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On the zeros of the derivative of Dedekind zeta-functions

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1. Introduction

Speiser [2] discovered a correspondence between the distributions of the complex zeros of the Riemann zeta-function $\zeta(s)$ and of its derivative (as usual, $s=\sigma+it$ with $\sigma, t\in\mathbb{R}$, $i=\sqrt{-1}$). He showed that Riemann's hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ in the left-half of the critical strip. Yildirim generalized this result for the Dirichlet L-functions $L(s,\chi)$. Assuming Generalized Riemann hypothesis he proved [3] that, if χ is a character mod$q$ with $\chi(-1)=1$ and $q\geqslant 216$, then $L'(s,\chi)$ has exactly one real zero in the left of the critical strip at $\frac{1}{q}+\mathrm{O}\left(\frac{\log\log q}{\log^2 q}\right)$, and that, if $\chi(-1)=-1$ and $q\geqslant 23$, then $L'(s,\chi)$ has no zeros in the left of the critical strip. In this paper we extend these results to some class of Dedekind zeta-functions.

Let $\mathbb{K}=\mathbb{Q}(\alpha)$ be a number field of degree $n\geqslant 2$. Then, for $\sigma>1$, the Dedekind zeta-function of $\mathbb{K}$ is given by

$$\zeta_{\mathbb{K}}(s)=\sum_{\mathcal{A}}\frac{1}{N(\mathcal{A})^s}=\prod_{\mathcal{P}}\left(1-\frac{1}{N(\mathcal{P})^s}\right)^{-1},$$

where the sum is over all integral ideals $\mathcal{A}\neq 0$, resp. the product is over all prime ideals $\mathcal{P}\neq 0$ of the maximal order. The Dedekind zeta-function satisfies the following functional equation

$$\left(\frac{\sqrt{|D|}}{2^{r_2}\pi^{\frac{n}{2}}}\right)^s\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_{\mathbb{K}}(s)$$
$$=\omega\left(\frac{\sqrt{|D|}}{2^{r_2}\pi^{\frac{n}{2}}}\right)^{1-s}\Gamma\left(\frac{1-s}{2}\right)^{r_1}\Gamma(1-s)^{r_2}\zeta_{\mathbb{K}}(1-s), \tag{1}$$

where ω is a constant with $|\omega|=1$, D is the discriminant of $\mathbb{K}$, r_1 is the number of real conjugates and $2r_2$ is the number of complex conjugates of α; note that $n=r_1+2r_2$. Thus, $\zeta_{\mathbb{K}}(s)$ has an analytic continuation except for a simple pole at $s=1$.

We are interested in the zero distribution of the derivative of Dedekind zeta-functions. In [1] Hinz proved that the number $N_m(T)$ of zeros $\rho_m = \beta_m + i\gamma_m$ of $\zeta_{\mathbb{K}}^{(m)}(s)$ with $0 < \gamma_m \leqslant T$ is asymptotically

$$N_m(T) = \frac{nT}{2\pi}\log T + \frac{T}{2\pi}\log\frac{D}{2\pi e M} + O_m(\log T),$$

where M is the smallest norm of a proper integral ideal.

Our aim is to investigate the number and location of the zeros of the derivative Dedekind zeta-functions of real-quadratic number fields in the left half of the critical strip; our results can easily be extended to higher degrees.

2. Statement of results

Let $\mathbb{K}$ be a real-quadratic number field over $\mathbb{Q}$, i.e., there exists a squarefree positive integer d such that $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. One can show that also $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, where

$$D = \begin{cases} d, & \text{if } d \equiv 1 \bmod 4, \\ 4d, & \text{if } d \equiv 2, 3 \bmod 4, \end{cases}$$

is the discriminant of $\mathbb{K}$. In this case the Dedekind zeta-function has a representation

$$\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi_D),$$

where χ_D is a real primitive Dirichlet character modulo D, defined by

$$\chi_D(2) = \begin{cases} 0, & \text{if } D \equiv 0 \bmod 4, \\ +1, & \text{if } D \equiv 1 \bmod 8, \\ -1, & \text{if } D \equiv 5 \bmod 8, \end{cases}$$

and $\chi_D(p) = \left(\frac{D}{p}\right)$ for primes $p \neq 2$, where $\left(\frac{D}{p}\right)$ is the Legendre symbol. From this and the functional equations for the Riemann zeta-function and for the appearing Dirichlet L-function one can deduce that $\zeta_{\mathbb{K}}(s)$ satisfies the functional equation

$$\zeta_{\mathbb{K}}(1-s) = i^{\delta} D^{\frac{s-1}{2}} \tau(\chi_D) 4^{1-s} \pi^{-2s} \cos\frac{\pi}{2}(s-\delta) \cos\frac{\pi s}{2} \Gamma^2(s) \zeta_{\mathbb{K}}(s), \tag{2}$$

where $\delta = \frac{1}{2}(1 - \chi_D(-1))$ and $\tau(\chi_D)$ is the Gauss sum associated to χ_D; note that this coincides with the more general identity (1) in case of real-quadratic number fields.

We shall prove

Theorem 1. *Assume the truth of the Riemann hypothesis for $\zeta_{\mathbb{K}}(s)$ (i.e., $\zeta_{\mathbb{K}}(s) \neq 0$ for $\sigma > \frac{1}{2}$). If $\chi_D(-1) = 1$ and $D > 46367$, then $\zeta'_{\mathbb{K}}(s)$ has exactly one simple zero ρ' in the strip $0 \leqslant \operatorname{Re}\rho' < \frac{1}{2}$, located at $\frac{1}{\log D} + O\left(\frac{\log\log D}{\log^2 D}\right)$. If $\chi_D(-1) = -1$ and $D > 2003$, then $\zeta'_{\mathbb{K}}(s)$ has no zeros in the left of the critical strip.*

3. Proof of the theorem

Let us define

$$\xi_{\mathbb{K}}(s) = s(s-1)D^{\frac{s+\delta}{2}}\pi^{-s-\frac{\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right)\Gamma\left(\frac{s}{2}\right)\zeta_{\mathbb{K}}(s).$$

Then $\xi_{\mathbb{K}}(s)$ is an entire function of oder 1, which vanishes exactly at the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_{\mathbb{K}}(s)$, but does not vanish at $s = 0$ and $s = 1$. By Hadamard's factorization theorem we have

$$\xi_{\mathbb{K}}(s) = \mathrm{e}^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)\mathrm{e}^{\frac{s}{\rho}},$$

where A, B are certain constants. Logarithmic differentiation gives us

$$\frac{\xi'_{\mathbb{K}}}{\xi_{\mathbb{K}}}(s) = B + \sum_{\rho}\left(\frac{1}{\rho}+\frac{1}{s-\rho}\right). \tag{3}$$

On the other hand, by definition of $\xi_{\mathbb{K}}(s)$ we get

$$\frac{\xi'_{\mathbb{K}}}{\xi_{\mathbb{K}}}(s) = \frac{1}{s}+\frac{1}{s-1}+\log\frac{\sqrt{D}}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s). \tag{4}$$

By the functional equation (2), we see

$$\mathrm{Re}\, B = -\sum_{\rho}\mathrm{Re}\,\frac{1}{\rho}.$$

Thus, taking real parts in (3) and (4) we find

$$\begin{aligned}\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) &= \sum_{\rho}\frac{\sigma-\beta}{|s-\rho|^2}-\frac{\sigma}{|s|^2}-\frac{\sigma-1}{|s-1|^2}-\log\frac{\sqrt{D}}{\pi}\\ &\quad -\frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)-\frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right).\end{aligned}$$

Assume that the Riemann hypothesis for $\zeta_{\mathbb{K}}(s)$ is true, then all complex zeros lie on the critical line $\sigma = \frac{1}{2}$. Hence,

$$\begin{aligned}\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) &= \sum_{\rho}\frac{\sigma-\frac{1}{2}}{|s-\rho|^2}-\frac{\sigma}{|s|^2}-\frac{\sigma-1}{|s-1|^2}-\log\frac{\sqrt{D}}{\pi}\\ &\quad -\frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)-\frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right).\end{aligned}$$

Now take a T which is not an imaginary part of a zero of $\zeta_{\mathbb{K}}(s)$ and form the contour $\mathcal{L}$ by a rectangle with corners $\pm iT$, $\frac{1}{2} \pm iT$, making small left-semicircular indentations around the zeros on the critical line and around the origin.

In the case $\delta = 0$ we have

$$\operatorname{Re} \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \sum_{\rho} \frac{\sigma - \frac{1}{2}}{|s-\rho|^2} - \frac{\sigma}{|s|^2} - \frac{\sigma - 1}{|s-1|^2} - \log \frac{\sqrt{D}}{\pi} - \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right). \tag{5}$$

Since

$$\frac{\Gamma'}{\Gamma}(z) = \log z + \mathrm{O}\left(\frac{1}{|z|}\right), \tag{6}$$

we see that $\operatorname{Re} \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ on the horizontal edges of the contour for T large enough. On the small left-semicircular indentations around the non-trivial zeros ρ the first term on the right hand-side of (5) produces arbitrary large negative values with decreasing radius. For $s = \frac{1}{2} + it$ we get $\operatorname{Re} \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$, if

$$-\log \frac{\sqrt{D}}{\pi} - \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + i\frac{t}{2}\right) < 0.$$

From (6) it follows that

$$\operatorname{Re} \frac{\Gamma'}{\Gamma}(z) \geqslant \frac{\Gamma'}{\Gamma}(\operatorname{Re} z).$$

Hence, we have that $\operatorname{Re} \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ on the right of $\mathcal{L}$ if

$$D > \pi^2 \mathrm{e}^{-2\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right)} = 46367,76\ldots. \tag{7}$$

On the small left-semicircle around the origin we have for $s = \varepsilon e^{i\varphi}$, $\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$,

$$-\frac{\sigma}{|s|^2} = -\frac{\cos\varphi}{\varepsilon}, \quad -\frac{1}{2}\operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = \frac{\cos\varphi}{\varepsilon} + \frac{1}{2}C + \mathrm{O}(\varepsilon),$$

where C is the Euler-Mascheroni constant. Thus, the sum of the second, third and fifth term of (5) produces large negative values as $\varepsilon \to 0$. For $s = it$ we obtain

$$\operatorname{Re} \frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = -\frac{1}{2} \sum_{\rho} \frac{1}{|s-\rho|^2} + \frac{1}{|s-1|^2} - \log \frac{\sqrt{D}}{\pi} - \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{it}{2}\right).$$

Hence, the second term of (5) is less than 1 and

$$\operatorname{Re} \frac{\Gamma'}{\Gamma}(it) \geqslant \lim_{t \to 0} \operatorname{Re} \frac{\Gamma'}{\Gamma}(it) = -C,$$

we get $\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ on the left of $\mathcal{L}$, if

$$D > \pi^2 e^{2(1+C)} = 231,24\ldots.$$

With regard to (7) we obtain $\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ for $D > 46367$ on the whole contour $\mathcal{L}$. Since $\zeta_{\mathbb{K}}(s)$ has a zero at $s = 0$, by the argument principal there must be exactly one zero of $\zeta'_{\mathbb{K}}(s)$ inside the contour.

Differentiation of (2) with respect to s yields

$$\zeta'_{\mathbb{K}}(1-s) = 4\tau(\chi_D)D^{-1/2}\left\{\left(\frac{D}{4\pi^2}\right)^s \cos\frac{\pi s}{2}\Gamma(s)\left\{\log\frac{D}{4\pi^2}\cos\frac{\pi s}{2}\Gamma(s)\right.\right.$$
$$\left.\left.-\pi\sin\frac{\pi s}{2}\Gamma(s) + 2\cos\frac{\pi s}{2}\Gamma'(s)\right\}\zeta_{\mathbb{K}}(s) + \left(\frac{D}{4\pi^2}\right)^s \cos^2\frac{\pi s}{2}\Gamma(s)^2\zeta'_{\mathbb{K}}(s)\right\}.$$

Now assume that $\zeta'_{\mathbb{K}}(1-s) = 0$ for $|s-1| \ll \frac{1}{\log D}$, i.e., $\zeta'_{\mathbb{K}}(s)$ has a zero close to the origin, then

$$0=\left\{\log\frac{D}{4\pi^2}\cos\frac{\pi s}{2}\Gamma(s)-\pi\sin\frac{\pi s}{2}\Gamma(s)+2\cos\frac{\pi s}{2}\Gamma'(s)\right\}\zeta_{\mathbb{K}}(s)+\cos\frac{\pi s}{2}\Gamma(s)\zeta'_{\mathbb{K}}(s),$$

and

$$\frac{\zeta'}{\zeta}(s) + \frac{L'}{L}(s,\chi_D) = -\log\frac{D}{4\pi^2} + \pi\tan\frac{\pi s}{2} - 2\frac{\Gamma'}{\Gamma}(s).$$

Here we note that, for $|s-1| \ll \frac{1}{\log D}$, we have

$$\frac{\Gamma'}{\Gamma}(s) \ll 1, \quad \tan\frac{\pi s}{2} = \frac{1+O(|s-1|^2)}{-\frac{\pi(s-1)}{2}+O(|s-1|^3)}, \quad \frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1).$$

By the Riemann hypothesis, one has (see Lemma 2 of [3])

$$\frac{L'}{L}(s,\chi_D) \ll \log\log D \quad \text{for} \quad |s-1| \ll \frac{1}{\log D},$$

as $D \to \infty$. Therefore, we obtain

$$s = 1 - \frac{1}{\log D} + O\left(\frac{\log\log D}{\log^2 D}\right).$$

This shows that $\zeta'_{\mathbb{K}}(s) = 0$ for $|s| \ll \frac{1}{\log D}$ is satisfied with

$$s = \frac{1}{\log D} + O\left(\frac{\log\log D}{\log^2 D}\right).$$

In the case $\delta = 1$ we have

$$\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \sum_{\rho} \frac{\sigma - \frac{1}{2}}{|s-\rho|^2} - \frac{\sigma}{|s|^2} - \frac{\sigma - 1}{|s-1|^2} - \log\frac{\sqrt{D}}{\pi}$$
$$-\frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right) - \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right).$$

On the horizontal sides of the contour and on the left-semicircular indentations around the non-trivial zeros of $\zeta_{\mathbb{K}}(s)$ and around the origin we can argue in a similar manner as before. For $\sigma = \frac{1}{2}$ the inequality $\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ holds if

$$-\log\frac{\sqrt{D}}{\pi} - \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{3}{4} + i\frac{t}{2}\right) - \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + i\frac{t}{2}\right) < 0.$$

In view of (6) it follows that $\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ if

$$D > \pi^2 e^{-\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right)\right)} = 2003,73\ldots. \tag{8}$$

On the left side of the contour $\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ if

$$\frac{1}{|s-1|^2} - \log\frac{\sqrt{D}}{\pi} - \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + i\frac{t}{2}\right) - \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + i\frac{t}{2}\right) < 0.$$

For $|t| \geqslant 3$, $-\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + i\frac{t}{2}\right) < 0$, and for $|t| > 3$, $\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + i\frac{t}{2}\right) \geqslant \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)$. Remembering (6), we have $\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ on the left of the contour, if

$$D > \pi^2 e^{-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) + C - 1} = 46,06\ldots.$$

So, taking into account (8), we guarantee $\mathrm{Re}\,\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) < 0$ throughout the contour taking $D > 2003$. On and inside of the contour $\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)$ is analytic. Thus $\zeta'_{\mathbb{K}}(s)$ has no zeros in $0 \leqslant \sigma < \frac{1}{2}$.

References

[1] J. Hinz, Über Nullstellen der m-ten Ableitung der Dedekindschen Zetafunktion, *Journal of Number Theory*, **9**, 535–560 (1977).
[2] A. Speiser, Geometrisches zur Riemannschen Zetafunktion, *Math. Annalen*, **110**, 514–521 (1934).
[3] C.Y. Yildirim, Zeros of derivatives of Dirichlet L-functions, *Tr. J. of Mathematics*, **20**, 521–534 (1996).

Apie Dedekindo dzeta funkcijos išvestinės nulius

R. Šleževičienė

Straipsnyje nagrinėjamas Dedekindo dzeta funkcijos išvestinės nulių pasiskirstymas juostoje $0 \leqslant \mathrm{Re}\,s < \frac{1}{2}$ realiųjų kvadratinių skaičių kūnų atveju.