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MATHEMATICS STUDY PROGRAMME

A Quantitative Analysis of Risk Measures in Finance: Volatility, Value at Risk (VaR), and Expected Shortfall

Kiekybinė finansinės rizikos matų analizė: kintamumas, rizikuojamoji vertė (VaR) ir tikėtinas deficitas

Master’s Thesis

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SUMMARY

The study provides a quantitative description of significant financial risk measures, i.e. volatility, Value at risk (VaR) and Expected shortfall (ES) in one probabilistic model. The motivation of the study is that the traditional measures of dispersion used are constrained in explaining extreme market behaviour especially when the distributions of returns are believed to have heavy tails.

The analysis commences with the formulation of a strict mathematical basis to the asset based financial risk where asset returns are modeled as the random variables in a probability space. Logarithmic returns are computed and utilized to estimate empirical properties, such as, mean, variance, skewness and kurtosis. The findings show that its normality assumptions are grossly violated, and are skewed and overly kurtosis, which substantiates the fact that the financial data have heavy tails.

A GARCH(1,1) process models the volatility and conditional time-dependent variance and volatility clustering. Although it has been found to be useful in the process of measuring the dynamics of dispersion, volatility has been found to be inadequate in the endeavor to measure downside risk. VaR is then historically and parametrically tested. Whereas VaR gives a sensible amount of the possible losses, it is less on the heavy-tailed conditions and not receptive in the magnitude of the maximum losses.

Proposed alternative is Tail-Sensitive and consistent: Expected Shortfall. Both parametric and historical ES measure are more extreme market behavior sensitive and give a more detailed description of tail risk. The results show that ES beats VaR in the occurrence and magnitude of extreme losses in heavy tailed environments.

Altogether, the article underlines the significance of tail-sensitive risk indicators and shows the inefficiency of the traditional methods to the correct quantification of the financial risk under the real market conditions.

SANTRAUKA

Šiame tyrime pateikiamas kiekybinis pagrindinių finansinės rizikos matų, tokių kaip volatilumas, rizikos vertė (VaR) ir tikėtinas nuostolis (ES), aprašymas vieningoje tikimybinėje sistemoje. Tyrimo motyvacija grindžiama tuo, kad tradiciniai sklaidos matai yra riboti aiškinant ekstremalų rinkos elgesį, ypač kai grąžų pasiskirstymai pasižymi storomis uodegomis.

Analizė pradedama nuo griežtos matematinės finansinės rizikos sistemos formulavimo, kur turto grąžos modeliuojamos kaip atsitiktiniai kintamieji tikimybinėje erdvėje. Apskaičiuojamos logaritminės grąžos ir naudojamos empirinėms savybėms įvertinti, tokioms kaip vidurkis, dispersija, asimetrija ir kurtosis. Rezultatai rodo reikšmingus nukrypimus nuo normalumo prielaidų, įskaitant asimetriją ir padidintą kurtosį, kas patvirtina storų uodegų buvimą finansiniuose duomenyse.

Volatilumas modeliuojamas naudojant GARCH(1,1) procesą, siekiant atspindėti sąlyginę laiko kintančią dispersiją ir volatilumo klasterizaciją. Nors šis metodas yra veiksmingas sklaidos dinamikai apibūdinti, nustatyta, kad volatilumas nėra pakankamas nuostolių rizikai įvertinti. VaR analizuojamas tiek parametriniu, tiek istoriniu būdu. Nors VaR suteikia aiškią galimų nuostolių ribą, jis nuvertina riziką storų uodegų sąlygomis ir nepakankamai atspindi ekstremalių nuostolių mastą.

Kaip alternatyva siūlomas uodegoms jautrus ir nuoseklus matas – tikėtinas nuostolis (ES). Tiek parametrinis, tiek istorinis ES yra jautresni ekstremaliam rinkos elgesiui ir suteikia išsamesnį uodegos rizikos įvertinimą. Rezultatai rodo, kad storų uodegų aplinkoje ES geriau nei VaR atspindi tiek ekstremalių nuostolių dažnį, tiek jų dydį.

Apskritai, tyrimas pabrėžia uodegoms jautrių rizikos matų svarbą ir atskleidžia tradicinių metodų ribotumą tiksliai vertinant finansinę riziką realiomis rinkos sąlygomis.

CHAPTER 1: INTRODUCTION

1.1 Mathematical Framework of Financial Risk

Measures of financial risk are created within the context of the probability theory. Financial market is a modelling on the probability space where uncertainty is the random outcomes. The financial value of a position (investment portfolio) in the future time is, therefore, described as a random variable [1]. The current risk theory does not assume the value of risk as variability. Instead, it considers it to be the capital which is required to make a financial position acceptable as per the requirements. This shift towards functional analysis rather than the descriptive statistics is the primary development of modern financial mathematics [2]. Such a framework takes the form of a risk measure being a mapping where each admissible financial position is assigned a real number. The mapping under consideration is regulated by structural attributes which reflect the economic intuition of behaviour. In particular, it is a prerequisite that a sound risk measure has to be used, which complies coherently with the basic tenets of diversification and monotonicity.

1.2 Probability Space and Function Spaces

We begin by defining the probabilistic setting.

The underlying probability space is defined as

$$(\Omega, F, P)$$

where Ω denotes the state space, F is a σ-algebra, and P is a probability measure.

The financial position at the terminal time, T, is taken to be a random variable,

$$X_T\colon \Omega \to R.$$

We assume that (X_T) belongs to the Lebesgue space

$$X_T \in L^p(\Omega, F, P), \quad 1 \le p < \infty,$$

where

$$L^p = \{X\colon \Omega \to R \mid E[|X|^p] < \infty\}.$$

A random variable which is bounded almost surely is called and essentially bounded random variable [10]. In many risk-theoretic models, the analytical interest is limited to essentially bounded positions,

$$X_T \in L^{\infty}(\Omega, F, P).$$

1.3 Risk Measure as a Functional

In the field of quantitative finance, risk is defined with respect to a financial position; the position is normally described by a random variable, often written X_T, the value of a stochastic portfolio at the terminal horizon T. The random variable is then assessed by a risk measure, and the outcome is a numerical measure that indicates the capital buffer that is necessary to make the position compliant with a given regulatory or institutional requirement [9].

A risk measure is any functional acting on financial positions:

$$\rho: L^p \to R.$$

Instead of defining the concept of risk directly, modern theory proposes the concept of an acceptance set. A set of accepted positions is a set of positions that can be considered financially acceptable [3]. The above definition assumes that the value of the riskless asset will be the same over the period of time that one is considering it. Generally, with capital capable of being invested at riskless rate, the discounted capital that is to be used should be treated in accounting to the present value, which is that it is invested and generates interests up to time *T*,

$$\rho_A(X) = \inf\{\, m \in R \mid X + m \in A\}.$$

This measure represents the lowest amount of capital input needed to make a position acceptable.

1.4 Volatility and Quadratic Risk

Variance is one of the base quantitative measures of financial risk, which strictly measures the degree to which asset returns are not in accordance with its expected value [6]. It is an unequivocal measure of the deviation of asset returns as compared to their expectation.

Let R_T denote the return of a financial asset at time T. For variance to be well defined, we assume that the return belongs to the space $L^2(\Omega, F, P)$, meaning that the second moment is finite.

The expected return is defined as

$$\mu = E[R_T].$$

The variance is defined as

$$Var(R_T) = E(R_T - \mu)^2.$$

Volatility is the square root of variance

$$\sigma = \sqrt{E(R_T - \mu)^2}.$$

Variance and volatility provide easy analytical measures of risk, but they equally penalise both positive and negative risks about the mean. In addition, variance only describes the second moment of the distribution and, accordingly, it does not explain salient characteristics of the return distribution, e.g. skewness or heavy tails.

1.5 Value at Risk

To correct for the shortfalls of quadratic risk measures like variance which are symmetric in both positive and negative deviations and ignore the importance of tail events, scholars have provided quantile-based risk measures [5].

Whenever X is the terminal value of a financial position, the Value at Risk at significance level of $0 < \alpha < 1$ is defined as follows:

$$VaR_\alpha(X) = \inf\{\, m \in R \mid P(X + m < 0) \le \alpha\}.$$

This value is the value of capital that needs to be added to position X (invested in a riskless asset), such that there is no higher than a probability of suffering a loss than the level α, where α is usually selected to be small. Some authors express this in terms of the confidence level 1−α, which is close to 1.

Equivalently, Value at Risk may be presented as a quantile of the loss distribution. Let $F_X(x) = P(X \le x)$ denote the distribution of X. Value at Risk can be then expressed as

$$VaR_\alpha(X) = -F_X^{-1}(\alpha).$$

Here $F_X^{-1}(\alpha)$ denotes the **generalised inverse (quantile function)** defined by

$$F_X^{-1}(\alpha) = \inf\{\, x \in R \mid F_X(x) \ge \alpha\}.$$

Value at Risk focuses on the lower end of the distribution. A risk measure ρ is said to be **subadditive** if for any financial positions X and Y,

$$\rho(X + Y) \le \rho(X) + \rho(Y).$$

The property encodes the principle of diversification, i.e. the risk of an aggregate portfolio must not be greater than the risk sum.

1.6 Expected Shortfall

Another extension of Value at Risk is Expected Shortfall which measures the average amount of losses that are set to exceed VaR. Again, we are looking at the bottom (left) end of the distribution of the present model of profit/loss where negative values are losses.

For a confidence level $0 < \alpha < 1$, the Expected Shortfall is defined as

$$ES_\alpha(X) = -E[X|X \leq -VaR_\alpha(X)] + (\lambda - 1)\big(-E[X|X \leq -VaR_\alpha(X)] - VaR_\alpha(X)\big),$$

where λ = P($X \leq -VaR_\alpha(X))/\alpha \geq 1$.

Acerbi and Tasche (2002) formalised the Expected Shortfall (ES) in a manner that has coherence and in particular, the inadequacies of VaR are mitigated.

They define Expected Shortfall at level $\alpha \in (0{,}1)$ as

$$ES_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \; VaR_u(X)\, du.$$

The formulation emphasises the fact that Expected Shortfall sums up all the lower quantiles to level alpha, hence summarising the entire tail behaviour of the loss distribution.

Expected Shortfall may be represented as a conditioned expectation in the case of a continuous distribution as

$$ES_\alpha(X) = -E[X \mid X \leq -VaR_\alpha(X)].$$

In the literature, this representation is commonly known as the Tail Conditional Expectation (TCE) or Conditional Value at risk (CVaR) and highlights that it quantifies the amount of expected loss that happens upon exceeding the Value at risk level.
Expected Shortfall is an example of a coherent risk measure. The following four properties are required for a risk measure ρ to be coherent. For any risks *X* and *Y*, the following axioms need to hold:

Monotonicity:

$$X \leq Y \Rightarrow \rho(X) \geq \rho(Y).$$

Translation invariance:

$$\rho(X+m)=\rho(X)-m$$

for any constant $m \in R$.

Positive homogeneity:

$$\rho(\lambda X)=\lambda\rho(X), \quad \lambda \geq 0.$$

Subadditivity:

$$\rho(X+Y) \leq \rho(X)+\rho(Y).$$

Subadditivity yields the principle of diversification, in other words, there should not be an increase in total risk when two positions are combined. Coherent risk measures have a dual expression involving a family of probability measures. Specifically, it is possible to describe a coherent risk measure ρ in the form of

$$\rho(X) = \sup \; E_Q[-X],$$

where the supremum is taken over some family $\mathcal{P}$ of probability measures Q∈ $\mathcal{P}$.

1.7 Heavy Tails and Tail Behaviour

The distributions of empirical financial returns tend to possess heavy-tailed behaviour. One should distinguish between the fat and heavy-tailed distributions [10]. The fat-tailed distributions are those whose tails are heavier than those of the normal distribution, and the heavy-tailed distributions are those that have an infinite moment generating function for all its arguments *t*>0.

Informally, a heavy-tailed distribution is a distribution in which the likelihood of large losses is relatively slow decreasing in comparison to the light-tailed distributions such as the exponential or normal distribution [14]. Regularly varying distributions provide a typical form of heavy-tailed model in financial modelling. Their tail behaviour is characterised asymptotically in terms of behaviour

$$P(X>x) \sim x^{-\xi}L(x), \quad x \to \infty,$$

where $\xi > 0$ is called the tail index, and $L(x)$ is a slowly varying function, meaning that

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad \text{for all } t > 0.$$

The heavy tails have significant effects in the quantification of risk. More specifically, when the distribution is heavy-tailed, higher moments could not exist. For regularly varying distributions with tail index $\xi > 0$, the variance is infinite when $\xi < 2$, and even the mean is infinite when $\xi < 1$. For example, in the case of a Pareto distribution with tail index $\xi = 1.5$, the mean will exist but the variance will be infinite [15]. Risk measures which are volatility based are not defined and thus are not appropriate to reflect the amount of risk in such cases.

1.8 Research Aim

This thesis is a vigorous quantitative comparison of volatility, Value at risk and Expected shortfall on the basis of a single probabilistic framework. Their structural mathematical properties, sensitivity to heavy-tailed return distributions, their behaviour under extreme market conditions, and their implications to capital allocation are evaluated in the study.

CHAPTER 2: LITERATURE REVIEW

2.1 Evolution of Contemporary Financial Risk Measurement

The more recent research on financial risk measurements has continued to shift toward the tail-sensitive and structurally robust approaches from dispersion-based financial risk measurement. Adrian, Boyarchenko, and Giannone showed that degradation of downside tail conditions has been believed to be the major cause of macro-financial vulnerability in comparison to central distributional behaviour [1]. The severe downside realisations help make the systems fragile and refer to the reality that financial instability is tightly linked with tail dynamics [2]. Glasserman and Young mentioned that during stressful periods, contagion mechanisms will also enhance the loss, and this would be in favour of the view that risk is a tail phenomenon and not a second-moment property [7]. Brownlees, Lillo, and Pelizzon also stated that capital requirements have been observed to be interacting directly with the systemic tail exposure in stressed environments [20]. The relevance of these studies is that they construe the risk in terms of extreme downside consequences rather than average behaviour. As shown by Adrian et al. [1] and Brownlees and Engle [2], tail events cause instability, which is directly relevant to the topic of this research. Glasserman and Young [7] and Brownlees et al. [20] also treat tail risk as contagion risk and a capital requirement, hence the emphasis on systemic tail exposure.

2.2 Volatility Modelling and Its Structural Limitations

The key point in financial econometrics is volatility since it is manageable and effective in prediction. Francq and Zakoïa stated that conditional heteroskedastic models create large degrees of persistence and grouping of financial returns volatility [12]. Such models are highly applicable with regard to predicting the short-term variances and are yet to be uncommon in the practice of risk management. However, the empirical returns distributions tend to be heavy-tailed and not normally distributed [11]. When the behaviour is heavy-tailed, the higher moments may become unstable, limiting the ability to explain metrics based on variance. Kratz and Mittnik mentioned that the extreme value methods provide superior approximations of tail probabilities than the Gaussian assumptions, particularly at the time of crisis [13]. In cases where dispersion-based modelling is considered, an additional complication of the problem can be observed due to multivariate extreme behaviour, as extreme co-movements may not be reflected in measures of second moment [4]. This literature underscores the importance of volatility not being sufficient in itself. Heavy tails invalidate measures depending on variance, as shown by Francq and Zakoïan

[12], Embrechts et al. [11], and Kratz and Mittnik [13]. Multivariate extremes [4] also indicate that extreme co-movements are obviated. The works advocate going past volatility in this study.

2.3 Value at Risk and Quantile-Based Risk Measures

Value at Risk is used due to the fact that it is easy to interpret as a lower quantile of the loss distribution. Brownlees and Engle explained that the systemic risk models that use quantiles have been proven to be of practical use in identifying the institutional vulnerability to stress [2]. However, structural problems still exist. Cont and Scaillet mentioned that dynamic risk models highlight the problems of time consistency in a situation where quantile-based measures are approximated over a sequence of periods [8]. Risk aggregation of the capital requirement using quantile measures will lead to unstable capital requirements when correlation structures are not known [9]. Moreover, aggregate quantile behaviour is also highly affected by multivariate extreme dependence, meaning it is sensitive to model specification [4]. The examination of stress testing indicates that the tail dependence structures fail to modify the way the losses are distributed with respect to quantile limits alone [3]. The above studies bring out the applicability of VaR and its limitations. Its usefulness in practice is illustrated by Brownlees and Engle [2], its time inconsistency and instability are illustrated by Cont and Scaillet [8] and Puccetti and Ruehschdorf [9]. Cont et al. [3] and Embrechts et al. [4] show that not all tail dependence is taken care of. This is what necessitates more powerful action.

2.4 Expected Shortfall and Coherent Risk Measures

ES is becoming more popular due to its application for the better business decisions and has gained adoption among regulators and governments. It does not only capture the threshold of the extreme losses but also the average magnitude and hence captures additional tail data [5]. Coherent risk measures are quite compatible with convex optimisation models, and the idea of diversification is upheld [6]. Spectral and distortion risk measures are generalisations of expected shortfall in a broader mathematical structure and indicate that tail weighting is an essential component of capital measurement [10]. The worst-case expectations have a dual representation using convex risk measures, and this improves the strength of coherent measures in uncertain environments [16]. Law-independent coherent risk measures have been considered again and simplified, and structural properties and their practical use in pragmatic modelling have been developed [18]. Empirically, expected shortfall methodologies that are in compliance with the present regulatory requirements have been backtested to address implementation challenges [14]. In addition, it has proposed strong estimation techniques to limit instability on the finite samples [19]. These sources must be included to provide stronger alternatives to VaR. Tasche [5] and McNeil and Peters [6] are consistent, and in [10,16,18], extensions are introduced to make them more flexible. Empirical work can be found in the implementation literature [14, 19]. The two of them underpin the study of tail-sensitive and coherent measures.

2.5 Heavy Tails, Extreme Value Theory, and Systemic Stress

One of the characteristics of financial returns is the heavy-tailed behaviour. It has been empirically observed that extreme losses, which are power-law decaying and not Gaussian, are usually ubiquitous [11]. The extreme value methods provide the tail approximation which is asymptotically justified and also improves the accuracy of the estimation at the periods of the crisis [13]. Dependence structures with heavy tails are far less likely to provide the same capital estimates as elliptical models of stress testing [3]. Tail co-movement leads to systemic vulnerability, especially during the slump of the market [17]. Left-tail vulnerability is closely associated with macroeconomic recessions, and this argues for the necessity of tail modelling [1]. The basis of modelling extremes is found in this literature. Embrechts et al. [11] and Kratz and Mittnik [13] show that financial losses exhibit heavy-tailed distributions. Dependence in extremes is pointed out by Cont et al. [3] and Glasserman et al. [17], whereas Adrian et al. [1] attribute it to macro risk. The texts should be given particular tail modelling.

2.6 Risk Aggregation and Dependence Uncertainty

The assumptions of aggregation are highly necessary in calculating the risk at the portfolio level. Acute delimitations derived using the partial dependence statistics indicate the presence of substantial changes in the aggregated level of risk estimation [9]. The distortion-based approaches are coherent and can be freely used to model tail weighting of aggregated portfolios [10]. It is also demonstrated by multivariate extreme modelling that dependence structures are also influential on tail behaviour at the portfolio level [4]. Systemic capital requirements are also highly sensitive to aggregated tail exposure, especially during stressed market conditions [20]. These studies show that aggregation involves many dependence assumptions. Variability in results is noted by Puccetti and Rueschndorf [9], and distortion and extreme models [4, 10] can be used to cope with it. According to Brownlees et al. [20], this is associated with systemic capital. This underpins the need to incorporate dependence on risk analysis.

2.7 Identified Research Gap

The literature has concluded that volatility is highly effective in the conditional dispersion but inefficient as far as severity, which is heavy-tailed, is concerned [12]. Value at risk measures risk as quantiles but has been proven to be unstable when it depends on uncertain and dynamic environments [9]. The expected shortfall satisfies the coherence properties as well as the strong dual representations in the case of model uncertainty [18]. Heavy-tailed dependence causes a huge impact on the estimation of capital and susceptibility of systems [3]. Despite these advances, no agreement exists on the empirical comparison of volatility, value at risk and expected shortfall under the same heavy tailed circumstances. The literature available is usually focused on single measurements or even specified modelling environments rather than a consistent structural and empirical study of dispersion-based and tail-based models.

CHAPTER 3: RESEARCH METHODOLOGY

3.1 Probabilistic Framework

This study is conducted in a discrete-time probabilistic setting.

Let, (Ω, F, P) be a probability space, where Ω denotes the set of states of the world, F is a σ-algebra of events, and $P$is a probability measure, as defined in Chapter 1.

Time is indexed discretely by $t = 1,2, \dots, T$, where T represents the fixed investment horizon introduced in Chapter 1. At each time t, the financial return of an asset is modelled as a random variable

$$R_t : \Omega \to \mathbb{R}.$$

The sequence $\{R_t\}_{t=1}^{T}$thereby describes the process of analysing the history of returns over the observation period. This construction can be used to take a dynamic view, whilst we are still consistent with the terminal framework that was described above.

To obtain the risk measures we consider to be well-defined, we use conditions of integrability on returns. In particular, we suppose that

$$R_t \in L^p(\Omega, F, P), \quad p \geq 1,$$

where $L^p(\Omega, F, P)$is defined as in Chapter 1.

In particular, volatility-based measures refer to the case $p = 2$, and tail-risk measures (Value at Risk and Expected Shortfall) to the case $p = 1$.

The dispersion-based and tail-based risk measures modelling is based on this probabilistic framework as return distribution functional.

3.2 Return Construction

Let P_t denote the price of the asset at time t. The return process is defined using logarithmic returns as

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right), t = 1,2, \dots, T.$$

This specification is consistent with the probabilistic framework introduced in Section 3.1, where returns are modelled as a stochastic process $\{R_t\}_{t=1}^{T}$.

The first and second moments of returns are defined analogously to the expressions introduced in Chapter 1, with the terminal variable replaced by the time-indexed return R_t.

The unconditional mean return is given by

$$\mu = E[R_t],$$

and the unconditional variance is defined as

$$Var(R_t) = E[(R_t - \mu)^2].$$

Volatility is then defined as the standard deviation of returns,

$$\sigma = \sqrt{Var(R_t)}.$$

These quantities provide the basis for dispersion-based risk measurement and will be used in the subsequent modelling of conditional volatility.

Higher order moments, i.e. skew and kurtosis, are also tested in order to identify deviation from the normality.

Skewness:

$$Skew\ (R_t) = \frac{E[(R_t - \mu)^3]}{\sigma^3}$$

Kurtosis:

$$Kurt\ (R_t) = \frac{E[(R_t - \mu)^4]}{\sigma^4}$$

3.3 Volatility Modelling

The clustering of volatility of the series of financial returns is not a new phenomenon, and the high variation periods are followed by more high variation periods and the low-volatility periods are followed by more low-volatility periods. Models that have constant variance cannot capture this empirical feature. A conditional heteroskedastic model is needed to explain this.

In this study, volatility dynamics are modelled using a GARCH(1,1) specification, which provides a parsimonious yet effective representation of time-varying variance in financial returns.

Let $t = 1,2, \ldots, T$ denote the discrete time index introduced earlier. The return process is specified as

$$R_t = \mu + \varepsilon_t,$$

where $\mu = E[R_t]$ is the unconditional mean and ε_t represents the innovation at time t.

The innovations are defined as

$$\varepsilon_t = \sigma_t Z_t,$$

where σ_t^2 is the conditional variance and Z_t is an independent and identically distributed sequence.

$$Z_t \sim N(0,1).$$

The conditional variance evolves according to the GARCH(1,1) recursion:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, t = 1,2, \ldots, T,$$

where $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$ are model parameters.

The GARCH(1,1) model captures past shocks via ε_{t-1}^2 and volatility persistence via σ_{t-1}^2, allowing it to reflect volatility clustering in returns.

For the process to be covariance stationary, it is required that

$$\alpha + \beta < 1.$$

3.4 Parametric Value at Risk

Under the assumption of conditional normality, the return at time t is given by

$$R_t = \mu + \sigma_t Z_t, \;\; Z_t \sim N(0,1),$$

where $\mu = E[R_t]$ and σ_t is the conditional standard deviation from GARCH.

VaR_α (with α close to 1) is defined as the lower α-quantile of returns:

$$VaR_\alpha(R_t) = -\inf\{x \in \mathbb{R}: P(R_t \leq \; x) \geq 1 - \alpha\}.$$

Under the normality assumption, we can standardise returns as

$$\frac{R_t - \mu}{\sigma_t} \sim N(0,1).$$

From the standardisation, we have

$$P(R_t \leq x) = P\left(\frac{R_t - \mu}{\sigma_t} \leq \frac{x - \mu}{\sigma_t}\right) = \Phi\left(\frac{x - \mu}{\sigma_t}\right).$$

Setting this equal to $1 - \alpha$, we obtain

$$\frac{x - \mu}{\sigma_t} = z_{1-\alpha} = -z_\alpha,$$

where $z_\alpha = \Phi^{-1}(\alpha)$and $\Phi$is the standard normal cumulative distribution function.

Solving for $x$yields

$$x = \mu + \sigma_t z_{1-\alpha}.$$

Therefore, the parametric Value at Risk is

$$VaR_{\alpha,t}^{par}(R_t) = -(\mu + \sigma_t z_{1-\alpha}).$$

In practice, μ is often negligible relative to volatility, so the expression simplifies.

$$VaR_{\alpha,t} \approx -z_{1-\alpha}\sigma_t = z_\alpha \sigma_t.$$

3.5 Historical Value at Risk

The empirical distribution function is used by the historical estimator.

$$VaR_\alpha^{hist} = -F_T^{-1}(1 - \alpha)$$

In this case, F_T^{-1} is the quantile function (generalised inverse) of the empirical distribution function of returns.

3.6 Expected Shortfall

Let $R_t$denote the return at time $t = 1,2, \ldots, T$. Expected Shortfall at level $\alpha \in (0,1)$is defined as

$$ES_{\alpha,t}(R_t) = -E[R_t \mid R_t \leq -VaR_{\alpha,t}(R_t)].$$

Parametric Expected Shortfall

Assume

$$R_t = \mu + \sigma_t Z_t, \quad Z_t \sim N(0,1).$$

Using standardisation and the properties of the normal distribution, we obtain

$$ES_{\alpha,t}(R_t) = -E\big[\, R_t \mid R_t \leq -\, VaR_{\alpha,t} \,\big]$$

$$\begin{aligned}
&= -\mathbb{E}\left[\mu_t + \sigma_t Z_t \mid \mu_t + \sigma_t Z_t \leq -\,\mathrm{VaR}_{\alpha,t}\right] \\
&= -\mathbb{E}\left[\mu_t + \sigma_t Z_t \mid Z_t \leq z_{1-\alpha}\right] \\
&= -\mu_t - \sigma_t\, \mathbb{E}\left[Z_t \mid Z_t \leq z_{1-\alpha}\right]
\end{aligned}$$

where due to symmetry of ϕ,

$$E[Z_t | Z_t \leq z_{1-\alpha}] = \frac{1}{\alpha}\int_{-\infty}^{z_{1-\alpha}} z\,\phi(z)\,dz = -\frac{\phi(z_{1-\alpha})}{\alpha} = -\frac{\phi(z_\alpha)}{\alpha}$$

$$ES_{\alpha,t}(R_t) = -\mu_t + \sigma_t \frac{\phi(z_\alpha)}{\alpha}$$

where

$$z_\alpha = \Phi^{-1}(\alpha), \qquad \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

If the mean is assumed negligible, this simplifies to

$$ES_{\alpha,t}^{par}(R_t) \approx \sigma_t \frac{\phi(z_\alpha)}{\alpha}.$$

Historical Expected Shortfall

The historical Expected Shortfall is defined as

$$ES_{\alpha,t}^{hist}(R_t) = -E[R_t \mid R_t \leq q_{1-\alpha}],$$

where $q_{1-\alpha}$is the empirical $(1-\alpha)$-quantile (historical VaR).

3.7 Backtesting Framework

Let $R_t$denote returns at time $t = 1,2, \ldots, T$, and let $VaR_{\alpha,t}(R_t)$be the estimated Value at Risk at level α. Under correct model specification and a continuous distribution of R_t, the probability of a violation is α.

Violation Indicator

Let the violation indicator be defined as

$$I_t = \begin{cases} 1, \text{if } R_t < -VaR_\alpha(R_t); \\ 0, \text{otherwise} \end{cases}$$

Assuming that the return distribution is continuous, it follows from the definition of Value at Risk that

$$\mathbb{P}\left(R_t < -\mathrm{VaR}_{\alpha,t}(R_t)\right) = \alpha.$$

Therefore, the indicator $I_t$follows a Bernoulli distribution with parameter α, i.e.

$$I_t \sim \text{Bernoulli}(\alpha),\ with\ \mathbb{E}[I_t] = \alpha.$$

The total number of violations over the sample is given by

$$N = \sum_{t=1}^{T} I_t.$$

By linearity of expectation, we obtain

$$\mathbb{E}[N] = \sum_{t=1}^{T} \mathbb{E}[I_t] = \alpha T.$$

The violation ratio is defined as

$$VR = \frac{N}{\alpha T},$$

which compares the observed number of violations to its theoretical expectation. A value close to 1 indicates that the model accurately captures the tail probability, while deviations from 1 suggest model misspecification.

3.8 Heavy-Tail Assessment

The tail behaviour of returns is analysed through regularly varying distributions. Specifically, the upper tail is characterised by

$$P(R_t > x) \sim x^{-\xi} L(x), x \to \infty,$$

where $\xi > 0$is the tail index and $L(x)$ is a slowly varying function.

The existence of moments is dependent on the value of ξ. Specifically, the variance is finite when $\xi > 2$ and infinite when the value is $\xi \leq 2$. This has a direct implication on risk measures (volatility) which are based on second moments and thus can be destabilized under heavy-tailed distributions.

In quantile-based measures, Value at Risk is as well defined since it only relies on the distributional quantile. But, when tails are heavy, VaR grows substantially, indicating greater chances of extreme losses.

The tail behaviour is more sensitive to Expected Shortfall, which reflects the average loss over VaR. In heavy-tailed distributions, ES grows more rapidly than VaR, and can be infinite when the tail index is too small (e.g., $\xi \leq 1$).

These observations emphasize the fact that VaR is a quantile-based statistic, but can understate tail risk, and Expected Shortfall is a better view of the extent of extreme risk in the presence of heavy-tailed returns.

CHAPTER 4: EMPIRICAL RESULTS AND ANALYSIS

4.1 Distributional Characteristics of Financial Returns

The chapter provides detailed empirical research on volatility, Value at risk and Expected shortfall in respect of daily logarithmic returns. The data, utilized in the study, was gathered daily of the price of the Bitcoin in Yahoo Finance in the sample of September 2014 to 2026. Logarithmic returns which are made based on close prices over a day are analysed.

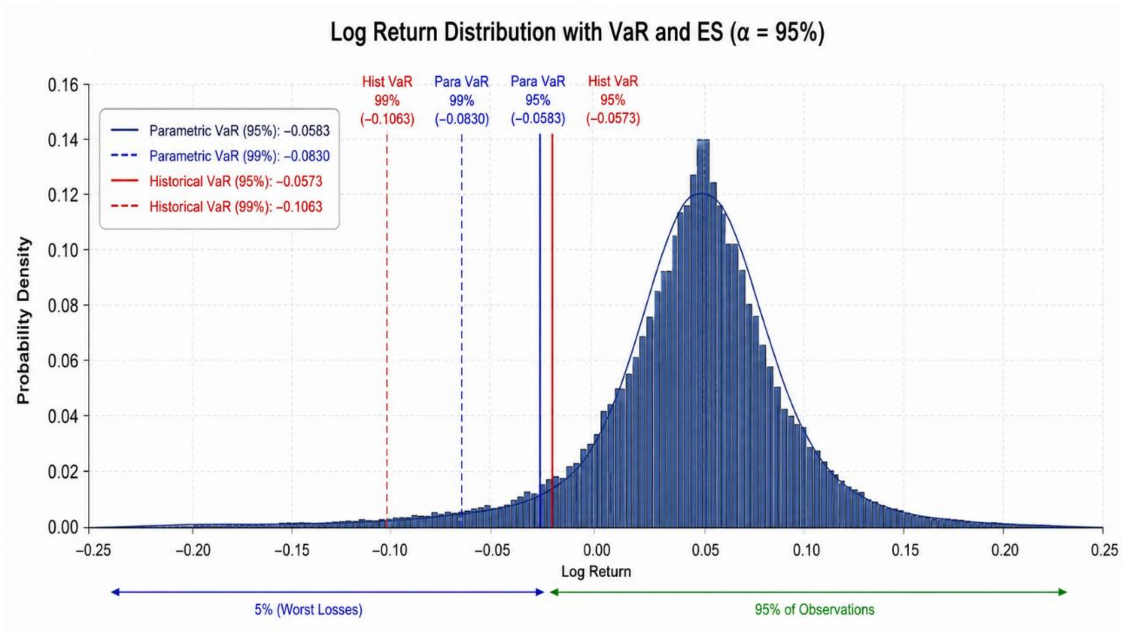

Figure 1: Logarithmic returns (Source: Author)

The computation of logarithmic returns is as follows:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right), \quad t = 1,2,\dots,T$$

The approximate mean and the approximate variance are unconditional, and they are represented as follows:

The evidence of the empirical data indicates that the mean returns per day are low relative to the variance of returns. This implies that there is a replacement of drift by uncertainty in the short horizon financial dynamics. More specifically, empirical distribution is very dissimilar to the normality. It gives too much kurtosis, and this just attests to the fact that there are extreme returns that are prevalent than what the Gaussian assumption assumes. This is behaviour that is consistent with the empirical evidence in the financial markets which is heavy tailed [11].

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} R_t\,, \quad \widehat{\sigma^2} = \frac{1}{T-1}\sum_{t=1}^{T}(R_t - \hat{\mu})^2$$

Skew is calculated as being -0.724072 and kurtosis is calculated as being 11.469646.

$$\widehat{Skew}(R_t) = \frac{1}{T}\sum_{t=1}^{T}\frac{(R_t - \hat{\mu})^3}{\widehat{\sigma^3}}, \quad \widehat{Kurt}(R_t) = \frac{1}{T}\sum_{t=1}^{T}\frac{(R_t - \hat{\mu})^4}{\widehat{\sigma^4}}$$

The empirical mean of the daily logarithmic returns is approximately equal to 0.001410 and the empirical variance of the daily logarithmic returns is approximately equal to 0.001317. The ratio of the mean to variance is quite small and indicates that uncertainty is more important than drift in the short-horizon dynamics of Bitcoin returns. The empirical distribution of Bitcoin returns has a large deviation from normality. The skewness (-0.724) of the return series is negative, which means that it is skewed in the large downside direction and that the negative shocks are stronger as compared to the positive gains. The skewness coefficient is -0.724 indicating the distribution of returns has moderate negative asymmetry. With a large sample size (from 2014-2026), the skewness differs statistically from zero, suggesting that the distribution of daily Bitcoin returns is highly asymmetric with a downside bias. Also, the kurtosis (11.47) is much higher than Gaussian (3), indicating that there are heavy tails and extreme values in the return distribution.

4.2 Conditional Volatility Dynamics

The clustering of volatility of returns series of financial returns suggests that big shocks will be followed by big shocks. This aspect is achieved using the estimated GARCH(1,1) model.

$$\mathrm{R_t} = \mu + \varepsilon_\mathrm{t}, \quad \varepsilon_\mathrm{t} = \sigma_\mathrm{t}\mathrm{Z_t}, \quad \sigma_\mathrm{t}^2 = \omega + \alpha\varepsilon_{\mathrm{t}-1}^2 + \beta\sigma_{\mathrm{t}-1}^2 \text{ (see Section 3.3)}$$

The estimated parameters are: $\omega \approx 0.0000658$, $\alpha \approx 0.1000$, and $\beta \approx 0.8500$, so that

$$\sigma_\mathrm{t}^2 = 0.0000658 + 0.1000\,\varepsilon_{\{\mathrm{t}-1\}}^2 + 0.8500\,\sigma_{\{\mathrm{t}-1\}}^2$$

The approximations of the parameters indicate that the degree of persistence in the conditional variance is high because the sum of the ARCH and GARCH coefficients is nearly equal to unity. The implication of this persistence is that volatility shocks would fade away over time something which has been reported extensively in financial econometrics [12]. During a state of crisis, the conditional volatility increases dramatically and is supported on a dramatic level. However, volatility is not paid attention to the positive or negative returns direction. Thus, although volatility is increased during the periods of market stress, it is not independent of the intensity of the downside risk.

4.3 Historical Value at Risk

The embarkation of historical VaR is also computed as follows using empirical distribution:

$$VaR_{\alpha}^{hist} = -F_T^{-1}(1-\alpha)$$

Confidence Level	Parametric VaR
95%	-0.0583
99%	-0.0830

Table 1: Parametric VaR Results (Source: Author)

The parametric VaR estimates suggest that there's significant downside risk in Bitcoin markets. The daily loss threshold is estimated at >8% at 99% confidence level, a high threshold for cryptocurrency assets, and this is the reason for the high volatility of these assets. This instability implies that it is sensitive to sample composition that is also a phenomenon in the study of risk aggregation [9].

Confidence Level	Historical VaR
95%	0.0573
99%	0.1063

Table 2: Historical VaR Results (Source: Author)

The historical VaR was significantly larger than the parametric VaR at 99% confidence level. This means that the Gaussian assumption underpredicts extreme downside risk in Bitcoin returns and also underpredicts for heavy-tailed behaviour.

4.4 Expected Shortfall and Tail Severity

Expected shortfall is estimated on both parametric and historical basis.

PARAMETRIC Var (95%)

$$VaR_{0.95}^{par} = -0.001410 + (1.645)(0.036291)$$

$$VaR_{0.95}^{par} = -0.001410 + 0.059698$$

$$VaR_{0.95}^{par} = 0.058288$$

$$VaR_{0.05}^{par} = 5.83\%$$

HISTORICAL Var (95%)

$$VaR_{0.95}^{hist} = 0.057307$$

$$VaR_{0.95}^{hist} = 5.73\%$$

Expected Shortfall (95%)

$$ES_{0.95}^{hist} = 0.088669$$

$$ES_{0.95}^{hist} = 8.87\%$$

PARAMETRIC Var (99%)

$$VaR_{0.99}^{par} = -0.001410 + (2.326)(0.036291)$$

$$VaR_{0.01}^{par} = -0.001410 + 0.084412$$

$$VaR_{0.99}^{par} = 0.083002$$

$$VaR_{0.99}^{par} = 8.30\%$$

HISTORICAL Var (99%)

$$VaR_{0.99}^{hist} = 0.106347$$

$$VaR_{0.99}^{hist} = 10.63\%$$

VaR and Expected Shortfall are reported to be lower-tail return thresholds as they are computed as a result of the analysis of the return distributions. As a result, both measures will be in the form of negative values and the more negative the value, the higher the downside risk. Corresponding values would be reported as positive percentages in case they are reported as magnitudes of losses.

Expected Shortfall (99%)

$$ES_{0.99}^{hist} = 0.146434$$

$$ES_{0.99}^{hist} = 14.64\%$$

Parametric Expected Shortfall Formula

$$ES_{\alpha,t}^{par}(R_t) = -\mu_t + \sigma_t \frac{\phi(z_\alpha)}{\alpha}$$

95% Parametric ES Substitution

$$ES_{0.95,t}^{par}(R_t) = -0.001410 + 0.036291 \frac{0.103}{0.05}$$

$$ES_{0.95,t}^{par}(R_t) = -0.001410 + 0.036291(2.06)$$

$$ES_{0.95,t}^{par}(R_t) = -0.001410 + 0.074759$$

$$ES_{0.95,t}^{par}(R_t) = 0.073349$$

$$ES_{0.95,t}^{par}(R_t) = 7.33\%$$

The empirical results reveal that the returns of Bitcoin have a large downside risk and fat tails. The parametric VaR(95%) was calculated to be 5.83% and the historical VaR(95%) was calculated to be 5.73%. These estimates are comparable and this gives a good indication that the Gaussian approximation can capture moderate fluctuations downward. But, the Expected Shortfall increased to (8.87%) indicating that losses were considerably more if the VaR had not been achieved.

The separation increases at a 99% confidence level. The Value at Risk (parametric) is estimated at 8.30% and the historical Value at Risk is 10.63%. Thus, the assumption of normality of the distribution overestimates the extreme downside risks in Bitcoin returns. Furthermore, Expected Shortfall rose to 14.64% in response to the level of losses that occurred in extreme market environment.

Results show that overall, there is evidence for heavy tails and a significant tail risk of returns of Bitcoin. In very harsh market conditions, Expected Shortfall will provide a more accurate picture of downside risk than Value at Risk.

4.5 Heavy-Tail Evidence and Systemic Implications

The tail behaviour of returns is characterised by regular variation:

$$\mathbb{P}(R_t > x) \sim x^{-\xi} L(x), \quad x \to \infty,$$

where $\xi > 0$is the tail index.

The empirical distribution of Bitcoin returns is very heavy tailed and exhibits high systemic risk. This can be seen from very high kurtosis (11.47) and very high asymmetry of the distribution downside and very large difference between VaR and ES estimates at 99% confidence level. The greater empirical tail than the Gaussian benchmark suggests that extreme market losses are more likely than the Gaussian model suggests. Therefore, regular risk models might underestimate the risk and impact of financial crises. The results show that Bitcoin markets do exhibit large tail dependence and sensitivity to extreme downward shocks.

This is not compatible with the normal distribution considered earlier, since the normal distribution is light-tailed, whereas regular variation assumes heavy tails (see Section 2.5).

Under heavy-tailed distributions, the Value at Risk behaves asymptotically as

$$VaR_\alpha \sim C\,(1-\alpha)^{-1/\xi}, \alpha \to 1,$$

for some constant $C > 0$. This indicates that quantiles increase at a polynomial rate as the tail probability decreases.

For Expected Shortfall, the asymptotic relation is given by

$$ES_\alpha \sim \frac{\xi}{\xi - 1} VaR_\alpha, \alpha \to 1, \xi > 1.$$

(see Embrechts et al. [11]; Kratz and Mittnik [13]).

Using the estimated tail index, the relationship can be approximated as

$$ES_\alpha \approx \frac{\hat{\xi}}{\hat{\xi} - 1} VaR_\alpha.$$

$$VaR_{0.99}^{hist} = 10.63\%, \qquad ES_{0.99}^{hist} = 14.64\%$$

$$\frac{ES_\alpha}{VaR_\alpha} \approx \frac{\xi}{\xi - 1}$$

$$\frac{14.64}{10.63} \approx \frac{\xi}{\xi - 1}$$

$$1.377 \approx \frac{\xi}{\xi - 1}$$

$Multiplying\ both\ sides\ by (\xi - 1),$

$$1.377(\xi - 1) = \xi$$

$$1.377\xi - 1.377 = \xi$$

$$1.377\xi - \xi = 1.377$$

$$0.377\xi = 1.377$$

$$\hat{\xi} \approx \frac{1.377}{0.377}$$

$$\hat{\xi} \approx 3.65$$

The empirical correlation between Historical Expected Shortfall and Historical Value at Risk at the 99% level of significance was considered in order to get a rough estimate of the tail heaviness. The implied tail index will be ξ = 0.65 by using the Historical VaR (10.63%) and Historical ES (14.64%). This should be regarded with a dose of skepticism as tail-index estimation is a mathematically challenging task and there are more sophisticated approaches in the literature, such as the Hill estimator, or extreme value theory methods. Nevertheless, the result is consistent with the presence of the heavy-tailed behaviour of the Bitcoin returns that indicates a higher likelihood of the extreme losses than in the case of the normal distribution. The variance is finite since the tail index is estimated to be bigger than 2.

4.7 Integrated Comparative Analysis

The relative results demonstrate the seeming structural differences. Volatility takes into account all the dispersion, but it does not distinguish between downside dispersion. One type of parametric VaR is that which trades off volatility in a systematic fashion but makes underestimates of extreme losses with heavy-tailed distributions. The historical VaR is more accurate in the empirical component but threshold-based. Expected Shortfall is the most accurate analysis of tail exposure. It is very susceptible to the periods of stress and provides more low budget estimates of capitals. It has behaviour consistent with theory of coherence and heavy tails empirically. Capital allocation In terms of dispersion, it would not be sufficient to develop buffers in case of systemic crises. The tail-sensitive measures have an increased coverage of tail-risks like extreme losses.

CHAPTER 5: DISCUSSION AND CONCLUSION

5.1 Theoretical Interpretation of Empirical Findings

The empirical analysis of the Bitcoin returns fits well with the theoretical framework presented in the thesis. The returning distribution was found to have a high level of departure from normality with a negative skewness of -0.724 and very high kurtosis of 11.47. These findings corroborate the heavy tails and downside asymmetry, with the result that extreme negative returns happen more often than one would expect based on Gaussian assumptions. The lowest daily return for the day was -46.47% further reinforcing the idea that there are severe tail events and increased systemic vulnerabilities in cryptocurrency markets.

There was also a high level of dispersion (and “clustering”) of returns using the volatility estimates. Volatilities can capture time varying uncertainty in the market well, but fail to reflect the extent of large losses during a crisis. A comparison of Value at Risk using a parametric model and Value at Risk using a historical model was done to highlight this restriction. Both measures resulted in relatively similar results at the 95% confidence level, indicating that moderate market fluctuations are possibly partially captured by Gaussian approximations. However, at the 99% confidence level, the historical VaR (-10.63%) was much bigger than the parametric VaR (-8.30%) and this indicates that this normal distribution assumption does not adequately reflect extreme downside risk in the case of heavy tails.

Expected Shortfall was the best measure of tail risk. The 99% EWF estimate was -14.64% as compared to the corresponding VaR estimates. This reflects that, when tail events occur, losses get much worse than suggested by the quantiles-based risk measures alone. This means that the empirical results support the conclusion that Expected Shortfall is a better, more stable and informative way to represent financial risks in the downside in heavy tailed markets like Bitcoin.

5.2 Structural Comparison of Risk Measures

The empirical distribution of the Bitcoin returns was found to have high negative skewness, heavy tailedness and downside risk. The estimated value of kurtosis was 11.47, much greater than a Gaussian model and hence indicating that there were extreme tail events, the returns were not normally distributed [16, 18]. But the volatility may only be a mirror image of the time varying dispersion of the market and not extreme losses, either directionally or in magnitude. Especially during the downtimes of these markets and stress periods the downside moves were much greater than the volatility would be indicative of. The Parametric VaR in comparison with the Historical VaR also revealed certain drawbacks of quantile-based measures.

At the 95% confidence level, both the estimates of VaR provided results that were fairly similar, indicating that, to a certain extent, Gaussian approximations may be representative of moderate market fluctuations. However, it can be observed that the historical VaR (-10.63%) is higher than the parametric VaR (-8.30%) at 99% confidence level, which indicates that hypothesis of normal distribution was not completely valid and that for heavy-tailedness the distribution of potential extreme downside exposures is underestimated [3].

Expected Shortfall was the best measure of downside exposure. The 99% ES estimate was -14.64% which was significantly below both VaR estimates. This means that in addition to being a measure of the probability of the extreme losses, Expected Shortfall also reflects the average size of the losses when the tail starts. As a result, coherent tail sensitive measures are more indicative of financial risk in heavy tailed cryptocurrencies, such as Bitcoin [16].

5.3 Implications for Heavy-Tailed Financial Markets

The results with the bitcoin data have been quite empirical, the heavy tailedness can be observed in the results and there is high level of systemic risk. The basic properties, like high kurtosis, negative skewness and extreme downside observations indicate that if we assume Gaussianity, large losses can happen more often than one might imagine [17]. The significant difference between VaR and ES at 99% confidence level also revealed that tail sensitive measures are better in giving signals during times of financial stress. These findings are indicative of the underestimation of systemic risk using dispersion based risk measures in heavy-tailed markets. The more money is on the line, the more losses increase than those that are evident from the normal volatility calculations. Tail-average measures of downside exposure, however, offer a more stable measure of downside exposure during crisis times [1]

5.4 Capital Allocation and Regulatory Considerations

The empirical results are significant in regard to capital allocation and financial regulation. Expected Shortfall also offered a better indication of the risk of the loss, its magnitude, and volatility based measures and quantiles based measures underestimated losses from the downside in the extreme market conditions [14]. The results confirm the increasing use of ES in the modern regulatory frameworks. While easy to comprehend, VaR is not a good indicator of the magnitude of losses exceeding the quantile. But ES is a more complete risk measurement in terms of the capital that is needed to deal with stressed market conditions and heavy tailed returns [19].

5.5 Contribution of the Study

The thesis is of double value: firstly, it links theory with practice as the theoretical study of risk measures is complemented by an empirical estimation of the tail risk of BTC; secondly, it introduces a new empirical approach to estimating tail risk of BTC. One empirical framework has been used throughout the study which includes volatility modelling and VaR, along with the ES and heavy-tail analysis. Empirical results showed the impact of different structural properties of the risk measures on their real-life performance in heavy tailed market conditions. The results in particular validated the use of the coherent tail-sensitive measures in the period studied, as opposed to the traditional dispersion-based measures [5] for extreme downside risk.

5.6 Limitations and Future Research

Several limitations remain. The analysis was performed with just one asset class and one return series. The framework could be potentially utilized in future works in multi-asset portfolio, stochastic volatility models and multivariate extreme value analysis. Other research can also examine dependence structures, contagion and the interactions between the networks on a systemic level during crises. Furthermore, volatility models [18] show that non-Gaussian innovations help to improve the tail risk estimation particularly under extreme market conditions.

5.7 Concluding Remarks

This thesis's results support the that Bitcoin returns have significant heavy-tailedness, downside asymmetry and high tail risk. The downsides for dispersion-based measures are that they do not reflect the amount of extreme losses and that quantile-based measures give up the sense of the severity of the tail, but they retain the sense of the threshold probabilities. Expected Shortfall (ES) was the most comprehensive indicator of downside risk especially in stressed markets [16]. Lastly, the empirical findings are quite encouraging with respect to the usefulness of coherent tail-sensitive risk measures in current applications in financial risk management. The results make it obvious that in the context of heavy-tailed financial markets, the risk frameworks ought to be able to account explicitly for the size and persistence of extreme downside events.

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