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On the influence of the arithmetical character of the parameters for the Lerch zeta-function

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1. Introduction

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, $i^2 = -1$, be a complex variable. In 1887 M. Lerch introduced the function, defined on the half-plane $\sigma > 1$ by the following absolutely convergent Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}.$$

The function $L(\lambda, \alpha, s)$ depends on two fixed parameters $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$. Without loss of generality we consider $0 < \alpha \leqslant 1$. The investigations of this function in different aspects show that its properties are strongly effected by the arithmetical character of the parameters λ and α. In this paper we restrict ourselves to the discrete value-distribution of $L(\lambda, \alpha, s)$.

Let $h \in \mathbb{R}_+$ be a fixed number, and $k, N \in \mathbb{N} \cup \{0\}$. We consider probability measures of the following form

$$\mu_N(\ldots) = \frac{1}{N+1} \# \{k \in [0, N]: \ldots\},$$

where in place of dots some condition satisfied k is to be written. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Let $D = \{s \in \mathbb{C}: \sigma > 1/2\}$. We write $H(D)$ for the space of analytic on D functions equipped with the topology of uniform convergence on compacta. The notation $M(D)$ means the space of meromorphic on D functions equipped with the topology as was stated above. Consider the following three probability measures:

$$\begin{aligned}
P_{1N}(A) &= \mu_N\left(L(\lambda, \alpha, \sigma + ikh) \in A\right), \quad A \in B(\mathbb{C}), \quad \lambda \notin \mathbb{Z},\\
P_{2N}(A) &= \mu_N(L(\lambda, \alpha, s + ikh) \in A), \quad A \in B(H(D)), \quad \lambda \notin \mathbb{Z},\\
P_{3N}(A) &= \mu_N(L(\lambda, \alpha, s + ikh) \in A), \quad A \in B(M(D)), \quad \lambda \in \mathbb{Z}.
\end{aligned}$$

In recent years the discrete value-distribution for the Lerch zeta-function with a transcendental parameter α was studied by the author. The main results are as follows.

Define γ to be the unit circle on $\mathbb{C}$, and

$$\Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$, $m = 0, 1, 2, \ldots$. There exists the probability Haar measure m_{1H} on $(\Omega_1, \mathcal{B}(\Omega_1))$. Let $\omega_1(m)$ be the projection of $\omega_1 \in \Omega_1$ to γ_m. Define the $\mathbb{C}$-valued random element on $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$ by

$$L(\lambda, \alpha, \sigma, \omega_1) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m} \omega_1(m)}{(m+\alpha)^{\sigma}}, \quad \sigma \in D, \quad \omega_1 \in \Omega_1.$$

Theorem 1. *Suppose that $\lambda \notin \mathbb{Z}$, α is a transcendental number, and* $\exp\{2\pi/h\}$ *is a rational number. Then the probability measure P_{1N} converges weakly to the distribution of $L(\lambda, \alpha, \sigma, \omega_1)$ as $N \to \infty$.*

For the proof see [2].

We obtain an $H(D)$-valued random element on $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$ for s in place of σ in $L(\lambda, \alpha, \sigma, \omega_1)$.

Theorem 2. *Under the assumptions of Theorem* 1 *the probability measure P_{2N} converges weakly to the distribution of $L(\lambda, \alpha, s, \omega_1)$ as $N \to \infty$.*

For the proof see [1] and the references given there.

Theorem 3. *Suppose that $\lambda \in \mathbb{Z}$, α is a transcendental number, and* $\exp\{2\pi/h\}$ *is a rational number. Then the probability measure P_{3N} converges weakly to the distribution of $L(\lambda, \alpha, s, \omega_1)$ as $N \to \infty$.*

See [3] for more details.

For a rational parameter α the results are as follows.

Theorem 1′. *Suppose that $\lambda \notin \mathbb{Z}$, $\alpha \in \mathbb{Q}$, and* $\exp\{2\pi k/h\}$, *$k \in \mathbb{Z}$, $k \neq 0$, is an irrational number. Then there exists a probability measure P_1 on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that P_{1N} converges weakly to P_1 as $N \to \infty$.*

Theorem 2′. *Under the assumptions of Theorem* 1′ *there exists a probability measure P_2 on $(H(D), \mathcal{B}(H(D)))$ such that P_{2N} converges weakly to P_2 as $N \to \infty$.*

Theorem 3′. *Suppose that $\lambda \in \mathbb{Z}$, $\alpha \in \mathbb{Q}$, and* $\exp\{2\pi k/h\}$, *$k \in \mathbb{Z}$, $k \neq 0$, is an irrational number. Then there exists a probability measure P_3 on $(H(D), \mathcal{B}(H(D)))$ such that P_{3N} converges weakly to P_3 as $N \to \infty$.*

The explicit form of the limit measure is obtained for $\alpha = a/b, 1 \leqslant a \leqslant b, (a,b) = 1$. Set

$$\Omega_2 = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all prime numbers p. There exists the probability Haar measure m_{2H} on $(\Omega_2, \mathcal{B}(\Omega_2))$. Let $\omega_2(p)$ stand for the projection of $\omega_2 \in \Omega_2$ to γ_p. For $m \in \mathbb{N} \cup \{0\}$, set

$$\omega(m) = \prod_{p^\alpha || m} \omega^\alpha(p), \quad \alpha \in \mathbb{N}.$$

Define the $\mathbb{C}$-valued random element on $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$ by

$$L(\lambda, \alpha, \sigma, \omega_2) = \omega_2(b) b^\sigma \mathrm{e}^{-2\pi i \lambda a/b} \sum_{\substack{m=1 \\ m \equiv a (\mathrm{mod}\ b)}}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m/b} \omega_2(m)}{m^\sigma}, \quad \sigma \in D, \quad \omega_2 \in \Omega_2.$$

Theorem 1″. *Suppose that* $\lambda \notin \mathbb{Z}$, $\mathbb{Q} \ni \alpha = a/b$, $1 \leqslant a \leqslant b$, $(a,b) = 1$, *and* $\exp\{2\pi k/h\}$, $k \in \mathbb{Z}$, $k \neq 0$, *is an irrational number. Then the probability measure* P_{1N} *converges weakly to the distribution of* $L(\lambda, \alpha, \sigma, \omega_2)$ *as* $N \to \infty$.

It follows easily that for s in place of σ in $L(\lambda, \alpha, \sigma, \omega_2)$ we obtain an $H(D)$-valued random element $L(\lambda, \alpha, s, \omega_2)$ on $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$.

Theorem 2″. *Under the assumptions of Theorem* 1″ *the probability measure* P_{2N} *converges weakly to the distribution of* $L(\lambda, \alpha, s, \omega_2)$ *as* $N \to \infty$.

Theorem 3″. *Suppose that* $\lambda \in \mathbb{Z}$, $\mathbb{Q} \ni \alpha = a/b$, $1 \leqslant a \leqslant b$, $(a,b) = 1$, *and* $\exp\{2\pi k/h\}$, $k \in \mathbb{Z}$, $k \neq 0$, *is an irrational number. Then the probability measure* P_{3N} *converges weakly to the distribution of* $L(\lambda, \alpha, s, \omega_2)$ *as* $N \to \infty$.

In this article we present the proof of Theorem 3″. The other results in the case of a rational parameter α may be proved in much the same way.

2. Proof of Theorem 3″

For $\lambda \in \mathbb{Z}$ the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function

$$\zeta(\alpha, s) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

This function is analytically continuable over the whole complex plane except for a simple pole at the point $s = 1$ with the residue 1. We may interpret the function $\zeta(\alpha, s)$ as

$$\zeta(\alpha, s) = \frac{f_2(\alpha, s)}{f_1(s)},$$

where

$$f_1(s) = 1 - 2^{1-s},$$
$$f_2(\alpha, s) = \zeta(\alpha, s) f_1(s).$$

Moreover, for $\mathbb{Q} \ni \alpha = a/b$, $1 \leqslant a \leqslant b$, $(a, b) = 1$, the function $f_2(\alpha, s)$ can be interpreted as a product of entire functions, i.e.,

$$\begin{aligned} f_2(\alpha, s) &= f_1(s) \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s} = f_1(s) \sum_{m=0}^{\infty} \frac{1}{(m+a/b)^s} \\ &= f_1(s) b^s \sum_{m=0}^{\infty} \frac{1}{(mb+a)^s} = f_1(s) b^s \sum_{\substack{m=1 \\ m \equiv a (\mathrm{mod}\ b)}}^{\infty} \frac{1}{m^s} \stackrel{def}{=} f_1(s) g_1(s) g_2(s), \end{aligned}$$

where

$$g_1(s) = b^s, \quad g_2(s) = \sum_{\substack{m=1 \\ m \equiv a (\mathrm{mod}\ b)}}^{\infty} \frac{1}{m^s}.$$

By the same method as in [1] it follows that the probability measures

$$\mu_N(f_1(s+ikh) \in A), \quad \mu_N(g_i(s+ikh) \in A), \quad A \in \mathcal{B}(H(D)), \quad i = 1, 2,$$

converge weakly to $P_{f_1}, P_{g_i}, i = 1, 2$, respectively, as $N \to \infty$, where

$$\begin{aligned} P_{f_1}(A) &= m_{2H}\left(\omega_2 \in \Omega_2\colon f_1(s, \omega_2) \in A\right), \\ P_{g_i}(A) &= m_{2H}\left(\omega_2 \in \Omega_2\colon g_i(s, \omega_2) \in A\right), \end{aligned}$$

$i = 1, 2$, $s \in D$, $A \in \mathcal{B}(H(D))$, and

$$\begin{aligned} f_1(s, \omega_2) &= (1 - 2^{1-s}) \omega_2(2), \\ g_1(s, \omega_2) &= b^s \omega_2(b), \quad g_2(s, \omega_2) = \sum_{\substack{m=1 \\ m \equiv a (\mathrm{mod}\ b)}}^{\infty} \frac{\omega_2(m)}{m^s}. \end{aligned}$$

The rest of the proof runs as in [3], p. 16–20, with the auxiliary function u: $H(D) \times H(D) \times H(D) \to H(D)$ defined by the formulae

$$u(h_1, h_2, h_3) = h_1 * h_2 * h_3, \quad h_1, h_2, h_3 \in H(D).$$

We obtain that

$$\begin{aligned}&\mu_N\left(f_2(\alpha, s+ikh) \in A\right)\\ &\quad = \mu_N\left(f_1(s+ikh)g_1(s+ikh)g_2(s+ikh) \in A\right), \quad A \in \mathcal{B}(H(D)),\end{aligned}$$

converges weakly to

$$P_{f_2}(A) = m_{2H}\left(\omega_2 \in \Omega_2\colon\ f_1(s,\omega_2)g_1(s,\omega_2)g_2(s,\omega_2) \in A\right),$$

$s \in D$, $\omega_2 \in \Omega_2$, $A \in \mathcal{B}(H(D))$, as $N \to \infty$.

Finally, applying the auxiliary function v: $H(D) \times H(D) \to M(D)$ defined by the formulae

$$v(h_1, h_2) = \frac{h_1}{h_2}, \quad h_1, h_2 \in H(D),$$

we obtain the assertion of the theorem.

References

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[2] J. Ignatavičiūtė, On statistic properties of the Lerch zeta-function, *Liet. Matem. Rink.*, **41**(4), 424–440 (2001) (in Russian).

[3] J. Ignatavičiūtė, On statistic properties of the Lerch zeta-function, II. *Liet. Matem. Rink.*, (to appear(2002)).

Parametrų aritmetinės prigimties įtaka Lercho dzeta funkcijai

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Įrodoma diskreti ribinė teorema Lercho dzeta funkcijai su parametrais $\alpha = a/b$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, $(a, b) = 1$, ir $\lambda \in \mathbb{Z}$ meromorfinių pusplokštumėje $\sigma > 1/2$ funkcijų erdvėje. Pateikiami analogiški rezultatai kompleksinėje plokštumoje bei analizinių pusplokštumėje $\sigma > 1/2$ funkcijų erdvėje.