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The distributions of additive functions with finite supports

Jonas ŠIAULYS (VU)
e-mail: jonas.siaulys@maf.vu.lt

1. Introduction

Let f_x be a set ($x \geqslant 2$) of integer-valued strongly additive functions (s. a. f) and $f_x(p) \in \{0,1\}$ for each prime number p. Let

$$\nu_x\bigl(f_x(m) < u\bigr) = [x]^{-1}\#\{m \leqslant x,\ f_x(m) < u\}$$

be the distribution function of a s. a. f. f_x from this set. The distributions $\nu_x(f_x(m) < u)$ we will call having finite support if

$$\lim_{x\to\infty} \nu_x\bigl(f_x(m) > c\bigr) = 0,$$

for some constant c.

In this work we will show that finite support of distributions $\nu_x(f_x(m) < u)$ separate the values $f_x(p)$ for small primes p ($p \leqslant \text{const}$) from the values $f_x(p)$ for large primes p ($p \geqslant x^\alpha$).

Theorem. *Let f_x, $x \geqslant 2$, be a set of s. a. f., $f_x(p) \in \{0,1\}$ for each prime number. The next two conditions are equivalent.*

(a) *It exists a constant c such that*

$$\lim_{x\to\infty} \nu_x\bigl(f_x(m) > c\bigr) = 0.$$

(b) *There exists constants D and $\alpha \in (0,1]$ for which*

$$\lim_{x\to\infty} \sum_{\substack{D<p\leqslant x^\alpha\\ f_x(p)=1}} \frac{1}{p} = 0.$$

2. Preliminary results

Lemma 1 [1]. *Let $h(m)$ be an arbitrary real-valued additive function. There is an absolute constant c_1 such that*

$$\sum_{\substack{m\leqslant x\\ h(m)=a}} 1 \leqslant c_1 x\Big(\sum_{\substack{p\leqslant x\\ h(p)\neq 0}} \frac{1}{p}\Big)^{-1/2}.$$

Lemma 2 [2]. *Let f_x be a set of s. a. f. and $f_x(p)\in\{0,1\}$ for each prime number p. Let $\hat{x}_n$ be an arbitrary unbounded increasing sequence. The distribution $\nu_{\hat{x}_n}(f_{\hat{x}_n}(m)<u)$ converges weakly if and only if the limits*

$$\lim_{n\to\infty} \sum_{p_1\leqslant\hat{x}_n}{}^{*} \sum_{\substack{p_2\leqslant\hat{x}_n\\ p_2\neq p_1}}{}^{*} \cdots \sum_{\substack{p_{l-1}\leqslant\hat{x}_n\\ p_{l-1}\neq p_1,p_2,\ldots,p_{l-2}}}{}^{*} \sum_{\substack{p_l\leqslant\hat{x}_n\\ p_1p_2\cdots p_l\leqslant\hat{x}_n\\ p_l\neq p_1,p_2,\ldots,p_{l-1}}}{}^{*} \frac{1}{p_1p_2\ldots p_l} = g_l$$

exist for each natural number l. Here the superscript * over the sign of sum means that the summation is expanded over primes for which $f_{\hat{x}_n}(p)=1$. Moreover, the limiting distribution has characteristic function

$$1+\sum_{l=1}^{\infty}\frac{g_l}{l!}\left(e^{it}-1\right)^l.$$

3. Proof of Theorem

I. Suppose that condition (a) is satisfied. Let x_n be unbounded increasing sequence. There exists a subsequence $\hat{x}_n$ such that $\nu_{\hat{x}_n}(f_{\hat{x}_n}(m)<u)$ converges weakly to some distribution function $F(u)$. From Lemma 2 we obtain that limits

$$\lim_{n\to\infty} \sum_{p_1\leqslant\hat{x}_n}{}^{*} \sum_{\substack{p_2\leqslant\hat{x}_n\\ p_2\neq p_1}}{}^{*} \cdots \sum_{\substack{p_{l-1}\leqslant\hat{x}_n\\ p_{l-1}\neq p_1,p_2,\ldots,p_{l-2}}}{}^{*} \sum_{\substack{p_l\leqslant\hat{x}_n\\ p_1p_2\cdots p_l\leqslant\hat{x}_n\\ p_l\neq p_1,p_2,\ldots,p_{l-1}}}{}^{*} \frac{1}{p_1p_2\ldots p_l} = \varphi_l$$

exist for each $l=1,2,\ldots$, and $F(u)$ has the characteristic function

$$1+\sum_{l=1}^{\infty}\frac{\varphi_l}{l!}\left(e^{it}-1\right)^l.$$

It follows from condition (a) that for $L=[c]$

$$\varphi_{L+1}=\varphi_{L+2}=\ldots=0.$$

Hence

$$\lim_{n\to\infty} \sum_{p_1\leqslant \hat{x}_n}^{*} \sum_{\substack{p_2\leqslant \hat{x}_n\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_L\leqslant \hat{x}_n\\ p_L\neq p_1,p_2,\dots,p_{L-1}}}^{*} \sum_{\substack{p_{L+1}\leqslant \hat{x}_n\\ p_1p_2\cdots p_Lp_{L+1}\leqslant \hat{x}_n\\ p_{L+1}\neq p_1,p_2,\dots,p_L}}^{*} \frac{1}{p_1p_2\cdots p_{L+1}} = 0.$$

Since x_n is an arbitrary unbounded increasing sequence, we obtain

$$\lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_L\leqslant x\\ p_L\neq p_1,p_2,\dots,p_{L-1}}}^{*} \sum_{\substack{p_{L+1}\leqslant x\\ p_1p_2\cdots p_{L+1}\leqslant x\\ p_{L+1}\neq p_1,p_2,\dots,p_L}}^{*} \frac{1}{p_1p_2\cdots p_{L+1}} = 0. \tag{1}$$

Here and later the superscript * over the sign of sum means that the sumation is expanded over primes for which $f_x(p) = 1$.

Let now d be natural number and

$$a_d = \limsup_{x\to\infty} \#\{p\leqslant d:\ f_x(p) = 1\}.$$

If $d^{L+1}\leqslant x$, we have

$$\sum_{p_1\leqslant d}^{*} \sum_{\substack{p_2\leqslant d\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_{L+1}\leqslant d\\ p_{L+1}\neq p_1,p_2,\dots,p_L}}^{*} \frac{1}{p_1p_2\cdots p_{L+1}}$$
$$\leqslant \sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_L\leqslant x\\ p_L\neq p_1,p_2,\dots,p_{L-1}}}^{*} \sum_{\substack{p_{L+1}\leqslant x\\ p_1p_2\cdots p_{L+1}\leqslant x\\ p_{L+1}\neq p_1,p_2,\dots,p_L}}^{*} \frac{1}{p_1p_2\cdots p_L}.$$

Hence from (1) we obtain

$$\lim_{x\to\infty} \sum_{p_1\leqslant d}^{*} \sum_{\substack{p_2\leqslant d\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_{L+1}\leqslant d\\ p_{L+1}\neq p_1,p_2,\dots,p_L}}^{*} \frac{1}{p_1p_2\cdots p_{L+1}} = 0.$$

It follows from the last equality that $a_d \leqslant L$. The sequence a_d is not decreasing and bounded. Consequently, it exists

$$\lim_{d\to\infty} a_d = a^*.$$

Since the sequence a_d is integer-valued, there exists a natural number D for which $a_d = a_D = a^*$ if $d\geqslant D$. Hence for each $d\geqslant D$ we have

$$\limsup_{x\to\infty} \#\{p\leqslant d:\ f_x(p) = 1\} = \limsup_{x\to\infty} \#\{p\leqslant D:\ f_x(p) = 1\} \leqslant L.$$

Using the last equality we obtain that

$$\lim_{x\to\infty} f_x(p) = 0$$

for each fixed prime number $p > D$. Thus

$$\lim_{x\to\infty} \max_{\substack{D<p\leqslant x\\ f_x(p)=1}} \frac{1}{p} = 0. \tag{2}$$

Let as above x_n is the unbounded increasing sequence. There exists a subsequence $\hat{x}_n$ such that $\nu_{\hat{x}_n}(f_{\hat{x}_n}(m) < u)$ converge weakly. According to the Lemma 1

$$\nu_{\hat{x}_n}(f_{\hat{x}_n}(m) = l) \ll \Big(\sum_{\substack{p\leqslant \hat{x}_n\\ f_{\hat{x}_n}(p)=1}} \frac{1}{p}\Big)^{-1/2}$$

for $l = 0, 1, \ldots, L$.

From (a) follows the existence of l^* for which

$$\lim_{n\to\infty} \nu_{\hat{x}_n}\big(f_{\hat{x}_n}(m) = l^*\big) \geqslant \frac{1}{L+1}.$$

Consequently,

$$\limsup_{n\to\infty} \sum_{\substack{p\leqslant \hat{x}_n\\ f_{\hat{x}_n}(p)=1}} \frac{1}{p} \ll (L+1)^2.$$

Since x_n is an arbitrary unbounded increasing sequence, then the last inequality implies

$$\limsup_{x\to\infty} \sum_{p\leqslant x}{}^{*} \frac{1}{p} \ll (L+1)^2. \tag{3}$$

Using the equality (2) and the estimation (3) we have

$$\limsup_{x\to\infty} \sum_{D<p_1\leqslant x^{1/L+1}}{}^{*} \cdots \sum_{D<p_i\leqslant x^{1/L+1}}{}^{*} \cdots \sum_{\substack{D<p_j\leqslant x^{1/L+1}\\ p_j=p_i}}{}^{*} \cdots \sum_{D<p_{L+1}\leqslant x^{1/L+1}}{}^{*} \frac{1}{p_1p_2\cdots p_{L+1}}$$

$$\leqslant \limsup_{x\to\infty} \max_{\substack{D<p\leqslant x\\ f_x(p)=1}} \frac{1}{p} \Big(\sum_{p\leqslant x}{}^{*} \frac{1}{p}\Big)^{L} = 0$$

for every pair i, j, $1 \leqslant i < j \leqslant L+1$. Thus we get that

$$\limsup_{x\to\infty} \Big(\sum_{D<p\leqslant x^{1/L+1}}{}^{*} \frac{1}{p}\Big)^{L+1}$$

$$\leqslant \limsup_{x\to\infty} \sideset{}{^*}\sum_{p_1\leqslant x^{1/L+1}} \sideset{}{^*}\sum_{\substack{p_2\leqslant x^{1/L+1}\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_{L+1}\leqslant x^{1/L+1}\\ p_{L+1}\neq p_1,p_2,\ldots,p_L}} \frac{1}{p_1p_2\cdots p_{L+1}}$$

$$\leqslant \limsup_{x\to\infty} \sideset{}{^*}\sum_{p_1\leqslant x} \sideset{}{^*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_{L+1}\leqslant x\\ p_{L+1}\neq p_1,p_2,\ldots,p_L\\ p_1p_2\cdots p_{L+1}\leqslant x}} \frac{1}{p_1p_2\cdots p_{L+1}}.$$

The condition (b) follows now from the last estimation and equality (1).

II. Suppose now that condition (b) holds. Let l be a natural number. For large x ($D^l \leqslant x$) we have

$$\frac{1}{x}\sum_{\substack{m\leqslant x\\ f_x(m)>l}} 1 \leqslant \frac{1}{l!}\sum_{m\leqslant x} f_x(m)\big(f_x(m)-1\big)\ldots\big(f_x(m)-l+1\big)$$

$$= \frac{1}{l!}\sideset{}{^*}\sum_{p_1\leqslant x} \sideset{}{^*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_{l-1}\leqslant x\\ p_{l-1}\neq p_1,p_2,\ldots,p_{l-2}}} \sideset{}{^*}\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\ldots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}} \frac{1}{p_1p_2\cdots p_l}$$

$$\ll_l \sum_{j=1}^{l}\Big(\sideset{}{^*}\sum_{D<p\leqslant x^\alpha}\frac{1}{p}\Big)^j\Big(\max\Big(\sideset{}{^*}\sum_{p\leqslant D}\frac{1}{p},\ \sideset{}{^*}\sum_{x^\alpha<p\leqslant x}\frac{1}{p}\Big)\Big)^{l-j}$$

$$+\sum_{j=0}^{l}\sideset{}{^*}\sum_{p_1\leqslant D}\sideset{}{^*}\sum_{\substack{p_2\leqslant D\\ p_2\neq p_1}}\cdots\sideset{}{^*}\sum_{\substack{p_j\leqslant D\\ p_j\neq p_1,p_2,\ldots,p_{j-1}}}\frac{1}{p_1p_2\cdots p_j}$$

$$\times \sideset{}{^*}\sum_{x^\alpha<p_{j+1}\leqslant x}\cdots\sideset{}{^*}\sum_{\substack{x^\alpha<p_l\leqslant x\\ p_1p_2\cdots p_l\leqslant x}}\frac{1}{p_{j+1}\cdots p_l}.$$

Since

$$\sideset{}{^*}\sum_{p\leqslant D}\frac{1}{p} \ll \ln\ln D, \qquad \sideset{}{^*}\sum_{x^\alpha<p\leqslant x}\frac{1}{p} \ll \ln\frac{1}{\alpha},$$

then the last estimation and condition (b) imply that

$$\limsup_{x\to\infty}\nu_x\big(f_x(m)>l\big)$$

$$\ll_l \sum_{j=0}^{l}\limsup_{x\to\infty}\sideset{}{^*}\sum_{p_1\leqslant D}\sideset{}{^*}\sum_{\substack{p_2\leqslant D\\ p_2\neq p_1}}\cdots\sideset{}{^*}\sum_{\substack{p_j\leqslant D\\ p_j\neq p_1,p_2,\ldots,p_{j-1}}}\frac{1}{p_1p_2\cdots p_j}$$

$$\times \sideset{}{^*}\sum_{x^\alpha<p_{j+1}\leqslant x}\cdots\sideset{}{^*}\sum_{\substack{x^\alpha<p_l\leqslant x\\ p_1p_2\cdots p_l\leqslant x}}\frac{1}{p_{j+1}\cdots p_l}.$$

If $l = 2\big(\max(\pi(D), [\frac{1}{\alpha}]) + 2\big)$ all terms of the last sum are equal zero. Hence the condition (a) holds with $c = 2\big(\max(\pi(D), [\frac{1}{\alpha}]) + 2\big)$. This proves the theorem.

References

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Adityviųjų funkcijų skirstiniai su baigtinėmis atramomis

J. Šiaulys

Darbe tiriama, kokią įtaką skirstinių $\nu_x(f_x(m) < u)$ koncentracija baigtinėje gardelėje turi stipriai adityviųjų funkcijų šeimos f_x reikšmėms pirminiuose skaičiuose. Nagrinėjamos tos stipriai adityviosios funkcijos, kurioms $f_x(p) \in \{0, 1\}$ visiems pirminiams skaičiams p.