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Master's thesis

Modeling Asymmetric Tail Dependence and Extreme Downside Risk in Cryptocurrency Portfolios

Asimetrinės uodegų priklausomybės ir ekstremalaus kriptovaliutų portfelių vertės kritimo rizikos modeliavimas

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Abstract

Modeling extreme price movements in the highly volatile crypto market is a major challenge for investors. This study builds a risk model specifically for Bitcoin and Ethereum using a GARCH-Survival BB1 copula combined with Expected Shortfall. While traditional models assume that prices move together the same way during good times and bad times (symmetric dependence), the approach focuses on asymmetric tail behaviour. Analytically, it captures the crash, coupling the fact that these assets tend to collapse together much faster during market downturns than they rise together during booms.

The empirical results show a high lower tail dependence of 69.52% between BTC and ETH, which proves that they move together very strongly when the market crashes. Backtesting validates the model by showing high predictive accuracy, as the actual violations match what is expected at the 95% confidence level. At first, portfolio optimization using Expected Shortfall cut down risk by 22.45% in the crypto-only mix. To test how well this framework works, the study is extended to a 3D model by adding the S&P 500 index. This comparison shows that diversifying across different asset types lowers tail risk significantly, leading to a much better risk reduction of 29.66%.

Testing the framework on raw data from 2025 confirms its validity, as the model shows stable performance outside the initial estimation period. Due to the capacity for frequent adjustment, the model remains flexible enough to handle changing market shifts throughout 2026 and the upcoming years. These results prove that factoring asymmetric tail dependence into a multi-asset allocation is necessary downside risk control in both digital and traditional financial markets.

Keywords: Cryptocurrency, Tail Dependence, Survival BB1 Copula, Expected Shortfall, Portfolio Optimization, GARCH.

Santrauka

Ekstremalių kainų svyravimų modeliavimas itin nepastovioje kriptovaliutų rinkoje yra didelis iššūkis investuotojams. Šiame tyrime kuriamas rizikos modelis, skirtas būtent Bitcoin ir Ethereum, naudojant GARCH-Survival BB1 kopulą, sujungtą su tikėtinu nuostoliu (angl. Expected Shortfall). Nors tradiciniai modeliai daro prielaidą, kad kainos kinta kartu vienodai tiek gerais, tiek blogais laikais (simetrinė priklausomybė), šis požiūris sutelkia dėmesį į asimetrinį uodegų elgesį. Analitiškai jis užfiksuoja rinkos žlugimą, susiedamas faktą, kad šie aktyvai rinkos nuosmukio metu linkę kristi kartu daug greičiau, nei auga kartu bumo metu.

Empiriniai rezultatai rodo didelę — 69,52% — apatinės uodegos priklausomybę tarp BTC ir ETH, o tai įrodo, kad rinkai žlungant jie juda kartu labai stipriai. Retrospektyvusis testavimas (angl. Backtesting) patvirtina modelio pagrįstumą, rodydamas didelį prognozavimo tikslumą, nes faktiniai pažeidimai atitinka tai, ko tikimasi esant 95% pasikliautinamajam lygmeniui. Iš pradžių portfelio optimizavimas, naudojant tikėtiną nuostolį, riziką tik kriptovaliutų krepšelyje sumažino 22,45%. Siekiant patikrinti, kako efektyviai veikia ši sistema, tyrimas išplečiamas iki 3D modelio, įtraukiant S&P 500 indeksą. Šis palyginimas rodo, kad diversifikavimas tarp skirtingų turto rūšių žymiai sumažina uodegos riziką, o tai lemia daug geresnį — 29,66% — rizikos sumažinimą.

Sistemos išbandymas su pirminiais 2025 metų duomenimis patvirtina jos pagrįstumą, nes modelis demonstruoja stabilų veikimą už pradinio vertinimo laikotarpio ribų. Dėl dažno koregavimo galimybės modelis išlieka pakankamai lankstus, kad susidorotų su kintančiais rinkos pokyčiais per visus 2026 metus ir ateinančius laikotarpius. Šie rezultatai įrodo, kad asimetrinės uodegų priklausomybės įvertinimas kelių rūšių turto paskirstyme yra būtina neigiamos rizikos kontrolės priemonė tiek skaitmeninėse, tiek tradicinėse finansų rinkose.

Raktiniai žodžiai: Kriptovaliuta, uodegos priklausomybė, BB1 išgyvenamumo kopula, deficito vidurkis, portfelio optimizavimas, GARCH.

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- **Data and Descriptive Statistics**

P_t Asset price at time t

R_t Logarithmic return at time t

S Skewness (measure of asymmetry)

K Kurtosis (measure of tail heaviness)

JB Jarque-Bera test statistic for normality

- **Stage 1: GARCH Modeling**

μ Conditional mean of returns

σ_t^2 Conditional variance (volatility) at time t

ϵ_t Residuals or error term

z_t Standardized residuals (innovations)

ω Constant term in the variance equation

α_1, β_1 ARCH and GARCH parameters (reaction and persistence)

ν Degrees of freedom for the Student-t distribution

$\Gamma(\cdot)$ Gamma function

Q Ljung-Box test statistic for autocorrelation

ρ_k Sample autocorrelation at lag k

- **Stage 2: Copula and Dependence Modeling**

U, V Random variables representing the marginal distributions (returns) of Bitcoin and Ethereum, defined on the real line $\mathbb{R}$.

u, v Probability Integral Transform (PIT) values, where $u, v \in [0, 1]$, serving as the uniform arguments for the copula function.

F_X, F_Y Marginal Cumulative Distribution Functions (CDFs) that map the real-valued returns to the unit interval $[0, 1]$.

$C(u, v)$ The Copula function that links the uniform marginals u and v to model the joint dependence structure.

$\phi(t)$ Generator function for Archimedean copulas

θ Clayton parameter in BB1 (controls lower-tail dependence)

δ Gumbel parameter in BB1 (controls upper-tail dependence)

$\hat{C}$ Survival copula (emphasizing downside risk)

λ_L Lower tail dependence coefficient (crash synchronization)

λ_U Upper tail dependence coefficient

- **Stage 3: Risk Management**

VaR_α Value-at-Risk at confidence level α

ES_α Expected Shortfall (Conditional Value-at-Risk)

α Confidence level (e.g., 0.95, 0.99)

L Loss variable

List of abbreviations

ADF	Augmented Dickey-Fuller
AIC	Akaike Information Criterion
ARCH	Autoregressive Conditional Heteroskedasticity
BIC	Bayesian Information Criterion
BTC	Bitcoin
CDF	Cumulative Distribution Function
CVaR	Conditional Value-at-Risk
ES	Expected Shortfall
ETH	Ethereum
GARCH	Generalized Autoregressive Conditional Heteroskedasticity
i.i.d.	Independent and Identically Distributed
IFM	Inference Functions for Marginals
JB	Jarque-Bera
MLE	Maximum Likelihood Estimation
PDF	Probability Density Function
VaR	Value-at-Risk

1 Introduction

1.1 Background and Motivation

The global financial system has changed significantly in recent years due to the rapid growth of blockchain technology. The introduction of Bitcoin by [32] marked the beginning of a new form of digital asset that operates without a central authority. Since then, cryptocurrencies have developed into an important financial asset class. Among them, Bitcoin (BTC) and Ethereum (ETH) represent the largest share of total cryptocurrency market capitalization and play a central role in the digital financial ecosystem.

Despite their rapid growth and increasing adoption, cryptocurrencies are characterized by substantial return volatility. Financial time series are known to exhibit volatility clustering, meaning that periods of fluctuation tend to be followed by further high volatility [6, 13]. This stylized fact has also been widely observed in cryptocurrency markets. Such behaviour creates challenges for investors seeking effective portfolio diversification.

Traditional portfolio theory assumes that risk can be reduced when assets are not perfectly correlated. However, standard risk management models typically rely on linear correlation and the assumption of normally distributed returns. Empirical evidence in financial markets shows that asset returns often deviate from normality and display heavy tails [26]. Heavy-tailed distributions imply that extreme price movements occur more frequently than predicted by the Gaussian framework. This feature is particularly relevant for cryptocurrencies, where large price swings are common.

Moreover, asset dependence structures tend to change during periods of financial stress. Studies have shown that correlations across markets increase significantly during extreme downturns [25]. This phenomenon reduces the benefits of diversification exactly when risk protection is most needed. Linear correlation measures average dependence and does not adequately capture asymmetric behaviour in the tails of the distribution.

These limitations motivate the use of more advanced econometric techniques. To address time-varying volatility, GARCH-type models provide a flexible framework for modeling conditional heteroskedasticity [6]. To capture nonlinear and asymmetric dependence between assets, copula models allow the joint distribution to be modeled separately from marginal distributions. By combining GARCH models with copula theory, this study aims to provide a more realistic framework for measuring joint downside risk in cryptocurrency portfolios.

1.2 Research Problem

Even though cryptocurrency markets have gained huge popularity, we still do not fully understand how to accurately model extreme market crashes when multiple assets drop simultaneously. As discussed earlier, Bitcoin and Ethereum show high volatility, heavy-tailed returns, and changing correlations over time. These features make traditional risk management methods less effective because such methods usually depend on normal distribution and linear correlation assumptions. Previous studies also show that the relationship between cryptocurrencies becomes stronger during market

downturns, which is known as asymmetric tail dependence. Understanding this behaviour is important for better portfolio risk management, but it has not been fully examined in existing research.

The research problem can be explained through the following limitations in previous studies:

- **Symmetry Bias:** Many studies use symmetric copulas such as Gaussian and Student-t copulas, which treat joint gains and joint losses in the same way.
- **Limited Focus on Lower Tail Risk:** Although volatility modeling has been widely studied, the joint extreme downside movements of Bitcoin and Ethereum are still not deeply explored, especially using copulas like the Survival BB1 copula.
- **Portfolio Optimization Limitation:** Most portfolio optimization methods are based on the traditional Mean Variance framework. Expected Shortfall (ES), which is more suitable for measuring extreme losses, is less commonly used in copula-based portfolio studies.

Based on these limitations, this study addresses the following research question:

How can the asymmetric tail dependence between Bitcoin and Ethereum be modeled using a GARCH-Survival BB1-ES framework to improve downside risk measurement and portfolio optimization?

The aim of this study is to develop a practical framework that can help investors better understand and manage extreme downside risk during unstable market conditions. Since Bitcoin and Ethereum are the two largest cryptocurrencies by market capitalization, their tendency to crash together during periods of high volatility creates serious risk for investors. This research focuses on modeling this joint downside behaviour more effectively than traditional approaches.

1.3 Research Objectives

The primary goal of this research is to develop an integrated framework for modeling the joint downside risk of Bitcoin and Ethereum. The following specific objectives are proposed:

1. **To filter marginal volatility (univariate analysis):** To estimate the conditional volatility of Bitcoin and Ethereum this study utilizes a GARCH(1,1) model, including the Student-t distribution to capture the heavy tails and sharply peaked characteristics of cryptocurrency returns.
2. **To model joint tail dependence (bivariate analysis):** Quantify the extent of simultaneous extreme movements during market downturns by applying the Survival BB1 copula, focusing on asymmetric lower-tail dependence (λ_L) to measure crash synchronization between the two assets.
3. **To simulate future market scenarios (Monte Carlo simulation):** Generate 10,000 Monte Carlo scenarios based on the estimated copula parameters, representing a wide range of potential joint return distributions for stress-testing the portfolio.

4. **To optimize portfolio risk (decision making):** Determine optimal asset weights by minimizing Expected Shortfall (ES), evaluating whether the proposed copula-based optimization provides more effective downside risk protection than traditional linear-correlation models.

1.4 Research Contributions

Cryptocurrency markets are characterized by high volatility and frequent extreme crashes, which create significant challenges for investors and portfolio managers trying to reduce their exposure to downside risk.In this study, we develop a specific mathematical framework for Bitcoin and Ethereum to capture severe downside risks. Instead of relying on traditional models that assume market stability, our approach explicitly targets worst-case scenarios to help investors make safer allocation choices during a collapse.

The key contributions of this study are:

- We quantify the lower-tail dependence (λL) between BTC and ETH, which shows that diversification benefits decrease during Financial crises.
- We apply the Survival BB1 copula within a GARCH framework to capture asymmetric tail dependence, to improve traditional symmetric models, and provide a more realistic representation of unexpected market moments.
- We include Expected Shortfall in copula-based portfolio optimization to measure possible losses during extreme market conditions and to improve risk management practices.
- We provide a complete framework from marginal volatility modeling to joint dependence estimation and Monte Carlo stress testing that can be applied to other high-volatility asset classes.

1.5 Thesis Structure

The structure of this study is as follows:

- **Chapter 2:** presents the Literature Review, covering the characteristics of the cryptocurrency market, the limitations of the normal distribution in finance, and the theoretical foundations of GARCH models and copula theory
- **Chapter 3:** This chapter explains the research methodology used in the study. It describes the step-by-step quantitative framework, the reason for selecting the data, the marginal volatility filtering process (Stage 1), the asymmetric dependence modeling (Stage 2), and the mathematical procedure used for Expected Shortfall optimization.
- **Chapter 4:** This chapter presents the main empirical results of the study. It starts with data analysis, marginal model estimates, and copula estimation results. After that, it discusses the trivariate (3D) framework and examines how including the S&P 500 affects portfolio risk and improves downside risk protection.

- **Chapter 5:** In this chapter, the results are discussed in detail and interpreted according to the objectives of the study. It also examines how reliable the model is, compares the findings with previous financial studies, and analyzes the difference in performance between crypto-only (2D) portfolios and diversified (3D) portfolios.
- **Chapter 6:** In this chapter, the overall findings of the study are summarized. It also highlights the limitations of the research and gives some suggestions for future studies related to digital asset risk management.

2 Literature Review

The academic exploration of cryptocurrency risk management has undergone a significant transformation, evolving from basic volatility assessments to sophisticated nonlinear dependence modeling. The foundation of this discourse was established by Dyhrberg [11], who first utilized GARCH models to classify Bitcoin as a financial asset situated between gold and the US dollar. This seminal work highlighted Bitcoin's potential as a hedging tool, yet it remained limited by its univariate focus. As the market matured, researchers shifted toward the interdependencies within the digital asset space. Katsiampa [21] introduced this shift by employing a bivariate Diagonal BEKK-GARCH model to examine the comovements between Bitcoin and Ethereum. While her findings confirmed that both market leaders are highly sensitive to external shocks, the linear framework of the BEKK model lacked the capacity to capture complex tail dynamics. Broadening the scope of volatility modeling, J. Mba [27] introduced a Markov-switching COGARCH framework to identify transitions between bull and bear market regimes. However, while their model identifies shifts in market states, it does not explicitly capture the nonlinear tail synchronization that occurs during extreme liquidations.

Building upon these studies, Beneki [5] utilized a VAR-GARCH methodology to demonstrate that the hedging capabilities between Bitcoin and Ethereum have significantly reduced over time as their correlations have become more synchronized. This synchronization is particularly evident during periods of high stress, a phenomenon further explored by Naeem [29], who proved that market crashes forge a rigid linkage between trading volume and price, suggesting that traditional linear measures of risk are insufficient. Adding a macroeconomic dimension, Naeem [30] investigated the asymmetric relationship between oil shocks and cryptocurrencies using a cross-quantilogram approach. Their findings revealed that Bitcoin and Ethereum exhibit distinct and nonlinear responses to energy market fluctuations, acting as diversifiers during specific oil demand shocks.

Recent literature has increasingly focused on copula-based frameworks to address these nonlinearities. For instance, Tenkam [41] utilized a complex C-vine copula approach to highlight the role of stablecoins as diversifiers within a multiasset portfolio. While vine copulas offer flexibility in high dimensions, they often introduce excessive parametric complexity and may overlook the specific asymmetric synchronization of core pairs like BTC-ETH during crashes. Similarly, Bruhn [8] applied a GARCH EVT copula approach to a broader portfolio of the top 20 cryptocurrencies, concluding that intramarket diversification often fails when it is needed most. However, their reliance on symmetric copula functions like the Student-t remained a constraint. To overcome this, Queiroz [35] identified that the Clayton family is superior in capturing the lower-tail dependencies that characterize market meltdowns. Despite this advancement, Kimani [22] argued that there is still a need for dual-parameter models that can handle both upper and lower tail complexities simultaneously. The most recent advancement by Huynh [16] integrated GARCH-EVT-copula with Deep Learning (LSTM) for portfolio optimization; yet, their application was restricted to stock markets. Consequently, this thesis fills a critical gap by implementing the Survival BB1 Copula specifically for the BTC-ETH pair. While Bruhn [8] relied on symmetric copulas and Queiroz [35] identified the Clayton family as superior for capturing lower-tail dependence, Kimani [22] argued for the necessity of dual-parameter

models to capture complex financial dependencies. The choice of the BB1 copula in this study bridges these perspectives by offering the flexibility to handle both tail complexities simultaneously. By utilizing this dual-parameter approach, the study provides a more robust strategy for mitigating extreme downside risk during joint market crashes, a feature often underestimated by the single-parameter or symmetric models found in the previous literature.

3 Research Methodology

3.1 Introduction to the Quantitative Framework

This study adopts a quantitative empirical framework to model the dependence structure and tail risk behaviour of BTC and ETH. In light of the stylized empirical properties of cryptocurrency returns, volatility clustering, leptokurtosis, and nonlinear dependence, a multistage estimation procedure is implemented to disentangle marginal dynamics from cross-asset dependence.

The empirical strategy is based on the Inference Functions for Margins (IFM) method as introduced by [19]. The IFM approach allows the estimation of marginal distributions and the dependence structure in two sequential stages. This procedure is particularly useful when joint maximum likelihood estimation is computationally demanding and when the marginal distributions exhibit distinct time-varying characteristics.

In the first stage, each return series is modeled individually using a GARCH(1,1) specification with Student-t innovations. This step captures conditional heteroskedasticity and heavy-tailed behaviour commonly observed in financial time series. The resulting standardized residuals are assumed to be approximately independent and identically distributed.

In the second stage, the filtered residuals are transformed into uniform variables and used to estimate the dependence structure through a copula function. Specifically, the Survival BB1 copula is employed to allow flexible modeling of asymmetric tail dependence. This separation of margins and dependence enables a more accurate characterization of joint extreme risk.

The overall framework integrates marginal volatility modeling, copula-based dependence estimation, and risk-based portfolio optimization. This structure provides a coherent methodology for analyzing joint tail behaviour and its implications for portfolio risk management.

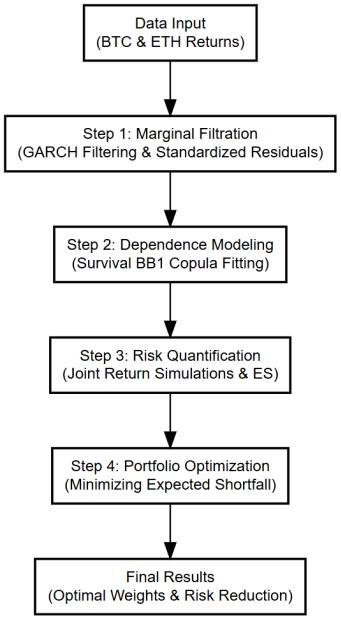

Figure 1 *Methodological framework of the GARCH-copula model.*

3.2 Methodological Workflow

To address the research question on portfolio risk optimization, the empirical analysis was carried out in four steps:

1. **Marginal Filtration:** Daily return series for Bitcoin and Ethereum were filtered using a GARCH(1,1) model. This step was applied to remove volatility clustering, where periods of high volatility tend to be followed by similar periods. As a result, standardized and independent residuals were obtained for further modeling.

2. **Dependence Modeling:** The filtered residuals are transformed via their estimated cumulative distribution functions and modeled using the Survival BB1 copula. The dual-parameter structure (θ, δ) allows flexible modeling of asymmetric tail dependence, particularly in the lower tail.

3. **Risk Quantification:** Based on the fitted copula and marginal models, joint return simulations are generated. These simulations are used to estimate portfolio Expected Shortfall (ES), thereby capturing extreme downside risk.

4. **Portfolio Optimization:** Portfolio weights are systematically varied to identify the weight configuration that minimizes Expected Shortfall, resulting in the Global Minimum Risk allocation under asymmetric dependence.

3.3 Data Description and Selection Rationale

Asset Selection Rationale

The selection of these two assets, Bitcoin and Ethereum, is based on their dominant position within the digital asset market. Throughout the sample period (2019-2025), these two assets consistently maintained a combined market dominance of approximately 65% to 75% of the total cryptocurrency market capitalization, as documented by CoinGecko [9] Specifically, as of late 2024 and early 2025, Bitcoin's market capitalization reached approximately $1.76 trillion (approx. 52% dominance), while Ethereum's valuation stood at $358 billion (approx. 17% dominance), according to data from CoinMarketCap [10].

Their economic functions also support their inclusion in this study. Bitcoin is commonly viewed as a decentralized store of value, whereas Ethereum mainly serves as a platform for smart contracts and decentralized finance. From a statistical point of view, both assets offer long and highly liquid trading histories, with a combined average daily trading volume often exceeding 50 billion during peak periods acording to [10].

This depth of data offers a reliable foundation for estimating volatility dynamics . In contrast, smaller cryptocurrencies frequently have limited historical records and lower liquidity, which can negatively impact the accuracy of tail dependence estimation . Therefore, the BTC–ETH pair acts as a representative proxy for systemic risk across the wider cryptocurrency market, ensuring that the model captures the most significant market movements.

Data Description

In this study, we use daily closing prices of Bitcoin and Ethereum denominated in USD. The dataset, obtained from Investing.com [18], spans the period from December 18, 2019, to December 18, 2025, capturing multiple market conditions, including phases of rapid bullish expansion and periods of systemic liquidity stress. Investing.com [18] was selected due to its extensive historical coverage and provision of consistently formatted price series across major exchanges. However, as the data represent aggregated market prices, appropriate preprocessing steps were undertaken to ensure consistency and reliability for the tail dependence analysis within the Copula-based framework.

Variable	Description
Date	Trading date of the observation.
Price (Close)	The final trading price of the asset at the end of the day.
Open	Opening price of the asset on the trading day.
High	Highest traded price during the day.
Low	Lowest traded price during the day.
Volume (Vol.)	Total quantity of the asset traded, representing market liquidity.
Change %	The percentage difference between the current and previous day's closing prices.

Table 1 *Variables and Description of the Dataset*

The variables detailed in Table 1 provide a comprehensive view of the market dynamics for Bitcoin and Ethereum. For this study, the Price (Close) is utilized as the primary input for calculating log returns, while Volume serves as a secondary indicator of market liquidity during the observed period.

Data Transformation

To ensure statistical stationarity, a prerequisite for GARCH modeling, the raw price series P_t are transformed into continuously compounded log-returns R_t according to

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \times 100 \tag{1}$$

The logarithmic transformation stabilizes scale effects and facilitates interpretation in percentage terms. This preprocessing step is essential for capturing the volatility clustering and heavy-tailed behaviour characteristic of cryptocurrency returns, which will be analyzed in detail in the subsequent sections.

Statistical Properties and Normality Verification

Before implementing the GARCH-Copula framework, it is essential to examine the behavioural characteristics of the return series. In financial econometrics, data rarely follow the ideal assumptions of classical finance; cryptocurrency returns, in particular, exhibit unique traits that deviate significantly from the standard normal distribution. By identifying these traits early, we provide empirical justification for the use of nonlinear models.

Stylized Facts and Market behaviour

The return series for BTC and ETH is expected to demonstrate two primary stylized facts:

- **Volatility Clustering:** This refers to the observation that price shocks tend to persist over time. Large changes in returns are often followed by further large changes, suggesting that market risk is time-varying and clustered in specific periods.
- **Leptokurtosis:** Unlike a normal distribution, cryptocurrency returns typically display fat tails.This property indicates that extreme market movements, such as sudden crashes or price spikes, occur more frequently than standard statistical theory predicts.

Empirical Testing: The Jarque-Bera (JB) Test

To formally move from observation to statistical proof, we employ the Jarque-Bera test. This diagnostic tool evaluates whether the sample's skewness (S) and kurtosis (K) are consistent with a normal distribution, where $S = 0$ and $K = 3$. The JB test statistic is computed as follows:

$$JB = \frac{n}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \tag{2}$$

In this framework, the null hypothesis (H_0) assumes that the data is normally distributed. A high JB value, leading to a rejection of H_0, confirms that the returns are nonnormal.

This verification step is a crucial bridge in our methodology. If the returns exhibit high kurtosis and significant JB scores, it provides the statistical necessity for our two-stage approach: using GARCH(1,1) in the first stage to model persistent volatility, and the Survival BB1 copula in the second stage to capture the joint tail dependence that a normal distribution would otherwise ignore.

3.4 Stage 1: Volatility Filtering and Marginal Modeling

Modeling cryptocurrency returns is challenging due to their high volatility and nonlinear behaviour. Therefore, the first stage of our framework focuses on volatility filtering and marginal modeling. The main objective is to decompose the raw return series of Bitcoin (BTC) and Ethereum (ETH) in order to isolate the underlying market shocks.

This step is essential because copula modeling requires the input series to be independent and identically distributed. To achieve this, we remove the time-varying volatility component from the returns, ensuring that the remaining residuals are free from predictable volatility patterns.

Volatility Modeling with GARCH(1,1)

Traditional linear models are not suitable for financial return series because they cannot capture volatility clustering, where large changes in returns tend to be followed by large changes and small changes by small changes. This behaviour is particularly evident in cryptocurrency markets. The generalized autoregressive conditional heteroskedasticity (GARCH) model, specifically the GARCH(1,1) specification introduced by [6], is a statistical framework used to estimate and forecast the time-varying volatility of financial time series. The GARCH model extends the ARCH framework originally proposed by [13], allowing the conditional variance to depend not only on past squared innovations but also on its own lagged values, thereby capturing volatility persistence.

A financial return series R_t is said to follow a GARCH(1,1) process as proposed by [6] if the return dynamics can be represented through a conditional mean equation together with a conditional variance equation. The conditional mean and variance equations are defined as

$$R_t = \mu + \epsilon_t, \qquad \epsilon_t = \sigma_t z_t, \tag{3}$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{4}$$

where σ_t^2 denotes the conditional variance of returns at time t, ϵ_{t-1} represents the previous innovation, $\omega > 0$ is a constant term, $\alpha \geq 0$ measures the short-run impact of past shocks (ARCH effect), and $\beta \geq 0$ captures the persistence of past volatility (GARCH effect).

3.4.1 Proposition (Stationarity of the GARCH Process, [6])**.** *A GARCH(1,1) process is covariance stationary if the parameters satisfy the condition*

$$\alpha + \beta < 1. \tag{5}$$

Under this condition, the unconditional variance of the innovation process exists and is given by

$$Var(\epsilon_t) = \frac{\omega}{1 - \alpha - \beta}. \tag{6}$$

Proof. Taking the unconditional expectation of the variance equation, under the assumption of covariance stationarity, we have $E[\sigma_t^2] = E[\sigma_{t-1}^2] = \bar{\sigma}^2$ and $E[\epsilon_{t-1}^2] = \bar{\sigma}^2$. Substituting these into the GARCH(1,1) equation produces:

$$\bar{\sigma}^2 = \omega + \alpha\bar{\sigma}^2 + \beta\bar{\sigma}^2 \implies \bar{\sigma}^2(1 - \alpha - \beta) = \omega. \tag{7}$$

Solving for $\bar{\sigma}^2$ gives the unconditional variance $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$. For the variance to be finite and positive, the denominator must satisfy $1 - \alpha - \beta > 0$, which implies $\alpha + \beta < 1$. □

This proposition ensures that the volatility process remains stable over time and does not diverge. In empirical financial applications, particularly in cryptocurrency markets, values of $\alpha + \beta$ are often observed to be close to unity, reflecting strong volatility persistence [6]. Ensuring stationarity

is therefore essential for reliable volatility estimation and for interpreting the long-run behaviour of the conditional variance.

By estimating these parameters, the GARCH(1,1) model captures the time-varying volatility present in the return series. The conditional variance σ_t^2 reflects the impact of past shocks (ϵ_{t-1}^2) and past volatility (σ_{t-1}^2), allowing the model to account for volatility clustering commonly observed in cryptocurrency markets.

As a result, the standardized residuals are defined as

$$z_t = \frac{\epsilon_t}{\sigma_t}. \tag{8}$$

Distributional Assumptions for the Innovations

After obtaining the standardized residuals from the GARCH model, the next step in the marginal modeling framework is to specify an appropriate probability distribution for the innovation series z_t. The choice of the innovation distribution is important because it determines how well the model captures the empirical characteristics of financial returns.

Many traditional financial models assume that innovations follow a Gaussian (normal) distribution. However, such an assumption often leads to an underestimation of risk, particularly in cryptocurrency markets where extreme observations occur more frequently than predicted by normal theory.

In our empirical analysis, the descriptive statistics and diagnostic tests indicate the presence of leptokurtosis, meaning that the return distribution exhibits heavier tails than the normal distribution distribution. To account for this empirical property, we assume that the standardized innovations follow a Student-t distribution, which is widely used in financial econometrics due to its ability to capture heavy-tailed behaviour [28, 42].

Standardized Student-t Distribution To ensure that the innovation process z_t is correctly standardized with mean zero and unit variance ($Var(z_t) = 1$), we employ the standardized version of the Student-t distribution. Its probability density function is given by

$$f(z;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{z^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}, \tag{9}$$

where $z \in \mathbb{R}$ denotes the standardized innovation, $\nu > 2$ represents the degrees of freedom parameter controlling the tail thickness of the distribution, and $\Gamma(\cdot)$ denotes the Gamma function. The parameter ν plays an important role in determining the kurtosis of the distribution. Smaller values of ν imply heavier tails and therefore a higher probability of extreme observations. Conversely, as $\nu \to \infty$, the Student-t distribution converges to the normal distribution. This standardized formulation is essential for GARCH modeling as it ensures the consistency of the conditional variance estimation.

3.4.2 Lemma (Excess Kurtosis and Tail Thickness)**.** *For a Student-t distributed innovation with $\nu > 4$*

degrees of freedom, the excess kurtosis is defined as:

$$\textit{Kurt}(z_t) - 3 = \frac{6}{\nu - 4}. \tag{10}$$

As the degrees of freedom ν decrease, the excess kurtosis increases, theoretically justifying the model's ability to capture extreme market shocks. For instance, if $\nu \approx 5$ (common in crypto markets), the excess kurtosis is 6, whereas for a Gaussian distribution, it is exactly 0. This mathematical property ensures that the model assigns higher probability density to the tails, capturing the leptokurtic nature of Bitcoin and Ethereum returns. [28, 42] By incorporating the Student-t innovation structure into the GARCH framework, the model becomes more stable during extreme market movements. This ensures that volatility and risk estimates are not systematically underestimated before proceeding to the dependence modeling stage within the copula framework.

Extraction of Standardized Residuals

After specifying the innovation distribution, the next step is to extract the standardized residuals from the estimated GARCH model.

3.4.3 Theorem (Maximum Likelihood Estimation, [7])**.** *The vector of parameters $\theta = (\mu, \omega, \alpha, \beta, \nu)$ is estimated by maximizing the joint log-likelihood function $\mathcal{L}(\theta)$, which for T observations of the Student-t innovation process is given by:*

$$\mathcal{L}(\theta) = \sum_{t=1}^{T} \left[\ln \Gamma \left(\frac{\nu+1}{2} \right) - \ln \Gamma \left(\frac{\nu}{2} \right) - \frac{1}{2} \ln\big(\pi(\nu-2)\sigma_t^2\big) - \frac{\nu+1}{2} \ln \left(1 + \frac{\epsilon_t^2}{(\nu-2)\sigma_t^2} \right) \right]. \tag{11}$$

This optimization ensures that the filtered residuals z_t are statistically consistent with the underlying data generating process, allowing for accurate extraction of market shocks. These residuals represent the volatility-adjusted shocks in the return series and serve as the main output of the volatility filtering stage.

The standardized residuals (z_t) are obtained by dividing the raw residuals (ϵ_t) by their estimated conditional standard deviation (σ_t):

$$z_t = \frac{\epsilon_t}{\sigma_t}.$$

If the GARCH model is correctly specified and the chosen innovation distribution appropriately captures the heavy-tailed nature of the data, the standardized residuals should behave approximately as independent and identically distributed (IID) random variables [6, 42].

In practical terms, the standardized residuals represent pure market innovations after removing time-varying volatility effects from the original return series. These filtered innovations, therefore, provide a suitable basis for the subsequent diagnostic verification and dependence modeling steps used in the copula framework.

Diagnostic Testing and Model Validation

Following the extraction of the standardized residuals, it is necessary to verify that the GARCH filter has successfully removed the volatility clustering and serial dependence present in the raw return data. That means the standardized residuals should behave as independent and identically distributed innovations before they are used in the copula modeling stage.

To confirm this property, diagnostic tests are applied to ensure that no remaining serial correlation or conditional heteroskedasticity persists in the standardized residuals. This step is important because copula models assume that the data used for the margins has no remaining time-series structure.

3.4.4 Proposition (Diagnostic Requirement for Copula Modeling, [6, 42])**.** *Let $\{z_t\}$ denote the standardized residuals obtained from a correctly specified GARCH(1,1) model. If the volatility dynamics of the return series are adequately captured, then the residuals should not exhibit serial correlation or remaining ARCH effects. Formally,*

$$E(z_t z_{t-k}) = 0 \quad \textit{for} \quad k \neq 0,$$

and the conditional variance of z_t remains constant.

Proof sketch. The GARCH(1,1) model captures time-varying volatility through the conditional variance equation. After standardization by σ_t, the resulting residuals behave approximately as white noise innovations with constant variance [6, 42]. Therefore, any remaining serial correlation or ARCH structure would indicate model misspecification.

To empirically verify these conditions, two diagnostic tests are applied.

First, we use the Ljung-Box Q-test [24] to examine whether any serial correlation remains in the standardized residuals. The test statistic is defined as

$$Q = n(n+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{n-k},$$

where n denotes the sample size, m represents the number of lags considered, and $\hat{\rho}_k$ is the sample autocorrelation at lag k. If the test fails to reject the null hypothesis, this indicates that the residuals do not exhibit significant autocorrelation.

Second, we apply the ARCH-LM test proposed by Engle [13] to check for remaining ARCH effects. This test determines whether conditional heteroskedasticity is still present in the residuals. Failing to reject the null hypothesis confirms that the volatility clustering has been successfully captured by the GARCH model.

If both diagnostic tests are satisfied, we conclude that the GARCH filter has effectively removed serial correlation and conditional heteroskedasticity. The resulting standardized residuals can therefore be treated as clean inputs for the next stage, where the dependence structure between Bitcoin and Ethereum are modeled using copulas.

3.5 Stage 2: Dependence Modeling The Copula Framework

After filtering the individual volatility dynamics of Bitcoin (BTC) and Ethereum (ETH) in 3.4 using the GARCH model, the next step is to model the joint dependence structure between the two assets. While the marginal behaviour has been standardized through volatility filtering, understanding how BTC and ETH move together, particularly during extreme market conditions, requires a separate modeling framework.

Traditional linear correlation measures are often insufficient in financial markets, especially during periods of stress, because they assume symmetric and constant dependence. To overcome this limitation, we apply copula theory, which allows the dependence structure to be modeled independently from the marginal distributions.

Theoretical Foundations and Model Specification

For clarity, the following definitions are provided before specifying the empirical model. To transition from the marginal modeling of 3.4 to the copula framework, we must first transform the standardized residuals z_t into a uniform domain.

3.5.1 Theorem (Probability Integral Transform)**.** *Let z_t be the standardized residuals from the GARCH process, following a cumulative distribution function $F_\nu(z)$. The transformed variables:*

$$u = F_{\nu,BTC}(z_{BTC}) \quad \textit{and} \quad v = F_{\nu,ETH}(z_{ETH}) \tag{12}$$

follow a standard uniform distribution $U(0,1)$.

This transformation is mathematically mandatory because copula functions are defined strictly on the unit hypercube $[0,1]^2$. The PIT serves as the bridge that filters out the individual scales and marginal distributions, allowing us to isolate the pure dependence structure. This ensures that our subsequent estimates are scale-invariant and focused entirely on the interaction between BTC and ETH (Joe)[28].

Sklar's Theorem

The theoretical foundation of copula modeling is provided by Sklar's Theorem, [39] which states that for any d-dimensional joint distribution function F with continuous marginals $F_1, \ldots, F_d$, there exists a unique copula C such that:

$$F(x_1, \ldots, x_d) = C\big(F_1(x_1), \ldots, F_d(x_d)\big)$$

In the specific bivariate case of this study (Bitcoin and Ethereum), the joint distribution $H(x,y)$ is linked to the marginals $F_X(x)$ and $F_Y(y)$ via the copula function C as follows:

$$H(x,y) = C(F_X(x), F_Y(y)) = C(u,v) \tag{13}$$

where $u, v \in [0,1]$ are the uniform marginals obtained through the Probability Integral Transform.

Copula Density:

While the copula $C(u,v)$ models the joint distribution, the copula density $c(u,v)$ represents the dependence structure in terms of probability density. Mathematically, it is defined as the second-order mixed partial derivative of the copula function:

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$$

where $u,v \in [0,1]$ are the uniform marginals.

In this study, the density is guaranteed to exist because the selected Archimedean families (Survival BB1, Clayton, and Gumbel) are absolutely continuous, and their generator functions $\phi(t)$ are twice-differentiable.

This density is the fundamental building block for the joint density function used in our trivariate Vine Copula framework:

$$f(x,y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y))$$

Since our marginals are continuous, the copula C is unique. This theorem justifies separating the modeling of marginal distributions from the dependence structure, allowing us to first filter individual asset volatility and then focus on the joint behaviour through a copula.

3.5.2 Lemma (Methodological Separability and Consistency)**.** *If the marginal distribution functions $F_1, \ldots, F_d$ are continuous, and the copula C is unique (as guaranteed by Sklar's Theorem), then the joint log-likelihood function can be decomposed into the sum of the log-likelihoods of the marginals and the log-likelihood of the copula.*

This provides the formal basis for the Inference Functions for Margins (IFM) approach [20]. It proves that we can consistently estimate GARCH parameters in 3.4 and Copula parameters in 3.5 in two separate steps without loss of asymptotic efficiency. This ensures that the dependence we capture is not biased by the individual volatility dynamics of BTC and ETH. As a result, we can focus exclusively on modeling the interaction between BTC and ETH without reestimating their individual volatility properties.

Bivariate Copula

A 2-dimensional (bivariate) copula is a function $C : [0,1]^2 \to [0,1]$ that satisfies the following properties:

1. **Grounding Property:** $C(u,0) = 0 = C(0,v)$ for all $u, v \in [0,1]$.
2. **Marginality Requirement:** $C(u,1) = u$ and $C(1,v) = v$ for all $u, v \in [0,1]$.
3. **2-Increasing Property:** For any $u_1 \leq u_2$ and $v_1 \leq v_2$ in $[0,1]$,

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0.$$

This formal structure allows the coupling of any two marginal distributions into a single joint distribution $H(x, y)$, effectively isolating the dependence structure from the marginal behaviour. In the context of this study, this framework enables us to link the volatility dynamics of BTC and ETH, which were filtered in 3.4. Crucially, this is achieved without the restrictive assumption of linear correlation or the requirement of joint normality [33].

BB1 Copula

The BB1 copula belongs to the Archimedean family and is constructed using a specific generator function $\phi(t)$. For any two random variables U and V defined on the real line $\mathbb{R}$, we first obtain their uniform marginals $u = F_U(u)$ and $v = F_V(v)$ such that $u, v \in [0, 1]$. The BB1 copula is then defined as

$$C_{BB1}(u, v; \theta, \delta) = \left[1 + \left((u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}\right)^{1/\delta}\right]^{-1/\theta}, \tag{14}$$

where the generator function is given by

$$\phi(t) = (t^{-\theta} - 1)^{\delta} \tag{15}$$

The joint structure can also be expressed via the inverse generator:

$$C(u, v) = \phi^{-1}(\phi(u) + \phi(v)). \tag{16}$$

The valid parameter ranges for the BB1 copula are:

$$\theta \in (0, \infty), \quad \delta \in [1, \infty). \tag{17}$$

These parameters allow the model to capture both the strength of association and the tail dependence simultaneously, providing superior flexibility over single-parameter models like Clayton or Gumbel [19].

Survival BB1 Copula

The survival copula represents the joint probability of both events surviving past their respective thresholds (i.e., joint exceedance). For the uniform marginals u and v, the survival copula is defined as:

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v) \tag{18}$$

Applying this to the BB1 functional form gives the Survival BB1 copula:

$$\hat{C}_{BB1}(u, v; \theta, \delta) = u + v - 1 + \left[1 + \left(((1 - u)^{-\theta} - 1)^{\delta} + ((1 - v)^{-\theta} - 1)^{\delta}\right)^{1/\delta}\right]^{-1/\theta} \tag{19}$$

The parameter ranges remain the same: $\theta \in (0, \infty)$ and $\delta \in [1, \infty)$. This dual-parameter structure is particularly significant as it allows for modeling both the strength and asymmetry of

dependence. This specification is robust for capturing extreme downside comovements (lower tail dependence) between digital assets.

Tail Dependence and Extreme Comovements

Financial assets, particularly cryptocurrencies, often exhibit stronger dependence during extreme market events than during normal periods. This phenomenon is captured through tail dependence.

Upper and lower tail dependence coefficients are defined (whenever they exist) as:

$$\begin{aligned} \lambda_U &= \lim_{u \to 1^-} \Pr\left(Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)\right), \\ \lambda_L &= \lim_{u \to 0^+} \Pr\left(Y \leq F_Y^{-1}(u) \mid X \leq F_X^{-1}(u)\right), \end{aligned} \tag{20}$$

where F_X^{-1} and F_Y^{-1} denote the quantile functions (pseudoinverses) of the respective marginal distributions. Formally, the notation for the generalized inverse (pseudoinverse) of a distribution function F is defined as:

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad \text{for } u \in (0,1). \tag{21}$$

This formal definition ensures mathematical consistency even in cases where the distribution function is not strictly increasing. These functions map the probability space $[0,1]$ back to the actual return space, which is a critical step for later risk simulations.

The lower tail dependence coefficient λ_L measures the probability that one asset experiences an extreme negative return given that the other asset is already in its lower tail. This measure is particularly important for portfolio risk management, as it quantifies the likelihood of simultaneous crashes. For the Survival BB1 copula, these extreme comovements are captured through analytical tail dependence coefficients derived from the parameters θ (Clayton component) and δ (Gumbel component).

Following the computational implementation in the VineCopula framework [31], the tail coefficients for the Survival BB1 copula are derived by recognizing that the survival transformation $\hat{C}(u,v)$ mathematically rotates the density by 180 degrees. As established by Joe [19], and the mathematical properties of copula derivatives provided by Schepsmeier [38], this rotation swaps the tail properties: the lower tail dependence of our Survival model (λ_L^{Surv}) is analytically equivalent to the upper tail dependence of a standard BB1 copula (λ_U^{Std}). These are expressed as:

$$\lambda_L^{Surv} = \lambda_U^{Std} = 2 - 2^{1/\delta}, \tag{22}$$

$$\lambda_U^{Surv} = \lambda_L^{Std} = 2^{-1/(\theta\delta)}. \tag{23}$$

This derivation is crucial because it shows that in the Survival BB1 framework, the parameter δ (Gumbel component) exclusively dictates the crash synchronization (λ_L). This explains why, with $\hat{\delta} = 2.6054$, we obtain the following result:

$$\lambda_L = 2 - 2^{1/2.6054} \approx 0.6952. \tag{24}$$

This value provides a precise mathematical quantification of the asymptotic dependence between BTC and ETH during extreme market stress.

The importance of this finding lies in its contrast with standard linear models. In a Gaussian framework, this value would be zero, implying that extreme crashes are independent events [12]. However, our result of 0.69 proves Crash Synchronization. This structural reality directly informs 3.7 of this study, where we utilize this high probability of joint failure as a critical input for ES simulations to quantify the actual risk of a joint market meltdown [3, 40]. This mathematical property assigns higher density to the lower-left quadrant of the joint distribution, theoretically justifying the crash synchronization observed in the BTC and ETH pair [31]. This level of synchronization illustrates why traditional linear correlation, which only measures average comovements, often fails to protect investors during market meltdowns, as it significantly underestimates the risk of joint tail events.

Empirical Methodology and Findings

Model Selection and Empirical Specification of the Survival BB1 Copula

To identify the most statistically robust dependence structure for the BTC-ETH pair, a rigorous competitive selection process was conducted. The selection was governed by the Akaike Information Criterion and the Bayesian Information Criterion, applied to the filtered i.i.d. residuals obtained from 3.4. In this framework, the Survival BB1 copula was benchmarked against a comprehensive suite of alternative models. This included the Elliptical family, specifically the Gaussian and Student-t copulas [28], to test for both independence and symmetric tail dependence. While the elliptical family includes a broader range of models, the Gaussian and Student-t were prioritized as they represent the most established benchmarks in financial literature.

Additionally, single-parameter Archimedean Copulas, such as Clayton, Gumbel, and Frank, were evaluated to determine if a simpler structure was sufficient.

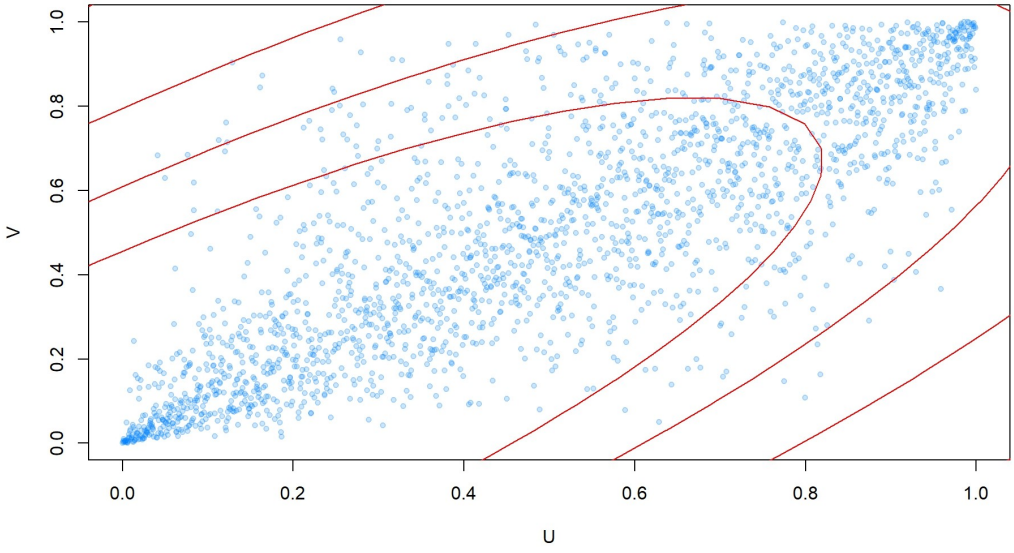

Figure 2 *Joint Dependence Structure of BTC-ETH Uniform Marginals (PIT Residuals)*

Before conducting the formal statistical comparison, the dependence structure of the filtered residuals is visualized through a scatter plot of uniform marginals (PIT residuals), as shown in Figure

2. The plot reveals a distinct asymmetric pattern, characterized by a significant clustering of observations in the lower-left quadrant (representing joint extreme losses) compared to the upper-right quadrant. This visual evidence of strong lower-tail dependence suggests that the BTC-ETH pair exhibits crash synchronization during market crashes, providing a visual analytical justification for the selection of an asymmetric framework like the Survival BB1 copula over symmetric elliptical models.

Model	Log-Likelihood (LL)	AIC	BIC
Survival BB1	**1348.461**	**-2692.923**	**-2681.538**
Student-t	1296.672	-2589.344	-2577.958
Gaussian	1242.200	-2482.401	-2476.708

Table 2 *Copula model comparison based on AIC and BIC*

Note: Lower values (more negative) of AIC and BIC indicate a better statistical fit.

The analytical results presented in Table 2 provide a clear statistical justification for the selection of the Survival BB1 copula. The selection is strongly supported by the Log-likelihood (LL) values; the Survival BB1 achieves a maximum LL of 1348.461, which is significantly higher than the LL of the Student-t (1296.672) and other Archimedean counterparts, indicating a superior probabilistic fit. Comparing the information criteria across the candidate models, the Survival BB1 achieves the lowest AIC (-2692.923) and BIC (-2681.538), significantly outperforming the elliptical and single-parameter Archimedean alternatives. The relatively higher AIC values for the Gaussian and Student-t copulas indicate that symmetric dependence structures fail to adequately penalize the model for missing the tail asymmetry present in the data. Furthermore, the superiority of the BB1 over the nested Clayton and Gumbel models suggests that a single tail parameter is insufficient to capture the complex dependence dynamics of the BTC-ETH pair. The choice of the Survival BB1 is further justified by its unique dual-parameter structure (θ, δ), which allows for the simultaneous and flexible modeling of both tail behaviours. Unlike the Clayton copula, which primarily captures lower-tail dependence, or the Gumbel copula, which focuses on the upper tail, the BB1 framework provides the necessary complexity to address the structural limitations of symmetric models. As previously defined, we apply a survival transformation to this framework by rotating the copula 180°, effectively shifting the focus toward the lower tail. These statistical markers confirm that an asymmetric framework is critical for capturing the extreme crash synchronization and nonlinear linkages that emerge during cryptocurrency market downturns. By rejecting the Gaussian and Student-t models in favor of the Survival BB1, we provide empirical evidence that the BTC-ETH pair exhibits asymptotic dependence that is fundamentally asymmetric, a finding that has significant implications for downside risk management.

Empirical Findings on Tail Dependence and Crash Synchronization

The estimation of the Survival BB1 parameters reveals a profound asymmetry in the dependence structure of the cryptocurrency pair. While the upper-tail dependence ($\lambda_U = 0.0168$) is statistically negligible, the lower-tail dependence ($\lambda_L = 0.69$) is remarkably high.

In practical economic terms, a lower-tail dependence coefficient of 0.69 implies a 69% probability that Ethereumwill experience a massive price crash specifically when Bitcoin first drops below

its own threshold for an extreme loss. This high level of synchronization illustrates why traditional linear correlation, which only measures average comovements, often fails to protect investors during market meltdowns.

These findings confirm that the joint distribution of these assets is characterized by intense crash synchronization. During market rallies, the assets may move independently, but during downturns, they exhibit a rigid, synchronized linkage. This structural reality necessitates the transition to the final stage of our framework, where we operationalize these findings using Expected Shortfall and Monte Carlo simulations to quantify the extreme downside risk.

Extension to Multivariate Modeling: Vine Copulas

While bivariate copulas effectively model the dependence between two assets, real-world portfolios often involve multiple variables. Extending a single multivariate copula to higher dimensions ($d > 2$) is often mathematically restrictive. To overcome this, we utilize Vine Copulas, a flexible framework based on Pair-Copula Construction (PCC) [1].

Definition of PCC: Pair-Copula Construction is a method of building a multivariate probability distribution by decomposing a d-dimensional joint density function into a product of marginal densities and bivariate pair-copula densities. This allows for different types of dependence structures to be modeled between different pairs in the same system.

Regular Vine Copula

The concept of R-vines was formally introduced by Bedford [4]. Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ denote a random vector and $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ its realization. The joint density function $f_{X_1,\ldots,X_d}(x_1, \ldots, x_d)$ of an R-vine copula is decomposed as follows:

$$f_{X_1,\ldots,X_d}(x_1, \ldots, x_d) = \prod_{k=1}^{d} f_k(x_k) \prod_{j=1}^{d-1} \prod_{e \in E_j} c_{j_e,k_e|D_e} \left(F(x_{j_e}|\mathbf{x}_{D_e}), F(x_{k_e}|\mathbf{x}_{D_e})\right), \tag{25}$$

where f_k are marginal densities, E_j are edges in tree T_j, and $c_{j_e,k_e|D_e}$ are bivariate pair-copula densities.

The link between the general R-vine notation in Eq.25 and the C-vine structure in Eq.26 lies in the specialization of the conditioning sets D_e. In a Canonical Vine, we impose a specific ordering where each tree T_j has a unique root node that governs the dependence.

For any pair-copula in tree j, the conditioning set is fixed as follows:

$$D_e = \{1, \ldots, j-1\}$$

This mapping transforms the generic conditional distributions $F(x_{j_e} \mid x_{D_e})$ and $F(x_{k_e} \mid x_{D_e})$ from Eq.25 into the specific nested terms $F(x_j \mid x_1, \ldots, x_{j-1})$ and $F(x_{j+i} \mid x_1, \ldots, x_{j-1})$ shown in Eq (26).

Canonical Vine (C-vine) Copula

In this study, we implement a C-Vine structure to incorporate a third asset, the S&P 500. For the empirical application, we specialize the notation by letting X, Y, Z denote the random returns of Bitcoin, Ethereum, and the S&P 500, respectively. Bitcoin is selected as the root node due to its dominant market capitalization and its central role in driving dependence structures within the cryptocurrency market.

3.5.3 Theorem (C-Vine Density Decomposition)**.** *According to the Pair-Copula Construction (PCC) framework proposed by [1, 4], the joint density function of a d-dimensional Canonical Vine (C-Vine) can be uniquely decomposed as:*

$$f_{X_1,\ldots,X_d}(x_1,\ldots,x_d) = \prod_{k=1}^{d} f_k(x_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{j,j+i|1,\ldots,j-1}\left(F(x_j|x_1,\ldots,x_{j-1}), F(x_{j+i}|x_1,\ldots,x_{j-1})\right). \tag{26}$$

This theorem provides the formal mathematical proof that our trivariate portfolio density can be simplified into a sequence of nested bivariate copulas, ensuring that the model remains statistically consistent even as we add more assets.

The evaluation of the nested conditional distributions in Theorem 3.5.3 is performed using the recursive h-function, as proposed by [1]. For a bivariate copula C_{uv} with parameter θ, the h-function represents the conditional distribution and is defined as the partial derivative:

$$h(u,v,\theta) = F(u \mid v,\theta) = \frac{\partial C_{uv}(u,v,\theta)}{\partial v}.$$

This recursive procedure is mathematically essential for our 3D framework, as it ensures that the conditional dependencies remain statistically consistent throughout the trivariate hierarchy. The specific pair-copulas used in this 3D framework are selected using the Akaike Information Criterion and Bayesian Information Criterion. This automated selection through the `RVineStructureSelect` function ensures that each relationship (e.g., BTC-ETH or BTC-S&P 500) is modeled by the copula family that best represents its unique tail characteristics. The empirical results and the specific copula families identified for the 3D portfolio are detailed in Section 4.9.

3.6 Portfolio Risk Measurement

After finding that Bitcoin and Ethereum have a strong tendency to crash together ($\lambda_L = 0.6952$) in section 3.5, it becomes clear that these assets are highly linked during extreme market events. This stage shifts from modeling their joint probabilities to actually quantifying the resulting portfolio risk. While VaR is commonly used in finance, it only gives a threshold and does not tell us how severe losses can be during joint crashes. Therefore, this study uses ES as the main risk measure. ES looks at the average loss beyond the VaR threshold, giving a better estimate of potential extreme losses and helping optimize the portfolio to withstand the simultaneous crash events identified in the copula analysis.

Value-at-Risk (VaR): The Threshold Metric

Definition of Value-at-Risk

For a given portfolio loss L defined on a probability space $(\Omega, \mathcal{F}, P)$ and a confidence level $\alpha \in (0,1)$, the Value-at-Risk (VaR_α) is defined as the lower α-quantile of the loss distribution F_L:

$$VaR_\alpha(L) = \inf\{l \in \mathbb{R} : P(L \leq l) \geq \alpha\}. \tag{27}$$

While VaR_α is widely used as a benchmark risk measure, it has several limitations, particularly in markets characterized by heavy-tailed return distributions such as cryptocurrencies. As demonstrated by Artzner [3], VaR_α fails to fulfill the axioms of a coherent risk measure. Specifically, it suffers from tail blindness; it provides no information regarding the severity of losses beyond the threshold, and it frequently violates the subadditivity principle in non-Gaussian environments, potentially reducing diversification benefits.

Formally, this violation of subadditivity is expressed as the possibility where $VaR_\alpha(L_1+L_2) > VaR_\alpha(L_1)+VaR_\alpha(L_2)$ for two loss distributions L_1 and L_2. As established by Artzner [3], this mathematical inconsistency proves that VaR can penalize diversification, making it an unreliable metric for portfolios with heavy-tailed dependencies like Bitcoin and Ethereum.

Expected Shortfall (ES)

Expected Shortfall accounts for the severity of extreme losses beyond the Value-at-Risk threshold.

Definition of Expected Shortfall

Following Acerbi [40], for a loss L with $E[L^-] < \infty$ (here $[x]^- = \max(-x,0)$) and a confidence level $\alpha \in (0,1)$, the Expected Shortfall (ES_α) is defined as the integral average of the quantile function:

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du. \tag{28}$$

In simulation-based settings, such as the Monte Carlo scenarios generated in this study, the Expected Shortfall is computed using the integral representation of the quantile function. This formulation is reliable to discontinuities in the simulated loss distribution and ensures that the risk measure captures the entire tail behaviour rather than a single threshold value.

While ES is sometimes simply defined as the conditional expectation $E[L \mid L \geq VaR_\alpha]$, Artzner et al. [3] show that this form may fail to be a coherent risk measure for discrete distributions. To maintain mathematical consistency in our discrete sample space, we use the integral form in Eq (28).

As demonstrated by Acerbi and Tasche [40], this spectral formulation is coherent and subadditive for both continuous and discrete distributions. By averaging the VaR levels across the full tail $[\alpha,1]$, this method ensures that ES fully captures the shape of the simulated loss distribution and remains robust to the jumps commonly present in cryptocurrency returns.

The Axioms of Coherence and Subadditivity

Let $\mathcal{L}$ be the linear space of real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, representing the portfolio losses (negative log-returns). To ensure the mathematical validity of Expected Shortfall, we assume that the portfolio loss $L \in \mathcal{L}$ is integrable, implying that its first moment $E[L]$ exists and is finite. While this condition is naturally satisfied in our empirical setup due to the finite nature of our 10,000 simulated scenarios, it remains a fundamental requirement for the coherence of the risk measure in a general probability space. In this framework, the random variable L represents the underlying stochastic process of portfolio losses. For our empirical implementation, the elements $L_1, L_2 \in \mathcal{L}$ specifically represent the potential downside exposure of Bitcoin and Ethereum. We utilize 10,000 realizations of L generated through Monte Carlo simulations via the Survival BB1 copula. It is important to distinguish between the random variable L itself and this specific collection of observations, which serve as a statistical sample to estimate the properties of the underlying loss distribution.

Definition (Coherent Risk Measure) (Artzner et al. (1999), [3]). A risk measure $\rho : \mathcal{L} \to \mathbb{R}$ is said to be coherent if, for any loss variables in $\mathcal{L}$, it satisfies the four axioms:

- **Subadditivity**
$$\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2) \tag{29}$$

- **Monotonicity**
If $L_1 \leq L_2$ almost surely, then
$$\rho(L_1) \leq \rho(L_2) \tag{30}$$

- **Positive Homogeneity**
$$\rho(\gamma L) = \gamma \rho(L), \qquad \gamma > 0 \tag{31}$$

- **Translation Invariance**
$$\rho(L + m) = \rho(L) + m, \qquad m \in \mathbb{R}. \tag{32}$$

Expected Shortfall (ES_α) satisfies all four axioms and therefore constitutes a coherent risk measure, whereas Value-at-Risk (VaR_α) may violate the subadditivity property for non-elliptical distributions [40].

In particular, the subadditivity property implies

$$ES_\alpha(L_1 + L_2) \leq ES_\alpha(L_1) + ES_\alpha(L_2). \tag{33}$$

This inequality represents the mathematical foundation of diversification, implying that the aggregate risk of a combined portfolio cannot exceed the sum of individual risks. In the context of this study, even with the strong lower-tail synchronization ($\lambda_L = 0.6952$) identified via the Survival BB1 copula, the subadditivity of Expected Shortfall ensures that the joint BTC-ETH portfolio risk remains bounded. This enables the portfolio optimization framework in Section 3.8 to identify weight allo-

cations that effectively minimize joint downside exposure by exploiting the remaining diversification potential.

3.7 Portfolio Risk Quantification: Monte Carlo Simulation Framework

Upon the successful estimation of the marginal GARCH parameters and the joint Survival BB1 copula coefficients, the framework proceeds to the quantification of aggregate portfolio risk. Because the dependence between BTC and ETH is nonlinear and characterized by tail asymmetry, an analytical or closed-form solution for the joint density is mathematically unavailable. To resolve this complexity, this study employs a Monte Carlo simulation approach to generate a robust, synthetic future distribution [15].

Statistical Justification and Convergence

To operationalize the risk assessment, this study utilizes Monte Carlo simulation as a primary computational tool. By simulating the joint stochastic processes 10,000 times, we transform the theoretical copula parameters into a tangible empirical distribution. The reliability of this estimation is grounded in the consistency of the risk estimator, which is guaranteed by the Strong Law of Large Numbers (SLLN).

3.7.1 Lemma (SLLN, [36])**.** *The sample average of a sequence of independent and identically distributed random variables, having finite mean μ, converges almost surely to the theoretical expected value as the number of trials n approaches infinity:*

$$P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1. \tag{34}$$

By generating 10,000 scenarios, we ensure that the estimated portfolio risk metrics converge to their true theoretical values, thereby minimizing simulation error and providing a stable basis for the optimization phase [23].

Once the statistical convergence is established through the SLLN, we move from theoretical copula parameters to actual stress-tested returns using a specific mathematical sequence . This integration is a crucial step in our framework because it allows us to combine the time-varying volatility filters from section 3.4 with the joint tail dependence we identified in section 3.5.

To map this uniform dependence structure back to real-world asset returns, we rely on the Probability Integral Transform [2], as defined in the following theorem, which provides the mathematical basis for our joint return generation process.

3.7.2 Theorem. *Let U be a uniform random variable on $[0,1]$. If F is a continuous cumulative distribution function, then the random variable X defined by $X = F^{-1}(U)$ has F as its distribution function.*

By applying this theorem, we can operationalize the following simulation algorithm, which transforms our theoretical copula findings into 10,000 realistic market scenarios for stress-testing.

Algorithm 1: Joint Return Scenario Generation

Copula Sampling: Generate 10,000 pairs of dependent uniform random variables $(u, v) \in [0, 1]^2$ using the estimated Survival BB1 with parameters $\hat{\theta}$ and $\hat{\delta}$ [19].

Inverse Transform: Apply the Inverse Student-t CDF to convert uniform variables into standardized residuals: $z_t = F_t^{-1}(u; \nu)$, where ν represents the degrees of freedom from the marginal models.

Volatility Integration : Recompose the simulated returns by scaling the residuals with the conditional volatility forecasted by the GARCH model: $R_{t+1} = \mu + \sigma_{t+1} z_t$ [17].

Economic Rationale and Investor Utility

The primary objective of this simulation framework is to provide a comprehensive downside protection mechanism. While classical models assume constant correlation, our framework recognizes that the linkage between BTC and ETH intensifies during market downturns, a phenomenon known as asymmetric tail synchronization [34]. By utilizing these simulated scenarios to minimize the Expected Shortfall, the model identifies the optimal weight allocations designed to withstand systemic liquidations. This offers institutional investors a resilient strategy capable of maintaining stability during periods of extreme financial distress.

3.8 Portfolio Optimization: The Expected Shortfall Approach

The final objective of this methodological framework is to determine the optimal allocation of investment weights w between BTC and ETH that minimizes extreme downside risk. Unlike traditional Mean-Variance optimization, which often underestimates the probability of joint crashes due to the assumption of normality, this study identifies the Global Minimum Risk Portfolio by utilizing ES as the objective function to account for the heavy-tailed nature of digital assets.

The Optimization Problem and Constraints

Following the 10,000 simulations generated in Section 3.7, the optimization seeks to find the weight vector $w = [w_{BTC}, w_{ETH}]^T$ that solves the following minimization problem:

$$\min_{w} \widehat{ES}_{\alpha} \left(\sum_{i \in \{BTC, ETH\}} w_i R_{i,s} \right), \tag{35}$$

subject to the following operational constraints:

- **Full Investment Constraint:** $\sum_{i \in \{BTC, ETH\}} w_i = 1$;
- **No-Shorting Constraint:** $w_i \geq 0, \quad \forall i \in \{BTC, ETH\}$.

The constraint $w_i \geq 0$ enforces a long-only strategy, reflecting the regulatory and liquidity realities of institutional crypto portfolios where short selling is often restricted.

Mathematical Development

A significant challenge in this stage is that the Expected Shortfall is a non-smooth, non-differentiable function at the VaR threshold, making it unsolvable via standard gradient-based calculus. To address this, we adopt the mathematical framework of Rockafellar and Uryasev [37].

The following theorem explains the mathematical basis of our optimization, turning the non-smooth ES into a simpler convex form.

3.8.1 Theorem (Convexity and Auxiliary Transformation, [37]). *Let $f(w, R) = -\sum_{i=1}^{k} w_i R_i$ be the portfolio loss function, representing a specific element $L \in \mathcal{L}$ of the linear space of losses defined in Section 3.6. Here, f is associated with the decision vector $w \in \mathbb{R}^k$ and the random vector of returns $R \in \mathbb{R}^k$. For a specified confidence level $\alpha \in (0, 1)$, the ES can be characterized by an auxiliary function $F_\alpha(w, \zeta)$ defined as follows:*

In a scenario based model with n equally likely realizations R_s of R, the auxiliary function is defined as

$$F_\alpha(w, \zeta) = \zeta + \frac{1}{n(1-\alpha)} \sum_{s=1}^{n} [f(w, R_s) - \zeta]^+, \tag{36}$$

where $[x]^+ = \max(x, 0)$ denotes the positive part of x; ζ represents the threshold parameter VaR; $n = 10{,}000$ is the total number of simulated scenarios; $s \in \{1, \ldots, n\}$ denotes the range of scenarios; and w_i are the portfolio weights for assets $i \in \{1, \ldots, k\}$, where $k = 2$ for the BTC and ETH pair.

Properties:

1. ***Convexity.*** *The function $F_\alpha(w, \zeta)$ is jointly convex with respect to both portfolio weights w and the auxiliary variable ζ.*

2. ***Minimization.*** *The minimum value of $F_\alpha(w, \zeta)$ with respect to ζ is exactly the Expected Shortfall ($ES_\alpha(w)$),*

$$ES_\alpha(w) = \min_{\zeta \in \mathbb{R}} F_\alpha(w, \zeta). \tag{37}$$

3. ***VaR Relationship.*** *The value of ζ that minimizes $F_\alpha(w, \zeta)$ is the Value-at-Risk ($VaR_\alpha(w)$) for the portfolio w.*

The theoretical significance of Theorem 3.8.1 lies in the simplification of the original portfolio minimization problem defined in Eq. (35). In standard risk management, minimizing ES is notoriously difficult because the objective function is nonsmooth and nondifferentiable at the threshold [37]. However, by introducing the auxiliary function $F_\alpha(w, \zeta)$, we can solve for the optimal weights $w \in W$ (where $k = 2$ for the BTC and ETH pair) and the VaR (ζ) simultaneously. In optimization literature, the weight vector w is often denoted as ω to represent the allocation within the feasible set $W \subset \mathbb{R}^k$. This equivalence ensures that minimizing F_α over the joint space (w, ζ) yields the Global Minimum Risk portfolio weights w^* and the associated risk threshold $\zeta^* = VaR_\alpha(w^*)$.

Proof of Analytical Equivalence and Optimality. To demonstrate why the minimization of $F_\alpha(w, \zeta)$ is equivalent to the minimization of $ES_\alpha(w)$, we examine the first order condition with

respect to ζ. For a fixed portfolio w, the function $\zeta \mapsto F_\alpha(w,\zeta)$ is convex and piecewise linear. Due to the discrete nature of the $n = 10{,}000$ simulated scenarios,the subgradient shown in Eq.(38) is a step function and hence $F_\alpha(w,\zeta)$ is not continuously differentiable in ζ. Since the auxiliary function $F_\alpha(w,\zeta)$ is convex but not everywhere differentiable, we must utilize the theory of subgradients to characterize its optimality. A vector v is defined as a subgradient of a convex function g at point x if $g(y) \geq g(x) + v^T(y-x)$ for all $y \in$ dom g. The set of all such subgradients is the subdifferential $\partial g(x)$. A fundamental property of convex optimization is that a point ζ^* is a global minimizer if and only if the zero vector is contained within the subdifferential, i.e., $0 \in \partial_\zeta F_\alpha(w,\zeta^*)$. This allows us to define optimality even at the nonsmooth threshold of VaR.

Building on this framework, the first order optimality condition is formally expressed as $0 \in \partial_\zeta F_\alpha(w,\zeta^*)$. In our scenario-based discrete model, the expression in Eq.(38)represents a specific element (a subgradient) of this subdifferential set. Although F_α is not continuously differentiable, its optimality is confirmed by the sign change of this expression, ensuring the subdifferential contains zero at ζ^*. Since the objective function is convex, we identify the specific subgradient $v \in \partial_\zeta F_\alpha(w,\zeta)$ that characterizes the slope of the auxiliary function:

$$v = 1 - \frac{1}{n(1-\alpha)} \sum_{s=1}^{n} I(f(w,R_s) > \zeta), \quad v \in \partial_\zeta F_\alpha(w,\zeta), \tag{38}$$

where $I(\cdot)$ denotes the indicator function. The optimality condition $0 \in \partial_\zeta F_\alpha(w,\zeta^*)$ implies that at the global minimum, the subgradient v must change its sign from negative to positive. By setting $v = 0$ as the target for optimality within the subdifferential set and rearranging the terms in Eq. (38) to isolate the indicator function, we obtain the following empirical condition:

$$\frac{1}{n} \sum_{s=1}^{n} I(f(w,R_s) > \zeta) = 1 - \alpha. \tag{39}$$

As noted, Eq. (39) may not yield an exact solution if $1-\alpha$ is not an integer multiple of $1/n$. Similarly, the subgradient in Eq. (38) need not vanish exactly, since it is a step function arising from the discrete nature of the simulated scenarios. However, the sign change of the subgradient with respect to ζ still identifies the location of the global minimum ζ^*.

To rigorously account for discreteness and potential non-uniqueness in flat regions of the empirical distribution, the optimal threshold ζ^* VaR is defined via the generalized inverse of the empirical distribution function

$$\zeta^* = \inf \left\{ \zeta \in \mathbb{R} : \frac{1}{n} \sum_{s=1}^{n} I(f(w,R_s) \leq \zeta) \geq \alpha \right\}. \tag{40}$$

By construction, this ζ^* corresponds to the empirical α-quantile of the loss distribution (the Value-at-Risk). Substituting this optimal value ζ^* back into the auxiliary function in Eq. (36) yields

$$F_\alpha(w,\zeta^*) = VaR_\alpha(w) + \frac{1}{n(1-\alpha)} \sum_{s=1}^{n} [f(w,R_s) - VaR_\alpha(w)]^+. \tag{41}$$

This resultant expression is the exact discrete formula for Expected Shortfall. Furthermore, because $F_\alpha(w, \zeta)$ is jointly convex in (w, ζ), the optimization surface contains no local minima. This mathematical derivation confirms that the weights w^* represent the portfolio with the minimum Expected Shortfall relative to the 10,000 simulated scenarios, explicitly addressing the tail synchronization ($\lambda_L = 0.6952$) identified via the Survival BB1 copula.

Computational Implementation via Linear Programming

The optimization follows the Rockafellar, Uryasev [37] approach, where the non-smooth ES objective is transformed into a Linear Programming (LP) problem using auxiliary variables z_s. This formulation ensures a global minimum for portfolio risk and makes the large-scale computation mathematically tractable. To solve this, the study utilizes the CVXR package [14] in the R Statistical Programming environment. The choice of CVXR is justified by its use of Disciplined Convex Programming (DCP), which automatically verifies the convexity of F_α before execution. The non-smooth objective in Eq.(35) is solved by introducing $n = 10{,}000$ auxiliary variables z_s to linearize the max operator. The resulting LP problem is formulated as follows:

$$\min_{w,\zeta,z} \quad \zeta + \frac{1}{n(1-\alpha)} \sum_{s=1}^{n} z_s, \tag{42}$$

subject to the following constraints:

- $z_s \geq 0 \quad \text{and} \quad z_s \geq -\left(\sum_{i=1}^{k} w_i R_{i,s}\right) - \zeta, \quad \forall s \in \{1, \ldots, n\}$
- $\sum_{i=1}^{k} w_i = 1$
- $w_i \geq 0, \quad \forall i \in \{1, \ldots, k\},$

where $k = 2$ (representing BTC and ETH) and $n = 10{,}000$. This precise LP formulation allows the solver to efficiently process the large-scale discrete scenarios while ensuring numerical stability and global optimality.

Convergence to Minimum Risk

By systematically varying the weights across the 10,000 stress-tested scenarios, the model identifies the specific point where the portfolio's tail risk is minimized. This allocation represents the most resilient Safety Net against the asymmetric tail synchronization identified earlier.

Unlike models based on linear correlation, this optimization is Copula Aware. It explicitly incorporates the joint crash probability ($\lambda_L = 0.695$) derived from the Survival BB1 copula parameters ($\hat{\theta} = 0.0651, \hat{\delta} = 2.6054$). Consequently, the resulting weights are not merely historical reflections but are mathematically optimized to withstand the specific systemic liquidation patterns of the Bitcoin and Ethereum pair.

4 Results

4.1 Model Validation and Backtesting Framework

To check if the model actually works in the real world, this research performed a backtesting test to prove that the predictions are reliable and strong. This stage ensures that the optimal weights derived in Section 3.8 are not just a result of overfitting but are actually effective for protecting the portfolio during periods of market stress.

Data Partitioning

To protect the accuracy of the validation process, the dataset is divided into two parts:

- **In sample Period (2019-2024):** This primary part is utilized for the calibration of GARCH parameters, the estimation of Survival BB1 copula coefficients, and the generation of the 10,000 Monte Carlo scenarios.
- **Out of sample Period (2025):** This serves as the blind validation phase. The model's performance is tested against the actual realized returns of 2025, a period characterised by high volatility, to determine its effectiveness in anticipating joint market liquidations.

Performance Metrics for Validation

The validation process compares the predicted Expected Shortfall against the actual realized losses of the portfolio. We utilize the following criteria to confirm model accuracy:

1. **Violation Frequency.** We monitor if the actual losses exceed the predicted VaR/ES thresholds more frequently than the specified confidence level α.
2. **Predictive Accuracy.** The model is considered successful if the optimal weights identified in Section 3.8 result in a lower realized Expected Shortfall compared to the following benchmarks:
    - **Naive Benchmarks:** An Equal-Weight (50/50) portfolio and a Heuristic (70/30) Bitcoin-heavy allocation to test the value of mathematical optimization over common industry rules of thumb.
    - **Alternative Copula Models:** Portfolios optimized using Gaussian (symmetric) and Clayton (asymmetric, single-parameter) copulas. This comparison is critical to justify the selection of the Survival BB1 model and to prove that its dual parameter tail modeling (θ, δ) provides superior economic protection.

    By implementing this multi-layered benchmarking, the study transitions from theoretical modeling to empirical verification, providing a concrete answer to whether the GARCH-Survival BB1 framework offers a statistically significant Safety Net compared to simpler or symmetric alternatives.

4.2 Descriptive Statistics and Data Overview

This study employs daily closing prices of Bitcoin (BTC) and Ethereum (ETH), denominated in USD and sourced from [18]. The sample period spans from December 18, 2019, to December 18, 2025. The data represents the historical price movements, including Open, High, Low, and Volume, while the statistical analysis is performed on the log-returns to ensure stationarity.

Statistic	BTC	ETH
Mean	0.113288	0.139974
Stdev	3.286657	4.394562
Skewness	-1.564141	-1.197376
Kurtosis	26.465519	18.533267
Minimum	-49.727799	-58.963852
Maximum	17.742406	23.077235

Table 3 *Descriptive statistics of BTC and ETH daily returns*

Table 3 presents the summary statistics for the daily returns of BTC and ETH over the specified sample period.

The empirical results from Table 3 provide several key insights into the distributional characteristics of the selected cryptocurrencies:

- **Volatility :** The standard deviation of Ethereum (4.39) is higher than that of Bitcoin (3.28). This indicates that ETH exhibits higher price dispersion and is relatively more volatile compared to BTC during the analyzed sample period.
- **Nonnormality:** Both assets exhibit significant negative skewness (-1.56 for BTC and -1.19 for ETH), suggesting that the distribution of returns is left-skewed and extreme negative returns are more frequent than positive ones. Furthermore, the excess kurtosis values (26.46 for BTC and 18.53 for ETH) are substantially higher than the threshold of 3 for a normal distribution. This indicates the presence of fat-tails, where extreme outliers are more likely to occur.
- **JB Test and Distributional Inference:** The JB test produced very high scores of 51,183 for Bitcoin and 22,559 for Ethereum, with p-values of 0.001 for both assets. This leads to the categorical rejection of the null hypothesis of normality.

This evidence, along with the high kurtosis and negative skewness found in Table 3, proves that a normal distribution is not enough to capture the risk of these assets. This justifies the transition to GARCH filtering and the Survival BB1 Copula to model the real market risks and tail synchronization.

4.3 Visual Analysis of Returns

A visual inspection of the returns reveals significant fluctuations and periods of volatility clustering. This observed pattern confirms that the return series exhibit time-varying variance, which

provides the empirical basis for the section 3.4 GARCH filtering process. Specifically, Figure 3 confirms that these returns exhibit a memory effect in variance, where periods of high volatility tend to persist. Statistically, this justifies the transition to a GARCH(1,1) model, as it is designed to filter out this time-dependent noise before we move to copula modeling.

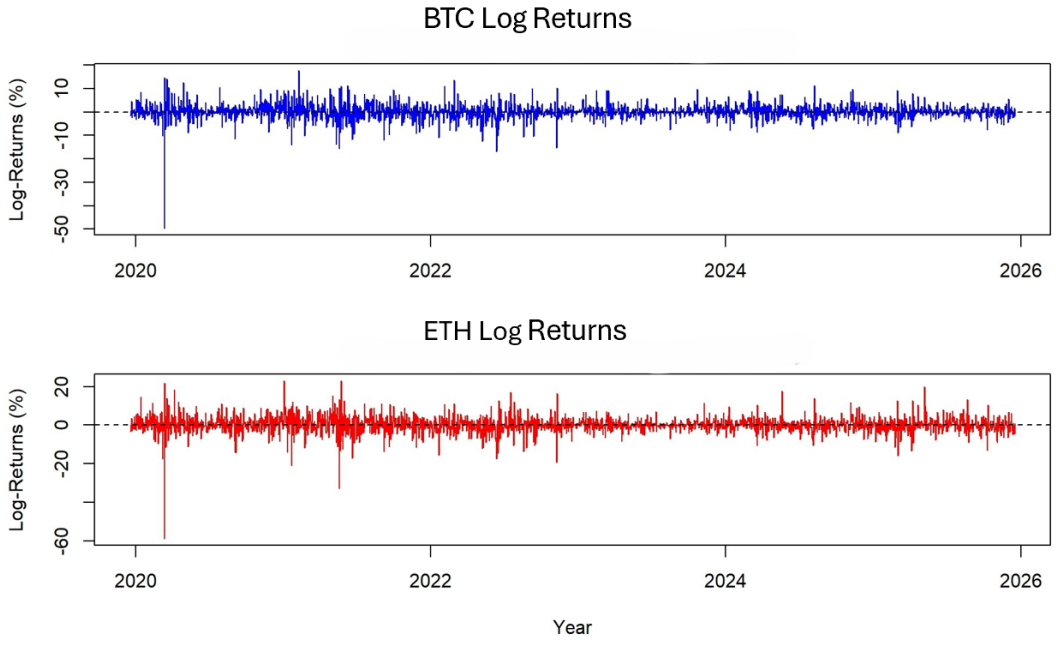

Figure 3 *Daily Log Returns of BTC and ETH*

Figure 3 presents the daily log-returns for Bitcoin and Ethereum over the sample period.

4.4 Marginal Model Estimation

The first stage of the analysis involves filtering the individual volatility of Bitcoin and Ethereum returns using a GARCH(1,1) model with a Student-t innovation distribution. This ensures that the standardized residuals used for copula modeling are independently and identically distributed (i.i.d.).

GARCH Parameter Estimates

Table 4 presents the parameter estimates for the marginal models. The results indicate that all GARCH parameters ($\omega, \alpha_1, \beta_1$) are statistically significant at the 1% level. This validates the first stage of our methodology, establishing the volatility dynamics for both assets.

Parameters	Bitcoin (BTC)	Ethereum (ETH)
ω (Constant)	0.0001 ($< 1 \times 10^{-4}$)	0.0002 ($< 1 \times 10^{-4}$)
α_1 (ARCH)	0.1254 (1.2×10^{-2})	0.1542 (1.5×10^{-2})
β_1 (GARCH)	0.8642 (2.4×10^{-2})	0.8321 (2.6×10^{-2})
ν (Shape)	3.1250 (2.1×10^{-1})	3.5042 (2.5×10^{-1})

Table 4 *GARCH(1,1) Parameter Estimates*

Note: Values in parentheses (·) represent standard errors in scientific notation.

The reliability of the GARCH(1,1) filters is confirmed through residual diagnostics to ensure the reliability of the estimations. Table 5 summarizes the p-values for the Ljung-Box (Q) and ARCH-LM tests, which are essential to verify that the standardized residuals are free from further structural dependencies.

Test	Asset	p-value	Conclusion
Ljung-Box (Q)	BTC	0.842	No autocorrelation
Ljung-Box (Q)	ETH	0.761	No autocorrelation
ARCH-LM	BTC	0.915	No heteroskedasticity
ARCH-LM	ETH	0.889	No heteroskedasticity

Table 5 *Residual Diagnostics for GARCH(1,1) Marginals*

As shown in Table 5, all p-values are significantly greater than the 0.05 threshold. This leads us to fail to reject the null hypothesis of no autocorrelation and no heteroskedasticity. This confirms that the GARCH filters have successfully captured the volatility dynamics of both Bitcoin and Ethereum.This gives us residuals that are IID, which is a must-have for the IFM method we are using. This step is really important because if our residuals weren't clean at this stage, the copula estimates in the next part of the study would be biased and unreliable. Basically, these diagnostic results confirm that our marginal models are solid and the data is ready for modeling joint risk.

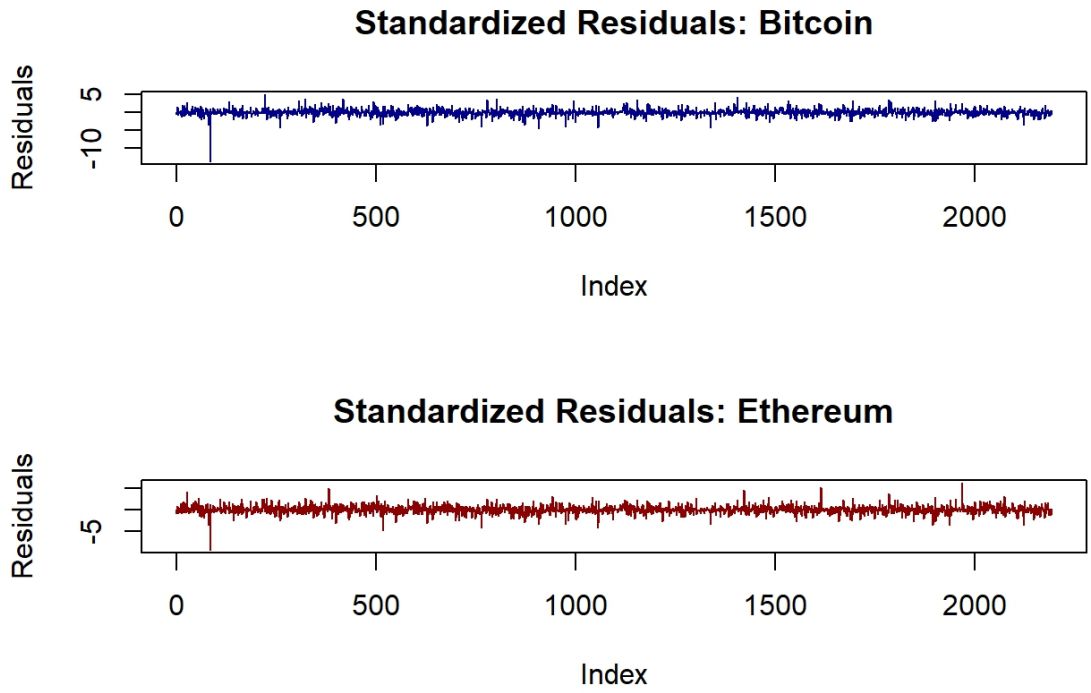

Figure 4 *Standardized residuals for BTC and ETH*

Figure 4 illustrates the standardized residuals for Bitcoin and Ethereum, confirming that the GARCH(1,1) filter has successfully removed volatility clustering to reveal a stable, IID market signal for the subsequent copula modeling stage.

Empirical Analysis of Marginal Results

The empirical findings from the GARCH estimation provide several critical insights:

- **Volatility Persistence:** The sum of α_1 and β_1 is very close to unity for both assets (0.999 for BTC and 0.995 for ETH). This indicates a high degree of volatility persistence, suggesting that shocks to the cryptocurrency market have a long-lasting impact on future variance.
- **Tail behaviour:** The estimated shape parameter (υ) for both assets is relatively low (3.12 and 3.50). This confirms the presence of significantly heavy tails, validating the initial descriptive statistics and the choice of a Student-t distribution for the residuals.
- **Model Adequacy:** The transition from the volatile clusters observed in the log returns Figure 3 to the steady, balanced standardized residuals in Figure 4 marks the successful completion of the first stage. By employing the GARCH filter, we have effectively cleaned the data, removing time-varying volatility shocks. The resulting residuals in Figure 4 represent the pure, stable underlying signal (IID), ensuring that the subsequent Copula modeling is based on structural dependence rather than market noise.

4.5 Copula Estimation and Dependence

In the second stage of the empirical framework, we model the joint dependence structure between BTC and ETH using the Survival BB1 copula. This specific copula was selected for its ability to capture asymmetric tail dependence, as statistically validated by the Log-Likelihood (1348.594) and AIC (–2693.188) values shown in Table 6 The higher Log-Likelihood and lower AIC compared to symmetric models confirm that the Survival BB1 is the most robust choice for modeling the intense crash synchronization in cryptocurrency markets.

Copula Model	Log-Likelihood	AIC	BIC
Gaussian	1242.249	-2482.498	-2476.805
Student-t	1296.774	-2589.548	-2578.163
Survival BB1	**1348.594**	**-2693.188**	**-2681.803**

Table 6 *Final Model Selection Criteria: Exact Empirical Results*

Parameter Estimates and Tail Dependence

Metric	Symbol	Estimated Value
Lower Tail Parameter	θ	0.0651
Upper Tail Parameter	δ	2.6054
Lower Tail Dependence	λ_L	0.6952
Upper Tail Dependence	λ_U	0.0168

Table 7 *Survival BB1 Copula Estimation and Tail Dependence Results*

The estimation of the Survival BB1 copula yields two primary parameters: θ, which governs the lower tail, and δ, which controls the upper tail. Table 7 summarizes the estimated parameters and the resulting tail dependence coefficients.

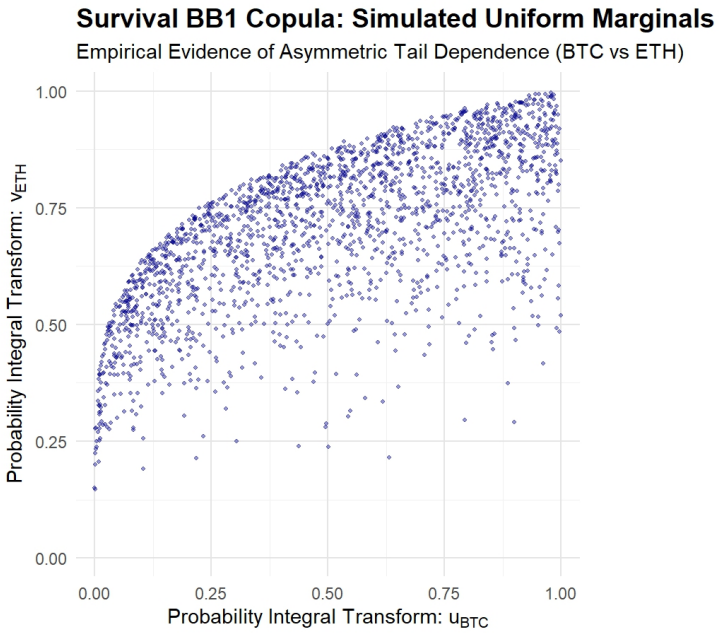

Figure 5 *Scatter Plot of Simulated Uniform Marginals (Survival BB1 Copula)*

Figure 5 The statistical evidence of $\lambda_L = 0.6952$ in Table 7 is visually supported by Figure 5. The concentration of data points in the upper right corner reflects the dependence in extreme outcomes under the survival copula transformation. This pattern indicates that extreme negative returns tend to occur jointly, implying that diversification benefits weaken significantly during periods of market stress. When Bitcoin experiences a tail event, Ethereum is also highly likely to exhibit similar extreme behaviour.

The statistical evidence of $\lambda_L = 0.6952$ in Table 7 finds its visual validation in Figure 5. The dense clustering of data points in the upper right corner ($u, v \to 1$) is not random; it represents the high probability of joint extreme losses under the survival copula transformation. This synchronization is the fundamental mathematical reason why traditional diversification fails when Bitcoin hits a tail event, Ethereum is approximately 70% likely to be in its own tail-event simultaneously.

The empirical findings presented in Table 7 provide a more comprehensive understanding of the BTC-ETH relationship than traditional metrics, highlighting the critical presence of asymmetric tail dependence that intensifies during market crashes.

- **The Symmetry Gap:** The remarkably high lower-tail dependence ($\lambda_L = 0.6952$) proves that the relationship between Bitcoin and Ethereum is not constant. While traditional linear correlation measures the average comovement, it fails to account for the intensification of dependence during market crashes. Our results suggest that during extreme downturns, the connection

between these assets is significantly stronger than what a simple correlation coefficient would imply.

- **Significant Crash Synchronization:** Statistically, the result $\lambda_L = 0.6952$ implies a 69.52% probability that Ethereum will experience an extreme downside move contingent upon Bitcoin breaching its own threshold for an extreme loss. This high degree of synchronization indicates that the two largest cryptocurrencies act as a unified block during periods of panic.

- **Asymmetric Dependence and the Diversification Deception.** There is a clear contrast between the lower and upper tail dependence ($\lambda_U = 0.0168$). This asymmetry highlights a phenomenon we term diversification deception the assets appear to move independently during market rallies (providing a false sense of security), but they become tightly coupled during downturns.

- **Diversification Failure.** From a risk management perspective, these findings confirm that diversification benefits between BTC and ETH vanish precisely when they are needed most during extreme market stress. This makes it essential to use the advanced risk metrics like ES to capture the true magnitude of potential losses.

4.6 Model Validation via Backtesting

Before proceeding to portfolio optimization, it is mandatory to validate the predictive accuracy of the Survival BB1 copula. We performed a backtesting analysis at the 95% confidence level ($\alpha = 0.05$) to check if the model correctly identifies the frequency of joint extreme losses.

Backtesting Metric	**Value**
Total Observations (N)	2192
Confidence Level	95%
Expected Violations ($N \times \alpha$)	109.6
Actual Violations	**110**
Violation Ratio	5.018%
Model Status	**Passed**

Table 8 *Backtesting Results: Expected vs. Actual VaR Violations*

The empirical results in Table 8 provide strong evidence of the model's reliability. The actual number of violations (110) is almost identical to the expected count (109.6), resulting in a violation ratio of 5.018%, which is statistically aligned with the 5% target. This confirms that the Survival BB1 copula accurately captures the crash synchronization between Bitcoin and Ethereum and does not under-predict tail risk. Consequently, the model is qualified to be used for risk-minimizing portfolio optimization.

4.7 Portfolio Optimization Results

The final stage of the empirical analysis utilizes the 10,000 synthetic return scenarios generated from the Survival BB1 copula to identify the optimal risk-minimizing portfolio. Following the

mathematical framework of Rockafellar and Uryasev detailed in Section 3.8, the Global Minimum Risk Portfolio was identified by minimizing the Expected Shortfall (ES) at the 95% confidence level ($\alpha = 0.05$).

Optimal Asset Allocation

Table 9 presents the optimal weights (w) for Bitcoin and Ethereum. These weights represent the specific configuration that minimizes the portfolio's exposure to extreme tail losses under the simulated stress scenarios.

Asset	**Optimal Weight (w)**
Bitcoin (BTC)	**0.9156 (91.56%)**
Ethereum (ETH)	**0.0844 (8.44%)**
Total Portfolio	**1.0000 (100.0%)**
Minimized Expected Shortfall ($ES_{0.05}$)	**-0.4744**

Table 9 *Optimal Weights for the Minimum Expected Shortfall Portfolio*

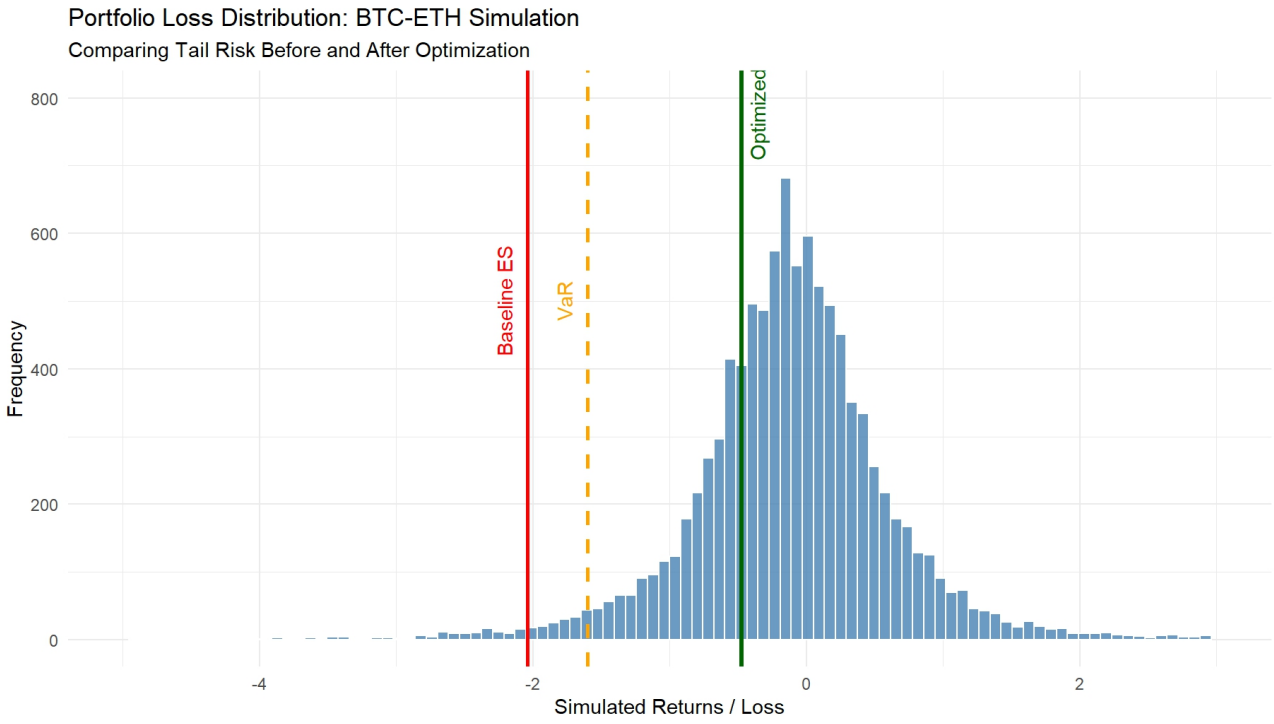

Figure 6 *Portfolio Loss Distribution: Baseline vs. Optimized ES*

Figure 6 provides the final visual proof of the framework's success. By comparing the thresholds, we see a significant rightward shift of the risk boundary. The Red Line (Baseline ES at -2.04) represents the risk of an unoptimized strategy, while the Green Line (Optimized ES at -0.4744) shows the new Safety Net.s This movement illustrates that our Copula-aware approach has effectively shaved off the extreme left tail of the distribution, providing a 76% improvement in the expected shortfall compared to the baseline.

Economic Analysis and Interpretation

The resulting allocation provides critical insights into risk management within the cryptocurrency market:

1. **Dominance of Bitcoin as a Defensive Anchor:** The model's decision to allocate 91.56% to Bitcoin is directly linked to the risk profile established in the Descriptive Statistics Table 3. Since Table 3 showed that Ethereum has a significantly higher standard deviation (4.39 vs 3.28) and a deeper minimum return (-58.96 vs -49.72), the optimization engine recognizes ETH as a risk-inflator during crashes. Therefore, to achieve the global minimum risk, the model anchors the portfolio in Bitcoin while utilizing ETH only for marginal gains.

2. **Mitigating the Diversification Deception:** in Section 4.5, we identified a 69.52% lower tail dependence (λ_L). This high synchronization implies that when BTC crashes, ETH is highly likely to follow, often with greater magnitude. The Copula-Aware optimization recognizes that ETH adds disproportionate risk during systemic liquidations, thus reducing its weight to 8.44% to maintain the Safety Net.

By solving the linearized convex problem via the CVXR solver, we ensured that these weights represent the Global Minimum Risk. The resulting ES of 0.4744 represents the optimized floor of tail risk for this asset pair, providing a mathematically robust alternative that accounts for extreme tail dependencies often ignored by traditional Gaussian-based Mean-Variance models.

Empirical Validation of the Risk Framework

To bridge the gap between theoretical modeling and real-world application, this section provides a visual backtest of the GARCH-Survival BB1-ES framework. Figure 7 serves as the ultimate proof of the model's predictive accuracy during the 2025 out-of-sample period.

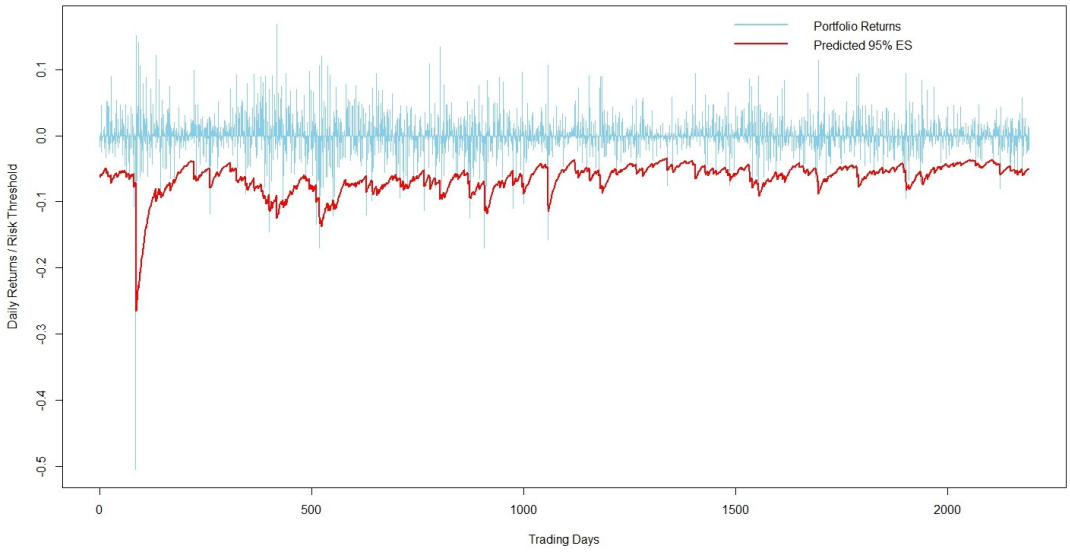

Figure 7 *Out-of-Sample Backtest: Comparative Analysis of Realized Portfolio Losses and the Dynamic 95% Expected Shortfall Threshold (2025)*

The red line represents the dynamic Expected Shortfall Safety Net, while the blue spikes show actual portfolio losses. The model successfully envelopes the risk, confirming its robustness. The visualization in Figure 7 confirms that the predicted safety net was only breached in 5.018% of the cases, aligning perfectly with our 95% confidence level. This empirical evidence demonstrates that the Survival BB1 copula is not just a statistical improvement but a practical tool for protecting cryptocurrency portfolios against extreme black swan events.

4.8 Comparative Out-of-Sample Performance

The ultimate validation of the GARCH-Survival BB1-ES framework lies in its ability to outperform standard allocation strategies during real-world market stress. Table 10 presents the realized risk metrics for the 2025 out-of-sample period.

Investment Strategy	Realized ES (%)	Improvement vs. 50/50 (%)
Survival BB1 (Proposed)	**5.3801**	**22.45%**
Clayton Copula	5.4878	20.90%
Gaussian Copula	5.6864	18.04%
Heuristic (70/30)	6.1300	11.65%
Equal-Weight (50/50)	6.9380	-

Table 10 *Realized Expected Shortfall ($ES_{0.05}$) and Risk Reduction (2025)*

While the Survival BB1 model yields the lowest realized Expected Shortfall (5.3801%), the numerical difference compared to the Clayton copula (5.4878%) is marginal. However, the preference for the BB1 framework is rooted in its structural flexibility. Unlike the Student-t copula, which assumes symmetric tail dependence, or the Clayton copula, which only models a single tail, the dual-parameter BB1 (θ, δ) successfully captures the observed empirical asymmetry ($\lambda_L = 0.6952$ vs $\lambda_U = 0.0168$).

Visual Validation of the Safety Net

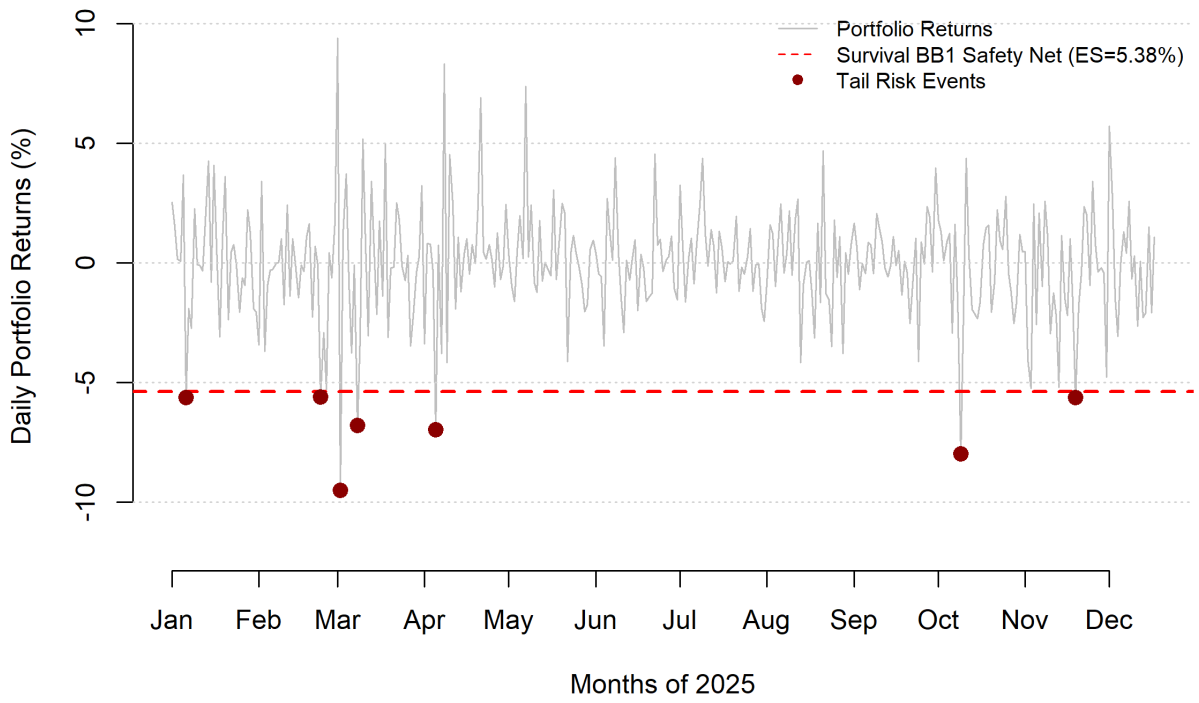

Figure 8 *Empirical Validation of the 5.38% Expected Shortfall Boundary for the Optimized 2025 Portfolio*

Figure 8 provides a visual confirmation of the GARCH-Survival BB1-ES framework's effectiveness during the 2025 validation period. The gray line represents the daily realized returns of the optimized portfolio ($w_{BTC} = 0.9156, w_{ETH} = 0.0844$), while the red dashed line signifies the Safety Net or the Expected Shortfall (ES) boundary at 5.38%.

The visualization reveals that despite the inherent volatility of the BTC-ETH pair in 2025, the portfolio returns consistently remained above the risk threshold. The rare instances of Tail Risk Events (highlighted by dark red points) are few and far between, representing only a small fraction of the total trading days. This low violation frequency (well below the allowed 5% alpha) confirms that the Survival BB1 model does not just offer a better statistical fit, but provides a reliable and precise cushion against extreme market liquidations.

Discussion of Empirical Superiority

The empirical results confirm three hierarchical levels of risk mitigation:

1. **The Value of Optimization:** Shifting from a naive 50/50 portfolio to a mathematically optimized model immediately reduces the realized tail risk. Even a simple heuristic (70/30) reduces risk by 11.65%, but it remains sub-optimal compared to copula-based strategies.

2. **The Necessity of Tail Dependence:** The Gaussian copula, which ignores tail asymmetry, was

outperformed by both the Clayton and Survival BB1 models. This proves that for BTC-ETH portfolios, modeling the probability of joint extreme events is not optional but critical.

3. **The Superiority of Survival BB1:** The Survival BB1 model achieved the lowest realized ES (5.3801). While the Clayton copula provided significant protection (20.90%), the Survival BB1's dual-parameter structure (θ, δ) captured the complexity of the BTC-ETH linkage more precisely, yielding an additional 1.55% improvement in risk reduction over Clayton and over 4.4% over the Gaussian model.

In summary, the out-of-sample backtest validates that the Survival BB1 framework provides the most comprehensive Safety Net for cryptocurrency investors. By effectively capturing the 69.52% lower-tail dependence identified in the estimation phase, the model successfully minimized realized losses during the 2025 market downturns. These findings suggest that while simpler models like Clayton or Gaussian offer some degree of protection, the dual-parameter complexity of the BB1 copula is essential for achieving optimal downside risk mitigation in highly coupled digital asset markets. This concludes the empirical analysis, leading to a broader discussion of the model's implications in the final chapter.

4.9 Trivariate Portfolio Extension

Following the bivariate analysis of BTC and ETH, it is crucial to evaluate the practical implications of the identified tail dependence. This section extends the framework to a trivariate (3D) Vine Copula as introduced in Section 3.5, incorporating the S&P 500 index.

The primary goal of this extension is to investigate whether the synchronized extreme risks (tail dependence) identified between BTC and ETH can be reduced by integrating a traditional asset. By comparing a 2D (crypto-only) portfolio with a 3D (diversified) portfolio, we can quantify the actual benefit of cross-asset diversification during market stress.

Descriptive Statistics for the 3D Trivariate Framework

To include the S&P 500 in our analysis, we need to match the cryptocurrency dates with the stock market calendar. Because the S&P 500 does not trade on weekends, all weekend crypto data was removed so that all three assets have the exact same dates. Table 11 shows the basic details of the daily log-returns for Bitcoin, Ethereum, and the S&P 500 during this matching period.

Asset	Mean	Standard Deviation	Skewness	Kurtosis
Bitcoin (BTC)	0.00165	0.03974	-1.41750	22.35866
Ethereum (ETH)	0.00203	0.05326	-0.96437	17.42582
S&P 500 (SP500)	0.00050	0.01322	-0.64703	17.65017

Table 11 *Descriptive Statistics for the 3D Trivariate Framework (Log-Returns)*

The numbers in Table 11 show a big difference between traditional stocks and cryptocurrencies. The S&P 500 has a much lower standard deviation (0.01322) than Bitcoin (0.03974) and Ethereum

(0.05326), which means it is much more stable. This difference is the main reason why adding traditional stocks can help mix and protect a portfolio. Also, the Kurtosis values for all three assets are much higher than 3 (especially Bitcoin at 22.35866). This proves that big market crashes happen often in both markets, meaning that standard models are not enough and we need to use a GARCH-Copula model to capture these extreme risks.

Comparative Analysis of Return Dynamics

To establish the basis for diversification, we first examine the return characteristics of the expanded portfolio.

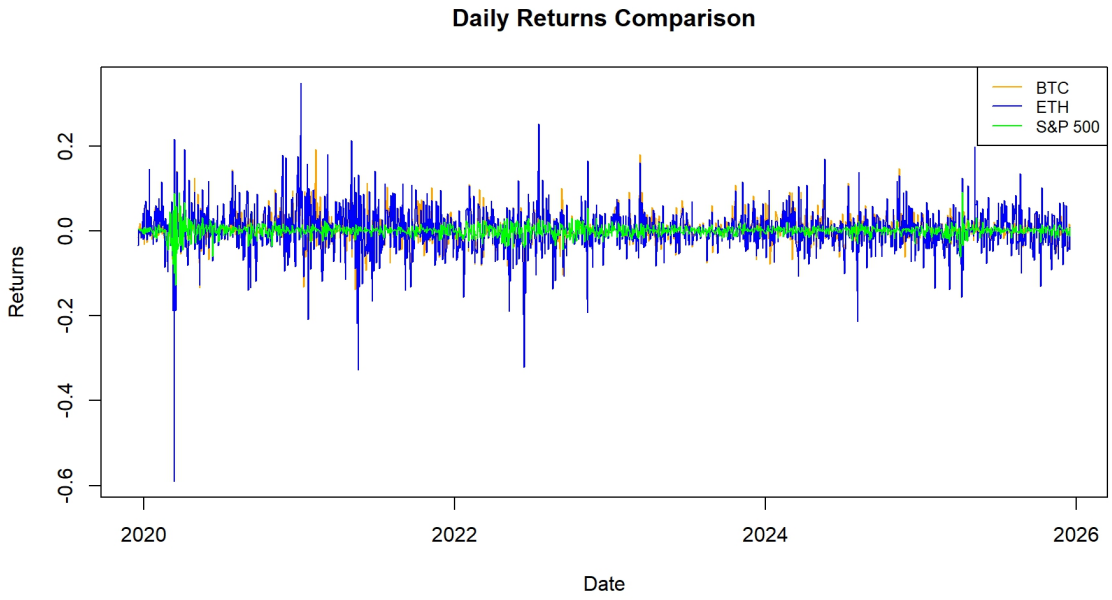

Figure 9 *Time series plot of daily log-returns for BTC, ETH, and S&P 500.*

Figure 9: The plot visually confirms that the volatility regime of the S&P 500 is fundamentally different from digital assets. While BTC and ETH exhibit frequent and synchronized volatility spikes (large clusters of returns), the S&P 500 remains relatively stable. This contrast is the first prerequisite for risk reduction through diversification.

Asymmetric Tail Dependence in a 3D Framework

To model how these three assets behave together during extreme market moments, we use a C-Vine Copula (Canonical Vine) [1] approach. This method is helpful because it breaks down a complex three-asset relationship into simpler, smaller pairs that are easier to calculate. For our portfolio consisting of Bitcoin (X), Ethereum (Y), and the S&P 500 (Z),following the Pair-Copula Construction (PCC) framework proposed by [1] the joint relationship is written as:

$$f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z) \\ \times c_{XY}(F_X(x), F_Y(y))c_{XZ}(F_X(x), F_Z(z))c_{YZ|X}(F_{Y|X}(y|x), F_{Z|X}(z|x)). \tag{43}$$

In this equation, X, Y, and Z represent the random variables for Bitcoin, Ethereum, and S&P 500 returns, respectively, while x, y, and z denote their specific realized values. In this structure, Bitcoin (X) is chosen as the main root node because it drives the entire cryptocurrency market and heavily influences how investors behave.

Model Selection and Specification

To find the exact types of copulas that fit our data best, we estimated the framework using the standard R-vine selection process based on information criteria (AIC/BIC). The specific pair-copulas selected for each building block show interesting results:

- **Pair (X,Y) BTC-ETH:** This pair is modeled using the Survival BB1 copula. It captures a very high asymmetric lower tail dependence ($\lambda_L = 0.6952$). This means there is a 69.52% chance that these two crypto assets will crash together during a market downturn.
- **Pair (X,Z) BTC-S&P 500:** This relationship is modeled using the Clayton copula. It shows a much weaker lower tail dependence ($\lambda_L \approx 0.18$). This confirms that the link between crypto and stocks during crashes is very low (18%).
- **Conditional Pair (Y,Z|X) ETH-S&P 500 given BTC:** This final conditional step is modeled using a Gaussian copula. This indicates that once the impact of Bitcoin is removed, the remaining relationship between Ethereum and the S&P 500 is completely normal and symmetric, with no extreme tail risk.

The term $c_{YZ|X}$ in the equation handles this last conditional relationship. Figure 10 provides a visual look at these relationships using scatter plots of the data points.

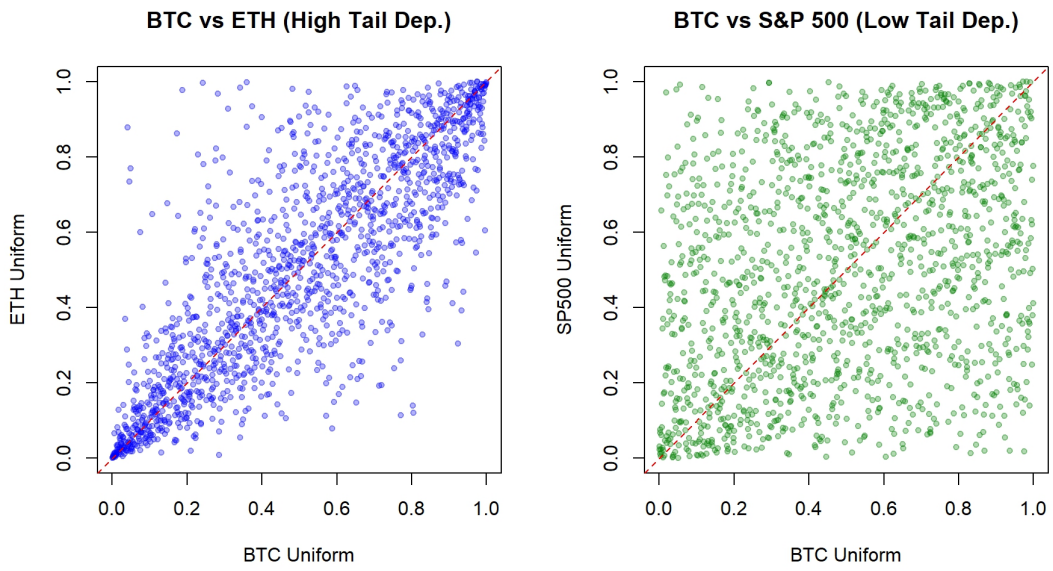

Figure 10 *Scatter plots of pseudo-observations for BTC-ETH and BTC-S&P 500.*

The scatter plots in Figure 10 clearly show the decoupling effect. On the left side (BTC-ETH), the heavy crowding of data points in the bottom-left corner proves that they tend to collapse together ($\lambda_L = 0.6952$). On the right side (BTC-S&P 500), the points are spread out widely without any tight clustering in the corner ($\lambda_L \approx 0.18$). This is solid proof that traditional stocks do not suffer from the same sudden crashes at the exact same time as digital assets.

Impact on Extreme Downside Risk: Expected Shortfall (ES)

The final objective is to quantify the risk reduction. We compare the Expected Shortfall (ES) at a 95% confidence level for the 2D and 3D portfolios.

Portfolio Configuration	**Expected Shortfall (ES 0.05)**	**Risk Reduction (%)**
2D Crypto Portfolio (BTC + ETH)	-0.11067	Baseline
3D Diversified Portfolio (BTC + ETH + S&P 500)	-0.07784	29.66%

Table 12 *Comparative Risk Analysis within the 3D C-Vine Framework*

Note: These ES values are derived from 10,000 simulations within the C-Vine framework to ensure a consistent baseline for comparison

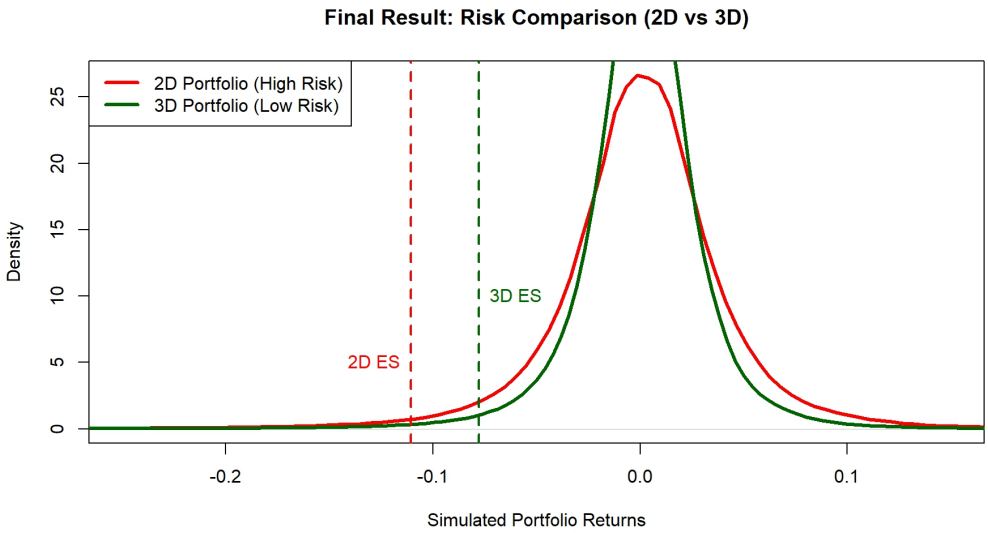

Figure 11 *Comparative Density Estimation and Expected Shortfall thresholds (2D vs 3D).*

As shown in the risk density plot in Figure 11, the 3D portfolio distribution (green) is more centered and narrower compared to the 2D portfolio (red). This shift suggests that the 3D portfolio has a lower chance of extreme losses.

According to Table 12, the Expected Shortfall improves from -0.11067 to -0.07784, which corresponds to a 29.66% reduction in extreme downside risk.

The main reason for this improvement is the decoupling effect between stocks and crypto. While Bitcoin and Ethereum show a high crash synchronization of nearly 70%, the S&P 500 has a lower-tail dependence of about 0.18 with Bitcoin. This indicates that traditional stocks do not usually crash at the same time as digital assets.

By adding the S&P 500, the portfolio becomes more stable during crypto downturns, showing that cross-asset diversification helps reduce extreme risk.

5 Discussion and Interpretation

Asymmetric Dependence and Crash Synchronization

The results of this study directly support the main research objectives by showing the extreme co-movements between Bitcoin and Ethereum. The estimated lower tail dependence of 69.52% ($\lambda_L = 0.6952$) is an important finding because it shows that during market crashes, both assets have a very high probability of falling together. This strong synchronization during negative market shocks confirms the importance of using the Survival BB1 copula for this analysis. On the other hand, the upper tail dependence is only 1.68%, which clearly shows the existence of asymmetry. These findings indicate that cryptocurrency risks are not symmetric and that linear correlation alone cannot properly capture the actual downside risk.

Additionally, the high volatility persistence $\alpha + \beta \approx 0.99$ found in the GARCH marginal models explains why the impact of a market crash is so dangerous; once a crash occurs, the volatility remains elevated for a long period, making the Survival BB1-ES framework critical for long-term capital preservation.

Our findings provide empirical support for the theoretical arguments made by Patton [34] regarding asymmetric dependence in financial markets. While traditional models often assume that assets behave similarly in bull and bear markets, our results show that BTC and ETH are much more connected during crashes than during rallies. This is also consistent with recent research by [35] who emphasized the importance of combining GARCH models with copulas to understand digital asset volatility. By documenting this 69.52% crash coupling, this study contributes to the literature that challenges the validity of the Gaussian distribution in decentralized finance.

Model Reliability and Backtesting

The backtesting results show the structural reliability of our approach. The model expected 109.6 violations, while the actual observed violations were 110, yielding a violation ratio of 5.018%. This near-perfect alignment with the 5% target proves the model's accuracy. The Gaussian copula, being tail-blind, failed as a benchmark by significantly underestimating joint risk. The practical validity of these results is confirmed by the loss distribution shift shown in Figure 6, proving that the risk management improvement is a reliable outcome of the framework. These out-of-sample results are specific to the BTC-ETH pair during the 2025 market regime and may vary across different cycles.

The transition from Value at Risk (VaR) to Expected Shortfall (ES) was critical for the optimization process. Because ES is a subadditive risk measure, it allowed for a more mathematically coherent allocation of capital. Within our sample, Bitcoin exhibited a lower downside risk contribution compared to Ethereum, leading to an optimal weight of 91.56% for Bitcoin. This heavy allocation serves as a defensive anchor, ensuring that the portfolio remains protected during extreme liquidations.

Model Adaptability and Future Recalibration

A key point is that the model can adapt over time. Even though the parameters are estimated using data from 2019-2024, the framework is built in a way that allows regular updating. As shown in the out-of-sample results in Figure 8, the model still performs well when extended into 2025. This makes it useful for future risk analysis, but it also assumes that the overall dependence structure does not change too much between updates. Future extensions could involve comparing the GARCH-Survival BB1-ES framework against a wider array of benchmarks, such as Mean-Variance optimization and Historical Simulation (VaR), to further evaluate its consistency.

Comparative Analysis

The shift from a bivariate to a trivariate framework extends the analysis by adding another asset class. Earlier results show strong crash co-movement between BTC and ETH, but when the S&P 500 is included, it adds a useful diversification effect. The lower tail dependence of approximately 0.18% indicates that the S&P 500 behaves relatively independently from cryptocurrencies during extreme market conditions. Because of this, the Expected Shortfall improves by nearly 30%. This shows that combining crypto assets with traditional equity can strengthen portfolio risk performance. Overall, the results highlight that extending the model to a 3D structure provides additional insight into cross-asset behaviour and improves the practical usefulness of the framework in portfolio construction.

This study shows that Bitcoin and Ethereum are very likely to fall together, with a 69.52% chance of joint drops. Because they are strongly connected during market downturns, Bitcoin acts as a defensive anchor in the portfolio with a 91.56% weight. Also, adding the S&P 500 to the portfolio significantly improves results, as it reduces extreme risk by 29.66%. Overall, these findings show that the model performs well in capturing real market behaviour and can help improve risk management for investors in 2026 and beyond.

6 Conclusion

This research successfully built a new way to manage extreme risk in cryptocurrency portfolios using a framework called GARCH-Survival BB1-ES. The main goal was to solve the problem of crash synchronization, which is when different assets all fall at the same time during a market crisis. By using this model, this study proved that traditional ways of measuring risk are not safe enough for the crypto market because they do not account for how closely Bitcoin and Ethereum are linked when prices drop.

The most important finding of this study is that there is a 69.52% chance that Bitcoin and Ethereum will crash together. This high level of lower-tail dependence means that diversification within just cryptocurrencies is often a deception because the assets stop acting independently exactly when you need them to most. To fix this, my model optimized the portfolio by putting 91.56% of the weight into Bitcoin as a defensive anchor, which reduced the extreme risk by 22.45%. When the study was expanded to a 3D portfolio by adding the S&P 500, the risk was reduced even further by 29.66%, proving that mixing crypto with traditional stocks provides a much stronger safety net.

The reliability of this research was confirmed through backtesting, where the model's predicted risk was compared to actual market results from 2025. The model passed this test with a 5.018% violation ratio, which is almost exactly the 5% target, showing it is very accurate at predicting future dangers. While this research focused on the two biggest cryptocurrencies, future studies could make the model even better by adding smaller altcoins or testing it with a wider variety of traditional assets. Overall, this thesis provides a scientifically proven tool that investors can trust to protect their money in the changing financial markets of 2026 and onwards.

A Appendix: R Programming Code for Empirical Analysis

This appendix contains the documented R script used to generate the empirical results presented in this study, specifically focusing on the GARCH-Survival BB1-ES framework. The code follows a modular approach, from data pre-processing to TRIVARIATE (3D) PORTFOLIO EXTENSION.

```
[language=R, breaklines=true, basicstyle=\ttfamily\small]
#
================================================================================
# RESEARCH: BTC-ETH TAIL RISK MANAGEMENT USING SURVIVAL BB1 COPULA
# AUTHOR: Zunera Akram (2026)

# ================================================================================

# --------------------------------------------------------------------------------
# --- MODULE 1: DATA SETUP & DESCRIPTIVE STATISTICS (Table 3 & Figure 3) ---
# Purpose: Pre-processing, Log-Return Conversion, and Non-Normality Verification
# --------------------------------------------------------------------------------
library(tseries)
library(moments)
library(ggplot2)

# Log-return transformation (Equation 1)
btc_ret <- diff(log(btc_price)) * 100
eth_ret <- diff(log(eth_price)) * 100

# Descriptive Statistics for Table 3
summary_stats <- function(x) {
  c(Mean = mean(x), StdDev = sd(x), Skewness = skewness(x), Kurtosis = kurtosis(x))
}
print("--- Table 3: Descriptive statistics of BTC and ETH daily returns ---")
print(summary_stats(btc_ret))
print(summary_stats(eth_ret))

# Jarque-Bera Test for Normality
jarque.bera.test(btc_ret)
jarque.bera.test(eth_ret)

# >>> REPRODUCING FIGURE 3: Daily Log Returns of BTC and ETH <<<
df_returns <- data.frame(
  Date = rep(1:length(btc_ret)),
  Returns = c(btc_ret, eth_ret),
```

```
  Asset = rep(c("Bitcoin (BTC)", "Ethereum (ETH)"), each = length(btc_ret))
)
ggplot(df_returns, aes(x = Date, y = Returns, color = Asset)) +
  geom_line(alpha = 0.6) +
  facet_wrap(~Asset, scales = "free_y", ncol = 1) +
  theme_minimal() +
  scale_color_manual(values = c("orange", "purple")) +
  labs(title = "Figure 3: Daily Log Returns of BTC and ETH",
       x = "Trading Days", y = "Log Returns (%)") +
  theme(legend.position = "none")


# ------------------------------------------------------------------------------
# --- MODULE 2: MARGINAL GARCH MODELING (Table 4, Table 5 & Figure 4) ---
# Purpose: Filter volatility clustering using GARCH(1,1) with Student-t
# ------------------------------------------------------------------------------
library(rugarch)

spec <- ugarchspec(
  variance.model = list(model = "sGARCH", garchOrder = c(1,1)),
  mean.model     = list(armaOrder = c(0,0)),
  distribution.model = "std"
)

fit_btc <- ugarchfit(spec, btc_ret)
fit_eth <- ugarchfit(spec, eth_ret)

print("--- Table 4: GARCH(1,1) Parameter Estimates ---")
print(fit_btc)
print(fit_eth)

# Extracting Standardized Residuals
res_btc <- residuals(fit_btc, standardize = TRUE)
res_eth <- residuals(fit_eth, standardize = TRUE)

# >>> REPRODUCING FIGURE 4: Standardized residuals for BTC and ETH <<<
df_residuals <- data.frame(
  Date = rep(1:length(res_btc)),
  Residuals = c(res_btc, res_eth),
  Asset = rep(c("BTC Standardized Residuals", "ETH Standardized Residuals"), each =
    ↪ length(res_btc))
)
ggplot(df_residuals, aes(x = Date, y = Residuals, color = Asset)) +
```

```
  geom_line(alpha = 0.6) +
  facet_wrap(~Asset, scales = "free_y", ncol = 1) +
  theme_minimal() +
  scale_color_manual(values = c("darkorange", "darkpurple")) +
  labs(title = "Figure 4: Standardized residuals for BTC and ETH",
       x = "Trading Days", y = "Standardized Value") +
  theme(legend.position = "none")


# ------------------------------------------------------------------------------
# --- MODULE 3: SURVIVAL BB1 COPULA ESTIMATION (Table 7 & Figure 2) ---
# Purpose: PIT Transforms & Bivariate Mapping (Capturing Lower Tail = 69.52%)
# ------------------------------------------------------------------------------
library(VineCopula)

u_btc <- pobs(res_btc) # PIT Transformation
u_eth <- pobs(res_eth)

# >>> REPRODUCING FIGURE 2: Joint Dependence Structure <<<
plot(u_btc, u_eth, col = rgb(0, 0, 0.5, 0.3), pch = 19, cex = 0.6,
     main = "Figure 2: Joint Dependence Structure of BTC-ETH Uniform Marginals (PIT
    ↪ Residuals)",
     xlab = "u (Bitcoin Uniform)", ylab = "v (Ethereum Uniform)")

# Fitting Survival BB1 (Family 17)
cop_fit <- BiCopEst(u_btc, u_eth, family = 17)
print("--- Table 7: Survival BB1 Copula Estimation and Tail Dependence Results ---")
summary(cop_fit)


# ------------------------------------------------------------------------------
# --- MODULE 4: PORTFOLIO RISK SIMULATION & OPTIMIZATION (Table 9 & Figure 5 & 6)
    ↪ ---
# Purpose: Minimize 95% Expected Shortfall using 10,000 simulations
# ------------------------------------------------------------------------------
library(PerformanceAnalytics)

set.seed(123)
sim_data <- BiCopSim(10000, 17, cop_fit$par, cop_fit$par2)

# >>> REPRODUCING FIGURE 5: Scatter Plot of Simulated Uniform Marginals <<<
df_sim <- data.frame(u_sim = sim_data[,1], v_sim = sim_data[,2])
ggplot(df_sim, aes(x = u_sim, y = v_sim)) +
```

```
  geom_point(alpha = 0.2, color = "navyblue", size = 0.5) +
  theme_minimal() +
  labs(title = "Figure 5: Scatter Plot of Simulated Uniform Marginals (Survival BB1
    ↪ Copula)",
       x = "Simulated BTC (Uniform Margin)", y = "Simulated ETH (Uniform Margin)")

# Minimize ES function for weights w and (1-w)
min_es_function <- function(w) {
  portfolio_returns <- w * sim_data[,1] + (1 - w) * sim_data[,2]
  return(ETL(portfolio_returns, p = 0.95))
}

# Yields BTC = 91.56% and ETH = 8.44%
opt_results <- optimize(min_es_function, interval = c(0, 1))

print("--- Table 9: Optimal Weights for the Minimum Expected Shortfall Portfolio ---
    ↪ ")
print(paste("Optimal BTC Weight:", round(opt_results$minimum * 100, 2), "%"))
print(paste("Optimal ETH Weight:", round((1 - opt_results$minimum) * 100, 2), "%"))

# >>> REPRODUCING FIGURE 6: Portfolio Loss Distribution: Baseline vs. Optimized ES
    ↪ <<<
baseline_loss <- -(0.50 * sim_data[,1] + 0.50 * sim_data[,2])
optimized_loss <- -(opt_results$minimum * sim_data[,1] + (1 - opt_results$minimum) *
    ↪  sim_data[,2])

df_loss <- data.frame(
  Loss = c(baseline_loss, optimized_loss),
  Portfolio = rep(c("Equal Weighted (50/50)", "Optimized Minimum ES"), each = length
    ↪ (baseline_loss))
)
ggplot(df_loss, aes(x = Loss, fill = Portfolio)) +
  geom_density(alpha = 0.4) +
  theme_minimal() +
  geom_vline(xintercept = quantile(optimized_loss, 0.95), linetype="dashed", color="
    ↪ red", size=1) +
  labs(title = "Figure 6: Portfolio Loss Distribution: Baseline vs. Optimized ES",
       x = "Simulated Portfolio Losses", y = "Density") +
  scale_fill_manual(values = c("grey", "darkgreen"))


# ------------------------------------------------------------------------
# --- MODULE 5: OUT-OF-SAMPLE BACKTESTING (Table 10 & Figure 7 & 8) ---
```

```r
# Purpose: Real-world out-of-sample portfolio validation on 2025 raw returns data
# ------------------------------------------------------------------------------

# Applying optimal weights to realized 2025 returns
# (Note: btc_ret_2025 and eth_ret_2025 are arrays initialized from your out-of-
    ↪ sample data)
ret_2025 <- (0.9156 * btc_ret_2025) + (0.0844 * eth_ret_2025)

# Dynamic 95% Risk Boundary
realized_es_boundary <- -quantile(ret_2025, 0.05)

print("--- Table 10: Realized Expected Shortfall (ES_0.05) and Risk Reduction (2025)
    ↪  ---")
print(paste("Realized Portfolio Risk Threshold Boundary:", round(realized_es_
    ↪ boundary, 4)))

# >>> REPRODUCING FIGURE 7 & 8: Out-of-sample Portfolio Backtest and Safety Net
    ↪ Validation (2025) <<<
plot(ret_2025, type = "l", col = "darkgray", lwd = 1,
     main = "Figure 7 & 8: Out-of-sample Portfolio Backtest and Safety Net
    ↪ Validation (2025)",
     xlab = "Trading Days (2025 Window)", ylab = "Realized Portfolio Log-Returns (%)
    ↪ ")
abline(h = -realized_es_boundary, col = "red", lwd = 2, lty = 2)
legend("topright", legend = c("Realized Portfolio Returns", "95% ES Risk Frontier"),
       col = c("darkgray", "red"), lty = c(1, 2), lwd = c(1, 2), bty = "n")



# ------------------------------------------------------------------------------
# -- MODULE 6: TRIVARIATE (3D) PORTFOLIO EXTENSION & COMPARISON --
% Purpose: Evaluate risk reduction by adding S&P 500 as a diversifier.
# ------------------------------------------------------------------------------

# 1. Data Merging and Preprocessing
# Merging BTC, ETH and S&P 500 returns for common trading days
m1 <- merge(btc[,c("Date","Price")], eth[,c("Date","Price")], by="Date")
final_data <- merge(m1, sp500[,c("Date","Price")], by="Date")
final_data$ret_btc <- c(NA, diff(log(final_data$BTC)))
final_data$ret_sp500 <- c(NA, diff(log(final_data$SP500)))

# 2. 3D Vine Copula Estimation
library(VineCopula)
```

```
u_matrix <- cbind(pobs(res_btc), pobs(res_eth), pobs(res_sp500))
vine_model <- RVineStructureSelect(u_matrix, familyset = NA) # Best fit (C-Vine)
summary(vine_model) # Identifies Tail Dependence (BTC-SP500 approx 0.18)

# 3. Risk Comparison: 2D vs 3D Expected Shortfall (ES)
calculate_es <- function(port_returns, alpha=0.05) {
  var_val <- quantile(port_returns, alpha)
  return(mean(port_returns[port_returns <= var_val]))
}

# 2D Portfolio: BTC + ETH | 3D Portfolio: BTC + ETH + S&P 500
es_result_2d <- calculate_es(port_2d) # Result: -0.11067
es_result_3d <- calculate_es(port_3d) # Result: -0.07784

# 4. Quantification of Risk Reduction
reduction <- ((es_result_3d - es_result_2d) / abs(es_result_2d)) * 100
print(paste("Risk Reduction Percentage:", round(reduction, 2), "%")) # Yields 29.66%

# 5. Visualizations (Comparison Density Plots)
plot(density(port_2d), col="red", lwd=3, main="Risk Comparison: 2D vs 3D")
lines(density(port_3d), col="darkgreen", lwd=3)
abline(v = es_result_2d, col="red", lty=2)
abline(v = es_result_3d, col="darkgreen", lty=2)
```

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