local correlations are neglected by assuming that every particle may interact with every other particle with equal probability. The system effectively becomes fully connected and may be represented by a complete graph containing N interacting particles, instead of predefined lattice or network. For the sake of simplicity, the focus will be on the ordering times in a fully connected case. Under the fully connected graph assumption, the probability of interacting with an agent (particle) holding a particular opinion (occupying certain state) is equal to the global fraction of agents (particle) holding that opinion (occupying that state). If $x(t)$ denotes the fraction of agents in opinion state 1, then the probability of selecting agent with opinion "1" is

$$P(s=1) = x(t), \tag{8}$$

while the probability of selecting an agent with the opposite opinion is

$$P(s=0) = 1 - x(t). \tag{9}$$

As a result, the dynamics are described using global quantities rather than actual microscopic state (opinion) configurations.

In this thesis, interactions are studied within the fully connected graph interaction topology, where each agent may interact with any other agent in the system. This represents the simplest possible interaction topology and significantly simplifies both numerical simulations and analytical treatment. Since explicit network structure and local neighborhoods are not considered, the dynamics may be analyzed using the mean-field approximation. Such an approach is frequently used as a benchmark when investigating voter-type models, allowing later comparisons with systems possessing more complex interaction topologies[16].

2.4 Lambiotte et al. latent voter model

The classical voter model assumes that agents may repeatedly change their opinion without any restrictions. After copying the opinion of a neighboring agent, the same agent may immediately participate in another interaction and change its state again. Although this assumption simplifies dynamics, it is often unrealistic when describing social or information-based systems. In many real situations, individuals require some time before becoming susceptible to influence again. Such behavior may arise due to memory effects, hesitation, social commitment or temporary resistance to changing recently adopted opinions.

To account for such mechanisms, several modifications of the voter model introducing internal time scales were proposed. One important example is the latent voter model introduced by Lambiotte and collaborators[10,17]. In this model, agents are characterized not only by their opinion state, but also by an additional activity state describing whether the agent is currently able to change its opinion. As a result, the state of each agent becomes two-dimensional.

Similarly to the classical voter model, agents possess binary opinions represented by spin variables

$$s_i \in \{0, 1\}, \tag{10}$$

Additionally, every agent may exist in one of two activity states: flexible or inflexible.

Only flexible agents are allowed to change their opinion during interactions. Inflexible agents preserve their current opinion, although they may still influence neighboring agents. This distinction introduces a memory-like mechanism into dynamics because recently updated agents have become temporarily resistant to further opinion changes.

The interaction dynamics remain based on stochastic imitation. During an elementary update step, a random agent i and one of its neighboring agents j are selected. If the selected agent is flexible, it copies the opinion of the neighboring agent with probability p,

$$s_i \to s_j. \tag{11}$$

If the copying event changes the opinion of the selected agent, the agent immediately becomes inflexible. Consequently, recently updated agents cannot instantly change their opinion through another interaction. Inflexible agents recover their activity stochastically. During later update steps, if an inflexible agent is selected, it becomes flexible again with probability λ[10]. The parameter λ therefore determines the average duration of the latent period. Small values of λ correspond to long inflexible periods, while large values lead to faster recovery of flexibility.

The latent voter model can be described with algorithm 2.

Algorithm 2. Latent voter model

1. Set the number of agents N, initial opinion fraction x_o, copying probability p, and flexibility recovery probability λ.
2. Initialize each agent with an opinion state and a flexibility state. At the beginning, all agents are flexible.
3. Select a random agent i.
4. If agent i is flexible, select a random interaction partner j.
5. With probability p, agent i copies the opinion of agent j.
6. If the opinion of agent i change, agent i becomes inflexible.
7. If agent i is inflexible, it cannot change opinion, but it may recover flexibility with probability λ.
8. Update the global opinion fraction $x(t)$.
9. Repeat Steps 3–8 until consensus is reached.

The introduction of latent states significantly modifies the ordering dynamics of the system. In the classical voter model, rapid local fluctuations may occur because agents can repeatedly change their opinion within short time intervals, while in latent model mechanism suppresses such rapid fluctuations by temporarily freezing recently updated agents[10]. As a result, the evolution of the system becomes slower and the formation of consensus may require substantially longer times.

Another important consequence of the latent mechanism is the breaking of magnetization conservation. In the classical voter model, the average magnetization remains conserved during the dynamics, which leads to the linear exit probability relation. However, previous studies demonstrated that the latent voter model no longer preserves this property[10]. The additional activity state therefore changes the probabilistic structure of the dynamics even though the interaction rules themselves remain symmetric.

The latent mechanism may also influence domain growth and coarsening processes. In lattice-based systems, inflexible agents reduce the mobility of domain boundaries because recently updated agents become temporarily resistant to further changes. This may slow down the elimination of competing opinion domains and alter the characteristic ordering behavior of the system [3].

From a physical perspective, the latent voter model introduces an additional internal time scale into the dynamics. The system evolution is determined not only by interaction topology and stochastic fluctuations, but also by the recovery dynamics of inflexible agents. Similar mechanisms appear in many nonequilibrium systems involving memory effects, delayed responses or internal relaxation processes [18].

Because of these properties, latent-state voter models became an important extension of classical voter dynamics. They provide a framework for investigating how delayed reactions and temporary inactivity influence consensus formation and collective ordering processes in stochastic interacting systems.

2.5 Reversed Lambiotte et al. latent voter model

The reversed latent voter model modifies the ordering of state transitions used in the standard formulation used by Lambiotte et al.. In the original latent voter model, an agent first changes its opinion and only afterwards becomes inflexible[10]. In the reversed version, the order of these processes is reversed: after a successful interaction, the agent first becomes inflexible while temporarily preserving its original opinion. The actual change occurs only after activity is restored.

As a result, opinion updates are no longer immediately visible to neighboring agents. During the inflexible period, the agent continues influencing its neighbors using its previous opinion state. The system therefore contains a delayed realization of microscopic state transitions. We introduce this novel variation of the latent voter model, to align it with conceptual idea behind the polled voter model, which will be discussed later.

The recovery of activity occurs stochastically with probability λ, similarly to the standard latent voter model. However, because pending opinion changes are temporarily hidden from the surrounding system, the macroscopic evolution becomes partially decoupled from the underlying interaction events.

The reversed latent voter model can be described with algorithm 3.

Algorithm 3. Reversed latent voter model

1. Set the number of agents N, initial opinion fraction x_o, copying probability p, and flexibility recovery probability λ.

2. Initialize each agent with an opinion state and a flexibility state. At the beginning, all agents are flexible.

3. Select a random agent i.

4. If agent i is flexible, select a random interaction partner j. With probability p, agent i accept the future opinion update but does not change its visible opinion immediately. If agent decides to accept the future opinion update, then agent i becomes inactive while still influencing other agents with its old opinion.

5. When inactive agent i is selected again, it recovers flexibility with probability λ. After recovery, the stored opinion update is applied and the visible opinion of agent i changes.

6. Update the global opinion fraction $x(t)$.

7. Repeat Steps 3–6 until consensus is reached.

This mechanism effectively introduces delayed information propagation into the dynamics. Local domains may temporarily preserve outdated opinions even after interaction events favor an opposite state. The reversed latency mechanism may alter fluctuation dynamics, slow down ordering processes, and modify consensus formation times[19].

From a physical perspective, the model represents a simple nonequilibrium system containing internal delayed-response dynamics. Similar mechanisms appear in systems with memory effects, relaxation processes or postponed state activation[18]. Although the modification itself is relatively simple, changing the order of inactivity and opinion updates may produce qualitatively different collective behavior compared to the standard latent voter model.

2.6 Polling and delayed-polling models

Polling-based voter models replace direct local interactions with collective information feedback. Instead of copying the opinion of a neighboring agent, individuals respond to periodically published measurements describing the global state of the system. Such mechanisms may represent public opinion surveys, popularity indicators or delayed information spreading in communication systems.

The dynamics are governed by a published opinion fraction x_p, obtained from a poll performed at discrete time intervals. During the evolution of the system, agents interact with a "virtual" neighbor generated according to the published polling result. If the published fraction of opinion 1 is equal to x_p, then the probability of generating the corresponding virtual state is

$$P(s=1)=x_p, \tag{12}$$

while the opposite state is generated with probability

$$P(s=0)=1\text{-}x_p. \tag{13}$$

The selected agent then copies the generated opinion with probability p.

In the standard polling model, poll results are published immediately after measurement. The interaction dynamics always depend on the most recent available information about the state of the system.

The delayed polling model introduces an additional time lag between poll measurement and publication. After a poll is conducted, its results become available only after a specified delay period. During this interval, agents continue interacting using previously published information rather than the current system state. Such mechanisms may represent situations where individuals react not to direct interpersonal interactions, but to delayed collective information distributed through media or public surveys.

This delayed feedback mechanism may substantially modify collective dynamics. Since the published information no longer exactly reflects the instantaneous microscopic configuration, the macroscopic evolution becomes partially decoupled from the underlying stochastic interactions. Delayed-response mechanisms of this type are known to generate oscillatory behavior and long-lived fluctuations in nonequilibrium systems[8].

The polling model can be described with algorithm 4.

Algorithm 4. Polling model

1. Set the number of agents N, initial opinion fraction x_o, copying probability p, and polling interval τ.
2. Initialize each agent with an opinion state.
3. Measure the initial global opinion fraction x_p and set it as the published polling result.
4. Select a random agent i.
5. Generate a virtual interaction state using the published polling result. The virtual state equals 1 with probability equal to the published opinion fraction.
6. With probability p, agent i copies the virtual interaction state.
7. After every τ update steps, measure the current opinion fraction and immediately publish the new polling result.
8. Update the global opinion fraction $x(t)$.
9. Repeat Steps 4–8 until consensus is reached.

The delayed polling model can be described with algorithm 5.

Algorithm 5. Delayed polling model

1. Set the number of agents N, initial opinion fraction x_o, copying probability p, and polling interval τ.
2. Initialize each agent with an opinion state.
3. Store the current opinion fraction as pending polling information x_p.
4. Select a random agent i.
5. Generate a virtual interaction state using the currently published polling result.
6. With probability p, agent i copies the virtual interaction state.
7. After every τ update step, publish the previously stored pending polling result.
8. Measure the current opinion fraction and store it as the new pending polling result.
9. Update the global opinion fraction $x(t)$.
10. Repeat Steps 4–9 until consensus is reached.

An important difference between polling models and latent voter models is the origin of the delay mechanism. In the model proposed by Lambiotte et al., delayed behavior arises from internal agent activity states while in polling models, the delay instead appears through collective information propagation dynamics. These properties, polling and delayed-polling models provide a useful framework for studying how delayed macroscopic feedback influences consensus formation and stochastic ordering processes[3].

2.7 Ordering times and exit probabilities

In this thesis, two main observables used to characterize voter-type dynamics are the ordering time and the exit probability[20]. These quantities describe how the system approaches consensus and which ordered state is eventually reached. If x_0 denotes the initial fraction of agents holding opinion 1, then let the exit probability P_E represent the probability that the system ultimately reaches the consensus state $x = 1$. In the classical voter model, conservation of average magnetization leads to the linear relation

$$P_E = E(x(t)) = x_0, \tag{14}$$

As a result, the probability of reaching a particular consensus state is directly determined by the initial system configuration[12].

Generalized voter-type models may lose this property. Additional mechanisms such as latent states, delayed interactions, or collective feedback can modify the probabilistic structure of the dynamics and produce deviations from the linear voter-model behavior[3] . The shape of the exit probability function therefore provides useful information about the influence of additional dynamical processes on consensus formation.

Another important observable is the ordering time T, often also referred to as the ordering time. This quantity describes the time required for the system to reach one of the fully ordered consensus states. In numerical simulations, ordering times are typically averaged over many independent realizations,

$$\langle T \rangle = \frac{1}{M} \sum_{k=1}^{M} T_k, \tag{15}$$

where T_k denotes the ordering time measured in the k-th realization and M is the total number of simulation runs.

Ordering times are particularly sensitive to delayed interactions and memory-like mechanisms[21]. Internal inactivity periods or delayed feedback may substantially slow down the propagation of information through the system and therefore increase the average time required to reach consensus [9,10].

In stochastic systems, ordering times are closely related to first-passage processes, where the dynamics are characterized by the time required to reach predefined thresholds (often implemented as absorbing states) for the first time [14]. Ordering times and exit probabilities are widely used to characterize stochastic voter-type dynamics and compare the behavior of different generalized models, including systems with delayed interactions or latent states.

2.8 Numerical simulation methods

Numerical simulations play an important role in the investigation of stochastic interacting systems, particularly when analytical treatment becomes difficult due to delayed interactions, latent states, or additional internal dynamical mechanisms [22].

Voter-type dynamics are commonly studied using Monte Carlo methods, where the system evolves through a sequence of stochastic elementary updates. During each update step, agents are selected randomly and modified according to the interaction rules of the corresponding model. Because the dynamics are probabilistic, individual realizations may produce substantially different trajectories even for identical initial conditions. Time is usually measured in Monte Carlo steps (MCS). One Monte Carlo step corresponds to N elementary update attempts, where N denotes the total number of agents in the system [23]. Such normalization allows comparison between systems of different sizes and interaction rates.

In mean-field simulations, interacting agents are selected randomly from the entire population. This removes spatial correlations associated with lattice geometry and allows investigation of the global stochastic behavior of the system. Since ordering processes are strongly influenced by fluctuations, measured observables are typically averaged over many independent realizations. Numerical simulations are therefore widely used to investigate quantities such as ordering times, exit probabilities, and stationary behavior in generalized voter-type models.

Monte Carlo approaches became one of the standard numerical tools in nonequilibrium statistical physics because they allow direct investigation of emergent collective dynamics generated by microscopic stochastic interaction rules[22].

3. Methodology

All models investigated were implemented numerically using Python. The simulations were performed using Monte Carlo dynamics, where during each elementary update step a random agent and a second interacting agent were selected from the system. Since the simulations were carried out making fully connected system assumption, every agent could interact with every other agent with equal probability.

The implementation was organized using a modular structure separating the simulation framework, model definitions and numerical experiments. Independent implementations were created for the classical voter model, latent voter model, reversed latent voter model, polling model and delayed polling model. This structure simplified modification of interaction rules and allowed direct comparison between different dynamical mechanisms.

In the classical voter model implementation, the selected agent copied the opinion of another randomly selected agent with probability p. The latent voter model additionally included an internal activity state. After changing its opinion, the agent became inflexible and could no longer update its state until activity was restored with probability λ. The reversed latent voter model was implemented using modified transition ordering. After a successful interaction, the selected agent first entered the inflexible state while temporarily preserving its previous opinion. The actual opinion change was applied only after reactivation. During the inflexible period, the agent continued influencing other agents using its old state.

Polling-based models were implemented using periodically measured global opinion fractions. Instead of copying directly from another agent, the selected agent interacted with a virtual state generated according to the published polling result. In the delayed polling model, newly measured polling information was stored and published only after a specified delay period.

Time was measured in Monte Carlo steps (MCS), where one Monte Carlo step corresponded to N elementary update attempts for a system containing N agents[22]. Different initial opinion fractions and parameter values were investigated in order to analyze how latency mechanisms and delayed information propagation influence the dynamics of the system.

For each parameter set, simulations were repeated multiple times yielding independent stochastic realizations of the model. Models were simulated according to **Algorithm 1-5** steps. The measured quantities included ordering time, exit probability and temporal evolution of the global opinion state. Numerical data analysis and visualization were performed in Python using NumPy and Matplotlib libraries.

4. Results and discussion

The following parameters were used in the simulations: N - particles(agents) count; x_o - initial fraction, x_i - initial published fraction; p – copying probability, λ - recovery probability; $\langle T \rangle / N$ - normalized mean ordering time; P_E – exit probability; τ – polling interval; t – number of elementary update steps; T – elementary steps for reaching final ordered state.

Before analyzing graphs, their interpretation needs to be understood.

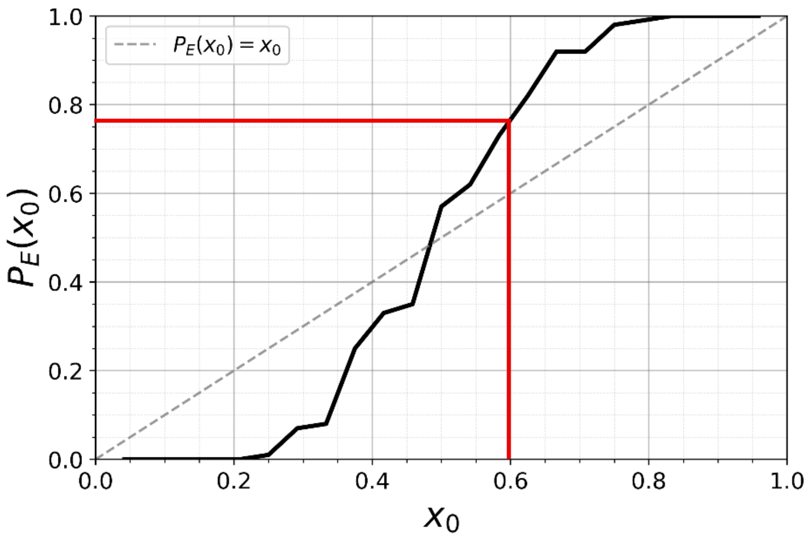

Figure 1 Example graph of exit probability

This graph (Fig. 1) illustrates how the exit probability depends on the initial fraction of agents holding opinion "1". The marked point (connection of red lines) shows that over some number of realizations when approximately 60% of agents initially support opinion "1", the probability of the system reaching the ordered state where all agents hold opinion "1" is about 70%. In other words, across many independent realizations, roughly 70% of simulations ended in consensus state "1". The dotted line represents theoretical exit probability of classical voter model.

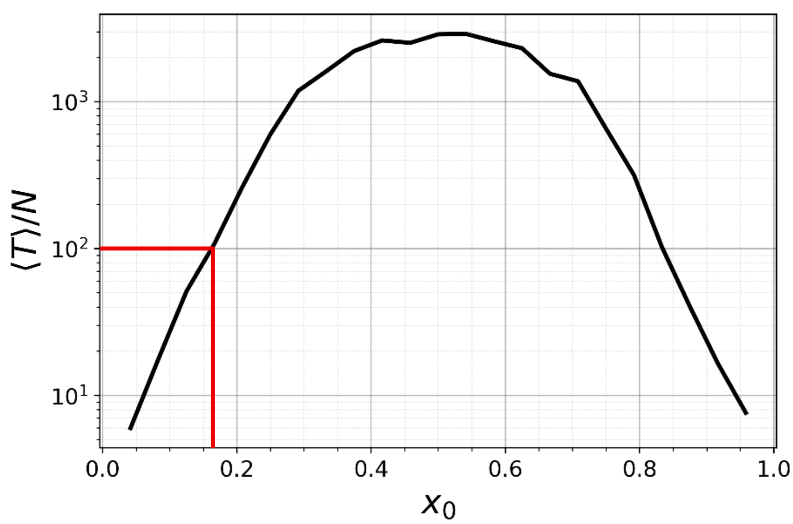

Figure 2 Example graph of mean ordering time

Figure 2 shows the dependence of the normalized mean ordering time $\langle T \rangle/N$ on the initial fraction x_0 of agents holding opinion "1". The marked point (connected with red lines) shows that over some number of realizations when approximately 16% of agents initially support opinion "1", the system requires on average about 100 normalized time units (steps) to reach one of the ordered consensus states (all states reach "0" or "1").

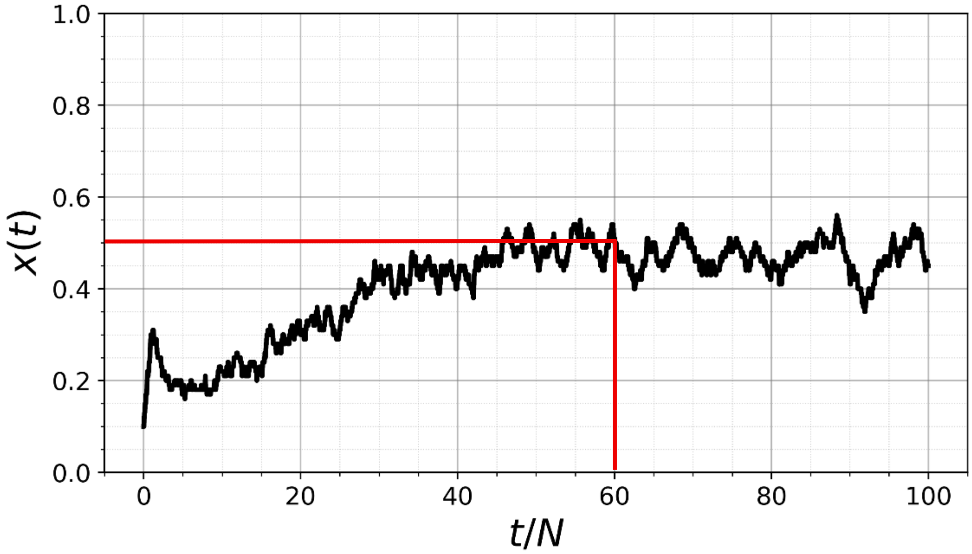

Figure 3 Example graph of trajectory

This trajectory graph (Fig. 3) shows the time evolution of the opinion fraction. The marked point (connected with red lines) shows that over some number of realizations after approximately 60 normalized time units, the fraction of agents holding opinion "1" fluctuates around 50%.

4.1 Classical voter model

In this work, the classical voter model was used as the starting point for analyzing the ordering dynamics of the system. The exit probability is expected to satisfy the relation

$$P_E = E(x(t)) = x_0, \tag{4}$$

meaning that the probability of reaching the final consensus state is approximately equal to the initial fraction of agents supporting that opinion[5,6].

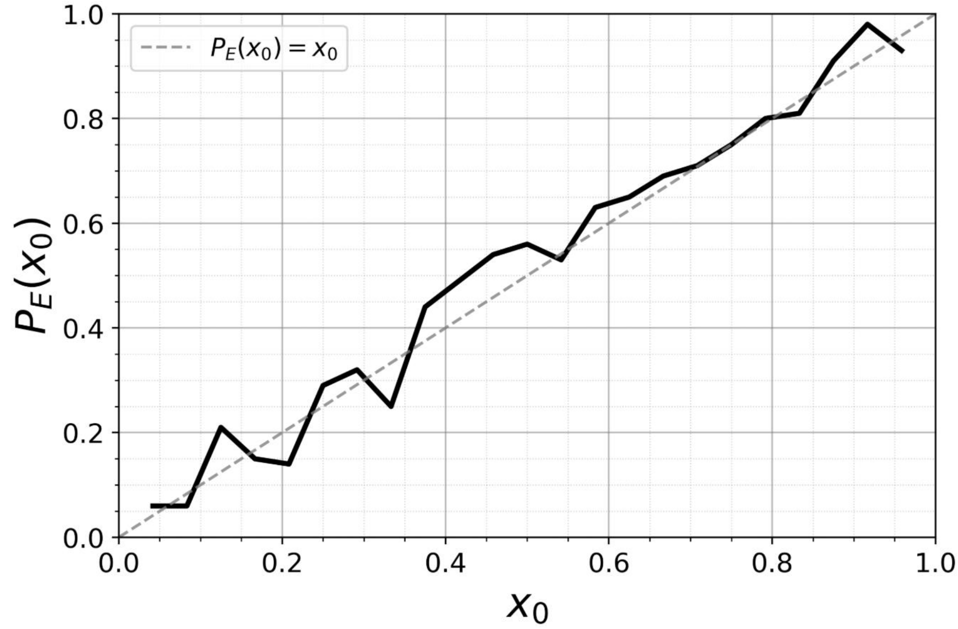

Figure 4 Exit probability P_E for the classical voter model. N = 24, Reported results were obtained by averaging over 100 realizations for each reported point.

The numerical results obtained show that the exit probability follows an almost linear dependence on the initial fraction x_0. This means that the dynamics of the model are symmetric and neither opinion has an advantage during the evolution of the system. Small deviations from the theoretical line appear because of stochastic fluctuations and the finite number of simulations used in the calculations. As for an infinite number of realizations and agents, the numerical results should converge to the theoretical linear dependence predicted by the classical voter model.

Another important quantity is the mean ordering time $\langle T \rangle$. In the simulations, the ordering time was divided by the number of agents N, allowing easier comparison between different realizations.

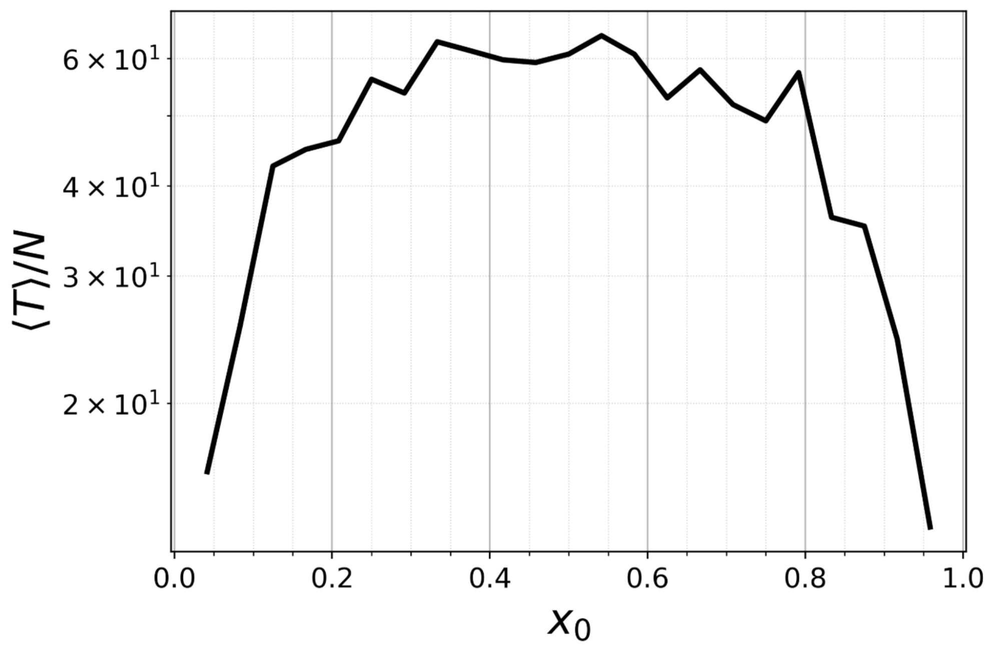

Figure 5 Mean ordering time $\langle T \rangle / N$ for the classical voter model. N = 24, Reported results were obtained by averaging over 100 realizations for each reported point.

The results shown in Fig. 5 represent that the ordering time becomes largest near $x_0 \approx 0.5$. In this case, both opinions initially have a similar number of supporters, therefore the system requires larger fluctuations before one opinion becomes dominant. When the initial condition strongly favors one opinion, consensus is reached much faster (system already starts close to the ordered state). Similar behavior is commonly observed in previous studies of voter dynamics [3,6].

The stochastic trajectories obtained during the simulations are illustrated in Fig. 6.

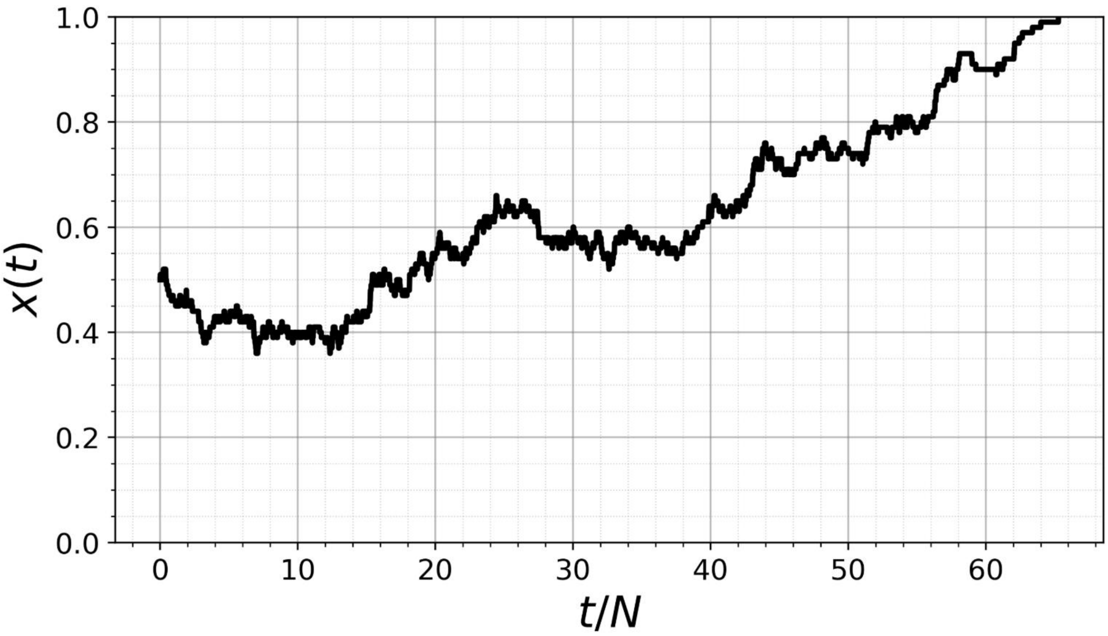

Figure 6 Trajectory x(t) for the classical voter model. N = 100, x_o = 0.5.

The trajectories fluctuate irregularly before eventually reaching one of the absorbing consensus states. This behavior reflects the stochastic nature of the voter model, where the dynamics are driven entirely by random imitation processes between interacting agents.

During voter model simulation, the obtained results reproduce the characteristic properties of the classical voter model and provide a reference point for comparing the more complicated models investigated later in this thesis. In particular, the linear exit probability and the ordering-time dependence serve as useful baseline properties when studying the influence of latent states and delayed interactions on consensus formation.

4.2 Lambiotte et al. latent voter model

After investigating the classical voter model, simulations were performed for the Lambiotte et al. model, which introduces an additional inflexible state into the dynamics. In this model, agents become temporarily inflexible after changing their opinion and therefore cannot immediately participate in another opinion update[10,24]. Because of this mechanism, the propagation of information through the system becomes slower compared to the classical voter model.

The first quantity investigated was the exit probability P_E .

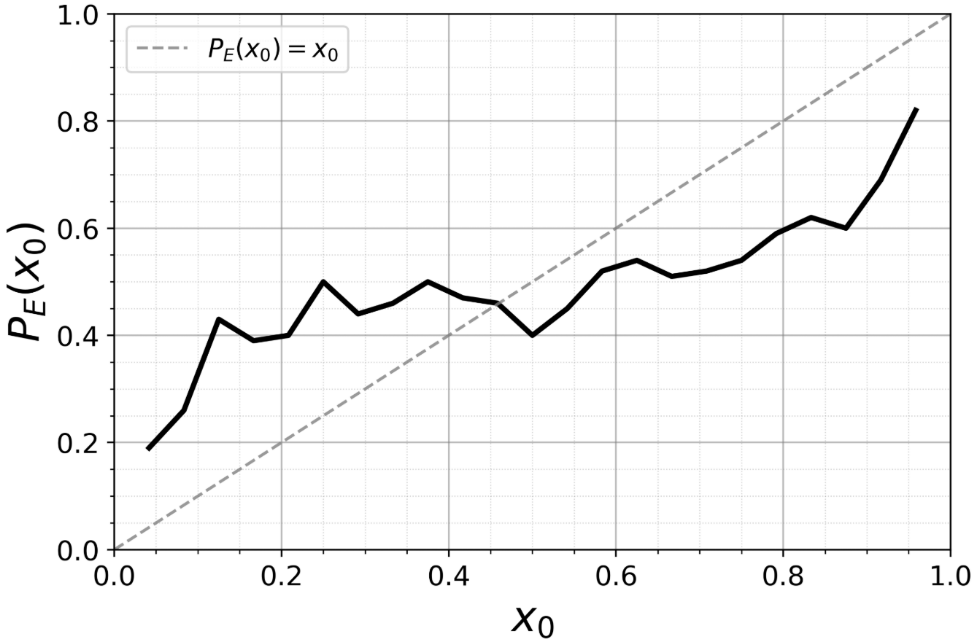

Figure 7 Exit probability P_E for the Lambiotte et al. latent voter model. Simulation parameters: p = 0.25, λ =0. 25, N = 24. Reported results were obtained by averaging over 100 realizations for each reported point.

The results obtained show that the exit probability remains approximately symmetric with respect to the initial condition x_0. However, compared to the classical voter model, larger fluctuations are visible in the numerical results. Exit probability tends to stay around 40% - 60% across a larger range of initial fraction (majority of agents in system have less impact on overall outcome). This behavior appears because inflexible agents temporarily reduce the number of possible interactions in the system, increasing the influence of stochastic effects during the evolution process.

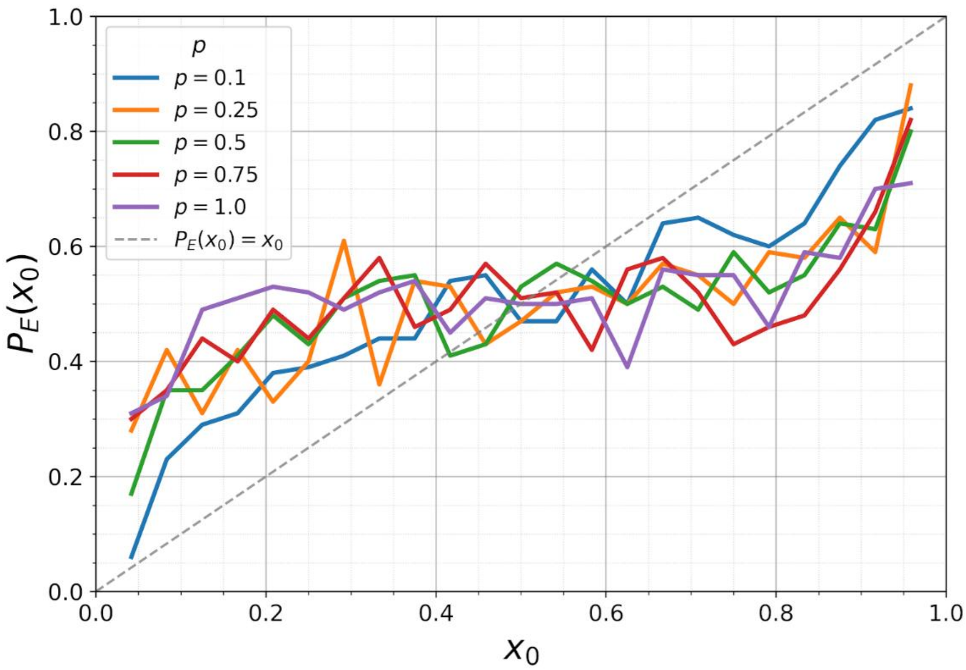

Figure 8 Exit probability P_E for the Lambiotte et al. latent voter model on influence of copying probability p. Simulation parameters: λ =0. 25, N = 100. Reported results were obtained by averaging over 100 realizations for each reported point.

By analyzing copying probability change effect on exit probability P_E (Fig. 8) we can observe minimal changes between each graph. Small differences between curves appear mainly near extreme values of x_o. Lower copying probability slightly influences exit probability closer to classical voter model.

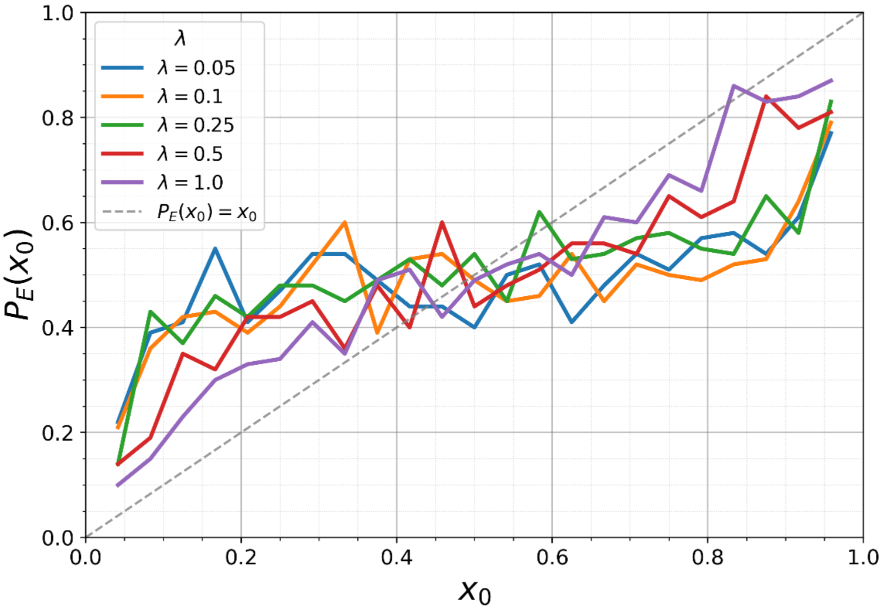

Figure 9 Exit probability P_E for the Lambiotte et al. latent voter model on influence of recovery probability λ. Simulation parameters: p =0. 25, N = 100. Reported results were obtained by averaging over 100 realizations for each reported point.

As for recovery probability λ, more noticeable influence can be observed on the exit probability. Smaller values of λ lead to flatter dependencies, indicating that the latent mechanism weakens the effect of the initial opinion distribution. As λ increases, the curves move closer to the classical voter model relation, meaning that the influence of the initial majority opinion becomes stronger.

The influence of latent states becomes much more visible in ordering-time dependence.

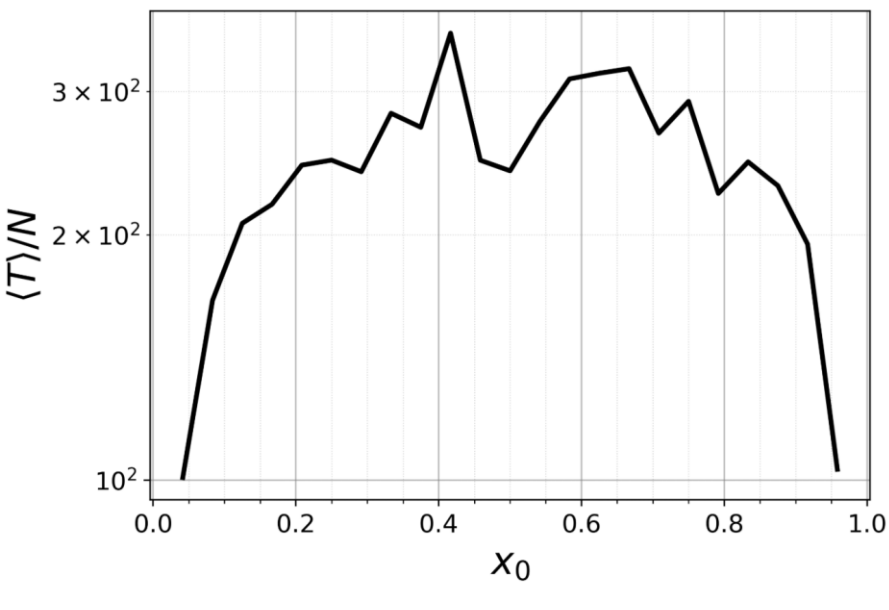

Figure 10 Mean ordering time $\langle T \rangle/N$ for the Lambiotte et al. latent voter model. Simulation parameters: p = 0.25, λ =0. 25, N = 24. Reported results were obtained by averaging over 100 realizations for each reported point.

The simulations (Fig. 10) demonstrate that the ordering times are considerably larger than in the classical voter model (Fig. 5). The longest ordering times are again observed near $x_0 \approx 0.5$, where both opinions initially have similar support. In this region, the inflexible-state mechanism suppresses rapid opinion changes and slows down the formation of large dominant domains. As a result, the system requires substantially more time to reach one of the absorbing consensus states.

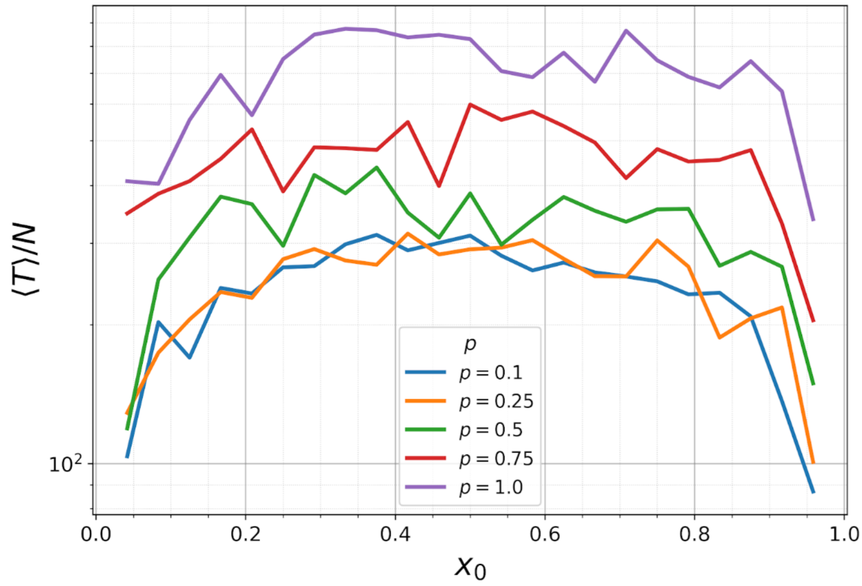

Figure 11 Dependence of the normalized ordering time ⟨T⟩/N on the copying probability p in the Lambiotte et al. latent voter model. N = 100, λ = 0.25. Reported results were obtained by averaging over 100 realizations for each reported point.

In Fig. 11 the dependence on copying probability p shows different behavior from the initial expectation. In the obtained simulations, larger values of p lead to longer ordering times. The shortest ordering times are observed for the smallest copying probabilities. As the copying probability increases toward 1, the system approaches the classical voter-model interaction mechanism, with the recovery probability remaining the main additional parameter distinguishing the latent model from the classical case. However, even for large p, the ordering times remain significantly higher (around 50 times) than in the classical voter. This suggests that frequent successful opinion-copying events, combined with the latent inactive state, may stabilize competing opinion domains and prolong fluctuations before one opinion becomes dominant. Since agents become temporarily inflexible after changing their opinion, rapid information propagation does not immediately produce consensus and may instead slow down the elimination of competing states. As a result, the interaction between strong copying dynamics and latent periods increases the characteristic ordering times of the system[10].

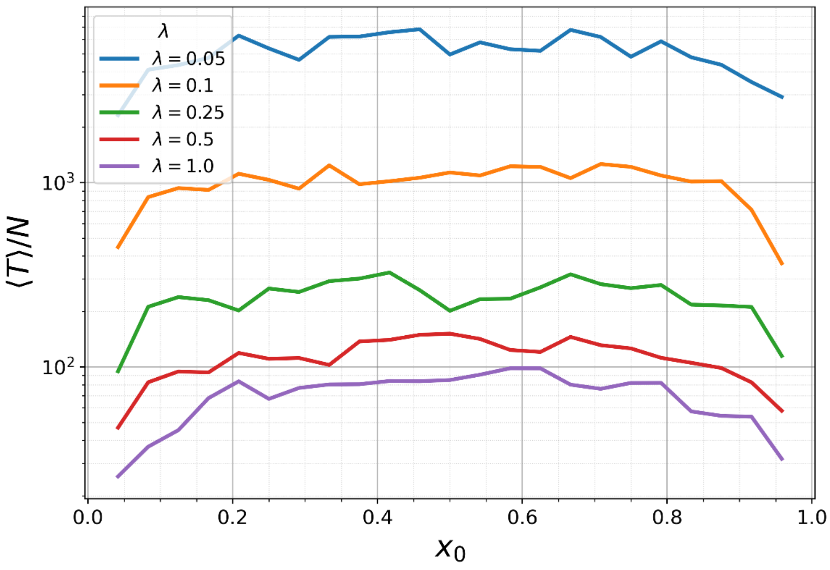

Figure 12 Dependence of the normalized ordering time ⟨T⟩/N on the recovery probability λ in the Lambiotte et al. latent voter model. N = 100, p = 0.25. Reported results were obtained by averaging over 100 realizations for each reported point.

Fig. 12 demonstrates that recovery probability strongly affects the temporal behavior of the system. Small values of λ produce freeze of the system. It creates very large ordering times because agents remain inflexible for longer periods after changing their opinion. When λ increases, inflexible agents recover more quickly, and the system gradually approaches the behavior of the classical voter model.

This behavior can also be observed in individual stochastic trajectories.

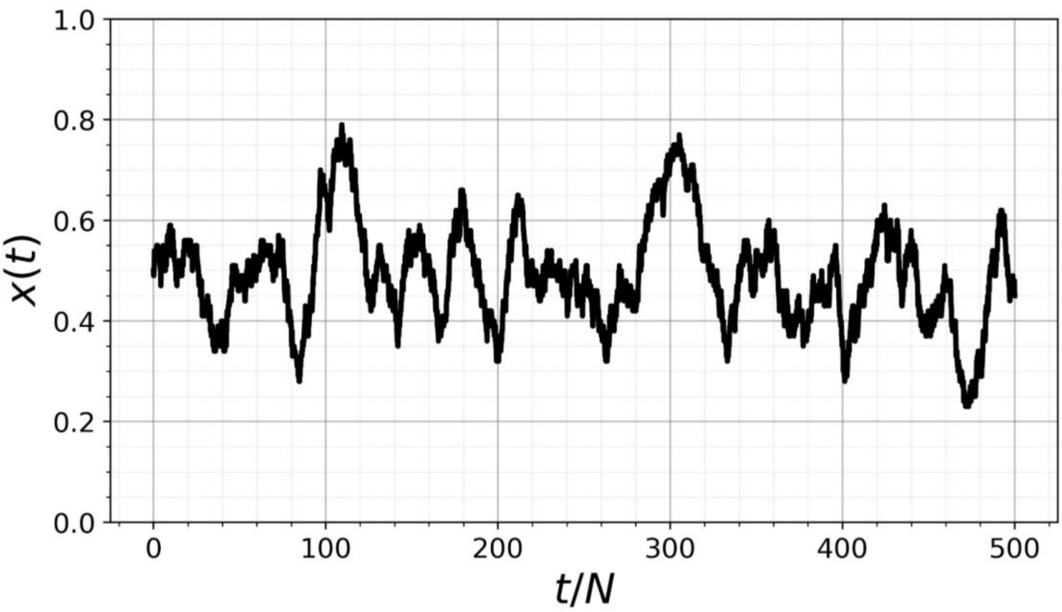

Figure 13 Trajectory x(t) for the Lambiotte et al. latent voter model. p = 0.25, λ =0. 25, N = 100, x_o = 0.5

Compared to the classical voter model, the trajectories evolve more slowly and exhibit longer periods of local fluctuations before consensus is reached. Since recently updated agents become temporarily inflexible, rapid reversals of opinion producing slower collective dynamics and delayed ordering processes.

Overall, the obtained results confirm that the introduction of latent states significantly changes the temporal behavior of the system. While the general structure of the exit probability remains similar to the classical voter model, the ordering dynamics become noticeably slower because the inflexible-state mechanism effectively delays information propagation between interacting agents[3,24].

4.3 Reversed Lambiotte et al. latent voter model

The next step was to investigate how the dynamics change when the order of the opinion update and inflexible period is reversed. In the standard Lambiotte et al. model, the agent first changes opinion and then becomes inflexible. However, in the reversed version, the agent first becomes inflexible while keeping its old visible opinion, and the new opinion is applied only after reactivation. This change is small in the update rule, but it can strongly affect the way information spreads through the system.

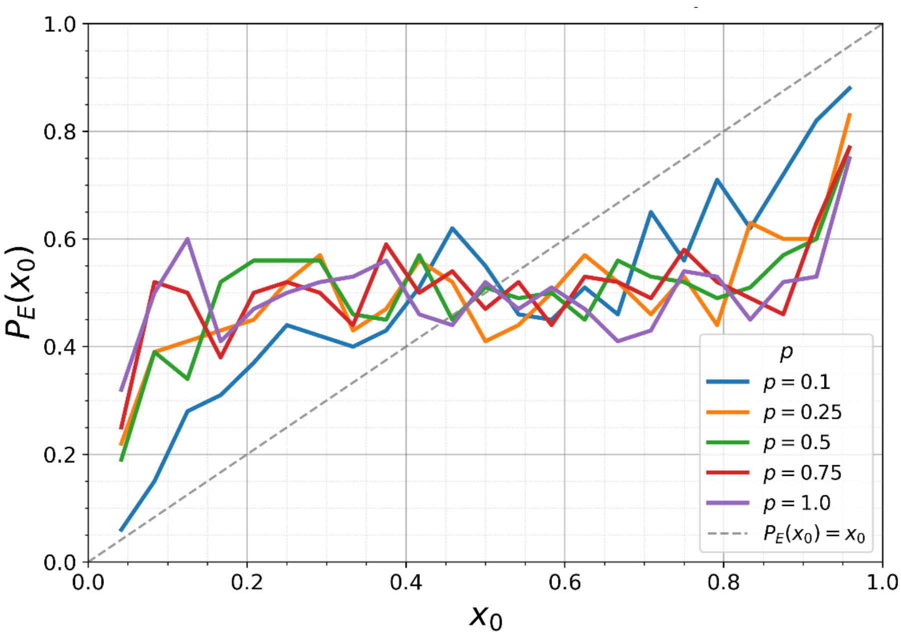

Figure 14 Exit probability P_E for the reversed Lambiotte et al. latent voter model on influence of copying probability p. Simulation parameters: $\lambda = 0.25$, $N = 24$. Reported results were obtained by averaging over 100 realizations for each reported point.

The dependence on copying probability p in Fig. 14 shows that the reversed Lambiotte model remains qualitatively similar to the standard latent voter model, because the exit probability curves are still flatter than the classical voter model prediction. Changing p does not fully restore the linear voter-model behavior, which means that the delayed realization of opinion changes remains important for all investigated copying probabilities. However, the curves become more irregular than in the standard Lambiotte case, suggesting that the combination of copying events and delayed visible opinion updates increases stochastic fluctuations in the final consensus outcome.

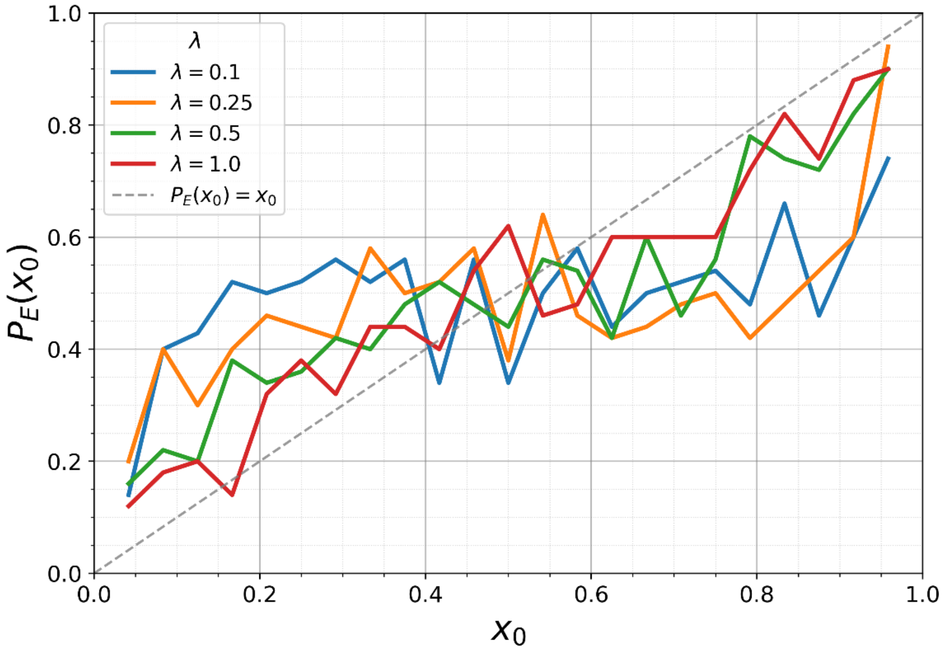

Figure 15 Exit probability P_E for the reversed Lambiotte et al. latent voter model on influence of recovery probability λ. Simulation parameter: p = 0. 25, N = 24. Reported results were obtained by averaging over 100 realizations for each reported point.

The recovery probability λ significantly influences the exit probability behavior. For smaller values of λ, the exit probability dependence remains relatively flat, indicating that the influence of the initial opinion distribution is reduced. However, increasing λ produces curves that are noticeably closer to the theoretical voter model, especially near extreme values of x_o, where the system becomes more sensitive to the initial majority opinion.

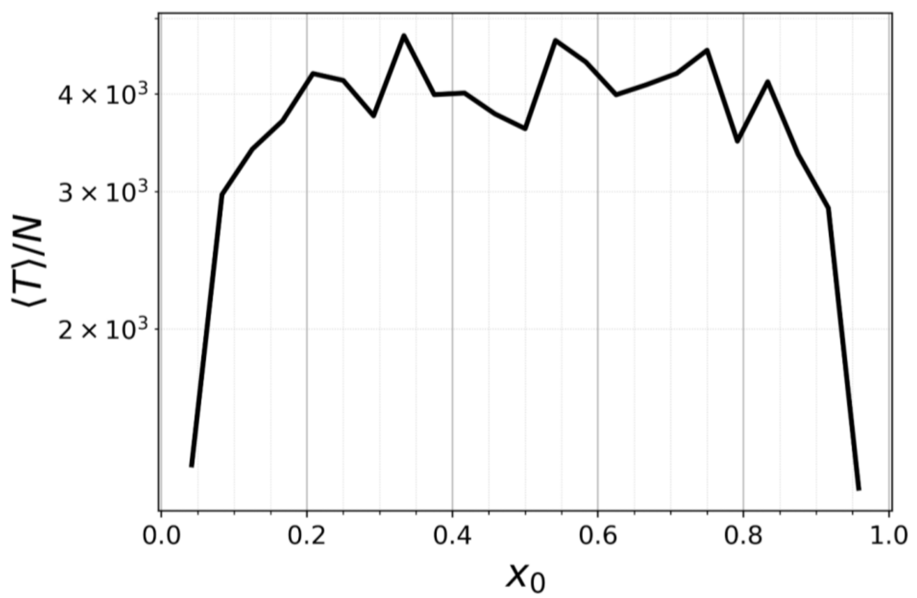

Figure 16 Mean ordering time ⟨T⟩/N for the reversed Lambiotte et al. latent voter model. p = 0.25, λ =0. 25, N = 24, Reported results were obtained by averaging over 100 realizations for each reported point.

The most important difference is observed in the ordering time. Compared with the classical voter and standard Lambiotte et al. latent voter models, the reversed latent voter model requires substantially longer times to reach consensus with ordering times being around 10 times larger than in the standard latent model and up to 500 times larger than in the classical voter model. The system remains in fluctuating intermediate states for significantly longer periods before eventually reaching one of the absorbing ordered states.

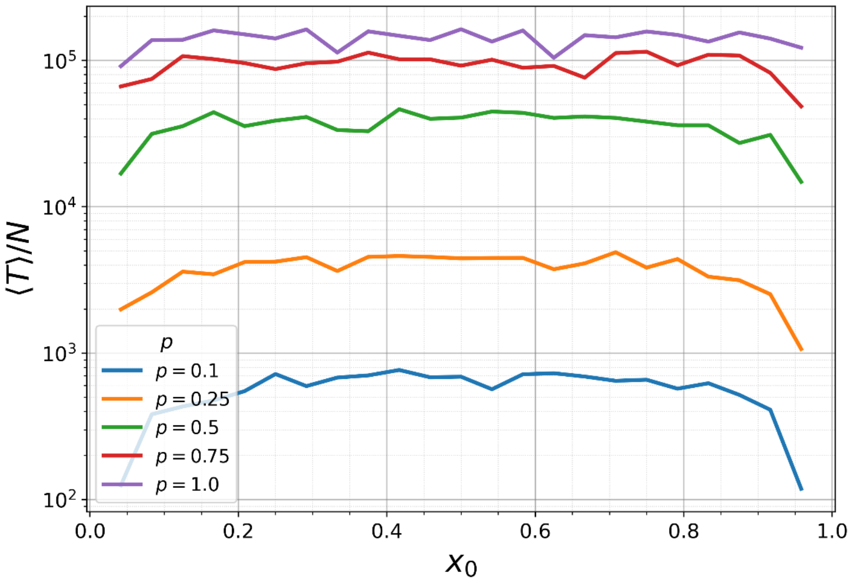

Figure 17 Dependence of the normalized ordering time ⟨T⟩/N on the copying probability p in the reversed Lambiotte et al. latent voter model. N = 24, λ = 0.25. Reported results were obtained by averaging over 100 realizations for each reported point.

The ordering-time dependence on copying probability p differs noticeably from the behavior observed in the standard latent voter model. In the reversed version (Fig. 17), larger values of p substantially increase the consensus time over the entire range of initial conditions. However, at p = 1 system reaches its longest ordering. Comparing Fig. 11 and Fig. 17, it can be observed that at 100% copying probability the reversed latent voter model exhibits a much stronger "freezing" effect on the ordering dynamics. The ordering times become approximately three orders of magnitude larger than in the standard latent voter model.

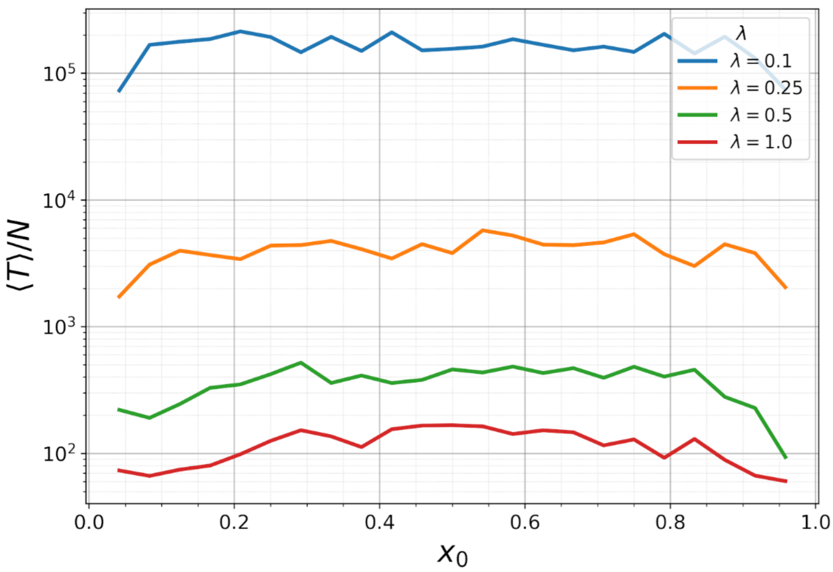

Figure 18 Dependence of the normalized ordering time ⟨T⟩/N on the recovery probability λ in the reversed Lambiotte et al. latent voter model. N = 24, p = 0.25. Reported results were obtained by averaging over 100 realizations for each reported point.

As for λ, the ordering time becomes larger as parameter decreases (Fig. 18). At 10% of recovery probability, the reversed latent model takes more than 100 times longer to reach consensus state.

Analysis of those graphs shows that the order of microscopic events is important. Even though the same basic processes are present (copying, inactivity, and reactivation) changing their sequence modifies the collective dynamics. The reversed mechanism introduces an additional delay between interaction and visible state change, which makes the ordering process slower and less direct.

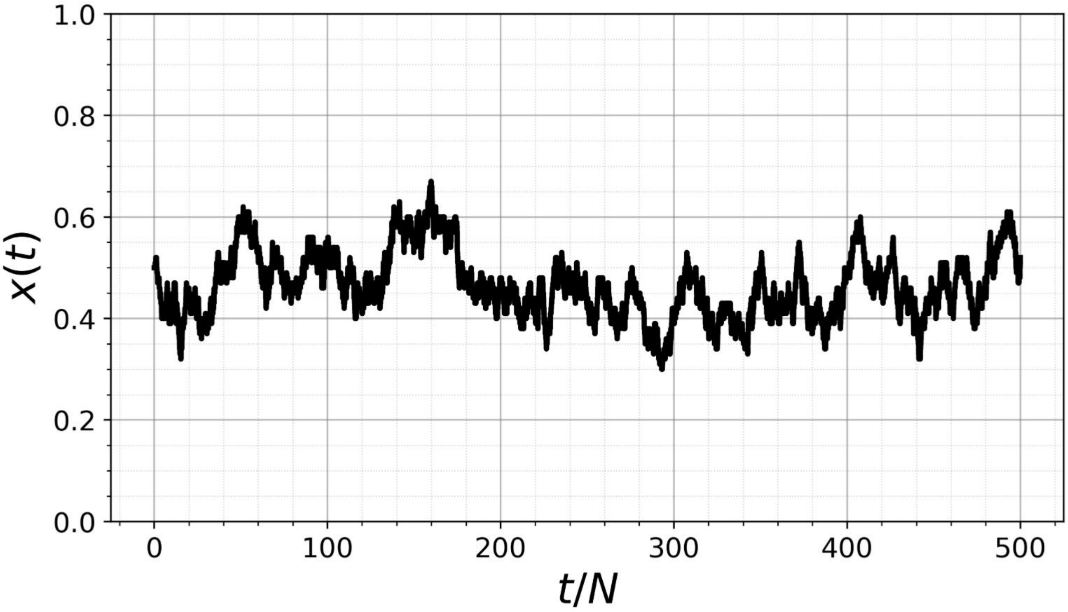

Figure 19 Example trajectory x(t) for the reversed Lambiotte et al. latent voter model, $p = 0.25$, $\lambda =0.25$, $N = 100$, $x_o = 0.5$

The trajectory illustrates this delayed behavior more clearly. Instead of rapidly moving toward one consensus state, the system may remain for a long time in an intermediate region where both opinions coexist. These long transient states indicate that delayed visibility of opinion changes can create memory-like effects in the dynamics.

Overall, the reversed latent voter model demonstrates that not only the presence of a latent period, but also its position in the update sequence, can strongly influence ordering dynamics. This result is important because it shows that apparently small changes in microscopic rules may lead to large differences in ordering times and stochastic evolution[5,18].

4.4 Polling and delayed polling models

The final part of the numerical investigation focused on polling-based voter models. Unlike the previous models, where agents interact directly with neighboring agents, polling models use global information about the state of the entire system. During the dynamics, agents react to a published opinion fraction obtained from periodic polling measurements.

The first investigated model was the standard polling model, where poll results are published immediately after measurement.

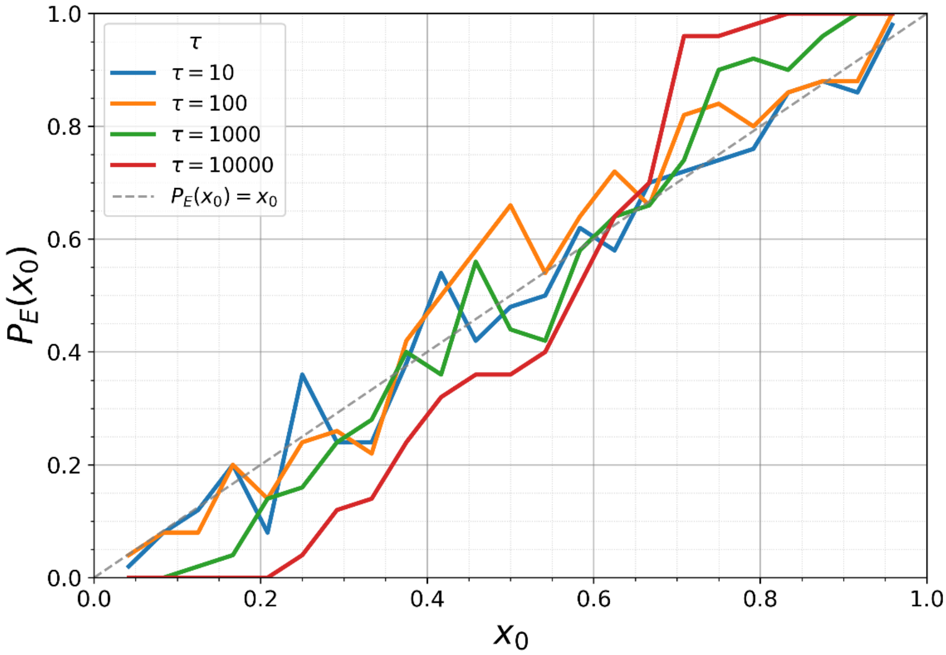

Figure 20 Exit probability P_E for the polling model. N = 24, p = 1. Reported results were obtained by averaging over 100 realizations for each reported point.

The results obtained show that the exit probability still increases with the initial fraction x_o, but the dependence becomes steeper compared to the classical voter model. In many simulations, the system reached consensus more rapidly once one opinion obtained a small majority. This behavior appears because the published global information reinforces the currently dominant opinion and reduces the importance of local stochastic fluctuations.

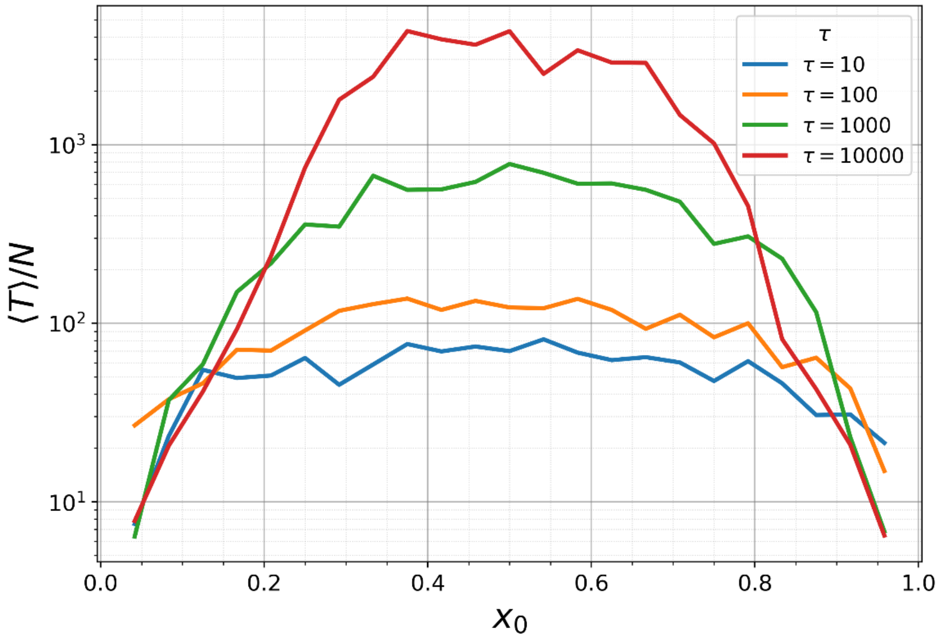

Figure 21 Mean ordering time ⟨T⟩/N for the polling model., N = 24, p = 1. Reported results were obtained by averaging over 100 realizations for each reported point

The ordering times are generally smaller than those of latent-state models. Since agents react to macroscopic information about the system rather than local neighboring states, consensus formation becomes more coordinated. The dynamics therefore exhibit reduced randomness and faster convergence toward one of the absorbing states.

The delayed polling model introduces an additional delay between measuring the opinion fraction and publishing the result to the system. During this delay period, agents continue interacting using outdated global information.

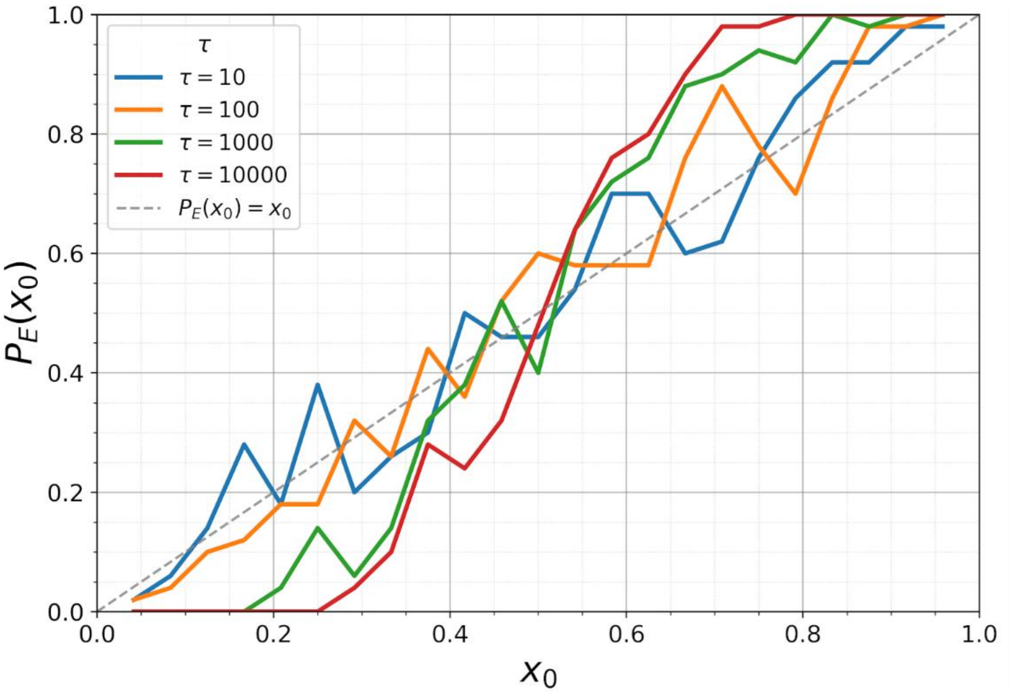

Figure 22 Exit probability P_E for the delayed polling model., N = 24, p = 1. Reported results were obtained by averaging over 100 realizations for each reported point.

The delayed feedback mechanism produces stronger deviations from the linear behavior observed in the classical voter model. Because agents react to previously measured information, the system may temporarily continue moving in a direction that no longer reflects the current state of the population.

Comparing polled model (Fig. 20) and delayed polled model (Fig. 22), we can notice that at low polling intervals results are similar. With increase of polling delayed model appears to have stronger majority effect on exit probability.

The influence of delayed information becomes less visibile in ordering-time dependence compared to standard polling models.

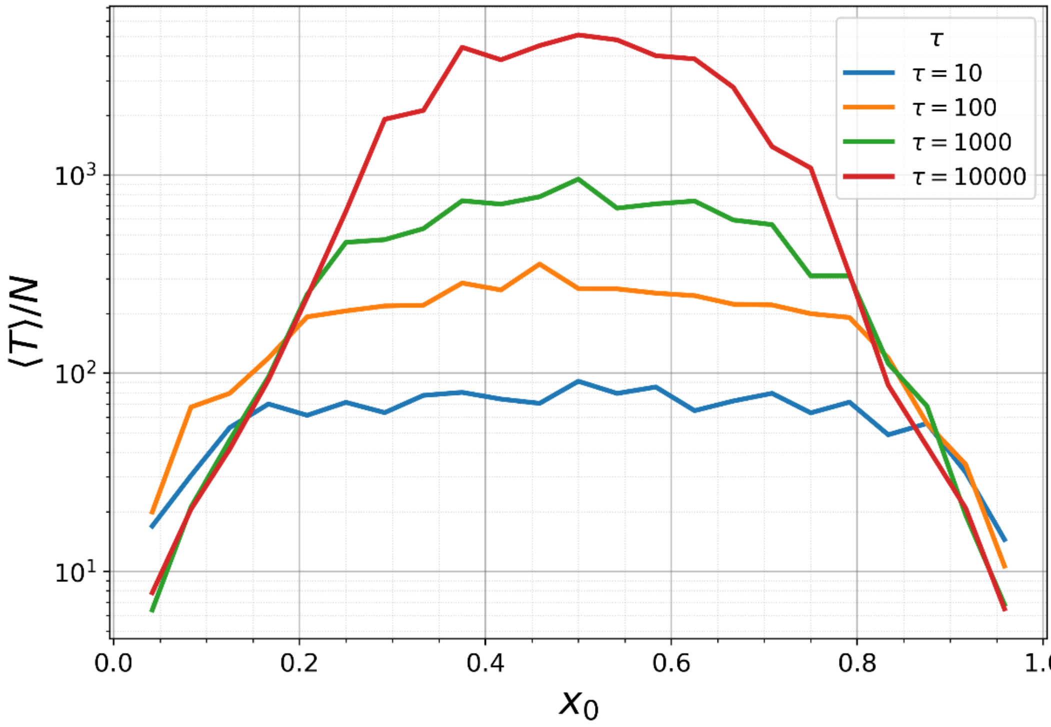

Figure 23 Mean ordering time ⟨T⟩/N for the delayed polling model., N = 24, p = 1. Reported results were obtained by averaging over 100 realizations for each reported point

The obtained results demonstrate that the polling interval τ influences delayed model's ordering time similarly like in polling-based voter models. In both the polling and delayed polling models, increasing τ leads to longer consensus times and stronger deviations from the classical voter model relation. Less frequent updates of global information increase the persistence of collective trends and strengthen the influence of the initial majority opinion on the final consensus state[9].

Despite the introduction of delayed information updates, the delayed polling model remains qualitatively similar to the standard polling model in terms of ordering times and exit probabilities. The results obtained suggest that the dominant mechanism controlling dynamics is the global feedback itself rather than the exact timing of information updates. Since both models are governed by the same periodically published collective information, their averaged macroscopic behavior remains close under the investigated conditions. The delayed mechanism mainly increases stochastic fluctuations and transient nonequilibrium effects, which are more clearly visible in individual stochastic trajectories than in averaged quantities such as exit probability or mean ordering time.

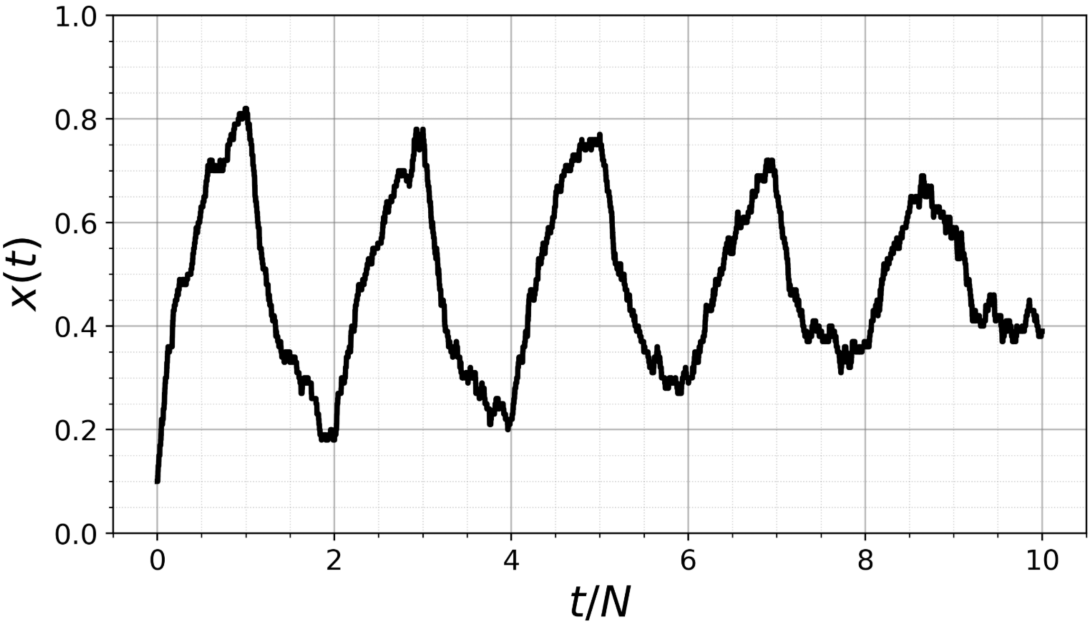

Figure 24 Trajectory x(t) for the delayed polling model. N = 100, τ = 1000, x_o = 0.1, x_i= 0.9. Reported results were obtained by averaging over 100 realizations for each reported point

The trajectories show large oscillations of the global opinion fraction caused by delayed system response. Since the published information no longer exactly matches the current state of the system, agents may continue reinforcing outdated trends even after the real macroscopic state has already changed. This delayed adaptation mechanism creates oscillatory behavior and slows down consensus formation[19].

Overall, the polling-based models demonstrate how global feedback mechanisms can significantly modify stochastic ordering dynamics. Immediate polling tends to stabilize the dominant opinion and accelerate consensus formation, while delayed polling introduces additional nonequilibrium effects caused by outdated information propagation. Similar delayed-feedback phenomena are commonly observed in stochastic systems containing memory and finite response times [3,9].

4.5 Comparison of ordering dynamics between models

For a more direct comparison of the interaction mechanisms investigated, the exit probabilities and normalized ordering times of all models were analyzed together. Such comparison makes it easier to observe how latent states, delayed updates, and global feedback mechanisms influence collective dynamics relative to the classical voter model.

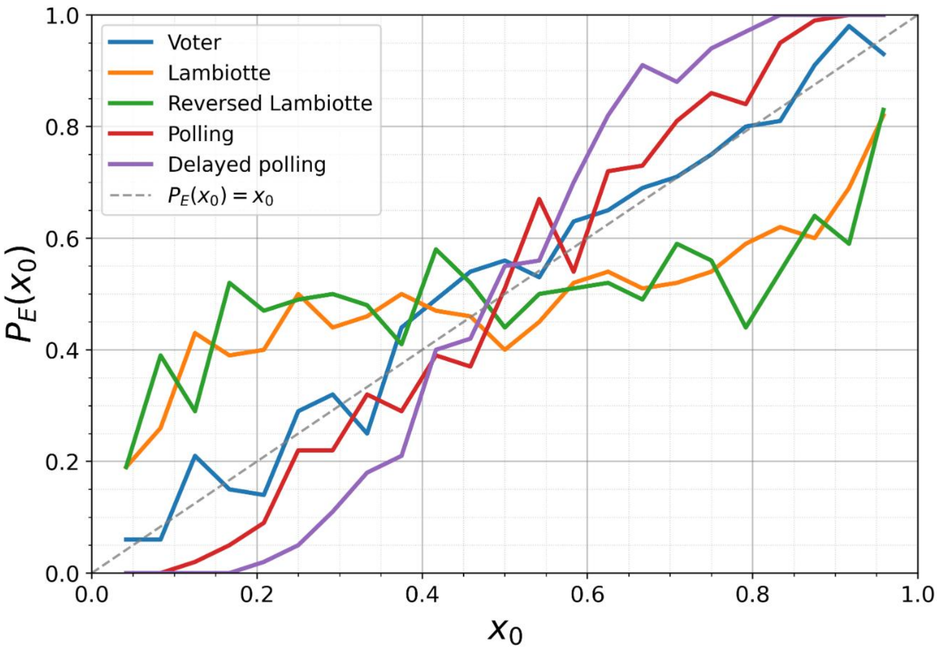

Figure 25 Comparison of exit probability P_E for all models investigated. N = 24, p = 0.25, λ =0. 25, p = 1 for polled and delayed polled models, , τ = 1000. Reported results were obtained by averaging over 100 realizations for each reported point

The comparison of exit probabilities shows that the classical voter model follows the expected approximately linear dependence $P_E=E(x(t))=x_0$, which is characteristic for unbiased voter dynamics[3]. The latent voter model and reversed latent voter model exhibit larger deviations from this linear behavior because inflexible states reduce the number of immediate interactions and introduce additional stochastic fluctuations into the system. The polling-based models display steeper transitions near balanced initial conditions, indicating that global feedback mechanisms strengthen the influence of the currently dominant opinion.

More significant differences become visible in the ordering-time comparison.

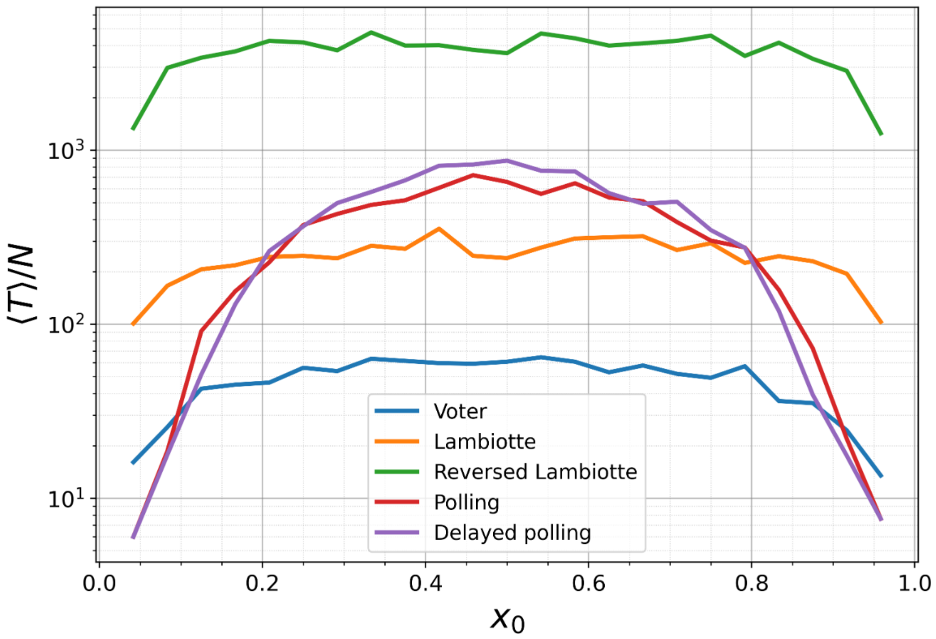

Figure 26 Comparison of normalized ordering time ⟨T⟩/N for all investigated models. N = 24, p = 0.25, λ =0. 25, p = 1 for polled and delayed polled models, τ = 1000. Reported results were obtained by averaging over 100 realizations for each reported point

The classical voter model produces the smallest ordering times among the investigated systems. Introducing latent states in the latent voter model increases the ordering time because inflexible agents temporarily stop participating in opinion updates[10]. The reversed latent voter model demonstrates the strongest slowdown of dynamics. Since agents continue influencing others using outdated visible opinions during the inflexible period, the system spends much longer fluctuating before consensus is finally reached. The polling model exhibits more coordinated dynamics because agents react to global information instead of direct local interactions. As a result, consensus formation becomes faster than in latent-state models. Delayed publication of polling information causes agents to react to outdated macroscopic states, producing stronger fluctuations and slower convergence toward consensus.

Overall, the comparison confirms that even relatively small modifications of microscopic interaction rules may strongly affect macroscopic ordering dynamics. Delayed interactions and latent mechanisms generally suppress rapid consensus formation and may substantially increase the characteristic ordering times of voter-type systems[3,5].

5. Conclusion

In this thesis, several voter-type delayed interaction models were investigated using Monte Carlo simulations to analyze ordering dynamics and consensus formation in stochastic interacting systems. The classical voter model was used as a reference system for comparing the influence of latent states and delays in information on collective behavior.

The results obtained showed that the classical voter model reproduces the expected linear exit probability dependence and relatively fast consensus formation. Introducing latent states in Lambiotte et al. latent voter model increased the average ordering times because inflexible agents temporarily effectively stopped updating their opinions. Interestingly, exit probability P_E tends to become close to 0.5 for intermediate values of x_o. This effectively implies that inflexibility of agents in the latent voter model induces memory loss in respect to the initial condition. This result aligns well with results obtained in the original paper by Lambiotte et al.[10]

While the original latent voter model assumes that particles first update their state, and only then become inflexible, we have investigated what happens if the particles first become inflexible and only then update their state. This reversal does not have a qualitative effect on the exit probabilities of the system, but the mean ordering time is significantly increased. Delaying the state change significantly slowed down the ordering process and produced long periods with ongoing transient fluctuations. These results demonstrated that the sequence of microscopic update events can strongly influence macroscopic system behavior.

While the latent voter models implement individual particle delay mechanisms, the polling and delayed polling models[9] do the opposite - they implement the delay on system level. Namely, the polling-based models explore the role of global information delay mechanisms. In the short polling period limit we can observe more linear relations with exit probability, mostly like for the classical voter model. Immediate release of the polling information accelerated the consensus (ordered state) formation, while delayed release of the polling information introduced oscillatory behavior and slower convergence caused by outdated information propagation.

By comparing latent and polling models, we can observe that latent and polling interaction mechanisms influence state ordering (consensus) formation in qualitatively different ways. In the Lambiotte et al. model, the exit probability curves imply that the influence of the initial opinion fraction (initial polarization) becomes weaker due to temporary inactivity of individual particles, independently whether the particle state updates are immediate or are delayed. In contrast, the polling-based models amplify the effect of initial polarization, as the exit probability curves become increasingly biased towards the respective ordered states (extremely polarized states) when the particles react with a delay to the global system state updates.

Overall, the performed simulations confirmed that latency and polling mechanisms substantially modify the ordering processes in elementary spin systems. Within the voter model framework, they may lead to a qualitatively different self-organization behavior.

Mantas Pupkevičius

UŽDELSTŲ SĄVEIKŲ POVEIKIS ELEMENTARIŲ SUKINIŲ SISTEMŲ NUSISTOVĖJIMO LAIKAMS

Santrauka

Šiame darbe nagrinėjama skirtingų uždelstų sąveikų mechanizmų įtaka tvarkos nusistovėjimo statistinėms savybėms elementariose sukinių sistemose. Sociofizikoje elementarios sukinių sistemos, dažnai modeliuojamos naudojant rinkėjo modelį, yra naudojamos siekiant suprasti ir aprašyti visuomenės nuomonės formavimosi procesus. Pagrindinis šio darbo tikslas yra ištirti skirtingų delsos mechanizų įtaką tvarkos būsenos (konsensuso) nusistovėjimo dinamikai, tikėtinam nusistovėjimo laikui ir nusistovėjusių būsenų tikimybėms. Buvo analizuojamas Lambiotte ir bendrautorių pasiūlytas uždelsto virsmo (angl. *latency*) rinkėjo modelis[10] ir jo "apgręžta" (angl. *reversed*) versija, taip pat apklausų delsos mechanizmu (angl. *polling delay mechanism*) grįstas rinkėjo modelis ir jo versija be informacijos delsos[9]. Atliekant modeliavimą taikytas Monte Karlo metodas pilnai sujungtoms sistemoms (t.y., visos dalelės galėjo sąveikauti su visomis kitomis dalelėmis). Išskirtinis dėmesys buvo skirtas parametrų valdančių delsos trukmę (t.y., būsenos kopijavimo tikimybė, lankstumo atsistatymo tikimybė, apklausų periodas) įtakos analizei.

Atlitkas modeliavimas ir modeliavimo rezultatų analizė parodė, kad uždelsto virsmo ir apklausų mechanizmai gali reikšmingai pakeisti elementarių sukinių sistemos nusistovėjimo statistines savybes, lyginant su klasikiniu rinkėjo modeliu. Lambiotte ir bendraautorių uždelsto virsmo rinkėjo modelyje laikinas agentų nelankstumas sumažina pradinės sąlygos svarbą. Uždelso virsmo rinkėjo modelyje, jei pradinė būsena nėra stipriai poliarizuota (t.y., nėra arti kraštinių tvarkos būsenų), tai ilgėjant delsos trukmei tikimybės pasiekti kraštines tvarkos būsenas tampa lygios. Tuo tarpu apklausų delsos mechanizmas sustiprina pradinės poliarizacijos įtaką ir padidina atitinkamų kraštinių tvarkos būsenų nusistovėjimo tikimybę. Taip pat buvo ištirta, kaip sistemos dinamiką veikia būsenos kopijavimo tikimybė, lankstumo atsistatymo tikimybė ir apklausų periodo trukmė. Didėjant būsenos kopijavimo tikimybei uždelsto virsmo rinkėjo modelyje sistemos nusistovėjimo laikas ilgėja dėl greitai susiformavusio didelio kiekio nelanksčių dalelių. Visuose nagrinėtuose modeliuose ilginant delsos trukmę (mažinant lankstumo atsistatymo tikimybę uždelsto virsmo rinkėjo modelyje arba ilginant apklausų periodą apklausomis grįstame rinkėjo modelyje) aprašyti nukrypimai nuo klasikinio rinkėjo modelio nusistovėjimo statistinių savybių didėja.

Atlikta analizė parodė, kad skirtingi nagrinėti delsos mechanizmai nulemia kokybiškai skirtingą tvarkos būsenos nusistovėjimą elementariose sąveikaujančių sukinių sistemose.

Mantas Pupkevičius

EFFECTS OF TIME-DELAYED INTERACTIONS ON ORDERING TIMES OF ELEMENTARY SPIN SYSTEMS

Abstract

This thesis investigates the influence of different delayed-interaction mechanisms on the statistical properties of ordering processes in elementary spin systems. In sociophysics, elementary spin systems, often modeled using the voter model, are used to understand and describe the formation of public opinion. The main goal of this work is to study how different delay mechanisms affect consensus formation dynamics, mean ordering times, and exit-state probabilities. The investigated models include the latent voter model proposed by Lambiotte et al. and its reversed version, as well as the polling-delay-based voter model and its version without information delay. Monte Carlo simulations were performed for fully connected systems, meaning that every particle could interact with every other particle. Particular attention was given to the influence of parameters controlling the delay duration, including copying probability, flexibility recovery probability, and polling period.

The performed simulations and analysis showed that latent-state and polling mechanisms can significantly modify the statistical properties of ordering in elementary spin systems compared with the classical voter model. In the Lambiotte et al. latent voter model, temporary agent inflexibility reduces the importance of the initial condition. When the initial state is not strongly polarized, increasing the delay duration causes the probabilities of reaching the two ordered states to become approximately equal. In contrast, the polling-delay mechanism strengthens the influence of initial polarization and increases the probability of reaching the corresponding ordered state. The influence of model parameters was also analyzed. Increasing the copying probability in the latent voter model increases the ordering time because a large number of inflexible particles forms rapidly. In all investigated models, increasing the delay duration, either by decreasing the flexibility recovery probability in the latent voter model or by increasing the polling period in the polling-based model, leads to stronger deviations from the statistical properties of the classical voter model.

The results demonstrate that different delay mechanisms can produce qualitatively different ordering behavior in elementary interacting spin systems.

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