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VALUE DISTRIBUTION OF LERCH AND PERIODIC HURWITZ ZETA-FUNCTIONS

Doctoral dissertation

Physical sciences, mathematics (01P)

Vilnius, 2018

Doctoral dissertation was written in 2013–2017 at Vilnius University

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VILNIAUS UNIVERSITETAS

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LERCHO IR PERIODINIŲ HURVICO DZETA FUNKCIJŲ REIKŠMIŲ PASISKIRSTYMAS

Daktaro disertacija

Fiziniai mokslai, matematika (01P)

Vilnius, 2018

Disertacija rengta 2013–2017 metais Vilniaus universitete.

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Notation

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{P}$	the set of primes
$\mathbb{C}$	the set of complex numbers
i	a complex number, satisfying $i^2 = -1$
$\lvert x \rvert$	the absolute value of x
$[x]$	the integer part of x
$\{x\}$	the fractional part of x
$f(x) = O(g(x))$	there exists a fixed $C > 0$ such that $\lvert f(x) \rvert \leqslant Cg(x)$ as $x \to \infty$
$f(x) \ll g(x)$	same as $f(x) = O(g(x))$
$f(x) = o(g(x))$	$\forall \varepsilon > 0$ exists N such that $\lvert f(x) \rvert \leqslant \varepsilon \lvert g(x) \rvert$ for all $x \geqslant N$
(r_n)	a sequence of numbers $r_n,\ n = 1, 2, 3, \ldots$
Γ	the Euler's gamma function
$\Re s$	Real part of complex number s
$\Im s$	Imaginary part of complex number s

1 Introduction

In this work the Lerch zeta-function, its derivative and the periodic Hurwitz zeta-function will be studied. Both functions are generalizations of the famous Riemann zeta-function [59] given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Here $s = \sigma + it$ denotes a complex number with $\sigma, t \in \mathbb{R}$ being real and imaginary parts respectively. This Dirichlet series converges absolutely for $\sigma > 1$. The Riemann zeta-function has a simple pole with residue 1 at $s = 1$ and is defined as the meromorphic continuation into the rest of the complex plane.

Let $0 < \lambda, \alpha \leqslant 1$ and denote by $r = (r_m)_{m=0}^{\infty}$, $r_m \in \mathbb{C}$, a periodic sequence with period k. Then, for $\sigma > 1$, the periodic Hurwitz zeta-function and the Lerch zeta-function are defined by the Dirichlet series

$$\zeta(s, \alpha; r) = \sum_{m=0}^{\infty} \frac{r_m}{(m+\alpha)^s} \quad \text{and} \quad L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$$

respectively.

In this dissertation, we investigate three related topics: the zero and a-value distribution of the periodic Hurwitz zeta-function; the zero distribution of the derivative of the Lerch zeta-function; the zero behavior of the Lerch zeta-function for the special case, when parameters are equal.

Chapter 1 contains major results attained in this study, together with the bibliographical data of our articles and the list of conferences and visits where thesis results were presented.

Literature review, found in Chapter 2, contains overview of main advancements related to the distribution of zeta-function zeros, motivation and main definitions.

Chapter 3 contains auxiliary results, used throughout this dissertation, together with their classical proofs.

Chapter 4 is dedicated for the distribution of the zeros of the periodic Hurwitz zeta-function. We find the asymptotic formula for the number of nontrivial zeros. Also, we explore nontrivial zeros distribution with respect to the critical line. Results were published in a joint paper with Ramūnas Garunkštis in [28]. In Chapter 5 these results were extended to a-values distribution of the periodic Hurwitz zeta-function.

In Chapter 6 the zero distribution of the derivative of the Lerch zeta-function is explored. We indicate zero-free regions; locate approximate positions of the trivial zeros; consider the asymptotic formula for the number of nontrivial zeros; explore the zero distribution with respect to the critical line. This is a joint work with Ramūnas Garunkštis and Raivydas Šimėnas, accepted for publication.

Chapter 7 is dedicated to the special case of the Lerch zeta-function when parameters λ and α are equal. Calculations show that the nontrivial zeros either lie extremely close to the critical line or are distributed almost symmetrically with respect to it. We investigate this phenomenon theoretically. These results were proved in the joint work with thesis adviser Ramūnas Garunkštis and, with the exception of Theorem 7.4, were published in [18]. Theorem 7.4 will appear in sepperate paper, which is submitted for publication.

1.1 Actuality and novelty

The results obtained in this dissertation are all original. Most of them are based on some classical results. Results related to the distribution of the Lerch zeta-function zeros complement Garunkštis and Steuding [25] findings. Since the Hurwitz zeta-function is a special case of the Lerch zeta-function, these results are also useful for a more profound understanding of the zero distribution of the derivative of the Hurwitz zeta-function.

Study of the zero distribution of the Lerch zeta-function, when the parameters λ and α are equal is motivated by computational findings and is in fact an interesting phenomenon worth investigating.

Results describing the periodic Hurwitz zeta-function zero and a-point distribution are generalized versions of similar results obtained for the Riemann zeta-function by R. J. Backlund, H. V. Mangoldt and N. Levinson.

1.2 Aims and main results

First aim of the thesis is to investigate zero and a-value distribution of the periodic Hurwitz zeta-function and zero distribution of the derivative of the Lerch zeta-function. By distribution here we mean two things: number of nontrivial zeros till the given size and their distances from the critical line. The second aim of the thesis is to investigate the special case of the Lerch zeta-function when parameters λ and α are equal.

The main results are summarized as follows:

1. We have identified zero and a-value free regions of the periodic Hurwitz zeta-function (see Theorems 4.4 and 5.4 respectively);

2. We have obtained the asymptotic formula for the number of nontrivial zeros of the periodic Hurwitz zeta-function and a-values (see Theorems 4.2 and 5.2 respectively). Also, we showed, that the periodic Hurwitz zeta-function zeros and a-values are clustered around the critical line (see Theorems 4.3 and 5.3 respectively);

3. We have identified zero-free regions of the derivative of the Lerch zeta-function (see Theorems 6.1, 6.2 and 6.3);

4. We have improved the asymptotic formula for the number of nontrivial Lerch derivative zeros (see Theorem 6.5) and showed that they are clustered around the critical line (see Theorem 6.6).

5. We showed, that in the upper half-plane nontrivial zeros of the Lerch zeta-function with equal parameters on average are symmetrically distributed with a small error term (see Theorem 7.3). We found, the Speiser type relation between zeros of the Lerch zeta-function and its derivative in the special case of equal parameters (see Theorem 7.4).

1.3 Methodology

The methods used in this dissertation mostly come from Complex Analysis. For completeness main classical theorems, lemmas and formulas which are used multiple times throughout this work, are discussed separately in Chapter 3. Most proofs consist of different forms of Dirichlet series and their analytical continuations, obtained using functional equations, together with classical techniques introduced by G. H. Hardy, J. E. Littlewood and J. Jensen. Even though new methods will not be introduced into the field, all results obtained in this work are new and original.

1.4 Dissemination of the thesis results

The results of this thesis will appear in 5 research papers. Two of them have already been published, one is accepted for publishing and other two are submitted for peer review.

1. R. Garunkštis and R. Tamošiūnas, Zeros of the periodic Hurwitz zeta-function, Šiauliai Math. Semin., 8(16):49–62, 2013.

2. R. Tamošiūnas, a-values of the periodic Hurwitz zeta-function, Šiauliai Math. Semin., 11(19):125-133, 2016.

3. R. Garunkštis and R. Tamošiūnas, Symmetry of zeros of Lerch zeta-function for equal parameters, Lith. Math. J., 57(4):433–440, 2017.

4. R. Garunkštis and R. Tamošiūnas, Zeros of the Lerch zeta-function and of its derivative for equal parameters, preprint, 2017.

5. R. Garunkštis, R. Tamošiūnas and R. Šimėnas, Zeros of derivative of Lerch's zeta-function, accepted for publication in Proceedings of Conference in Honor of Kohji Matsumoto's 60th Birthday, 2018.

All the results were presented in a series of seminars held in Vilnius University at Department of Probability Theory and Number Theory of Faculty of Mathematics and Informatics. Selected results were presented at the following conferences and events:

1. Zero distribution of the periodic Hurwitz zeta-function, poster presented at Diophantine Analysis Summer School, Germany, July 21-26, 2014.

2. Zero distribution of the periodic Hurwitz zeta-function, The 56th Conference of the Lithuanian Mathematical Society, Kaunas, Lithuania, June 16-17, 2015.

3. On a-value distribution of the periodic Hurwitz zeta-function, The 57th Conference of the Lithuanian Mathematical Society, Vilnius, Lithuania, June 20-21, 2016.

4. Zeros and a-values of the periodic Hurwitz zeta-function, The Sixth International Conference Analytic and Probabilistic Methods in Number Theory, Palanga, Lithuania, September 11-17, 2016.

5. Zero trajectories of the Lerch zeta-function and its derivative, The 58th Conference of the Lithuanian Mathematical Society, Vilnius, Lithuania, June 21–22, 2017.

6. Symmetry of zeros of the Lerch zeta-function for equal parameters, Vilnius Conference in Combinatorics and Number Theory, Vilnius, Lithuania, July 16-22, 2017.

1.5 Acknowledgments

I would like to thank my advisor, Prof. Ramūnas Garunkštis for his constant help and support. Special thanks to Prof. Juan Arias-de-Reyna for explaining how to do an X-ray with Sage and giving useful tips how to improve the Lerch zeta-function zero computation speed. I am grateful to Prof. Jörn Steuding for sharing historical results related to X-rays and inspiring to be interested in the history of mathematics. Last but not least I want to thank my fiancee, relatives, and friends for constant support.

2 Literature review

In this chapter the Riemann zeta-function and its generalizations will be presented together with results, which are either closely connected with thesis problem or provide motivational background to explore zero distribution. All results and findings in this chapter are not original and are given there only as a reference point.

2.1 Number of primes up to a given magnitude

One of the most famous results (prime number theorem) is due to Hadamard [31] and Poussin [73] who in 1896 independently proved that

$$\pi(x) \sim \frac{x}{\log x}, \tag{2.1}$$

where $\pi(x)$ denotes the number of primes up to x. This is the asymptotic law of distribution of prime numbers which states, that limit of the quotient of those two functions approach 1. The prime number theorem is equivalent to the statement that the n-th prime number p_n satisfies $p_n \sim n \log n$.

It is believed, that Carl Friedrich Gauss in the year 1792 conjectured, that

$$\pi(x) \sim \mathrm{li}(x),$$

where the logarithmic integral is given by

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{du}{\log u}.$$

Approximation by logarithmic integral is slightly better than (2.1) and approaches $\pi(x)$ faster when x is increasing, see figure 21. Plots suggest that for sufficiently large x exist upper and lower bounds for $\pi(x)$. P. Chebyshev in 1850 had proved that for sufficiently large x

$$0.921\ldots \leqslant \pi(x)\frac{\log x}{x} \leqslant 1.055\ldots$$

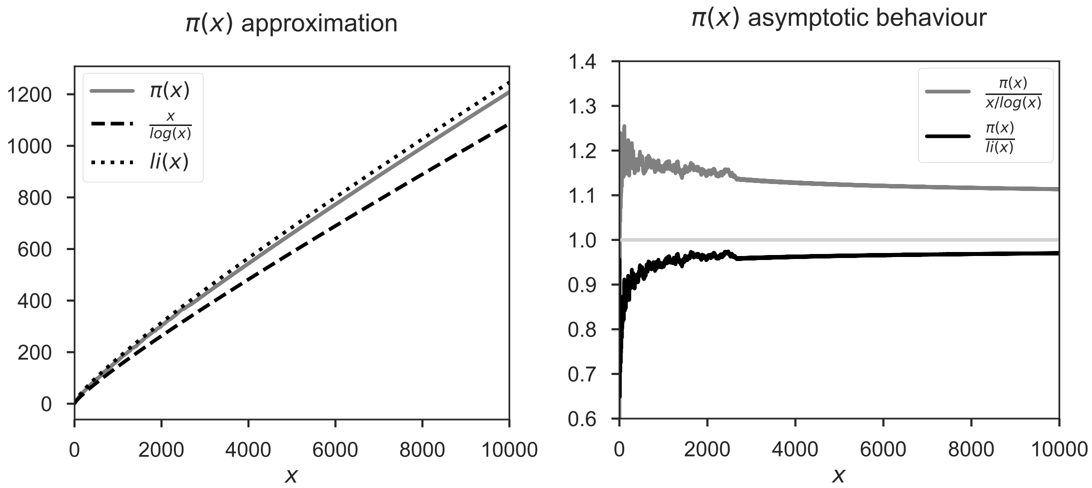

Figure 21: $\pi(x)$ approximation given by $x/\log x$ and $\mathrm{li}(x)$.

and if the limit

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x}$$

exists, then it is equal to 1.

As we will see in the next section, Riemann [59] introduced the zeta-function (see Equation (2.4) bellow) and was able to prove an important relationship between the Riemann zeta-function zeros and the distribution of the prime numbers using analytical methods. His result was critical to the proof of the prime number theorem which is equivalent to a statement, that there are no zeros of the Riemann zeta-function, which have real part equal to 1.

2.2 Short history of zeta-functions

It is well known that Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

is divergent, this fact dates back to 14th century (Nicole Oresme). Euler [16] in 1737 explored alternative series containing prime reciprocals

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

and observed, that using a method similar to the sieve of Eratosthenes, one obtains

$$\sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_p \left(1 - \frac{1}{p^x}\right)^{-1}, \tag{2.2}$$

where $x > 1$ and p denotes prime numbers and product is taken over all primes. This equation (Euler's product formula) turned out to be a starting point for numerous number theory results concerning prime distribution. As motivational example, we give the simple proof of the infinitude of the primes.

Sum on the left hand side of (2.2) can be approximated by

$$\int_1^\infty \frac{dt}{t^x} = \frac{1}{x-1}. \tag{2.3}$$

Let $x \to 1^+$, then (2.2) together with (2.3) yields

$$\sum_p \left(1 - \frac{1}{p}\right) = 0,$$

thus

$$\sum_p \frac{1}{p} = \infty.$$

Riemann [59] was the first to investigate the Euler product formula as a function of the complex variable. The Riemann zeta-function is a function of a complex variable $s = \sigma + it$, for $\sigma > 1$, given by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^{-s}}\right)^{-1}, \tag{2.4}$$

here, as in (2.2), p denotes prime numbers and product is taken over all primes.

It is easy to see that for $\sigma > 1$ the Riemann zeta-function does vanish. As mentioned in previous section, proof that Riemann zeta-function does not vanish on $\sigma = 1$ is due to Hadamard [31] and Poussin [73] and is equivalent to the prime number theorem.

The main tool which allowed Riemann to extend the zeta-function to whole complex plane (except the pole at $s = 1$) is the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ denotes Euler's gamma function defined as

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx, \quad \text{for } s > 0.$$

For more details about Euler's gamma function see Artin [3]. Denote

$$\xi(s) := \frac{1}{2}s(s-1)\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s),$$

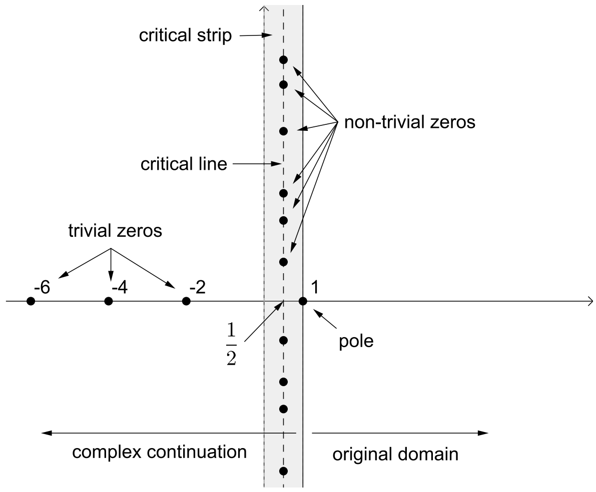

Figure 22: ζ function regions and definitions.

then, the functional equation can be expressed in symmetrical form $\xi(1-s) = \xi(s)$.

It follows from the functional equation and the properties of gamma function that $\zeta(s)$ vanishes in $\sigma < 0$ at $s = -2n,\ n \in \mathbb{N}$ (so-called trivial zeros). Also, the functional equation of zeta-function reveals symmetrical behavior of the nontrivial Riemann zeta-function zeros

$$\zeta\left(\overline{s}\right) = \overline{\zeta\left(s\right)},$$

which means that nontrivial zeros are distributed symmetrically with respect to vertical line $\sigma = \frac{1}{2}$ and real axis (see Figure 22).

Taking the logarithmic derivative of (2.4) yields

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} = \sum_p \sum_{m \geqslant 1} \frac{\log p}{p^{ms}}.$$

Next we will use discontinuous integral defined by

$$\frac{1}{2\pi i} \int_{s:\Re(s)=c} \frac{y^s}{s} ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

Notice that when $y = x/p^m$ this formula enables to take into account only members for which $p^m < x$, thus

$$\sum_{\substack{p,m\geqslant 1\\ p^m\leqslant x}} \log p = \frac{1}{2\pi i}\sum_{p,m\geqslant 1} \log p \int_{s:\Re(s)=c} \left(\frac{x}{p^m}\right)^s \frac{ds}{s}$$
$$= -\frac{1}{2\pi i}\int_{s:\Re(s)=c} \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds.$$

Now, using the Cauchy argument principle (see Theorem 3.1 in Chapter 3 bellow), we easily obtain

$$\sum_{\substack{p,m\geqslant 1\\ p^m\leqslant x}} \log p = x - \sum_{\rho:\zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}. \tag{2.5}$$

This statement links the number of primes till the given size and zeta-function zeros, which motivates to explore exact positions of the Riemann zeta-function zeros and their distribution.

For $0 < t \leqslant T$, denote the number of the Riemann zeta-function nontrivial zeros as $N(T)$ and the number of zeros of $\zeta(\frac{1}{2}+it)$ as $N_0(T)$. It is well known (see Titchmarsh [69, Theorem 9.2]), that as $T\to\infty$

$$N(T+1) - N(T) = O(\log T). \tag{2.6}$$

Riemann Hypothesis (RH) states that $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$. This hypothesis originated from famous Riemann paper [59]. There are numerous alternative formulations. If RH is true, then it is easy to see that

$$\left|\frac{x^\rho}{\rho}\right| \leqslant \frac{\sqrt{x}}{|\mathrm{Im}(\rho)|}.$$

Substituting this bound into (2.5) together with (2.6) leads to

$$\sum_{\substack{p,m\geqslant 1\\ p^m\leqslant x}} \log p = x + O\left(\sqrt{x}\log^2 x\right) \implies \pi(x) = \int_2^x \frac{dt}{\log t} + O\left(\sqrt{x}\log x\right).$$

Toward the Riemann Hypothesis we have the following results related to the number of zeros on the critical line. In 1914 Hardy [32] proved that there exists an infinity of nontrivial zeros on the critical line (for alternative proofs see Polya, Landau, Titchmarsh [69]). Later, in 1921 Hardy and Littlewood [33] proved that there exists such constant $c > 0$ for which $N_0(T) > cT$ for any T. This was improved by Selberg [61] in 1942 to $N_0(T) > cT\log T$. Next came multiple attempts to improve proportion bound between $N_0(T)$ and $N(T)$, for example Levinson [51] in 1974 showed, that $N_0(T) > \frac{1}{3}N(T)$. This was improved by Conrey [11] in 1989 up to $N_0(T) > \frac{2}{5}N(T)$ and at the same year together with Ghosh and Gonek [13] up to $N_0(T) \geqslant 0.40219N(T)$, for T large enough.

2.3 Zero distribution of the Riemann zeta-function

This section restates results on counting the number of zeros till the given size obtained by Backlund [4] and Mangoldt [72].

Let

$$\theta(t) = \arg\left(\pi^{-i\frac{t}{2}}\Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)\right)$$

and

$$S(t) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + it\right),$$

where the arguments are defined on the $\rceil$ shaped contour through points $2 + 0i$, $2 + it$ and $\frac{1}{2} + it$ (if segment from $2 + it$ to $\frac{1}{2} + it$ contains the Riemann zeta-function zero, then we take a limit $t + \varepsilon$, $\varepsilon \to \infty$). Then the number of the Riemann zeta-function zeros till T (denoted as $N(T)$) is

$$N(T) = 1 + \frac{\theta(T)}{\pi} + S(T).$$

Asymptotic expansion of $\theta(T)$, using Stirling series, yields

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \ldots,$$

thus

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right). \tag{2.7}$$

If $T \to \infty$, then $S(T)$ can be approximated by (see Titchmarsh [69, Theorem 9.4])

$$S(T) = O\left(\log T\right),$$

or, if we assume Riemann hypothesis, by the stronger bound (see Titchmarsh [69, Theorem 14.13])

$$S(T) = O\left(\log T/\log\log T\right).$$

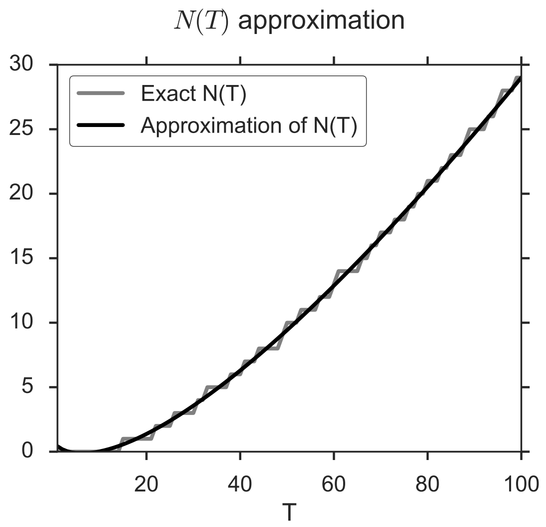

Figure 23: $N(T)$ approximation given by $\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}$.

Figure 23 reveals how well this approximation works even for small values of T.

Denote the number of zeros $\beta + it$ in the strip $\beta > \alpha$ and $0 < t \leqslant T$ by $N(\alpha, T)$. According to Bohr and Landau [8], for any fixed $\sigma > \frac{1}{2}$, we have

$$N(\sigma, T) = O(T).$$

For improved bound see Ivic [35]. From this and (2.7) we deduce that all but infinitesimal proportion of the zeros lie in the strip $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$ for any $\delta > 0$.

2.4 a-value distribution of the Riemann zeta-function

Norman Levinson [52] in 1975 proved the following theorem concerning the a-value distribution of the Riemann zeta-function.

Let a be a fixed complex number and $T^{1/2} \leqslant U \leqslant T$. Let $N^{(1)}(a; T, U)$ be the number of roots of $\zeta(s) = a$ in

$$\sigma > \frac{1}{2} + \frac{(\log\log T)^2}{\log T}, \quad T < t < T + U;$$

let $N^{(2)}(a; T, U)$ be those in

$$\sigma < \frac{1}{2} - \frac{(\log\log T)^2}{\log T}, \quad T < t < T + U;$$

and let $N^{(3)}(a;T,U)$ be those in

$$\frac{1}{2}-\frac{(\log\log T)^2}{\log T}\leqslant\sigma\leqslant\frac{1}{2}+\frac{(\log\log T)^2}{\log T},\quad T<t<T+U.$$

Theorem 2.1. *For large T we have*

$$N^{(3)}(a;T,U)=\frac{U}{2\pi}\log T+O\left(U\frac{\log T}{\log\log T}\right),$$
$$N^{(1)}(a;T,U)+N^{(2)}(a;T,U)=O\left(U\frac{\log T}{\log\log T}\right).$$

This states that almost all roots of $\zeta(s)=a$ are arbitrary close to $\sigma=\frac{1}{2}$.

Similar results to Theorem 2.1 were obtained by Steuding [66], [67] for the Epstein zeta-function. Results were further extended to the Selberg zeta-function by Garunkštis and Šimėnas [29]. In recent work Garunkštis and Steuding [27] extended some of Levinson's ideas about a-points of the Riemann zeta-function.

2.5 Zero distribution of the derivative of the Riemann zeta-function

Speiser [63] showed that the Riemann hypothesis is equivalent to the absence of the nontrivial zeros of the derivative of the Riemann zeta-function $\zeta(s)$ left of the critical line $\sigma=\frac{1}{2}$. Later on, Levinson and Montgomery [53] proved the quantitative version of the Speiser result, namely, that the Riemann zeta-function and its derivative have approximately the same number of zeros left of the critical line. The results from [53] were important for the Levinson's [51] proof that at least one-third of all the nontrivial zeros of $\zeta(s)$ are located on the critical line (see also Selberg [61] and [62]). Therefore, it is important to study the zeros of the derivatives of zeta-functions.

Speiser's result was extended to Dirichlet L-functions with primitive Dirichlet characters by Yıldırım [75] and the Selberg zeta-function on a compact Riemann surface by Luo [54], (see also Garunkštis [21]). In all these cases, an analog of the RH is expected or, as in the case of the Selberg zeta-function attached to compact Riemann surfaces, it is known to be true.

In [30] Speiser's result was extended to the extended Selberg class. This class contains zeta-functions for which the Riemann hypothesis is not true. From another side every zeta-function of this class satisfies the functional equation of the Riemann type. Thus every zeta-function has a symmetrical distribution of nontrivial zeros in respect of the critical line. This symmetry is an important ingredient in the proof of Theorem 1.2 in [30], which relates the nontrivial zeros of the element of the extended Selberg class with the zeros of its derivative in a half-plane $\sigma < \frac{1}{2}$.

Spira [64] explored what regions are free of zeros for each of the Riemann zeta-function derivative $\zeta^{(k)}(s)$ and gave bounds for these regions for the right and left-half planes. Using the Euler-McLaurin formula, Spira calculated positions of first and second zeta derivative zeros and found, that there is strange bunching effect - the imaginary parts of the zeros of ζ' fall between imaginary parts of zeros of ζ. He also conjectured that the number of k-th ζ derivative zeros up to T satisfies

$$N(T) = N_k(T) + \left[\frac{T\log 2}{2\pi}\right] \pm 1,$$

where $N_k(T)$ is the number of zeros of $\zeta^{(k)}\left(\sigma + it\right)$ for $0 < t \leqslant T$. This conjecture was later explored by Berndt [6]. He proved that

$$N(T) = N_k(T) + \frac{T\log 2}{2\pi} + O\left(\log T\right). \tag{2.8}$$

Conrey and Gosh [12] improved Levinson and Montgomery [53] results related to the zero positions of the Riemann zeta-function by prooving, that almost all zeros of the $\zeta^{(k)}(s)$ are in the region

$$\sigma > \frac{1}{2} - \frac{\phi(t)}{\log t}$$

for any $\phi(t)$ which goes to infinity with t. Also, for any $c > 0$, a positive proportion of zeros of $\zeta^{(k)}(s)$ are in the region $\sigma \geqslant \frac{1}{2} + \frac{c}{\log t}$. Assuming the RH, there are $\gg_\varepsilon T$ zeros of $\zeta^{(k)}(s)$ in the region

$$\frac{1}{2} \leqslant \sigma < \frac{1}{2} + \frac{(1+\varepsilon)\log\log T}{\log T}, \quad 0 < t < T$$

for any $\varepsilon > 0$, when $T \to \infty$.

2.6 The periodic Hurwitz zeta-function

There are multiple generalizations of the Riemann zeta-function, which share some similarities. Most generalizations have analogs to the functional equation (2.2), and some even have analogs to Euler product (2.4). Exploring these generalizations is important because they are useful in a variety of disciplines and even applications like fractals and dynamical systems. In this section we will introduce the Hurwitz zeta-function and the periodic Hurwitz zeta-function, which will be studied in this thesis.

Adolf Hurwitz [34] in 1881 invented a simple, but important generalization, the nowadays so-called Hurwitz zeta-function, given by

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^{s}},$$

where α is a real parameter satisfying $0<\alpha\leqslant 1$. Hurwitz himself considered only the case of rational α, which is sufficient for most number theory applications (for more details see [57]). Let $1\leqslant m\leqslant n$ be integers, then the following functional equation holds (see Hurwitz [34])

$$\zeta\left(1-s,\frac{m}{n}\right)=\frac{2\Gamma(s)}{(2\pi n)^{s}}\sum_{k=1}^{n}\cos\left(\frac{\pi s}{2}-\frac{2\pi km}{n}\right)\zeta\left(s,\frac{k}{n}\right). \tag{2.9}$$

While $\zeta(s)$ has no zeros in $\sigma>1$, the Hurwitz zeta-function has infinitely many zeros if $\alpha\neq\frac{1}{2}$ or 1. Even though the analogue of the Riemann hypothesis for the Hurwitz zeta-function is false (see Davenport and Heilbronn [14]), with rational parameters Hurwitz zeta-function has self-approximation property (see Pańkowski [58]).

Denote by $r=(r_m)_{m=0}^{\infty}$, $r_m\in\mathbb{C}$, a periodic with period k sequence. To avoid the trivial case we further assume that $r_k\neq 0$, for some k. The periodic Hurwitz zeta-function is defined by the Dirichlet series

$$\zeta(s,\alpha;r)=\sum_{m=0}^{\infty}\frac{r_m}{(m+\alpha)^{s}}\qquad(\sigma>1), \tag{2.10}$$

where $0<\alpha\leq 1$ is a fixed real number. If $k=1$ and $r_m=1$, then we obtain the Hurwitz zeta-function $\zeta(s,\alpha)$ and, for $\alpha=1$, we get the classical Riemann zeta-function $\zeta(s)$.

It is easy to see that, for $\sigma>1$,

$$\zeta(s,\alpha;r)=\frac{1}{k^{s}}\sum_{l=0}^{k-1}r_l\sum_{m=0}^{\infty}\frac{1}{(m+(l+\alpha/k))^{s}}=\frac{1}{k^{s}}\sum_{l=0}^{k-1}r_l\zeta\left(s,\frac{\alpha+l}{k}\right). \tag{2.11}$$

In view of the equation (2.11) and the functional equation (2.9) the function $\zeta(s, \alpha; r)$ has meromorphic continuation to the whole complex plane with possible simple pole at $s = 1$ with residue

$$R = \frac{1}{k}\sum_{l=0}^{k-1} r_l.$$

If $R = 0$, then $\zeta(s, \alpha; r)$ is an entire function.

In the sense of probabilistic limit theorems and the universality theory the periodic Hurwitz zeta-function were extensively investigated by Laurinčikas and and his pupils in the papers [36], [38], [37], [39], [41], [40], [46], [48].

2.7 The Lerch zeta-function

Mathias Lerch [50] in 1887 studied the infinite series which is nowadays well-known as the Lerch zeta-function, namely

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \tag{2.12}$$

for $\sigma > 1$ and $0 < \lambda, \alpha \leqslant 1$. Lerch [50] obtained analytic continuation to the whole complex plane, except a simple pole at $s = 1$. In this work we will use the following version of the functional equation formulation (Garunkštis and Steuding [25]):

$$\begin{aligned} L(\lambda, \alpha, 1-s) = (2\pi)^{-s}\Gamma(s)\Bigg(&\exp\left(\frac{\pi i s}{2} - 2\pi i \alpha\lambda\right) L(-\alpha, \lambda, s) \\ &+ \exp\left(-\frac{\pi i s}{2} + 2\pi i \alpha(1-\{\lambda\})\right) L(\alpha, 1-\{\lambda\}, s)\Bigg), \end{aligned} \tag{2.13}$$

where $0 < \lambda, \alpha \leqslant 1$. Various proofs of this functional equation can be found in Lerch [50], Apostol [1], Oberhettinger [56], Mikolás [55], Berndt [7], see also Lagarias and Li [42], [43]. The Lerch zeta-function has a second moment (Garunkštis, Laurinčikas, and Steuding [23]) and it is a universal function (Laurinčikas [45], Lee, Nakamura, Pańkowski [49]).

Notice that

$$\begin{gathered} L(1,1,s) = \zeta(s), \quad L(1,\alpha,s) = \zeta(s,\alpha), \quad L\left(\frac{1}{2},\frac{1}{2},s\right) = 2^s L(s,\chi), \\ L\left(\frac{1}{2},1,s\right) = (1-2^{1-s})\zeta(s), \quad L\left(1,\frac{1}{2},s\right) = (2^s-1)\zeta(s), \end{gathered} \tag{2.14}$$

where $L(s, \chi)$ is the Dirichlet L-function with the character χ mod 4, $\chi(3) = -1$. For these five cases, certain versions of the Riemann hypothesis (RH) can be formulated. For all the other cases, it is expected that the real parts of zeros of the Lerch zeta-function form a dense subset of the interval $(\frac{1}{2}, 1)$. This is proved for any λ and transcendental α (see Garunkštis [47, Theorem 4.7 in Chapter 8]).

In general, the Hurwitz zeta-function and the Lerch zeta-function has no Euler product expansion over the prime numbers and does not have a symmetrical version of the functional equation (analog to $\xi(1-s) = \xi(s)$). These are the main reasons why we need alternative techniques to explore zero-free regions and zero distributions in both of these cases.

Garunkštis and Laurinčikas [22] proved a theorem which estimates the number of the Lerch zeta-function zeros up to a fixed height. Let $N^+(\lambda, \alpha, T)$ denote nontrivial zeros in $0 < t < T$, and $N^-(\lambda, \alpha, T)$ in $-T < t < 0$.

Theorem 2.2. *When $T \to \infty$, then*

$$N^+(\lambda, \alpha, T) = \frac{T}{2\pi} \log \frac{T}{2\pi\alpha\lambda} + O(\log T),$$
$$N^-(\lambda, \alpha, T) = N^+(1-\lambda, \alpha, T).$$

In 2000 Garunkštis and Steuding [25] proved that almost all zeros of the Lerch zeta-function are clustered around the vertical line of the complex plane with the real part $\frac{1}{2}$. Let $s = \beta + i\gamma$ denote the nontrivial zero of $L(\lambda, \alpha, s)$.

Theorem 2.3. *For $0 < \lambda, \alpha \leqslant 1$ we have, as T tends to infinity,*

$$\sum_{|\gamma| \leqslant T} \left(\beta - \frac{1}{2}\right) = \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda(1-\{\lambda\})}} + O(\log T).$$

3 Methodology

This chapter contains main classical theorems, lemmas and formulas which which will be used together with classical techniques, introduced by G. H. Hardy, J. E. Littlewood and J. Jensen, in later chapters. For each theorem and lemma, a short proof will be given or references provided. All results in this chapter are not original and are given here only as a reference point.

3.1 Auxiliary results

Theorem 3.1 (Cauchy's argument principle). *If $f(z)$ is a meromorphic function inside and on some closed contour C, and f has no zeros or poles on C, then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P,$$

where N and P denote respectively the number of zeros and poles, of $f(z)$ inside contour C, with each zero and pole counted as many times as its multiplicity and order, respectively, indicate. Notice that if $f(z)$ is analytic inside C, then the left-hand side gives the number of zeros of $f(z)$ inside C.

Proof. Since the zeros and poles are isolated, we can deform C into a set of small disjoint circles, one centered at each zero or pole, and such that the center is the only such point within the circle. It follows that it is enough to prove the theorem for each circle separately and then add the results.

In such circle C_ε with center at z_0, we can write $f(z) = (z - z_0)^m h(z)$ where $m > 0$ if z_0 is a zero and $m < 0$ if z_0 is a pole, and where $h(z) \neq 0$. Then it is easy to see that

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{h'(z)}{h(z)},$$

thus

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{m}{z - z_0} = m,$$

since $\frac{h'(z)}{h(z)}$ is analytic inside C_ε. □

Rouché's theorem formulation and the proof are based on Titchmarch [68, §3.42].

Theorem 3.2 (Rouché's theorem). *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.*

Proof. Define
$$\Psi(t) := \frac{1}{2\pi i}\int_C \frac{f'(z)+tg'(z)}{f(z)+tg(z)}dz,$$
on the interval $0 \leqslant t \leqslant 1$. Notice that since

$$|f(z)+tg(z)| \geqslant ||f(t)| - t|g(t)|| \geqslant ||f(t)| - |g(t)|| > 0,$$

that denominator of the integrand is never zero. Also note that Ψ is continuous on the interval $[0,1]$ and is integer-valued, thus constant. Ψ is the number of zeros of $f+tg$ inside C, thus
$$\Psi(0) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$
is the number of f zeros in C, and

$$\Psi(1) = \frac{1}{2\pi i}\int_C \frac{f'(z)+f'(z)}{f(z)+g(z)}dz$$

is the number of zeros of $f+g$ in C. Since $\Psi(0) = \Psi(1)$, this completes the proof. ☐

Formulation and proof of Littlewood's lemma is based on Titchmarsh [69, §9.9].

Theorem 3.3 (Littlewood's lemma). *Let $f(s)$ be analytic and nonzero on the rectangle $\mathcal{C}$ with verices $\sigma_0, \sigma_1, \sigma_1+iT$ and σ_0+iT, here $\sigma_0 < \sigma_1$ and $T > 0$. Also denote $\rho = \beta + i\gamma$ zeros of f belonging to the region enclosed by $\mathcal{C}$, with $\beta, \gamma \in \mathbb{R}$. Then*

$$2\pi\sum_{\rho\in\mathcal{C}}(\beta-\sigma_0) = \int_0^T \log|f(\sigma_0+it)|\,dt - \int_0^T \log|f(\sigma_1+it)|\,dt \\ + \int_{\sigma_0}^{\sigma_1} \arg f(\sigma+iT)\,d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\,d\sigma.$$

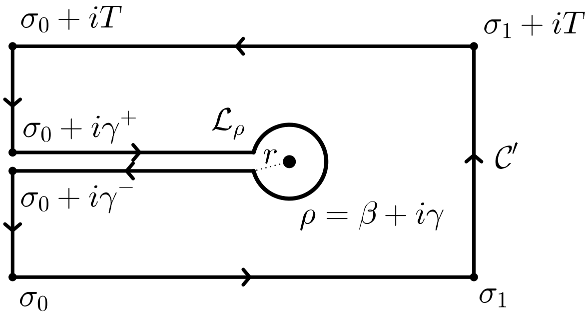

Figure 31: Contour $\mathcal{C}'$, with loop $\mathcal{L}_\rho$ around the f zero ρ.

Proof. Let $\mathcal{C}'$ denote a contour which, with the help of loops $\mathcal{L}_\rho$, cuts $\mathcal{C}$ in such a way, that all zeros ρ are lying outside of $\mathcal{C}'$ (see Figure 31). Then

$$\int_{\mathcal{C}} \log f(s)\, ds = \int_{\mathcal{C}'} \log f(s)\, ds + \sum_{\rho \in \mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds.$$

Since $\log f(s)$ is analytic and single-valued in $\mathcal{C}'$, the first integral on the right hand side is zero. It remains to estimate integrals which traverse path $\mathcal{L}_\rho$. If circular path in $\mathcal{L}_\rho$ has radius r, then

$$\begin{aligned} \int_{\mathcal{L}_\rho} \log f(s)\, ds = & \int_{\sigma_0}^{\beta - r} \log f\left(\sigma + i\gamma^+\right) d\sigma + \int_{\pi}^{-\pi} \log f\left(\rho + re^{i\theta}\right) ire^{i\theta} d\theta \\ & - \int_{\sigma_0}^{\beta - r} \log f\left(\sigma + i\gamma^-\right) d\sigma. \end{aligned}$$

It is easy to see that

$$\int_{\pi}^{-\pi} \log f\left(\rho + re^{i\theta}\right) ire^{i\theta} d\theta \to 0, \quad \text{as } r \to 0^+.$$

Since traveling around circular part of $\mathcal{L}_\rho$ we have gained 2π in the argument along γ, we obtain

$$\begin{aligned} \int_{\mathcal{L}_\rho} \log f(s)\, ds &= \int_{\sigma_0}^{\beta - r} \log f\left(\sigma + i\gamma^+\right) d\sigma - \int_{\sigma_0}^{\beta - r} \log f\left(\sigma + i\gamma^+\right) + 2\pi i d\sigma \\ &= -2\pi i \int_{\sigma_0}^{\beta - r} d\sigma. \end{aligned}$$

Hence, as $r \to 0^+$,

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds = -2\pi i \left(\beta - \sigma_0\right).$$

Now, we can estimate integral along rectangle $\mathcal{C}$ in two ways

$$\begin{aligned}\int_{\mathcal{C}} \log s ds =& -2\pi i \sum_{\rho\in\mathcal{C}} (\beta - \sigma_0)\\ &= \int_0^T \log f(\sigma_1 + it)\, idt + \int_{\sigma_1}^{\sigma_0} \log f(\sigma + iT)\, d\sigma\\ &+ \int_T^0 \log f(\sigma_0 + it)\, idt + \int_{\sigma_0}^{\sigma_1} \log f(\sigma)\, d\sigma.\end{aligned}$$

The theorem follows by expanding principle value of the logarithm and equating the imaginary parts. □

Jensen's formula and the proof are based on Rudin [60, §15.18].

Theorem 3.4 (Jensen's formula). *Let f be holomorphic inside and on the circle $|z| = R$ and $f(0) \neq 0$. Let $a_1, a_2, \ldots, a_m$ be the zeros of f inside the disk with multiplicities $p_1, p_2, \ldots, p_m$, and denote poles by $b_1, b_2, \ldots, b_n$ with multiplicities $q_1, q_2, \ldots, q_n$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| d\theta = \log|f(0)| + \sum_{k=1}^{m} p_k \log\left|\frac{R}{a_k}\right| - \sum_{k=1}^{n} q_k \log\left|\frac{R}{b_k}\right|.$$

That is, the distribution of zeros of $f(z)$ inside the circle is related to the mean of $\log|f(z)|$ on the circle.

Proof. Since $\Re \log z = \log|z|$

$$\begin{aligned}I :=& \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\log f(0) + \int_0^R \frac{d}{dr}\log f\left(re^{i\theta}\right) dr\right\} d\theta\\ &= \log|f(0)| + \frac{1}{2\pi}\Re\int_0^{2\pi}\int_0^R \frac{f'\left(re^{i\theta}\right)e^{i\theta}}{f\left(re^{i\theta}\right)} dr d\theta.\end{aligned}$$

Reversing the order of integration yields

$$\begin{aligned}I &= \log|f(0)| + \Re\int_0^R \frac{1}{2\pi i r}\int_0^{2\pi} \frac{f'\left(re^{i\theta}\right) ire^{i\theta}}{f\left(re^{i\theta}\right)} d\theta dr\\ &= \log|f(0)| + \Re\int_0^R \frac{1}{2\pi i r}\int_{C_r} \frac{f'(z)}{f(z)} dz dr,\end{aligned}$$

where C_r is the circle of radius r. Now, by the argument principle

$$\frac{1}{2\pi i}\int_{C_r} \frac{f'(z)}{f(z)} dz = \sum_{|a_k|<r} p_k - \sum_{|b_k|<r} q_k := n(r).$$

Since $n(r)$ is a real number, we conclude that

$$I = \log|f(0)| + \int_0^R \frac{n(r)}{r} dr.$$

Now, let $0 < j_1 \leqslant \ldots \leqslant r_{m+n} < R$ be the ordered magnitudes of the poles and zeros a_k, b_k. Then $n(r)$ is constant on the interval (r_j, r_{j+1}). Set $r_0 = 0$ and $r_{m+n+1} = R$ for simplicity. Denote its value on this interval by n_j. Then

$$\begin{aligned} I - \log|f(0)| &= \sum_{j=0}^{m+n} \int_{r_j}^{r_{j+1}} \frac{n(r)}{r} dr \\ &= \sum_{j=1}^{m+n} \int_{r_j}^{r_{j+1}} \frac{n_j}{r} dr \\ &= \sum_{j=1}^{m+n} n_j (\log(r_{j+1}) - \log(r_j)) \\ &= -n_1 \log j_1 + (n_1 - n_2) \log j_2 + \cdots + (n_{m+n-1} - n_{m+n}) \log r_{m+n} + n_{m+n} \log R. \end{aligned}$$

Now, $n_{j+1} - n_j$ is precisely the multiplicity of the zero and/or pole of radius r_{j+1}, because that's how much $n(r)$ changes by when the radius passes from $r < r_{j+1}$ to $r > r_{j+1}$. So then

$$\begin{aligned} I &= \log|f(0)| - \sum_{k=1}^{m} p_k \log|a_k| + \sum_{k=1}^{n} q_k \log|b_k| + n_{m+n} \log R \\ &= \log|f(0)| + \sum_{k=1}^{m} p_k \log\frac{R}{|a_k|} - \sum_{k=1}^{n} q_k \log\frac{R}{|b_k|}, \end{aligned}$$

where the last equality holds because of $n_{m+n} = \sum_k p_k - \sum_k q_k$. □

Following statements are given without proofs.

Minimum value of $\zeta(s)$ in certain parts of the critical strip was obtained by Landau [44] (see Titchmarch [69, §9.7]).

Lemma 3.1. *There is a constant A such that each interval $(T, T+1)$ contains a value of t for which*

$$|\zeta(s)| > t^{-A} \quad (-1 \leqslant \sigma \leqslant 2).$$

Similar bound is also true for the Lerch zeta-function (see Garunkštis [22]) and as we will see later, for the derivative of the Lerch zeta-function.

Lemma 3.2. *For any σ_0, $\sigma \geqslant \sigma_0$, we have*

$$L(\lambda, \alpha, s) = B_\lambda |t|^k, \quad k = k(\sigma_0).$$

Considering mean values of $\zeta(s)$ powers Titshmarch [69, §7.3] gave following theorem.

Theorem 3.5. *We have*

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = O(T\log T).$$

This result can be improved to asymptotic equality, see Hardy and Littlewood [33].

Lemma 3.3 (Stirling's formula). *If* $|\arg s| \leqslant \pi - \varepsilon$, *then*

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right). \tag{3.1}$$

For the proof see Titchmarsh [68, §4.42].

3.2 X-rays of the Hurwitz zeta-function

Hurwitz zeta-function when $\alpha = 1$ behaves very differently than with other α values and thus requires normalization. As a motivation, to use normalized Hurwitz zeta-function $\alpha^s\zeta(s,\alpha)$ in chapters 4 and 5, we draw several 'X-ray' pictures (cf. Arias-de-Reyna [2]) of the Hurwitz zeta-function for various fixed values of parameter α near 1.

Let $f(s)$ be a meromorphic function. The *real lines* of $f(s)$ are defined by the set

$$\{s \in \mathbb{C} : \Re f(s) = 0\}.$$

In figures of this section real lines are denoted by solid lines. Analogously, the *imaginary lines* of $f(s)$ are defined by the set

$$\{s \in \mathbb{C} : \Im f(s) = 0\}.$$

In figures, imaginary lines are denoted by dotted lines.

In the upper part of Figure 3.2, we see the unstable behavior of real lines of the Hurwitz zeta-function $\zeta(s,\alpha)$ when α tends to 1. In the upper part of Figure 3.2, the network of real and imaginary lines is very different when α is near to 0 and when α is near to 1. It seems that this instability is caused by the first term

α^s of the Dirichlet series of $\zeta(s,\alpha)$. The lower parts of Figures 3.2 and 3.2 shows that the network of real and imaginary lines is more stable for different values of α if we consider the normalized Hurwitz zeta-function $\alpha^s\zeta(s,\alpha)$. Both functions $\zeta(s,\alpha)$ and $\alpha^s\zeta(s,\alpha)$ have the same zeros. In later chapters similar normalization will be applied for periodic Hurwitz zeta-function and the Lerch zeta-function to obtain 1 as a first term in Dirichlet series.

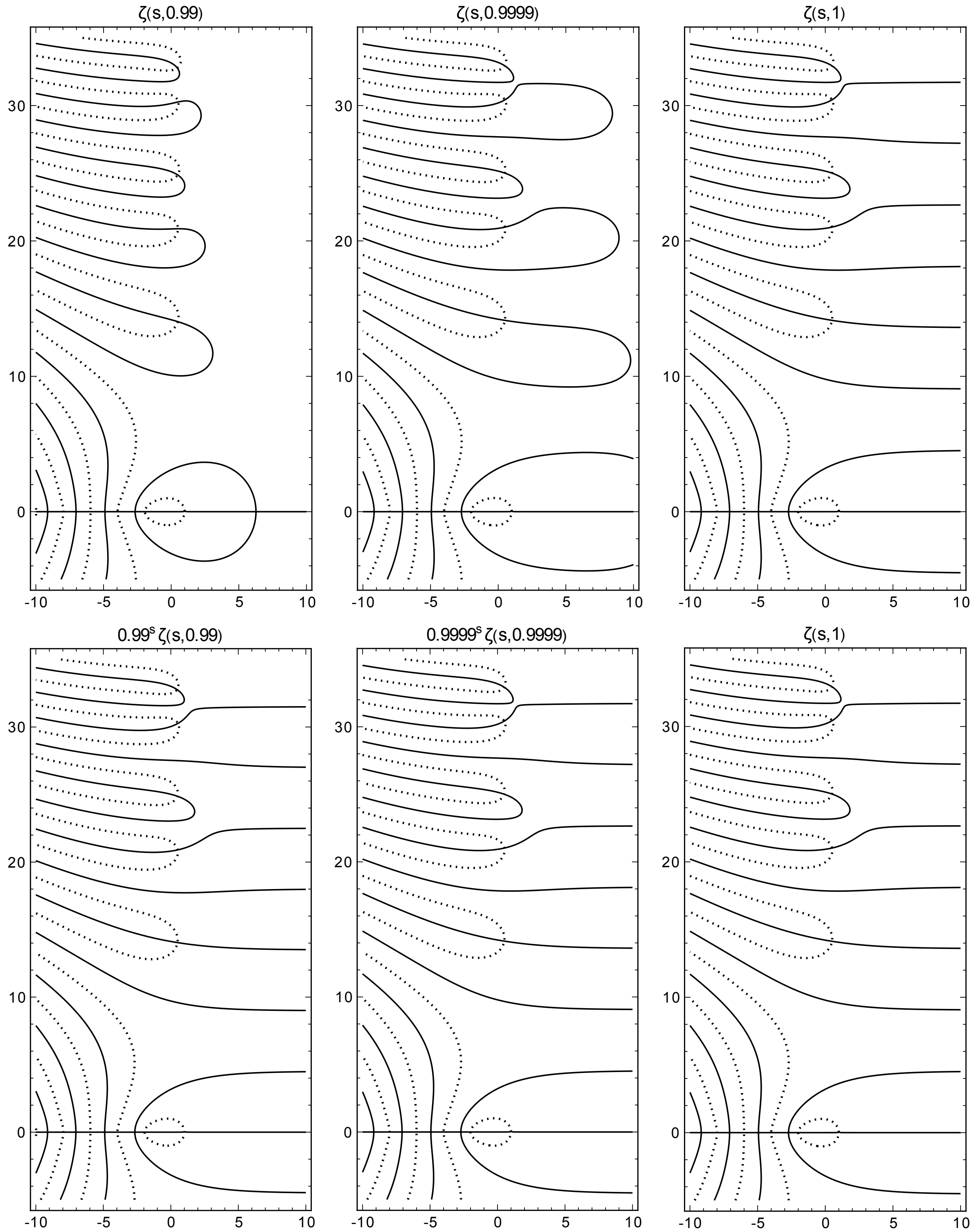

Figure 32: Real and imaginary lines of the function $\zeta(s, \alpha)$ compared to real and imaginary lines of the normalized function $\alpha^s \zeta(s, \alpha)$, for $\alpha = 0.99, 0.9999, 1$. In the graphs the horizontal axis is $\Re s$ and the vertical axis is $\Im s$.

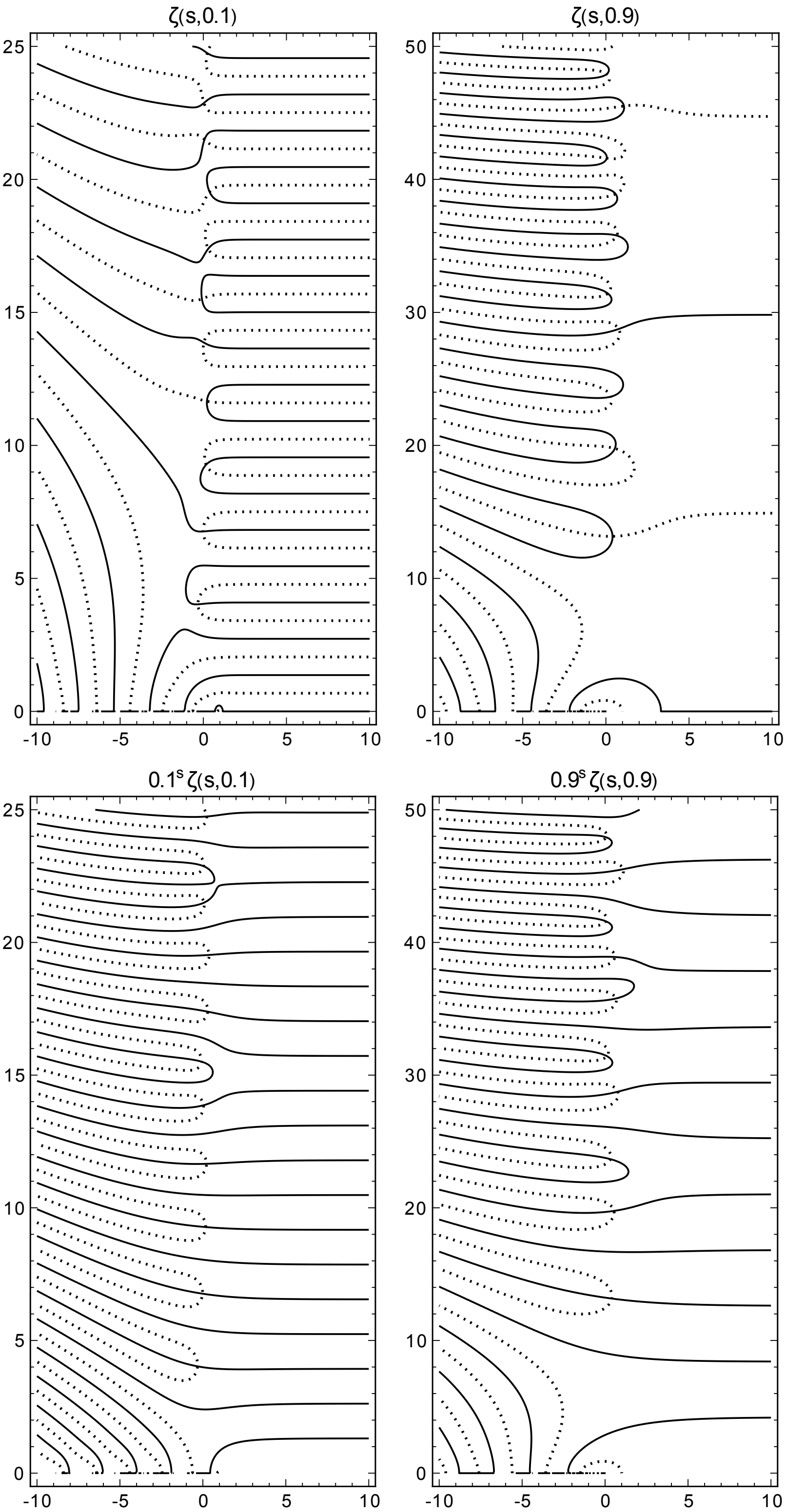

Figure 33: Real and imaginary lines of the function $\zeta(s,\alpha)$ compared to real and imaginary lines of the normalized function $\alpha^s\zeta(s,\alpha)$, for $\alpha = 0.1, 0.9$.

4 Zeros of the periodic Hurwitz zeta-function

Here we consider the zero distribution of the periodic Hurwitz zeta-function for a sequence $r = (r_m)_{m=0}^{\infty}$, $r_m \in \mathbb{C}$, with period k. In this chapter, we always suppose that T tends to plus infinity. Let

$$d = \min \{n \in \mathbb{N}_0 : r_n \neq 0\}. \tag{4.1}$$

The functions $\zeta(s, \alpha; r)$ and $r_d^{-1}\zeta(s, \alpha; r)$ have the same zeros. Without loss of generality we further suppose that $r_d = 1$. Let $\rho = \beta + i\gamma$ always denote a nontrivial zero of $\zeta(s, \alpha; r)$ (for the definition see Section 4.2 below). Let $N(T, k, \alpha)$ count the number of nontrivial zeros ρ of $\zeta(s, \alpha; r)$ with $|\gamma| \leqslant T$ (according to multiplicities).

By (2.11) and (2.9) the function $\zeta(s, \alpha; r)$ satisfies the following functional equation (for complete proof see Lemma 4.5)

$$\begin{aligned}\zeta(s, \alpha; r) = \frac{1}{k^s}\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Bigg(&\exp\left(\frac{-\pi i(1-s)}{2}\right)\zeta\left(s, 1; q^+\right)\\ &+\exp\left(\frac{\pi i(1-s)}{2}\right)\zeta\left(s, 1; q^-\right)\Bigg),\end{aligned} \tag{4.2}$$

where

$$q^{\pm}(n) = q^{\pm}(n, \alpha, d; r) = \sum_{l=0}^{k-d-1} r_{l+d} \exp\left(\pm 2\pi i n \frac{\alpha + l + d}{k}\right). \tag{4.3}$$

Define

$$j_1 = \min\left\{n \in \mathbb{N} : q^+(n) \neq 0\right\},\ j_2 = \min\left\{n \in \mathbb{N} : q^-(n) \neq 0\right\}. \tag{4.4}$$

Later we will show the existence of such j_1 and j_2 (see Lemma 4.6 in Section 4.3).

Constants j_1, j_2 and thus also $q^{\pm}(j_1), q^{\pm}(j_2)$ are easy to compute for any periodic sequence. For example, let $r_m = \exp\left(\frac{2}{3}\pi i m\right)$, then to locate j_1 we iterate through different values of n until $q^+(n) \neq 0$.

$$q^+(1) = \sum_{l=0}^{2} \exp\left(2\pi i l \frac{1}{3}\right) \exp\left(2\pi i \frac{\alpha + l}{3}\right) = 0,$$
$$q^+(2) = \sum_{l=0}^{2} \exp\left(2\pi i l \frac{1}{3}\right) \exp\left(4\pi i \frac{\alpha + l}{3}\right) = 3 \exp\left(4\pi i \frac{\alpha}{3}\right) \neq 0,$$

thus $j_1 = 2$. Similarly for q^- we have

$$q^-(1) = \sum_{l=0}^{2} \exp\left(2\pi i l \frac{1}{3}\right) \exp\left(-2\pi i \frac{\alpha + l}{3}\right) = 3 \exp\left(-2\pi i \frac{\alpha}{3}\right) \neq 0,$$

thus $j_2 = 1$.

4.1 Results

The next proposition will be the main tool in the investigation of zeros.

Proposition 4.1. *Let $0 < \alpha \leqslant 1$. Let d be defined by formula (4.1). For any sufficiently large real number b, we have*

$$\sum_{|\gamma|<T} (b+\beta) = \left(b + \frac{1}{2}\right) \frac{T}{\pi} \log \frac{T}{2\pi e} + \frac{T}{2\pi} \left(\log \left|q^+(j_1)\right| + \log \left|q^-(j_2)\right|\right)$$
$$+ \frac{T}{\pi} b \log \frac{k}{\alpha + d} + O(\log T)$$

In Proposition 4.1, substracting the case b from $b+1$, we obtain the following theorem.

Theorem 4.2. *Under the conditions of Proposition 4.1 we have*

$$N(T, k, \alpha) = \frac{T}{\pi} \log \frac{Tk}{2\pi e (\alpha + d)} + O(\log T).$$

The expression

$$\sum_{\tau<\gamma\leq T} \left(\beta - \frac{1}{2}\right) = \sum_{\tau<\gamma\leq T} (\beta + b) - \left(\frac{1}{2} + b\right) \sum_{\tau<\gamma\leq T} 1,$$

Theorem 4.2, and Proposition 4.1 with an appropriate b lead to the following statement.

Theorem 4.3. *Under the conditions of Proposition 4.1 we have*

$$\sum_{|\gamma|<T} \left(\beta - \frac{1}{2}\right) = -\frac{T}{2\pi} \log \frac{k}{(\alpha + d)} + \frac{T}{2\pi} \left(\log \left|q^+(j_1)\right| + \log \left|q^-(j_2)\right|\right) + O(\log T).$$

The last theorem shows that the zeros of the periodic Hurwitz zeta-function are clustered around the critical line $\sigma = \frac{1}{2}$.

If $r = (e^{2\pi i\lambda m})_{m=0}^{\infty}$, for rational number λ, then the periodic Hurwitz zeta-function becomes the Lerch zeta-function. Zeros of this function were investigated in [22], [17], [25], [47].

In the next section, we define nontrivial zeros of $\zeta(s,\alpha;r)$, Section 4.3 is devoted to lemmas, in Sections 4.4 and 4.5 we prove the theorem about zero-free regions and Proposition 4.1.

4.2 Zero-free regions and nontrivial zeros

By the Dirichlet series (2.10) we see that there is a positive σ_1 such that $\zeta(s,\alpha;r) \neq 0$ in the right half-plane $\sigma > \sigma_1$.

Let l be a line on the complex plane, $\varrho(s,l)$ stands for the distance of s from l, and let, for $\varepsilon > 0$,

$$L_\varepsilon(l) = \left\{ s \in \mathbb{C} : \ \varrho(s,l) < \varepsilon \right\}.$$

The next theorem gives zero-free regions on the left-hand side of the complex plane.

Theorem 4.4. *There exist constants $\sigma_0 \leqslant 0$ and $\epsilon_0 > 0$ such that $\zeta(s,\alpha;r) \neq 0$ for $\sigma < \sigma_0$ and*

$$s \notin L_{\epsilon_0}\left((\sigma-1)\log\frac{j_1}{j_2} - \pi t = \log\left|\frac{q^-(j_2)}{q^+(j_1)}\right| \right).$$

In view of above we say that the zero $\rho = \beta + i\gamma$ of $\zeta(s,\alpha;r)$ is *nontrivial* if $\sigma_0 \leq \beta \leq \sigma_1$. The zero z is called *trivial* if

$$z \in L_{\epsilon_0}\left((\sigma-1)\log\frac{j_1}{j_2} - \pi t = \log\left|\frac{q^-(j_2)}{q^+(j_1)}\right| \right).$$

Using the Rouché theorem 3.2 and arguing similarly as in the proof of Theorem 3 in [22] one can show that there are infinitely many trivial zeros.

Theorem 4.4 is proved in Section 4.4.

4.3 Lemmas

We will use the following functional equation.

Lemma 4.5. *If $\sigma < 0$, then*

$$\zeta(s,\alpha;r) = \frac{1}{k^s}\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(\exp\left(\frac{-\pi i(1-s)}{2}\right)\sum_{n=1}^{\infty}\frac{q^+(n)}{n^{1-s}}\right.$$
$$\left.+\exp\left(\frac{\pi i(1-s)}{2}\right)\sum_{n=1}^{\infty}\frac{q^-(n)}{n^{1-s}}\right).$$

Proof. The lemma follows by expression (2.11) in view of the functional equation for the Hurwitz zeta-function

$$\zeta(1-s,\alpha) = \frac{\Gamma(s)}{(2\pi)^s}\left(\exp\left(\frac{\pi i s}{2}\right)\sum_{n=1}^{\infty}\frac{e^{-2\pi i\alpha n}}{n^s} + \exp\left(-\frac{\pi i s}{2}\right)\sum_{n=1}^{\infty}\frac{e^{2\pi i\alpha n}}{n^s}\right).$$

□

Lemma 4.6. *There exist*

$$j_1 = \min\left\{n\in\mathbb{N} : q^+(n)\neq 0\right\} \quad \textit{and} \quad j_2 = \min\left\{n\in\mathbb{N} : q^-(n)\neq 0\right\}.$$

Proof. Easy to see that the sequence $(q^+(n))_{n=1}^{\infty}$ is periodic, with the period k. Contrary to the statement of Lemma 4.6 let us assume that

$$\begin{cases} q^+(1) = \sum_{l=0}^{k-1} r_l \exp\left(2\pi i\frac{\alpha+l}{k}\right) = 0 \\ q^+(2) = \sum_{l=0}^{k-1} r_l \exp\left(4\pi i\frac{\alpha+l}{k}\right) = 0 \\ \dots \\ q^+(k) = \sum_{l=0}^{k-1} r_l \exp\left(2\pi i(\alpha+l)\right) = 0. \end{cases}$$

The matrix

$$V = \begin{bmatrix} \exp\left(2\pi i\frac{\alpha}{k}\right) & \exp\left(2\pi i\frac{\alpha+1}{k}\right) & \cdots & \exp\left(2\pi i\frac{\alpha+k-1}{k}\right) \\ \exp\left(2\cdot 2\pi i\frac{\alpha}{k}\right) & \exp\left(2\cdot 2\pi i\frac{\alpha+1}{k}\right) & \cdots & \exp\left(2\cdot 2\pi i\frac{\alpha+k-1}{k}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(k\cdot 2\pi i\frac{\alpha}{k}\right) & \exp\left(k\cdot 2\pi i\frac{\alpha+1}{k}\right) & \cdots & \exp\left(k\cdot 2\pi i\frac{\alpha+k-1}{k}\right) \end{bmatrix}$$

is Vandermonde's matrix. We have

$$\det(V) = \prod_{1\leqslant j\leqslant k}\exp\left(2\pi i\frac{\alpha+j-1}{k}\right)\prod_{1\leqslant t<j\leqslant k}\left(\exp\left(2\pi i\frac{\alpha+j-1}{k}\right)\neq 0.\right.$$

Thus $r_n = 0,\ 0 \leqslant n \leqslant k-1$. This is a contradiction. Hence there exists

$$j_1 = \min \left\{ n \in \mathbb{N} :\ q^{+}(n) . \neq 0 \right\}.$$

In a similar way it is shown that there exists

$$j_2 = \min \left\{ n \in \mathbb{N} :\ q^{-}(n) \neq 0 \right\}.$$

□

Next two lemmas will be useful in the proof of Proposition 4.1.

Lemma 4.7. *For any sufficiently large real number b we have*

$$\int_{-T}^{T} \log \left| \exp\left(\frac{-\pi i (1+b-it)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{+}(n)}{n^{1+b-it}} + \exp\left(\frac{\pi i (1+b-it)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{-}(n)}{n^{1+b-it}} \right| dt$$
$$= \frac{\pi |T|^2}{4} + T \left(\log \left| q^{+}(j_1) \right| + \log \left| q^{-}(j_2) \right| \right) + O(1).$$

Proof. Let L denote the integrand in the last formula. In view of Lemma 4.6, for all sufficiently large b and any t, we have

$$\sum_{n=1}^{\infty} \frac{q^{\pm}(n)}{n^{1+b-it}} \neq 0.$$

Thus, for $t \geq 0$,

$$\begin{aligned} L &= \log \left| \exp\left(\frac{\pi i (1+b-it)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{-}(n)}{n^{1+b-it}} \right| + O(\exp(-\pi t)) \\ &= \frac{\pi t}{2} + \log \left| \sum_{n=1}^{\infty} \frac{q^{-}(n)}{n^{1+b-it}} \right| + O(\exp(-\pi t)). \end{aligned}$$

Further

$$\int_0^T \log \left| \sum_{n=1}^{\infty} \frac{q^{-}(n)}{n^{1+b-it}} \right| dt = T \log \left| q^{-}(j_2) \right| + \int_0^T \log \left| \left(1 + \sum_{n=j_1+1}^{\infty} \frac{q^{-}(n)/q^{-}(j_2)}{n^{1+b-it}} \right) \right| dt.$$

By expanding the logarithm on the right-hand side of the last formula, we get, for sufficiently large b,

$$\begin{aligned} &\int_0^T \log \left| \left(1 + \sum_{n=j_1+1}^{\infty} \frac{q^{-}(n)/q^{-}(j_2)}{n^{1+b-it}} \right) \right| dt \\ &= \int_1^T \operatorname{Re} \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{\sum_{n=j_2+1}^{\infty} \frac{q^{-}(n)}{n^{1+b-it}}}{q^{-}(j_2)} \right)^m \right) dt \\ &= \operatorname{Re} \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(q^{-}(j_2) \right)^{-m} \right) \sum_{n_1=j_2+1}^{\infty} \ldots \sum_{n_m=j_2+1}^{\infty} \frac{1}{(n_1 \ldots n_m)^{1+b}} \\ &\quad \times \frac{i \left((n_1 \ldots n_m)^{iT} - 1 \right)}{\log (n_1 \ldots n_m)} \left(q^{-}(j_2 n_1) \cdot \ldots \cdot q^{-}(j_2 n_m) \right) = O(1). \end{aligned}$$

Arguing similarly, for the case $t \leq 0$, we complete the proof of Lemma 4.7.

$\square$

Lemma 4.8. *For any sufficiently large real number a we have*

$$\int_{-T}^{T} \log \left| \zeta \left(a + it, \alpha ; r \right) \right| dt = -2Ta \log \left(\alpha + d \right) + O \left(1 \right).$$

Proof. The lemma is proved analogously to Lemma 4.7. $\square$

4.4 Proof of Theorem 4.4

Proof of Theorem 4.4. Let $\sigma < 0$. By the functional equation (see Lemma 4.5) we have $\zeta \left(s, \alpha ; r \right) \neq 0$ if

$$\exp \left(\frac{-\pi i \left(1 - s \right)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{+} \left(n \right)}{n^{1-s}} + \exp \left(\frac{\pi i \left(1 - s \right)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{-} \left(n \right)}{n^{1-s}} \neq 0.$$

Let l be a line

$$l : \ \left(\sigma - 1 \right) \log \frac{j_1}{j_2} - \pi t = \log \left| \frac{q^{-} \left(j_2 \right)}{q^{+} \left(j_1 \right)} \right|.$$

Assume a point $s_1 = \sigma_1 + it_1, \ \sigma_1 < 0$ lies over the line l and let $j_1 < j_2$. Then

$$-\pi t_1 \leqslant -\log \left| \frac{q^{+} \left(j_1 \right)}{q^{-} \left(j_2 \right)} \right| + \left(1 - \sigma_1 \right) \log \frac{j_1}{j_2}$$

We have

$$\begin{aligned}
&\left| \exp \left(\frac{\pi i \left(1 - s \right)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{-} \left(n \right)}{n^{1-s}} + \exp \left(\frac{-\pi i \left(1 - s \right)}{2} \right) \sum_{n=1}^{\infty} \frac{q^{+} \left(n \right)}{n^{1-s}} \right| \\
&\geqslant j_2^{\sigma - 1} \exp \left(\frac{\pi t}{2} \right) \left| q^{-} \left(j_2 \right) \right| - j_1^{\sigma - 1} \exp \left(\frac{-\pi t}{2} \right) \left| q^{+} \left(j_1 \right) \right| \\
&\quad - \exp \left(\frac{\pi t}{2} \right) \sum_{n=j_2+1}^{\infty} \frac{q^{-} \left(n \right)}{n^{1-\sigma}} - \exp \left(\frac{-\pi t}{2} \right) \sum_{n=j_1+1}^{\infty} \frac{q^{+} \left(n \right)}{n^{1-\sigma}} \\
&\geqslant j_2^{\sigma - 1} \exp \left(\frac{\pi t}{2} \right) \left| q^{-} \left(j_2 \right) \right| \left(1 - \left| \frac{q^{+} \left(j_1 \right)}{q^{-} \left(j_2 \right)} \right| \exp \left(-\pi t \right) \left(\frac{j_1}{j_2} \right)^{\sigma - 1} \right. \\
&\quad \left. - \frac{j_2^{1-\sigma}}{\left| q^{-} \left(j_2 \right) \right|} \left(\sum_{n=j_2+1}^{\infty} \frac{q^{-} \left(n \right)}{n^{1-\sigma}} + \exp \left(-\pi t \right) \sum_{n=j_1+1}^{\infty} \frac{q^{+} \left(n \right)}{n^{1-\sigma}} \right) \right).
\end{aligned}$$

Since $-\pi t_1 \leqslant -\log\left|\frac{q^+(j_1)}{q^-(j_2)}\right| + (1-\sigma_1)\log\frac{j_1}{j_2}$ we conclude that

$$\begin{aligned}
F_1(s_1) &:= \exp\left(-\pi t_1 + \log\left|\frac{q^+(j_1)}{q^-(j_2)}\right| - (1-\sigma_1)\log\left(\frac{j_1}{j_2}\right)\right) \\
&\quad + \frac{j_2^{1-\sigma_1}}{|q^-(j_2)|}\left(\sum_{n=j_2+1}^{\infty}\frac{q^-(n)}{n^{1-\sigma_1}} + \exp(-\pi t_1)\sum_{n=j_1+1}^{\infty}\frac{q^+(n)}{n^{1-\sigma_1}}\right) \\
&\leqslant \exp\left(-\pi t_1 + \log\left|\frac{q^+(j_1)}{q^-(j_2)}\right| - (1-\sigma_1)\log\left(\frac{j_1}{j_2}\right)\right) \\
&\quad + \frac{j_2^{1-\sigma_1}}{|q^-(j_2)|}\sum_{n=j_2+1}^{\infty}\frac{q^-(n)}{n^{1-\sigma_1}} + \left(\frac{j_2^{1-\sigma_1}}{|q^+(j_1)|} + \frac{j_1^{1-\sigma_1}}{|q^-(j_2)|}\right)\sum_{n=j_1+1}^{\infty}\frac{q^+(n)}{n^{1-\sigma_1}}.
\end{aligned}$$

Now, we choose the point s_1 such that the inequality

$$F(s_1) < 1$$

is satisfied. Let

$$A(s_1) = \left\{s \in \mathbb{C} :\ \sigma \leqslant \sigma_1,\ (\sigma - \sigma_1)\log\frac{j_1}{j_2} \leqslant \pi(t - t_1)\right\}.$$

The region $A(s_1)$ lies over the line

$$(\sigma - \sigma_1)\log\frac{j_1}{j_2} = \pi(t - t_1)$$

and on the left of the line $\sigma = \sigma_1$. If $s = \sigma + it \in A(s_1)$, then

$$\begin{aligned}
F_1(s) &= \exp\left(-\pi t + \log\left|\frac{q^+(j_1)}{q^-(j_2)}\right| - (1-\sigma)\log\left(\frac{j_1}{j_2}\right)\right) \\
&\quad + \frac{j_2^{1-\sigma}}{|q^-(j_2)|}\left(\sum_{n=j_2+1}^{\infty}\frac{q^-(n)}{n^{1-\sigma}} + \exp(-\pi t)\sum_{n=j_1+1}^{\infty}\frac{q^+(n)}{n^{1-\sigma}}\right) \\
&\leqslant F_1(s_1)
\end{aligned}$$

Thus $\zeta(s,\alpha;r) \neq 0$, for $s \in A(s_1)$.

Now, we consider a region beneath the line l and on the left from the imaginary axis. Let $s_2 = \sigma_2 + it_2$, with $\sigma_2 < 0$ and

$$-\pi t_2 \geq -\log\left|\frac{q^+(j_1)}{q^-(j_2)}\right| + (1-\sigma_2)\log\frac{j_1}{j_2}.$$

Let

$$B(s_2) = \left\{s \in \mathbb{C} :\ \sigma \leqslant \sigma_2,\ (\sigma - \sigma_1)\log\frac{j_1}{j_2} \geqslant \pi(t - t_2)\right\}.$$

Again, we can choose s_2 such that $\zeta(s,\alpha;r) \neq 0$, for $s \in B(s_2)$.

Let $\overline{r} = (\overline{r}_m)_{m=0}^{\infty}$. If $j_2 > j_1$ then the assertion of the theorem follows from the case $j_1 > j_2$ in view of the equality

$$\zeta(s, \alpha; r) = \overline{\zeta(\overline{s}, \alpha; \overline{r})}.$$

If $j_1 = j_2$ then the proof is similar as above, only simpler. Theorem 4.4 is proved.

$\square$

4.5 Proof of Proposition 4.1

Proof of Proposition 4.1. The proof follows the ideas of Levinson [52]. Let

$$G\left(s, \alpha; r\right) = \left(\alpha + d\right)^{s} \zeta\left(s, \alpha; r\right),$$

where d is defined by equality (4.1). The zeros of $G\left(s, \alpha; r\right)$ correspond exactly to the zeros of $\zeta\left(s, \alpha; r\right)$. Let $a > \sigma_1$ and $-b < \sigma_0$, where σ_0 and σ_1 are from Theorem 4.4. Applying Littlewood's lemma 3.3 to the function $G(s)$ on the rectangle with edges on the lines $t = -T$, $t = T$, $\sigma = a$, $\sigma = -b$ $(a > -b)$ we get

$$\begin{aligned}
2\pi \sum_{|\gamma|<T} (b + \beta) = & \int_{-T}^{T} \log\left|G\left(-b + it, \alpha; r\right)\right| dt - \int_{-T}^{T} \log\left|G\left(a + it, \alpha; r\right)\right| dt \\
& - \int_{-b}^{a} \arg G\left(\sigma - iT, \alpha; r\right) d\sigma + \int_{-b}^{a} \arg G\left(\sigma + iT, \alpha; r\right) d\sigma \\
=: & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Thirst we consider the integral I_1. By the functional equation (see Lemma 4.5) we obtain

$$\begin{aligned}
\log\left|\zeta\left(-b + it, \alpha; r\right)\right| = \log & \left| \frac{1}{k^{-b+it}} \frac{\Gamma\left(1 + b - it\right)}{\left(2\pi\right)^{1+b-it}} \right. \\
\times & \left. \left(\exp\left(\frac{-\pi i\left(1 + b - it\right)}{2}\right) \sum_{n=1}^{\infty} \frac{q^{+}\left(n\right)}{n^{1+b-it}} + \exp\left(\frac{\pi i\left(1 + b - it\right)}{2}\right) \sum_{n=1}^{\infty} \frac{q^{-}\left(n\right)}{n^{1+b-it}} \right) \right|.
\end{aligned}$$

Stirling's formula (3.1) gives

$$\log\left|\frac{1}{k^{-b+it}} \frac{\Gamma\left(1 + b - it\right)}{\left(2\pi\right)^{1+b-it}}\right| = \log\left|\frac{1}{k^{-b+it}}\right| + \left(b + \frac{1}{2}\right) \log\frac{|t|}{2\pi} - \frac{\pi\,|t|}{2} + O\left(\frac{1}{|t|}\right).$$

Then by Lemma 4.7 we get

$$\begin{aligned} I_1 &= -2Tb\log(\alpha+d) + 2Tb\log k + (1+2b)\,T\log\frac{T}{2\pi e} \\ &\quad + T\left(\log\left|q^{+}(j_1)\right| + \log\left|q^{-}(j_2)\right|\right) + O(1). \end{aligned}$$

Further Lemma 4.8 gives that

$$I_2 = 2Ta\log(\alpha+d) - 2Ta\log(\alpha+d) + O(1) = O(1).$$

We turn to integrals I_3 and I_4. By expression (2.11) we have, for $\sigma \geq -b$, there is a positive constant A such that

$$\zeta(\sigma \pm iT, \alpha; c) = O\left(T^{A}\right).$$

Then similarly as in Titchmarsh [69, §9.4] (see also proof of Theorem 1 in [25]) we obtain

$$\int_{-b}^{a} \left|\arg G(\sigma \pm iT, \alpha; r)\right| d\sigma = O(\log T).$$

Thus $I_4, I_3 = O(\log T)$. This proves Proposition 4.1.

$\square$

5 a-values of the periodic Hurwitz zeta-function

In this chapter, we obtain results similar to the results in the previous chapter regarding the zeros of the periodic Hurwitz zeta-function, but we consider a-values instead of zeros and treat upper and lower half of the complex plane individually. This section could be viewed as a generalization of the previous one. Quite a few results overlap, however, there are some significant differences. Roots of $\zeta(s, \alpha; r) = a$, for $a \in \mathbb{C}$, are called a-values of the periodic Hurwitz zeta-function. Throughout this chapter, we will assume that $a \neq 0$. We will reuse notations for $q^{\pm}, j_1, j_2, d$ defined by equations (4.2), (4.4) and (4.1).

In 1975, Levinson [52] showed that almost all a-values of the Riemann zeta-function are arbitrarily close to the line $\sigma = \frac{1}{2}$. Similar results are obtained by Steuding [66], [67] for the Epstein zeta-function. Results were further extended to the Selberg zeta-function by Garunkštis and Šimėnas [29]. Here we extend some of these results to the periodic Hurwitz zeta-function. The case of the Hurwitz zeta-function can be easily obtained from our results by taking $r_m = 1$.

Let $\varrho = \beta + i\gamma$ denote the nontrivial a-values of $\zeta(s, \alpha; r)$. For the definition of trivial a-values, see Section 5.2. Let $N(T, k)$ count the number of nontrivial a-values with $|\gamma| \leqslant T$ according to multiplicities. In this chapter, always $T \to \infty$.

5.1 Results

From the definition of $\zeta(s, \alpha; r)$ by a Dirichlet series, we see that there exists $\sigma'(\alpha; r)$ such that, for all $\sigma > \sigma' > 0$, the periodic Hurwitz zeta-function has no a-values.

Theorem 5.1. *Let $a \neq 0$ be a fixed complex number. For sufficiently large*

$b, c \in \mathbb{R}$, *we have*

$$2\pi \sum_{0<t<T} (b+\beta) = -T \log \left|a - r_0 \alpha^{-c}\right| + Tb \log k + \left(b + \frac{1}{2}\right) T \log \frac{T}{2\pi e}$$
$$- T(1+b) \log j_2 + T \log \left|q^{-}(j_2)\right| + O(\log T),$$

and

$$2\pi \sum_{-T<t<0} (b+\beta) = -T \log \left|a - r_0 \alpha^{-c}\right| + Tb \log k + \left(b + \frac{1}{2}\right) T \log \frac{T}{2\pi e}$$
$$- T(1+b) \log j_1 + T \log \left|q^{+}(j_1)\right| + O(\log T).$$

Now, we can use this with $b+1$ instead of b, and subtract the resulting formula from the one above. Therefore, denoting by $N^{+}(\lambda, \alpha, T)$ and $N^{-}(\lambda, \alpha, T)$ the number of nontrivial a-values of the function $\zeta(s, \alpha; r)$ in the regions $0 < t < T$ and $-T < t < 0$, respectively, we obtain

Theorem 5.2. *We have*

$$N^{+}(T, k) = \frac{T}{2\pi} \log \frac{Tk}{2\pi e j_2} + O(\log T),$$
$$N^{-}(T, k) = \frac{T}{2\pi} \log \frac{Tk}{2\pi e j_1} + O(\log T).$$

We multiply the last result by $b + \frac{1}{2}$ and subtract it from the formula of Theorem 5.1, to obtain the following corollary.

Theorem 5.3. *For sufficiently large* $c \in \mathbb{R}$, *the estimates*

$$2\pi \sum_{0<t<T} \left(\beta - \frac{1}{2}\right) = T \log \frac{\left|q^{-}(j_2)\right|}{\left|a - r_0 \alpha^{-c}\right| \sqrt{k j_2}} + O(\log T)$$

and

$$2\pi \sum_{-T<t<0} \left(\beta - \frac{1}{2}\right) = T \log \frac{\left|q^{+}(j_1)\right|}{\left|a - r_0 \alpha^{-c}\right| \sqrt{k j_1}} + O(\log T)$$

are true.

5.2 a-value free regions

In this section, we will find zero-free regions of the function $\zeta(s, \alpha; r) - a$ and define its trivial zeros.

Let $\varrho(s, l)$ be the distance of s from a line l. Let the line l be defined as

$$\sigma = 1 + \left(\log \left|\frac{q^{-}(j_2)}{q^{+}(j_1)}\right| + \pi t\right) \left(\log \frac{j_1}{j_2}\right)^{-1}.$$

Theorem 5.4. *There exist $\varepsilon(a,\sigma) > 0$ and $\sigma_0 = \sigma_0(r) < 0$ such that*

$$\zeta(s,\alpha;r) \neq a$$

if $\sigma < \sigma_0$ and $\varrho(s,l) > \varepsilon$.

Proof. The following proof is based on the functional equation (4.2). We have, for $\sigma > 1$,

$$\begin{aligned}
F(s) :=& \left|\exp\left(\frac{-\pi i(1-s)}{2}\right)\sum_{n=1}^{\infty}\frac{q^{+}(n)}{n^{1-s}} + \exp\left(\frac{\pi i(1-s)}{2}\right)\sum_{n=1}^{\infty}\frac{q^{-}(n)}{n^{1-s}}\right| \\
\geqslant& \exp\left(\frac{\pi t}{2}\right)\frac{|q^{-}(j_2)|}{j_2^{1-\sigma}} - \exp\left(-\frac{\pi t}{2}\right)\frac{|q^{+}(j_1)|}{j_1^{1-\sigma}} \\
&- \exp\left(-\frac{\pi t}{2}\right)\sum_{n=j_1+1}^{\infty}\frac{|q^{+}(n)|}{n^{1-\sigma}} - \exp\left(\frac{\pi t}{2}\right)\sum_{n=j_2+1}^{\infty}\frac{|q^{-}(n)|}{n^{1-\sigma}} \\
=& \exp\left(\frac{\pi t}{2}\right)\frac{|q^{-}(j_2)|}{j_2^{1-\sigma}}\left(1 - \exp(-\pi t)\frac{|q^{+}(j_1)|}{|q^{-}(j_2)|}\left(\frac{j_2}{j_1}\right)^{1-\sigma}\right. \\
&\left. - \exp(-\pi t)\frac{j_2^{1-\sigma}}{|q^{-}(j_2)|}\sum_{n=j_1+1}^{\infty}\frac{|q^{+}(n)|}{n^{1-\sigma}} - \frac{j_2^{1-\sigma}}{|q^{-}(j_2)|}\sum_{n=j_2+1}^{\infty}\frac{|q^{-}(n)|}{n^{1-\sigma}}\right) \\
=:& E(s)(1 - B(s)).
\end{aligned}$$

Now, let the point $s_1 = \sigma_1 + it_1$ with $\sigma_1 < 0$ lie below the line

$$l: \quad \sigma = 1 + \left(\log\left|\frac{q^{-}(j_2)}{q^{+}(j_1)}\right| + \pi t\right)\left(\log\frac{j_1}{j_2}\right)^{-1},$$

i.e.,

$$\sigma_1 \leqslant 1 + \left(\pi t_1 - \log\frac{|q^{+}(j_1)|}{|q^{-}(j_2)|}\right)\left(\log\frac{j_1}{j_2}\right)^{-1}.$$

Then

$$\begin{aligned}
B(s_1) =& \exp(-\pi t_1)\frac{|q^{+}(j_1)|}{|q^{-}(j_2)|}\left(\frac{j_2}{j_1}\right)^{1-\sigma_1} \\
&+ \exp(-\pi t_1)\frac{j_2^{1-\sigma_1}}{|q^{-}(j_2)|}\sum_{n=j_1+1}^{\infty}\frac{|q^{+}(n)|}{n^{1-\sigma_1}} + \frac{j_2^{1-\sigma_1}}{|q^{-}(j_2)|}\sum_{n=j_2+1}^{\infty}\frac{|q^{-}(n)|}{n^{1-\sigma_1}} \\
\leqslant& \exp\left(-\pi t_1 + \log\frac{|q^{+}(j_1)|}{|q^{-}(j_2)|} + (1-\sigma_1)\log\frac{j_2}{j_1}\right) \\
&+ \frac{j_1^{1-\sigma_1}}{|q^{+}(j_1)|}\sum_{n=j_1+1}^{\infty}\frac{|q^{+}(n)|}{n^{1-\sigma_1}} + \frac{j_2^{1-\sigma_1}}{|q^{-}(j_2)|}\sum_{n=j_2+1}^{\infty}\frac{|q^{-}(n)|}{n^{1-\sigma_1}}.
\end{aligned}$$

We can choose s_0 such that the right-hand side of the last inequality would be less than 1. This follows from the fact that the sum

$$\frac{j_1^{1-\sigma}}{\left|q^+ (j_1)\right|} \sum_{n=j_1+1}^{\infty} \frac{\left|q^+ (n)\right|}{n^{1-\sigma}} + \frac{j_2^{1-\sigma}}{\left|q^- (j_2)\right|} \sum_{n=j_2+1}^{\infty} \frac{\left|q^- (n)\right|}{n^{1-\sigma}}$$

is finite. Then $F (s_1) \geqslant E (s_1) (1 - B (s_1)) > 0$.

Using the same technique, for $s_2 = \sigma_2 + it_2$ lying above the line l, we can choose $\varepsilon (a, \sigma_2) > 0$ and $\sigma_0 (r) < 0$ such that, for $\sigma_2 < \sigma_0$ and $\varrho (s_2, l) > \varepsilon$,

$$\zeta (s_2, \alpha; r) \neq a.$$

The theorem is proved. □

We say that a zero of $\zeta (s, \alpha; r) - a$ is *trivial* if $\varrho (s, l) < \varepsilon$.

5.3 Proof of Theorem 5.1

Define

$$G (s, \alpha; r) = \zeta (s, \alpha; r) - a.$$

Obviously, the zeros of $G (s, \alpha; r)$ are exactly the a-values of $\zeta (s, \alpha; r)$.

Let $b, c \geqslant 3$ be constants. Then Littlewood's lemma 3.3 applied to $G (s, \alpha; r)$ on the rectangle with vertices $-b$, $-b + iT$, $c + iT$, c states

$$\begin{aligned} 2\pi \sum_{|\gamma|<T} (b + \beta) = & \int_0^T \log \left|G (-b + it, \alpha; r)\right| dt - \int_0^T \log \left|G (c + it, \alpha; r)\right| dt \\ & - \int_{-b}^{c} \arg G (\sigma, \alpha; r) \, d\sigma + \int_{-b}^{c} \arg G (\sigma + iT, \alpha; r) \, d\sigma \\ =: & I_1 - I_2 - I_3 + I_4. \end{aligned}$$

We have

$$\begin{aligned} I_2 &= \int_0^T \log \left|\zeta (c + it, \alpha; r) - a\right| dt \\ &= \int_0^T \log \left| a - \frac{r_0}{\alpha^{c+it}} - \sum_{n=1}^{\infty} \frac{r_n}{(\alpha + n)^{c+it}} \right| dt. \end{aligned}$$

It is clear that $\left|\sum_{n=1}^{\infty}\frac{r_n}{(\alpha+n)^{c+it}}\right|$ is small when c is sufficiently large, thus, using the bound $\log(1+w) = O(|w|)$, $w\to 0$, we get

$$I_2 = T\log\left|a - r_0\alpha^{-c}\right| + O(\log T).$$

To evaluate I_1 note that, by the functional equation (4.2), we have

$$\begin{aligned} I_1 &= \int_0^T \log|a-\zeta(-b+it,\alpha;r)|\,dt \\ &= \int_0^T \log\left|a - \frac{1}{k^{-b+it}}\frac{\Gamma(1+b-it)}{(2\pi)^{1+b-it}}\left(\exp\left(\frac{-\pi i(1+b-it)}{2}\right)\sum_{n=1}^{\infty}\frac{q^+(n)}{n^{1+b-it}}\right.\right. \\ &\quad \left.\left. + \exp\left(\frac{\pi i(1+b-it)}{2}\right)\sum_{n=1}^{\infty}\frac{q^-(n)}{n^{1+b-it}}\right)\right|\,dt. \end{aligned}$$

The definition of j_1 and j_2, in view of (4.4), implies

$$\begin{aligned} I_1 = \int_0^T \log\Bigg|a &- \frac{1}{k^{-b+it}}\frac{\Gamma(1+b-it)}{(2\pi)^{1+b-it}} \\ &\times\left(\exp\left(\frac{-\pi i(1+b-it)}{2}\right)\left(\frac{q^+(j_1)}{j_1^{1+b-it}} + \sum_{n=j_1+1}^{\infty}\frac{q^+(n)}{n^{1+b-it}}\right)\right. \\ &\left. + \exp\left(\frac{\pi i(1+b-it)}{2}\right)\left(\frac{q^-(j_2)}{j_2^{1+b-it}} + \sum_{n=j_2+1}^{\infty}\frac{q^-(n)}{n^{1+b-it}}\right)\right)\Bigg|\,dt. \end{aligned}$$

Denote

$$J_0 := \frac{1}{k^{-b+it}}\frac{\Gamma(1+b-it)}{(2\pi)^{1+b-it}},$$

$$J_1 := \exp\left(\frac{-\pi i(1+b-it)}{2}\right)\frac{q^+(j_1)}{j_1^{1+b-it}} + \exp\left(\frac{\pi i(1+b-it)}{2}\right)\frac{q^-(j_2)}{j_2^{1+b-it}}$$

and

$$\begin{aligned} J_2 :=& \exp\left(\frac{-\pi i(1+b-it)}{2}\right)\sum_{n=j_1+1}^{\infty}\frac{q^+(n)}{n^{1+b-it}} \\ &+ \exp\left(\frac{\pi i(1+b-it)}{2}\right)\sum_{n=j_2+1}^{\infty}\frac{q^-(n)}{n^{1+b-it}}. \end{aligned}$$

Then

$$I_1 = \int_0^T \log|a - J_0J_1 - J_0J_2|\,dt. \tag{5.1}$$

Notice that, for any fixed $t\in\mathbb{R}$,

$$\lim_{b\to\infty}\left|\frac{J_2}{J_1}\right| = 0. \tag{5.2}$$

By (5.1) and (5.2), we find

$$\log\left|1+\frac{J_2}{J_1}-\frac{a}{J_0J_1}\right|=O\left(\log T\right).$$

Thus, we obtain the estimate

$$I_1=\int_0^T\log\left|J_oJ_1\right|dt+O\left(\log T\right).$$

By the Stirling formula (3.1), we get, for $|t|>1$, that

$$\log\left|\frac{1}{k^{-b+it}}\frac{\Gamma\left(1+b-it\right)}{\left(2\pi\right)^{1+b-it}}\right|=\log\left|\frac{1}{k^{-b+it}}\right|+\left(b+\frac{1}{2}\right)\log\frac{|t|}{2\pi}-\frac{\pi\left|t\right|}{2}+O\left(\frac{1}{|t|}\right).$$

Thus, for fixed $\tau>1$,

$$\int_\tau^T\log\left|J_o\right|dt=Tb\log k+\left(b+\frac{1}{2}\right)T\log\frac{T}{2\pi e}-\frac{\pi T^2}{4}+O\left(\log T\right).$$

Now, for sufficiently large T, we have

$$J_1=\exp\left(\frac{\pi i\left(1+b-it\right)}{2}\right)\frac{q^-\left(j_2\right)}{j_2^{1+b-it}}+O\left(\exp\left(-\pi t\right)\right).$$

From this, we deduce

$$\begin{aligned}\int_0^T\log\left|J_1\right|dt=&\int_0^T\log\left|\exp\left(\frac{\pi i\left(1+b-it\right)}{2}\right)\frac{q^-\left(j_2\right)}{j_2^{1+b-it}}\right|dt+O\left(\log T\right)\\=&\int_0^T\frac{\pi t}{2}+\log\left|\frac{q^-\left(j_2\right)}{j_2^{1+b-it}}\right|dt+O\left(\log T\right)\\=&\frac{\pi T^2}{4}-T\left(1+b\right)\log\left|j_2\right|+T\log\left|q^-\left(j_2\right)\right|+O\left(\log T\right).\end{aligned}$$

Finally, it follows that

$$\begin{aligned}I_1=&Tb\log k+\left(b+\frac{1}{2}\right)T\log\frac{T}{2\pi e}\\&-T\left(1+b\right)\log\left|j_2\right|+T\log\left|q^-\left(j_2\right)\right|+O\left(\log T\right).\end{aligned}$$

It remains to estimate the horizontal integrals $I_3,\ I_4$. By analogy to [25], see also [10], in any bounded strip with a certain constant $c>0$

$$\zeta\left(s,\alpha\right)=O\left(|t|^c\right).$$

Using Theorem 2 from [37] which states that, for $\sigma>\frac{1}{2}$,

$$\frac{1}{T}\int_0^T\left|\zeta\left(\sigma+it,\alpha;r\right)\right|^2dt\ll 1$$

and Jensen's inequality, we get that

$$\int_0^T \log |\zeta (\sigma + it, \alpha; r)| dt = O(1).$$

Thus, we conclude that, for fixed $\sigma \geq -b$, there exist a positive constant A such that

$$\zeta (\sigma \pm iT, \alpha; r) = O\left(T^A\right).$$

By analogy to [69, §9.4], see also the proof of Theorem 1 in [25],

$$\int_{-b}^{a} |\arg G (\sigma \pm iT, \alpha; r)| \, d\sigma = O(\log T).$$

Collecting together the above estimates, we obtain the assertions of the theorem.

6 Zeros of the derivative of the Lerch zeta-function

Behavior of the derivate of the Riemann zeta function is well studied (see Section 2.5), as well as the zeros of the Lerch zeta-function (see Section 2.7). In this chapter we aim to improve the asymptotic formula for the number of nontrivial Lerch derivative zeros (obtained by Steuding and Garunkštis's [25]) and show, that they are clustered around the critical line. Results obtained in this chapter could be further extended by analyzing k-th derivate. This would provide an extension of Berndt results about the Riemann zeta-function [6], but will require more complicated machinery, thus is left for future studies.

In this chapter T always tends to plus infinity, and all implicit constants depend on parameters $0 < \lambda, \alpha \leqslant 1$. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined by (2.12). Using modified version of the functional equation (2.13), it can be continued analytically to the whole complex plane, except possibly the point $s = 1$ where it can have a simple pole.

The distribution of the zeros right of the critical line of $L(\lambda, \alpha, s)$, say if α is a transcendental number, is very different from that of $\zeta(s)$. Denote by $N(\sigma', T; \alpha, \lambda)$ the number of zeros of $L(\lambda, \alpha, s)$ in the region $\sigma > \sigma'$, $0 \leq t \leq T$. It is known (see Titchmarsh [69, §9.15]) that for the Riemann zeta-function

$$N(\sigma, T) := N(\sigma, T; 1, 1) = O(T),$$

if $\sigma > \frac{1}{2}$ and for the Lerch zeta-function

$$N(\sigma, T; \alpha, \lambda) \asymp T,$$

if $\frac{1}{2} < \sigma < 1 + 0.6\alpha$ and α is a transcendental number [47, §8.4].

In general, for the Lerch zeta-function, the RH is not true. Also, the functional equation is rather asymmetric, so we do not expect that the distribution of the nontrivial zeros is symmetric with respect to the critical line. However, Garunkštis and Steuding [25], using a similar technique as Levinson [52], showed that almost all nontrivial zeros of the $L(\lambda, \alpha, s)$ are clustered around the critical line (for other related results see section 2.7), we will explore this further for the derivative of the Lerch zeta-function. As mentioned in introduction Berndt (2.8) proved that

$$N(T) = N_k(T) + \frac{T \log 2}{2\pi} + O(\log T).$$

We will obtain similar bound for the derivative of the Lerch zeta-function.

We state main results and definitions in the following section. Section 6.2 contains proofs of theorems related to zero-free regions of $L'(\lambda, \alpha, s)$. Next, in Section 6.3, we explore the number of nontrivial zeros till the given size and their expected distance from line $\sigma = \frac{1}{2}$.

6.1 Results

Define

$$L'(\lambda, \alpha, s) = \frac{\partial}{\partial s} L(\lambda, \alpha, s).$$

Theorems 6.1, 6.2 and 6.3 identify the zero-free regions of the derivative of the Lerch zeta-function and explore the trivial zero locations. First, we locate zero-free regions on the right half-plane.

Theorem 6.1. *If* $0 < \alpha < 1$, $t \in \mathbb{R}$ *and*

$$\sigma > \max\left\{2, \left(\log\log\alpha^{-1} - \log\left(\frac{1}{2e} + \frac{3\log 2 + 2}{4}\right)\right)(\log\alpha)^{-1}\right\},$$

then $L'(\lambda, \alpha, \sigma + it) \neq 0$. *Also, for* $\sigma > 3.6$, *we have* $L'(\lambda, 1, \sigma + it) \neq 0$.

Notice that when α is close to 1, then the bound in Theorem 6.1 tends to infinity. For empirical evidence, see Figure 61, where we explore the trajectory of zero of the derivative of the Lerch zeta-function, when λ is fixed to $\frac{3}{4}$ and α varies.

Let l be the line defined by

$$l: \quad \sigma = 1 - \pi t\left(\log\left(\frac{\lambda}{1-\lambda}\right)\right)^{-1}, \quad \lambda \neq \frac{1}{2}, 1.$$

Let $d(s, l)$ be the distance of s from the line l. On the left half-plane, the zero-free regions are identified by the following theorems.

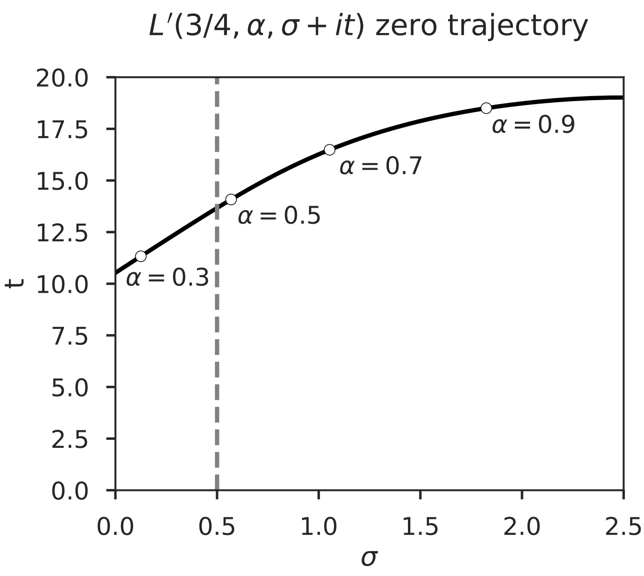

Figure 61: $L'\left(\frac{3}{4}, \alpha, s\right)$ zero real part tends to ∞ as $\alpha \to 1$.

k	Lerch (diff)	Lerch derivative (diff)
1	5.026747e-02	0.921093
5	1.068303e-04	0.611752
10	5.704532e-08	0.447896
15	3.002190e-11	0.384430
20	1.432145e-14	0.348708

Table 61: Absolute differences from s_k and trivial zeros of $L(\lambda, \alpha, s)$ and $L'(\lambda, \alpha, s)$.

Let

$$s_k := \begin{cases} \sigma_k + i\frac{1}{\pi}(\sigma_k - 1)\log\frac{1-\lambda}{\lambda}, & \text{for } \lambda \neq \frac{1}{2}, 1 \\ 2(1 - \alpha + k), & \text{otherwise} \end{cases}$$

where

$$\sigma_k = 1 + \frac{-2\alpha - 2k + 1}{1 + \pi^{-2}\log^2\frac{1-\lambda}{\lambda}}, \quad k \in \mathbb{Z}.$$

Notice that the numbers s_k lie on the line l. With the help of the Rouché's theorem (see Theorem 3.2), we show that the actual zeros lie in the rectangles which contain s_k.

Theorem 6.2. *Let $\lambda \neq \frac{1}{2}, 1$. For any $\varepsilon > 0$ there is $\sigma' = \sigma'(\varepsilon) < 0$ such that there is one zero in each parallelogram with vertices $(s_k + s_{k\pm1})/2 \pm \varepsilon$, for $\sigma_k < \sigma'$, and there are no other zeros on the half-plane $\sigma < \sigma'$.*

An analogous result is valid for $\lambda = \frac{1}{2}$ and $\lambda = 1$.

Theorem 6.3. *If $\lambda = \frac{1}{2}$ or 1, then there exists σ'' such that, for all $\sigma \leqslant \sigma''$, there is exactly one zero in each interval $(-2(n+\alpha)-1, -2(n+\alpha)+1)$ lying on the real axis, for $n > \frac{1}{2} - \sigma''/2 - \alpha$, $n \in \mathbb{N}$ and there are no other zeros on the half-plane $\sigma \leqslant \sigma''$.*

Fix $\varepsilon > 0$ and $\sigma'(\varepsilon)$ in Theorem 6.2. Let $\Re s_0 \leqslant \sigma'(\varepsilon)$ and $L'(\lambda, \alpha, s_0) = 0$, then s_0 is called trivial if $d(s_0, l) < \varepsilon$, for $\lambda \neq \frac{1}{2}, 1$. In case $\lambda = \frac{1}{2}$ or $\lambda = 1$, s_0 lies on the real axis.

Our empirical results (see table 61) show that the trivial zeros of the Lerch zeta-function and the derivative of the Lerch zera-function approach s_k as k increases. Notice that the derivative of the Lerch zeta-function does this much slower: for $k = 100$, the absolute difference from s_k to the actual zero of the Lerch zeta-function is still ≈ 0.23. Computations were executed using *mpmath*[1].

Let $N_{triv.}(\lambda, \alpha, R)$ and $N'_{triv.}(\lambda, \alpha, R)$ count the number of the trivial zeros of the Lerch zeta-function and the derivative of the Lerch zeta-function (according to multiplicities) which are at a distance $\leqslant R$ from the origin. From Theorem 6.2 and the proof of Theorem 3 in [22], it is easy to see that the following corollary is true.

Corollary 6.4. *Let $0 < \lambda, \alpha \leqslant 1$. The difference between the number of trivial zeros of the Lerch zeta-function and those of their derivative is bounded by*

$$\left| N_{triv.}(\lambda, \alpha, R) - N'_{triv.}(\lambda, \alpha, R) \right| \ll 1 \quad (R \to \infty).$$

Let $N'(\lambda, \alpha, T)$ count the number of the nontrivial zeros $\rho' = \beta' + i\gamma'$ (according to multiplicities) of the derivative of the Lerch zeta-function with $0 < \gamma' \leqslant T$.

Theorem 6.5. *Let $0 < \lambda, \alpha \leqslant 1$. We have*

$$N'(\lambda, \alpha, T) = \frac{T}{2\pi} \log \frac{T}{2\pi e \lambda ([\alpha] + \alpha)} + O(\log T).$$

This result contains the same leading term as shown in Steuding and Garunkštis's [25] proof sketch, but has an improved error term from $o(T)$ to $O(\log T)$.

The number of nontrivial zeros at $L'(\lambda, \alpha, s)$ in the lower half-plane can be obtained by the formula $L'(\lambda, \alpha, s) = \overline{L'(1 - \{\lambda\}, \alpha, \overline{s})}$.

[1]F. Johansson et. al., *mpmath: a Python library for arbitrary-precision floating-point arithmetic* (version 0.18), 2013. *http://mpmath.org/.*

The next theorem shows that nontrivial zeros of the derivative of the Lerch zeta-function are close to critical line $\sigma = \frac{1}{2}$. Let logarithmic integral be defined as

$$\mathrm{li}\,(x) = \int_2^x \frac{dt}{\log t}.$$

Theorem 6.6. *If* $0 < \lambda, \alpha \leqslant 1$, *then*

$$\sum_{0<\gamma'\leqslant T} \left(\beta' - \frac{1}{2}\right) = \frac{T}{2\pi} \log\log \frac{T}{2\pi\lambda} + \frac{T}{4\pi} \log \frac{[\alpha] + \alpha}{\lambda} - \frac{T}{2\pi} \log \left|\log\left([\alpha] + \alpha\right)\right| - \lambda \mathrm{li} \frac{T}{2\pi\lambda} + O\left(\log T\right).$$

This improves the result of Steuding and Garunkštis [25] result

$$\sum_{|\gamma'|\leqslant T} \left(\beta' - \frac{1}{2}\right) = \frac{T}{\pi} \log\log T + O(T).$$

6.2 Proofs of Theorems related to zero-free regions

In this section, we will prove the theorems concerning the zero-free regions and the trivial zero positions (see Theorems 6.1, 6.2 and 6.3).

Proof of Theorem 6.1. Let $\sigma > 1$ and $\alpha \neq 1$. Then

$$\begin{aligned} -\frac{\alpha^s}{\log \alpha} L'\left(\lambda, \alpha, s\right) &= \sum_{n=0}^{\infty} e^{2\pi i \lambda n} \frac{\log\left(n + \alpha\right)}{\log \alpha} \left(\frac{\alpha}{n+\alpha}\right)^s \\ &= 1 + \frac{\alpha^s}{\log \alpha} \sum_{n=1}^{\infty} e^{2\pi i \lambda n} \frac{\log\left(n+\alpha\right)}{\left(n+\alpha\right)^s} \\ &=: 1 + D\left(\lambda, \alpha, s\right). \end{aligned}$$

If $\left|D\left(\lambda, \alpha, s\right)\right| < 1$, then $L'\left(\lambda, \alpha, s\right) \neq 0$.

For $x > e^{1/\sigma} - \alpha$,

$$\left(\frac{\log\left(x+\alpha\right)}{\left(x+\alpha\right)^\sigma}\right)_x' = \frac{1 - \sigma \log\left(x+\alpha\right)}{\left(x+\alpha\right)^{\sigma+1}} < 0,$$

thus we have

$$\sum_{n=3}^{\infty} \frac{\log\left(n+\alpha\right)}{\left(n+\alpha\right)^\sigma} \leqslant \int_2^{\infty} \frac{\log\left(x+\alpha\right)}{\left(\alpha+x\right)^\sigma} dx = \frac{1 + (\sigma-1)\log\left(\alpha+2\right)}{\left(\sigma-1\right)^2 \left(\alpha+2\right)^{\sigma-1}}.$$

Then

$$|D(\lambda,\alpha,s)| \leqslant \frac{\alpha^{\sigma}}{\log\alpha^{-1}}\left(\frac{\log(1+\alpha)}{(1+\alpha)^{\sigma}}+\frac{\log(2+\alpha)}{(2+\alpha)^{\sigma}}+\frac{1+(\sigma-1)\log(2+\alpha)}{(\sigma-1)^{2}(2+\alpha)^{\sigma-1}}\right).$$

Now, we will find $\sigma_0=\sigma_0(\lambda,\alpha)$ such that, for all $\sigma>\sigma_0$, we have $|D(\lambda,\alpha,\sigma+it)|<1$.

Let $\sigma\geqslant 2$. The function $f(\alpha)=\frac{\log(1+\alpha)}{(1+\alpha)^{\sigma}}$ attains its maximum at $\alpha=\sqrt{e}-1$. Therefore,

$$\frac{\log(1+\alpha)}{(1+\alpha)^{\sigma}} \leqslant \frac{1}{2e}.$$

Clearly

$$\frac{\log(2+\alpha)}{(2+\alpha)^{\sigma}} \leqslant \frac{\log 2}{4}$$

and

$$\frac{1+(\sigma-1)\log(2+\alpha)}{(\sigma-1)^{2}(2+\alpha)^{\sigma-1}} \leqslant \frac{1+\log 2}{2}.$$

Substituting those bounds, we obtain

$$|D(\lambda,\alpha,\sigma+it)| \leqslant \frac{\alpha^{\sigma}}{\log\alpha^{-1}}\left(\frac{1}{2e}+\frac{3\log 2+2}{4}\right).$$

This gives that $|D(\lambda,\alpha,\sigma+it)|<1$ if

$$\sigma > \left(\log\log\alpha^{-1}-\log\left(\frac{1}{2e}+\frac{3\log 2+2}{4}\right)\right)(\log\alpha)^{-1}.$$

When $\alpha=1$, then arguing analogously we can see, that $L'(\lambda,1,s)$ has no zeros for $\sigma>3.6$. This proves Theorem 6.1. $\square$

Lemma 6.7. *We have*

$$L'(\lambda,\alpha,s) = -G(\lambda,\alpha,s)\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\exp(-2\pi i\alpha\lambda)\log s,$$

where, for $\sigma<0$,

$$G(\lambda,\alpha,s) = \left(\exp\left(\pi i\frac{1-s}{2}\right)\lambda^{s-1}+\exp\left(-\pi i\frac{1-s}{2}+2\pi i\alpha(1+[\lambda])\right)\right.$$
$$\left.(1-\{\lambda\})^{s-1}\right)\left(1+O\left(\frac{1}{\log|s|}\right)\right) \quad (|s|\to\infty).$$

Proof. Differentiation of the functional equation (2.13) yields

$$\begin{aligned}
L'(\lambda,\alpha,s) = &- \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left(\exp\left(\pi i \frac{1-s}{2} - 2\pi i\alpha\lambda\right)\left(L'(-\alpha,\lambda,1-s)\right.\right. \\
&+ L(-\alpha,\lambda,1-s)\left(\frac{\Gamma'}{\Gamma}(1-s) - \log 2\pi + \frac{\pi i}{2}\right)\Bigg) \qquad (6.1) \\
&+ \exp\left(-\pi i\frac{1-s}{2} + 2\pi i\alpha(1-\{\lambda\})\right)\left(L'(\alpha,1-\{\lambda\},1-s)\right. \\
&\left.\left.+ L(\alpha,1-\{\lambda\},1-s)\left(\frac{\Gamma'}{\Gamma}(1-s) - \log 2\pi - \frac{\pi i}{2}\right)\right)\right) \\
=: &- G(\lambda,\alpha,s)\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\exp(-2\pi i\alpha\lambda)\log s.
\end{aligned}$$

Here we define $\log s$ by choosing the principal branch on the real axis, as $\sigma \to +\infty$ and using analytic continuation for other values. Notice that $\Gamma(1-s)$ and $\log s$ has no zeros for $\sigma < 0$ and $\Gamma(1-s)$ has poles only when $s \in \{1,2,3,\dots\}$. Hence, for $\sigma < 0$, $L'(\lambda,\alpha,s)$ has zeros if and only if $G(\lambda,\alpha,s)$ is zero. For $\sigma \to -\infty$, by the formula (2.12) we have

$$L(\alpha,\lambda,1-s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i\alpha n}}{(n+\lambda)^{1-s}} = \lambda^{s-1} + O\left((1+\lambda)^{\sigma}\right); \qquad (6.2)$$

$$L'(\alpha,\lambda,1-s) = -\lambda^{s-1}\log\lambda + O\left((1+\lambda)^{\sigma}\right); \qquad (6.3)$$

uniformly in t. Also from the Stirling formula (3.1) $\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1})$, for $\Re(s) > 0$, and relation $\frac{\Gamma'}{\Gamma}(s) = \frac{\Gamma'}{\Gamma}(1-s) - \pi\cot(\pi s)$, we obtain that, for $\sigma < 0$, $t > 0$ and $|s| \to \infty$,

$$\frac{\Gamma'}{\Gamma}(1-s) - \log(2\pi) \pm \frac{\pi i}{2} = \log\left(\pm\frac{s}{2\pi i}\right) + O\left(\frac{1}{|s|}\right). \qquad (6.4)$$

In view of (6.2), (6.3) and (6.4) we obtain

$$\begin{aligned}
G(\lambda,\alpha,s) = &\exp\left(\pi i\frac{1-s}{2}\right)\left(-\lambda^{s-1}\frac{\log\lambda}{\log s} + O\left(\frac{(1+\lambda)^{\sigma}}{\log|s|}\right)\right. \\
&\left.+ \left(\lambda^{s-1} + O\left((1+\lambda)^{\sigma}\right)\right)\left(1 + O\left(\frac{1}{\log|s|}\right)\right)\right) \\
&+ \exp\left(-\pi i\frac{1-s}{2} + 2\pi i\alpha(1-\{\lambda\}+\lambda)\right)\left(-(1-\{\lambda\})^{s-1}\frac{\log(1-\{\lambda\})}{\log s}\right. \\
&\left.+ O\left(\frac{(2-\{\lambda\})^{\sigma}}{\log|s|}\right) + \left((1-\{\lambda\})^{s-1} + O\left((2-\{\lambda\})^{\sigma}\right)\right)\left(1 + O\left(\frac{1}{\log|s|}\right)\right)\right) \\
= &\exp\left(\pi i\frac{1-s}{2}\right)\lambda^{s-1}\left(1 + O\left(\frac{1}{\log|s|}\right)\right) \\
&+ \exp\left(-\pi i\frac{1-s}{2} + 2\pi i\alpha(1+[\lambda])\right)(1-\{\lambda\})^{s-1}\left(1 + O\left(\frac{1}{\log|s|}\right)\right).
\end{aligned}$$

□

It is left to consider the case when $\lambda = \frac{1}{2}$ and $\lambda = 1$. We will use the Rouché's theorem 3.2 to prove Theorems 6.2 and 6.3.

For $\sigma < -1$, denote

$$\begin{aligned} u(\lambda, \alpha, s) :=& \exp\left(-\pi i \frac{1-s}{2} + 2\pi i \alpha (1 + [\lambda])\right) (1 - \{\lambda\})^{s-1} \\ &+ \exp\left(\pi i \frac{1-s}{2}\right) \lambda^{s-1}; \end{aligned} \tag{6.5}$$

$$\begin{aligned} v(\lambda, \alpha, s) :=& G(\lambda, \alpha, s) - u(\lambda, \alpha, s) \\ =& O\left(\frac{|u(\lambda, \alpha, s)|}{\log |s|}\right) \quad (|s| \to \infty). \end{aligned} \tag{6.6}$$

Proof of Theorem 6.2. First, we show that there are zero-free regions above and below the line l, then we locate the trivial zeros of the derivative of the Lerch zeta-function more precisely.

Let $0 < \lambda < \frac{1}{2}$. Using asymptotic behavior and notations introduced in Lemma 6.7 and factoring out the leading nonzero term, we arrive at

$$\begin{aligned} G(\lambda, \alpha, s) =& \exp\left(\pi i \frac{1-s}{2}\right) \lambda^{s-1} \left(1 + \exp(-\pi i (1 - s - 2\alpha)) \left(\frac{1-\lambda}{\lambda}\right)^{s-1}\right. \\ &\left. + O\left(\frac{1 + \exp(-\pi t) \left(\frac{1-\lambda}{\lambda}\right)^{\sigma-1}}{\log |s|}\right)\right) \quad (|s| \to \infty). \end{aligned} \tag{6.7}$$

Let $s = \sigma + it$ be a point over the line l, with $t \geqslant \pi$ and $\sigma < 0$. Then

$$0 < \left|\exp(-\pi i (1 - s - 2\alpha)) \left(\frac{1-\lambda}{\lambda}\right)^{s-1}\right| = \exp(-\pi t) \left(\frac{1-\lambda}{\lambda}\right)^{\sigma-1} < 1$$

and from the error term in (6.7), it is clear that, for any $\varepsilon > 0$, there exists such $s_1 = \sigma_1 + it_1$, with sufficiently small negative σ_1, so that $G(\lambda, \alpha, s_1) \neq 0$ and $d(s_1, l) = \varepsilon$. Define the region to the left of σ_1 and above the line, which is parallel to l and goes through s_1, as follows

$$A(s_1) := \left\{s \in \mathbb{C} : \quad \sigma \leqslant \sigma_1,\ t \geqslant t_1 + \frac{1}{\pi} (\sigma - \sigma_1) \log \frac{1-\lambda}{\lambda}\right\}.$$

If $s \in A(s_1)$, then

$$\exp(-\pi t) \left(\frac{1-\lambda}{\lambda}\right)^{\sigma-1} \leqslant \exp(-\pi t_1) \left(\frac{1-\lambda}{\lambda}\right)^{\sigma_1-1}$$

and from (6.7) we conclude that $G(\lambda, \alpha, s) \neq 0$.

Next, we consider the zeros from the region below the line l. Similarly, as above we see that there is a number $s_2 = s_2(\varepsilon)$ such that the region

$$B\left(s_2\right) := \left\{ s \in \mathbb{C} : \quad \sigma \leqslant \sigma_2,\ t \leqslant t_2 + \frac{1}{\pi}\left(\sigma - \sigma_2\right) \log \frac{1-\lambda}{\lambda} \right\} \tag{6.8}$$

contains no zeros of $G\left(\lambda, \alpha, s\right)$.

When s lies on the line l, then from Lemma 6.7 and formulas (6.5)-(6.8) it is clear, that, for any $\varepsilon > 0$, we can choose $\sigma' = \sigma'(\varepsilon) < 0$ such that

$$\left|u\left(\lambda, \alpha, s \pm \varepsilon\right)\right| - \left|v\left(\lambda, \alpha, s \pm \varepsilon\right)\right| > 0, \tag{6.9}$$

if $\sigma \leqslant \sigma'$. Next, we will indicate horizontal segments such that $|u| > |v|$.

From Lemma 6.7 and definitions (6.5) and (6.6) we see that $L'\left(\lambda, \alpha, s\right)$ has the same zeros as $u\left(\lambda, \alpha, s\right) + v\left(\lambda, \alpha, s\right)$, for $\sigma < -1$. Let

$$s_k' := 1 - \frac{2\alpha + 2k}{\pi + \pi^{-1}\log^2 \frac{1-\lambda}{\lambda}} \left(\pi + i \log \frac{1-\lambda}{\lambda}\right) = \sigma_k' + it_k', \quad k \in \mathbb{Z},$$

then

$$\exp\left(\pi i \frac{1-s_k'}{2}\right) \lambda^{s_k'-1} = \exp\left(-\pi i \frac{1-s_k'}{2} + 2\pi i \alpha\right) (1-\lambda)^{s_k'-1}.$$

Let $\delta \in [-\varepsilon, \varepsilon]$, for $\varepsilon > 0$ small enough. When $\sigma < -1$, then $u\left(\lambda, \alpha, s\right)$ zeros are located at s_k and clearly

$$u\left(\lambda, \alpha, s_k' - \delta\right) \neq 0. \tag{6.10}$$

From the definitions (6.5) and (6.6) we have

$$\frac{v\left(\lambda, \alpha, s_k' - \delta\right)}{u\left(\lambda, \alpha, s_k' - \delta\right)} = O\left(\frac{1}{\log\left|s_k' - \delta\right|}\right) \quad (k \to \infty). \tag{6.11}$$

By (6.10) and (6.11) it is clear that there is $\sigma'' \leqslant -1$ such that, for all $\sigma_k' \leqslant \sigma''$, we have

$$\left|u\left(\lambda, \alpha, s_k' - \delta\right)\right| - \left|v\left(\lambda, \alpha, s_k' - \delta\right)\right| > 0. \tag{6.12}$$

From equations (6.9), (6.12) and the fact that u and v are holomorphic by Rouché's theorem 3.2, we find that $u\left(\lambda, \alpha, s\right)$ and $u\left(\lambda, \alpha, s\right) + v\left(\lambda, \alpha, s\right)$ have the same number of zeros in the parallelograms with vertices $\sigma_{k-1}' + \varepsilon + it_{k-1}'$, $\sigma_k' + \varepsilon + it_k'$, $\sigma_k' - \varepsilon + it_k'$ and $\sigma_{k-1}' - \varepsilon + it_{k-1}'$, where $\sigma_k' < \sigma_{k-1}' \leqslant \min\left(\sigma', \sigma''\right)$.

The case, when $\frac{1}{2} < \lambda < 1$, is derived using $L'\left(\lambda, \alpha, s\right) = \overline{L'\left(1-\lambda, \alpha, \overline{s}\right)}$. $\square$

Proof of Theorem 6.3. Let $\lambda = \frac{1}{2}$ or 1. First if $s = -2(m+\alpha) + i + \delta$ for $\delta \in [-1, 1]$. Then by Theorem 6.2 we can choose such $m' > 0$ that for all $m > m'$

$$|u(\lambda, \alpha, -2(m+\alpha) + i + \delta)| \geqslant |v(\lambda, \alpha, -2(m+\alpha) + i + \delta)|.$$

The case $s = -2(m+\alpha) - i + \delta$ is analogous. Next, for $k = n$ and $k = n+1$, we have

$$\left|\exp\left(\pi i (n+\alpha) + \frac{\pi}{2}\delta\right)\right| = \left|\exp\left(\pi i (n+1-\alpha) + \frac{\pi}{2}\delta\right)\right| = \exp\frac{\pi\delta}{2},$$

thus again arguing similarly as in Theorem 6.2 we can choose n' such that, for all $n > n'$, we have

$$|u(\lambda, \alpha, -2(n+\alpha) + 1 + \delta i)| \geqslant |v(\lambda, \alpha, -2(n+\alpha) + 1 + \delta i)|.$$

The Rouché's theorem is valid for the rectangle with vertices $-2(n+\alpha) \pm 1 \pm i$, for $n \geqslant \max(m', n')$. Notice that since $u(\lambda, \alpha, s)$ has only one zero in each rectangle, that zero has to lie on the real axis.

Now, let $|t| \geqslant 1$ and $\sigma < -1$. Then from $e^{-\pi t} < 1$ and Lemma 6.7 it is clear that we can choose $\sigma' < -1$, such that $G(\lambda, \alpha, s') \neq 0$. Finally, for all $\sigma \leqslant \sigma'$, we have $G(\lambda, \alpha, s) \neq 0$, since

$$\left|\frac{1 + \exp(\pi i (\sigma + it))}{\log(\sigma + it)}\right| \leqslant \left|\frac{1 + \exp(\pi i (\sigma' + it))}{\log(\sigma' + it)}\right|.$$

From this and proof of Lemma 6.7 the theorem statement follows with

$$\sigma'' \leqslant \min(\sigma', 1 - 2(\max(m', n') + \alpha)).$$

□

6.3 Proofs of Theorems related to the nontrivial zero distribution

In this section, we will prove Theorems 6.5 and 6.6. For the proofs, we will need the following proposition.

Proposition 6.8. *Let $b > 2$ be a constant such that all nontrivial zeros of $L'(\lambda, \alpha, s)$ are in $\Re(s) > -b$. Then*

$$\begin{aligned}2\pi \sum_{0<\gamma'\leqslant T} (b+\beta') =& bT \log \frac{T}{2\pi e\lambda\left([\alpha]+\alpha\right)} + \frac{T}{2}\log\frac{T}{2\pi e} \\ &+ T\log\log\frac{T}{2\pi\lambda} - T\log\left(\lambda\left|\log\left([\alpha]+\alpha\right)\right|\right) \\ &- 2\pi\lambda \mathrm{li}\frac{T}{2\pi\lambda} + O_b\left(\log T\right).\end{aligned}$$

Proof. Define

$$Z\left(\lambda,\alpha,s\right) = -\frac{\left(\alpha+[\alpha]\right)^s}{\log\left(\alpha+[\alpha]\right)} L'\left(\lambda,\alpha,s\right).$$

Notice that the zeros of $Z\left(\lambda,\alpha,s\right)$ are exactly the zeros of $L'\left(\lambda,\alpha,s\right)$. Let $b, c \geqslant 2$ be constants such that all nontrivial zeros have real parts $-b < \beta < c$ (existence of such constants follows from Theorems 6.2, 6.3). Let $N'\left(\sigma, T\right)$ denote the number of nontrivial zeros ρ of $L'\left(\lambda,\alpha,s\right)$ with $\beta > \sigma$ and $\gamma \leqslant T$. Also let T be such that $L'\left(\lambda,\alpha,s\right)$ is zero-free on the line joining the points $-b+iT$ and $c+iT$, otherwise we can take $T+\varepsilon$ for some small ε. Similarly, if $L'\left(\lambda,\alpha,s\right)$ has zero on the segment joining $-b$ and c, then without loss of generality we shift this segment by some fixed number in the imaginary direction, so that it is zero-free. Littlewood's Lemma 3.3 applied to $Z\left(\lambda,\alpha,s\right)$ on the rectangle $\mathcal{R}$ with vertices $-b,\ -b+iT,\ r+iT,\ r$ states

$$\int_{\mathcal{R}} \log Z\left(\lambda,\alpha,s\right) ds = -2\pi i \int_{-b}^{c} N'\left(\sigma,T\right) d\sigma,$$

where $\log Z\left(\lambda,\alpha,s\right)$ is defined by choosing principal branch of the logarithm on the real axis, as $\sigma \to \infty$ and obtaining other values by analytic continuation. Hence

$$\begin{aligned}2\pi \sum_{0<\gamma'\leqslant T}(b+\beta') =& \int_0^T \log\left|Z\left(\lambda,\alpha,-b+it\right)\right| dt - \int_0^T \log\left|Z\left(\lambda,\alpha,c+it\right)\right| dt \\ &- \int_{-b}^{c} \arg Z\left(\lambda,\alpha,\sigma\right) d\sigma + \int_{-b}^{c} \arg Z\left(\lambda,\alpha,\sigma+iT\right) d\sigma \\ =&: I_1 - I_2 - I_3 + I_4.\end{aligned}$$

We start with I_2

$$I_2 = \Re\frac{1}{i}\int_c^{c+iT} \log Z\left(\lambda,\alpha,s\right) ds.$$

Notice that $Z(\lambda,\alpha,s)\neq 0$ for $\sigma \geqslant c$. Then by Cauchy theorem 3.1

$$\int_{c}^{c+iT} \log Z\left(\lambda,\alpha,s\right) ds = \int_{c}^{c+\Delta} \log Z\left(\lambda,\alpha,s\right) ds$$
$$+ \int_{c+\Delta}^{c+\Delta+iT} \log Z\left(\lambda,\alpha,s\right) ds + \int_{c+\Delta+iT}^{c+iT} \log Z\left(\lambda,\alpha,s\right) ds.$$

Now, since

$$Z(\lambda,\alpha,s) = 1 + O\left(\left(\frac{\alpha}{1+\alpha}\right)^{\sigma}\right) \quad \left(\sigma \to +\infty\right),$$

for $\Delta \to \infty$ we obtain

$$\int_{c}^{\infty} \log Z\left(\lambda,\alpha,s\right) d\sigma \ll 1,$$

thus $I_2 = O\left(1\right)$.

We turn to I_1. Using the functional equation (6.1) and asymptotic formulas (6.2), (6.3), (6.4) similarly as in Lemma 6.7, we obtain

$$I_1 = \int_{0}^{T} \log\left|Z\left(\lambda,\alpha,-b+it\right)\right| dt = \int_{0}^{T} \log\left|\frac{\left(\alpha+[\alpha]\right)^{-b+it}}{\log\left(\alpha+[\alpha]\right)} \frac{\Gamma\left(1+b-it\right)}{\left(2\pi\right)^{1+b-it}}\right| dt$$
$$+ \int_{0}^{T} \log\left|\exp\left(\pi i \frac{1+b-it}{2}\right)\right| dt + \int_{0}^{T} \log\left|1 + O\left(\exp\left(-\pi t\right)\right)\right| dt$$
$$+ \int_{0}^{T} \log\left|\left(\lambda^{-b+it-1} + O\left(\left(1+\lambda\right)^{-b+it-1}\right)\right)\right.$$
$$\left.\times\left(\log\frac{-b+it}{2\pi i\lambda} + O\left(\frac{1}{|-b+it|}\right)\right)\right| dt$$

Using Stirling's formula (3.1) we get, for $|t|>1$,

$$\log\left|\Gamma(b+1-it)\right| = \left(b+\frac{1}{2}\right)\log|t| - \frac{\pi|t|}{2} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|t|}\right),$$

thus

$$\int_{\tau}^{T} \log\left|\frac{\left(\alpha+[\alpha]\right)^{-b+it}}{\log\left(\alpha+[\alpha]\right)} \frac{\Gamma\left(1+b-it\right)}{\left(2\pi\right)^{1+b-it}}\right| dt = -T\log\left|\log\left(\alpha+[\alpha]\right)\right|$$
$$- Tb\log\left(\alpha+[\alpha]\right) + \left(b+\frac{1}{2}\right) T\log\frac{T}{2\pi e} - \frac{\pi T^2}{4} + O\left(\log T\right).$$

For fixed $\tau > 2$, we have

$$\int_{\tau}^{T} \log\left|\log\frac{-b+it}{2\pi i\lambda}\right| dt = \int_{\tau}^{T} \log\left(\log\frac{t}{2\pi\lambda} + O\left(\frac{1}{t}\right)\right) dt$$
$$= T\log\log\frac{T}{2\pi\lambda} - 2\pi\lambda \mathrm{li}\frac{T}{2\pi\lambda} + O\left(\log\log T\right).$$

Finally using Cauchy theorem 3.1 the same way as in I_2 and collecting everything together, we obtain

$$\begin{aligned}I_1 =& bT\log\frac{T}{2\pi e\lambda\left([\alpha]+\alpha\right)}+\frac{T}{2}\log\frac{T}{2\pi e}\\&+T\log\log\frac{T}{2\pi\lambda}-T\log\left(\lambda\left|\log\left([\alpha]+\alpha\right)\right|\right)\\&-2\pi\lambda\mathrm{li}\frac{T}{2\pi\lambda}+O_b\left(\log T\right).\end{aligned}$$

It remains to estimate the horizontal integrals I_3 and I_4. We will use the same technique as in [25], with some minor changes, so we skip some details and demonstrate only the main idea. Define

$$f(z)=\frac{1}{2}\left(Z\left(z+it\right)+\overline{Z\left(\overline{z}+it\right)}\right),$$

then $f(\sigma)=\Re Z(\sigma+it)$. Now, suppose that $f(\sigma)$ has q zeros in the interval $-b<\sigma<c$ and let $T>2R=2(c+b)$. Denote the number of zeros of $f(z)$ in $|z-c|\leqslant r$ by $n(r)$. It is known (see Garunkštis [22]) that for any bounded strip there is a positive number a such that $L\left(\lambda,\alpha,\sigma+it\right)\ll t^a$. Thus, by the Cauchy formula for the derivative 3.1, it is also true that $L'\left(\lambda,\alpha,\sigma+it\right)\ll t^a$. It follows that $f(z)\ll t^a$, and using the Jensen formula 3.4 we obtain $n(R)=O(\log T)$. Interval $[-b,c]$ can be subdivided into at most $q+1$ subintervals in which $\Re Z(\sigma+iT)$ is of constant sign, thus $|\arg Z(\sigma+iT)|\leqslant(q+1)\pi\leqslant(n(R)+1)\pi$. Therefore $I_4=O(\log T)$ and $I_3=O(\log T)$. Finally, collecting estimates of I_1,I_2,I_3 and I_4 together we obtain theorem claim.

Notice that results, for the region $-T<\gamma'<-\tau$, can be easily obtained from $L'\left(\lambda,\alpha,s\right)=\overline{L'\left(1-\{\lambda\},\alpha,\overline{s}\right)}$. □

Proof of Theorem 6.5. We use proposition 6.8 with $b+1$ instead of b, and subtract the resulting formula from the original one thus obtaining theorem claim. □

Proof of Theorem 6.6. Multiplying equations from Theorem 6.5 by $b+\frac{1}{2}$ and subtracting them from the equation obtained in proposition 6.8 yields theorem claim. □

7 The Lerch zeta-function for equal parameters

For most values of parameters λ and α, the zeros of the Lerch zeta-function $L(\lambda, \alpha, s)$ are distributed very chaotically (see section 2.7). In this chapter, we consider the special case of equal parameters $L(\lambda, \lambda, s)$ and show by calculations that the nontrivial zeros either lie extremely close to the critical line $\sigma = \frac{1}{2}$ or are distributed almost symmetrically with respect to the critical line. We also investigate this phenomenon theoretically and show that there is a Speiser type relation between zeros of the Lerch zeta-function and its derivative.

Let $s = \sigma + it$. Denote by $\{\lambda\}$ the fractional part of a real number λ. In this chapter, ε is any positive real number and T always tends to plus infinity. In all theorems and lemmas, the numbers λ and α are fixed constants.

Let l be a straight line in the complex plane $\mathbb{C}$, and denote by $\varrho(s, l)$ the distance of s from l. Define, for $\delta > 0$,

$$L_\delta(l) = \left\{ s \in \mathbb{C} : \ \varrho(s, l) < \delta \right\}.$$

In Garunkštis and Laurinčikas [22], Garunkštis and Steuding [24], for $0 < \lambda < 1$ and $\lambda \neq \frac{1}{2}$, it is proved that $L(\lambda, \alpha, s) \neq 0$ if $\sigma < -1$ and

$$s \notin L_{\log \frac{4}{\pi}} \left(\sigma = \frac{\pi t}{\log \frac{1-\lambda}{\lambda}} + 1 \right).$$

For $\lambda = \frac{1}{2}, 1$, from Spira [65] and [22] we see that $L(\lambda, \alpha, s) \neq 0$ if $\sigma < -1$ and $|t| \geq 1$. Moreover, in [22] it is showed that $L(\lambda, \alpha, s) \neq 0$ if $\sigma \geq 1 + \alpha$. We say that a zero of $L(\lambda, \alpha, s)$ is *nontrivial* if it lies in the strip $-1 \leq \sigma < 1 + \alpha$ and we denote a nontrivial zero by $\rho = \beta + i\gamma$.

In this chapter, we investigate the zero distribution of the Lerch zeta-function $L(\lambda, \alpha, s)$ when the parameters are equal, i.e. $\lambda = \alpha$. The motivation for this are calculations which show that the first nontrivial zeros of $L(\lambda, \lambda, s)$ are often located almost on the critical line $\sigma = \frac{1}{2}$. Next are the first 4 zeros (rounded to two decimal numbers) for several parameter values.

$L(1/3, 1/3, s) : 0.50 + 3.99i,\ 0.50 + 7.28i,\ 0.50 + 9.54i,\ 0.50 + 12.18i.$

$L(1/3, 2/3, s) : 0.86 + 5.68i,\ 0.53 + 9.59i,\ 0.86 + 12.66i,\ 0.49 + 15.11i.$

$L(3/4, 3/4, s) : 0.50 + 9.69i,\ 0.50 + 15.26i,\ 0.50 + 18.65i,\ 0.50 + 23.05i.$

$L(1/4, 3/4, s) : 1.03 + 5.24i,\ 0.64 + 8.81i,\ 0.76 + 11.96i,\ 0.88 + 14.19i.$

For a rational number $\lambda \neq \frac{1}{2}, 1$ it is expected that the function $L(\lambda, \lambda, s)$ has many zeros off the critical line. Our calculations then show that the zeros are almost symmetrically distributed with respect to the critical line. For example, for $L(3/4, 3/4, s)$ we have the following zeros: $-0.10 + 120.60i$ and $1 + 0.10 + 120.60i$; $0.37 + 202.77i$ and $1 - 0.37 + 202.77i$. Usually, such symmetry of zeros can be explained by the shape of the functional equation. A typical example is the Heillbronn Davenport zeta-function. Possibly such symmetry forces zeros to stay on the critical line more often. More on this see, for example, Bombieri and Hejhal [9], Balanzario and Sánchez-Ortiz [5], Garunkštis and Šimėnas [30], Vaughan [74].

For $\lambda = \alpha$, we can rewrite the functional equation (2.13) as

$$\begin{aligned}
\overline{L(\lambda, \lambda, 1 - \overline{s})} =& (2\pi)^{-s}\Gamma(s)e^{-\pi i \frac{s}{2} + 2\pi i \lambda^2} L(\lambda, \lambda, s) \\
&+ (2\pi)^{-s}\Gamma(s)e^{\pi i \frac{s}{2} - 2\pi i (1-\lambda)\lambda} L(1 - \lambda, 1 - \{\lambda\}, s) \\
=& G(s)L(\lambda, \lambda, s) + P(s).
\end{aligned} \tag{7.1}$$

By the bound for the Lerch zeta-function and by the Stirling formula (3.1) we see that, for any vertical strip, $|P(s)| < t^A e^{-\pi t}$ and $|G(s)| \geq t^B$ (see Lemma 7.7 and its proof below). Thus the shape of the formula (7.1) suggests that the nontrivial zeros of $L(\lambda, \lambda, s)$ should be distributed almost symmetrically with the respect of the critical line. However, calculations in the next section show that this symmetry is not strict.

Denote by $N(\lambda, \alpha, T)$ the number of nontrivial zeros of the function $L(\lambda, \alpha, s)$ in the region $0 < t < T$. For $0 < \lambda, \alpha \leqslant 1$, we have [22]

$$N(\lambda, \alpha, T) = \frac{T}{2\pi} \log \frac{T}{2\pi e \alpha \lambda} + O(\log T). \tag{7.2}$$

7.1 Results

Propositions 7.1 and 7.2 will be proved for the parameters $0 < \lambda, \alpha \leqslant 1$. Both of these propositions, with a special case $\lambda = \alpha$, will be used in proofs of main theorems regarding symmetrical distribution (see 7.3) and Speiser type relation between zeros of the Lerch zeta-function and its derivative (see 7.4).

Proposition 7.1. *For* $0 < \lambda, \alpha \leqslant 1$,

$$\sum_{0<\gamma\leqslant T} \left(\beta - \frac{1}{2}\right) = \frac{T}{4\pi} \log \frac{\alpha}{\lambda} + O(\log T).$$

Note, that if $\lambda = \alpha$ we are left only with error term, thus we conclude, that in the upper half-plane nontrivial zeros of the Lerch zeta-function with equal parameters on average are symmetrically distributed with a small error term.

Now, we consider the symmetry of the individual zeros. Let ρ be a zero of $L(\lambda, \lambda, s)$. In view of (7.1) and Rouché's theorem 3.2 we see that $L(\lambda, \lambda, s)$ has an almost symmetrical zero in some small disc $|s - (1 - \overline{\rho})| < r$ if $P(s)$ is small and $L(\lambda, \lambda, s)$ is not very small on the edge of the disc. Thus we need a bound from below for $L(\lambda, \lambda, s)$ when s is close to a zero.

Proposition 7.2. *Let* $0 < \lambda, \alpha \leq 1$. *Let* $\sigma_0 \in \mathbb{R}$ *and* $\Re s \geq \sigma_0$. *Let* $L(\lambda, \alpha, s) \neq 0$ *and* d *be the distance from* s *to the nearest zero of* $L(\lambda, \alpha, s)$. *Then*

$$\frac{1}{|L(\lambda, \alpha, s)|} < \exp(C(|\log d| + 1) \log t),$$

where $C = C(\lambda, \alpha, \sigma_0)$ *is a positive constant.*

The proposition will help us to prove the following theorem.

Theorem 7.3. *Let* $0 < \lambda \leq 1$ *and* $A > 0$ *be such that* $AC < \pi$, *where* $C = C(\lambda, \alpha, -1)$ *is from Proposition 7.2. Let* $\rho = \beta + i\gamma$ *be a nontrivial zero of* $L(\lambda, \lambda, s)$. *If* γ *is sufficiently large, then there is a radius* r,

$$\exp(-A\gamma/\log\gamma) \leq r \leq \exp(-A\gamma/\log\gamma)\log^2\gamma,$$

such that the discs

$$|s - \rho| < r \quad \textit{and} \quad |s - (1 - \overline{\rho})| < r$$

contain the same number of zeros.

By the formula (7.2) it follows that there is a constant $D = D(T_0)$ such that, for $T > T_0$,

$$\left| N(\lambda, \alpha, T) - \frac{T}{2\pi} \log \frac{T}{2\pi e \alpha \lambda} \right| \leq D \log T.$$

Let $f : [T, T+U] \to \mathbb{R}$ be a continuous function. Let $N(T, U, f)$ (resp. $N'(T, U, f)$) be the number of nontrivial zeros of $L(\lambda, \lambda, s)$ (resp. $L'(\lambda, \lambda, s)$) in $T < t < T+U$, $\sigma < f(t)$.

Theorem 7.4. *Let $0 < \lambda \leq 1$ and $0 < U \leq T$. Assume that, for some T_0 and $0 < \varepsilon < 1$,*

$$D(T_0) < \frac{\varepsilon}{\log 4}. \tag{7.3}$$

Then, for sufficiently large T, there is a continuous function $f : [T, T+U] \to \mathbb{R}$ such that

$$\frac{1}{2} - \exp\left(-\frac{T^{1-\varepsilon}}{\log T} \right) \leq f(t) \leq \frac{1}{2},$$

and

$$N(T, U, f) = N'(T, U, f) + O(\log T).$$

We discuss the condition (7.3). For the Riemann zeta-function ($= L(1, 1, s)$) it is known that $D < 0.12 < 1/\log 4 = 0.72\ldots$ (Trudgian [70]). If $\lambda = \alpha = \frac{1}{2}$, then $D < 0.16$ (Trudgian [71]). Moreover, for the Riemann zeta-function the Lindelöf hypothesis implies that the constant D can be chosen as small as we please (Titchmarsh [69, Theorem 13.6(A)]). We expect the Lindelöf type hypothesis also for the Lerch zeta-function ([24], [20]). Similarly as in the case of the Riemann zeta-function (Titchmarsh [69, Sections 13.6 and 13.7]), it is possible to modify the proof of Theorem 3.2 in [47, Chapter 8] and to show that the Lindelöf type hypothesis for $L(\lambda, \alpha, s)$ implies that, for any $0 < \lambda, \alpha < 1$, the constant D can be chosen as small as we please.

In the next section we present the computer calculations related to Theorems 7.3 and 7.4. Sections 7.3, 7.4, and 7.5 contain proofs of Theorem 7.1, Proposition 7.2, and Theorem 7.3 respectively.

7.2 Computations

This section is devoted to the more precise calculations of the first nontrivial zeros. If a nontrivial zero ρ of $L(\lambda, \lambda, s)$ lies on the critical line, then by the functional equation (7.1) we have $L(1-\lambda, 1-\lambda, \rho) = 0$. Similarly, if $L(\lambda, \lambda, s)$ has symmetrical zeros ρ and $1-\overline{\rho}$ then again $L(1-\lambda, 1-\lambda, \rho) = 0$. Let $\rho_1 = 0.50...+9.69...i$, $\rho_2 = 0.50...+15.26...i$, $\rho_3 = 0.50...+18.65...i$, $\rho_4 = 0.50...+23.05...i$ be the first four zeros of $L(3/4, 3/4, s)$ indicated in the Introduction. We have

$|L(1/4, 1/4, \rho_1)| = 2.73...,$

$|L(1/4, 1/4, \rho_2)| = 0.13...,$

$|L(1/4, 1/4, \rho_3)| = 0.48...,$

$|L(1/4, 1/4, \rho_4)| = 1.15....$

Thus the zeros $\rho_1, \rho_2, \rho_3, \rho_4$ of $L(3/4, 3/4, s)$ do not lie on the critical line. Using arbitrary-precision floating-point arithmetic computations, we get that

$\Re\rho_1 = 0.5 + 7.16... \cdot 10^{-14},$

$\Re\rho_2 = 0.5 - 6.08... \cdot 10^{-23},$

$\Re\rho_3 = 0.5 - 4.53... \cdot 10^{-27},$

$\Re\rho_4 = 0.5 - 1.11... \cdot 10^{-32}.$

The last four lines were computed in the following two ways: one by using *findroot* and the other by computing the contour integral which encloses only one zero ρ of $L(3/4, 3/4, s)$. For more details on computation methodology see the end of this section.

In the upper half-plane, the first pair of almost symmetrical zeros of $L(3/4, 3/4, s)$ is $-0.10... + 120.59...i$ and $1.10... + 120.59...i$. These zeros are not strictly symmetrical, since

$|L(1/4, 1/4, 1.10... + 120.59...i)| = 3.94... \neq 0,$

$|L(1/4, 1/4, -0.10... + 120.59...i)| = 23.49... \neq 0.$

Further, we give a table (see Table 71) where the number of nontrivial zeros in $0 < t < 300$ is calculated for various cases of $L(\lambda, \lambda, s)$. For all those zeros, we have checked that $L(\lambda, \lambda, \rho) = 0$ implies $L(1-\lambda, 1-\lambda, \rho) \neq 0$ if $\lambda \neq \frac{1}{2}$. Thus, in Table 71, all zeros, except the case $\lambda = \frac{1}{2}$, are not strictly symmetrical with respect to the critical line.

Table 71: Distribution of the nontrivial zeros of $L(\lambda, \lambda, \sigma + it)$ in $0 < t < 300$. N_1 is the total number of the nontrivial zeros; N_2 is the number of the nontrivial zeros satisfying $\left|\Re\rho - \frac{1}{2}\right| > 10^{-9}$, these zeros appear in almost symmetrical pairs; in the last column of the table, we have $100N_2/N_1$.

λ	N_1	N_2	%
1/2	203	0	0.00
5/9	193	28	14.51
4/7	191	24	12.57
3/5	186	14	7.53
5/8	182	22	12.09
2/3	176	18	10.23
7/10	171	28	16.37
5/7	169	30	17.75
3/4	165	20	12.12
7/9	161	26	16.15
4/5	159	22	13.84
5/6	155	22	14.19
6/7	151	28	18.54
7/8	150	30	20.00
8/9	149	22	14.77
9/10	147	24	16.33

We turn to zero trajectories of $L(\lambda, \lambda, s)$ and its derivative. The idea to explore zero trajectories is inspired by Garunkštis and Steuding paper [26]. By $L^{(v)}(\lambda, \lambda, s)$ we denote that vth derivative of $L(\lambda, \lambda, s)$ with respect to s:

$$L^{(v)}(\lambda, \lambda, s) = \frac{\partial^v}{\partial s^v} L(\lambda, \lambda, s).$$

Suppose that $\rho = \rho(\lambda_0)$ is a zero of multiplicity m of $L(\lambda_0, \lambda_0, s)$ (i.e. $L^{(v)}(\lambda_0, \lambda_0, \rho(\lambda_0)) = 0$, $v = 0, 1, \ldots, m-1$, $L^{(m)}(\lambda_0, \lambda_0, \rho(\lambda_0)) \neq 0$). From the expression of the Lerch zeta-function by the Dirichlet series and the functional equation (2.13), it follows that, for any s, the function $f(\lambda) = L(\lambda, \lambda, s)$ is continuous in $\lambda \in (0, 1)$. By Rouché's theorem 3.2, we have for every sufficiently small open disc D with center at ρ in which the function $L(\lambda_0, \lambda_0, s)$ has no other zeros except for ρ, there exists $\delta = \delta(D) > 0$ such that each function $L(\lambda, \lambda, s)$, where $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, has exactly m zeros (counted with multiplicities) in the disc D (c.f. Theorem 1 in Balanzario and Sánchez-Ortiz [5] and Lemma 4.1 in Dubickas, Garunkštis, J. Steuding and R. Steuding [15]). If zero ρ is of multiplicity $m = 1$, then there exists a neighborhood of λ_0 and some function $\rho = \rho(\lambda)$, which is continuous at λ_0 and, in addition, satisfies the relation $L(\lambda, \lambda, \rho(\lambda)) = 0$. This way, we can speak about the continuous zero trajectory $\rho(\lambda)$. Similarly, the trajectories of the zeros of the derivative $L'(\lambda, \lambda, s)$ are understood.

In Figure 71, we see parametric plots of the trajectories of the zeros of $L(\lambda, \lambda, s)$ and its derivative, solid and dotted lines respectively.

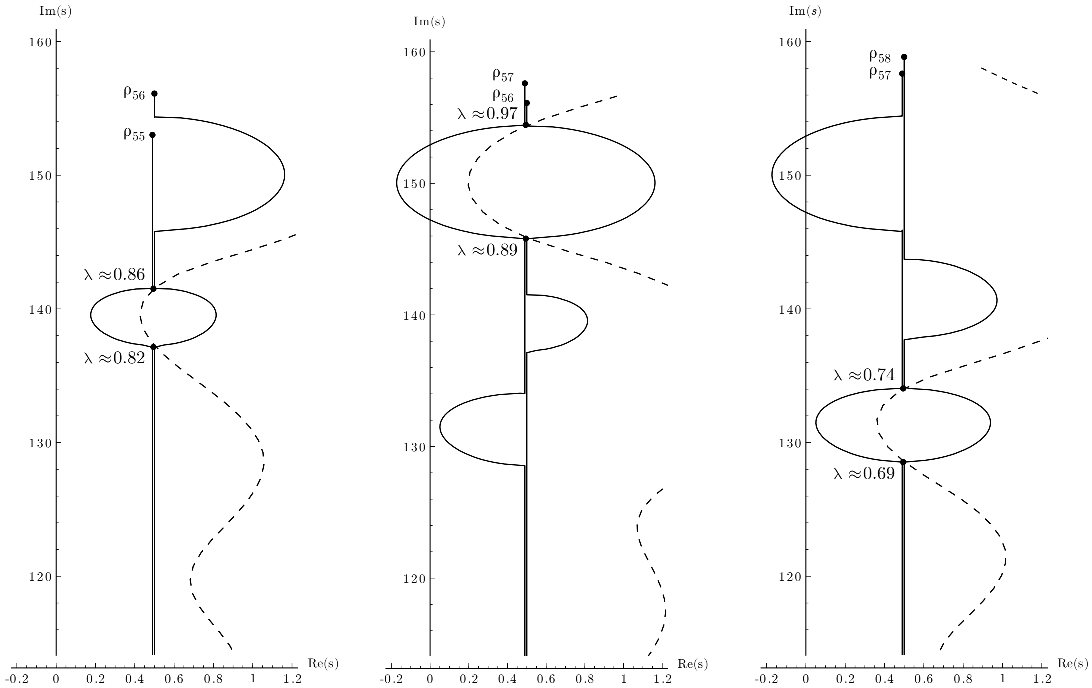

Figure 71: Parametric graphics of the several trajectories of the zeros with parameter $\frac{1}{2} \leq \lambda \leq 1$. Solid and dashed trajectories are the trajectories of the zeros of $L(\lambda, \lambda, s)$ and $L'(\lambda, \lambda, s)$ respectively. By $\rho_{55}, \ldots, \rho_{58}$ we denote 55th,..., 58th zeros of $\zeta(s) = L(1, 1, s)$. The trajectories of the zeros of the derivative correspond to the zeros $1.27 + 152.61i$ (left), $0.97 + 156.63i$ (middle), and $0.86 + 158.28i$ (right) of $\zeta'(s) = L'(1, 1, s)$.

Figure 71 can be compared to Figures 1 and 2 in Garunkštis and Šimėnas [25], where the trajectories of the zeros of the linear combination $f(s, \tau)$ of Dirichlet L-functions and its derivative $f'_s(s, \tau)$ is calculated. In Figure 71, we see that the trajectories of zeros approach the (almost) meeting point (after which the trajectories leave a neighborhood of the critical line) from the same direction, while the trajectories of the zeros of $f(s, \tau)$ approach the meeting point from the opposite directions. Notice that the nontrivial zeros of this linear combination of Dirichlet L-functions are distributed strictly symmetrically with the respect of the critical line. Because of this fact, the meeting point in Figures 1 and 2 in [25] is always a double zero $f(s, \tau)$.

To find the trajectories of the zeros, $\rho(\lambda)$ and $q(\lambda)$, $0 \le \lambda \le 1$,such that

$$L(\lambda,\lambda,\rho(\lambda)) = 0 \quad \text{and} \quad L'(\lambda,\lambda,q(\tau)) = 0,$$

we solve the differential equations numerically

$$\frac{\partial\rho(\lambda)}{\partial\lambda} = -\frac{\frac{\partial\ell(\lambda,\rho)}{\partial\lambda}}{\frac{\partial\ell(\lambda,\rho)}{\partial\rho}} \quad \text{and} \quad \frac{\partial q(\lambda)}{\partial\lambda} = -\frac{\frac{\partial^2\ell(\lambda,q)}{\partial q\partial\lambda}}{\frac{\partial^2\ell(\lambda,q)}{\partial q^2}},$$

where $\ell(\lambda,s) = L(\lambda,\lambda,s)$. As the initial conditions, some zeros of $L(1,1,s) = \zeta(s)$ and $L'(\lambda,\lambda,s)$, $\lambda = 0.86, 0.97, 0.74$ are used.

Computations were validated with the help of Python with *mpmath*[1] package. We used the following expression of the Lerch zeta-function for rational parameters

$$\begin{aligned} L\left(s,\frac{b}{d},\frac{b}{d}\right) &= \sum_{k=0}^{d-1}\sum_{m=0}^{\infty}\frac{\exp\left(2\pi i\frac{b}{d}\left(dm+k\right)\right)}{\left(dm+k+\frac{b}{d}\right)^s} \\ &= d^{-s}\sum_{k=0}^{d-1}\exp\left(2\pi i\frac{b}{d}k\right)\zeta\left(s,\frac{kd+b}{d^2}\right), \end{aligned}$$

where $\zeta(s,\alpha)$, $0<\alpha\le 1$, is the Hurwitz zeta-function. The function $\zeta(s,\alpha)$ is implemented by the command *zeta*. Zero locations were calculated using *findroot* with Muller's method. Note, that for non rational parameters approximate positions of the Lerch zeta-function zeros can be evaluated using approximation given in [19].

In this thesis, all computer computations should be regarded as heuristic because their accuracy was not controlled explicitly.

7.3 Proof of Proposition 7.1

In [25], it was proved that, for $0 < \lambda, \alpha \leqslant 1$,

$$\sum_{|\gamma|\leqslant T}\left(\beta-\frac{1}{2}\right) = \frac{T}{2\pi}\log\frac{\alpha}{\sqrt{\lambda(1-\{\lambda\})}} + O(\log T). \tag{7.4}$$

Proposition 7.1 can be derived from the proof of the formula (7.4). Namely, from the proof of Theorem 1 in [25] we derive the following lemma.

[1]Fredrik Johansson and others. mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 0.18), December 2013. http://mpmath.org/.

Lemma 7.5. *Let $b \geqslant 3$ be a constant. For $0 < \lambda, \alpha \leqslant 1$,*

$$\sum_{0<\gamma\leqslant T} (b+\beta) = \left(b+\frac{1}{2}\right)\frac{T}{2\pi}\log\frac{T}{2\pi e\alpha\lambda} + \frac{T}{4\pi}\log\frac{\alpha}{\lambda} + O(\log T).$$

Then the equality

$$\sum_{0<\gamma\leqslant T}\left(\beta-\frac{1}{2}\right) = \sum_{0<\gamma\leqslant T}(b+\beta) - (b+\frac{1}{2})\sum_{0<\gamma\leqslant T} 1$$

together with the zero counting formula (7.2) gives Proposition 7.1.

7.4 Proof of Proposition 7.2

We start from the following lemma.

Lemma 7.6. *If $f(s)$ is regular, and*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M$$

in $\{s : |s-s_0| \leqslant r\}$ with $M > 1$, then

$$\left|\frac{f(s_0)}{f(s)}\prod_\rho s - \frac{\rho}{s_0} - \rho\right| < e^{CM}$$

for $|s-s_0| \leqslant \frac{3}{8}r$, where C is some constant and ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leqslant \frac{1}{2}r$.

Proof. The lemma follows immediately from the proof of Lemma α in Titchmarsh [69, §3.9]. □

To apply Lemma 7.6, we need information about the growth of the Lerch zeta-function. For each σ, we define a number $\mu(\sigma) = \mu(\lambda, \alpha, \sigma)$ as the lower bound of numbers ξ such that $L(\lambda, \alpha, \sigma + iT) \ll T^\xi$.

Lemma 7.7. *Let $0 < \lambda, \alpha \leq 1$ and $\sigma_0 < 0$. Then*

$$\mu(\sigma) \leq \begin{cases} \frac{1}{2} - \sigma & \textit{if } \sigma_0 \leq \sigma \leq 0, \\ \frac{1}{2} + \left(\frac{64}{205} - 1\right)\sigma & \textit{if } 0 \leq \sigma \leq \frac{1}{2}, \\ \frac{64}{205}(1-\sigma) & \textit{if } \frac{1}{2} \leq \sigma \leq 1, \\ 0 & \textit{if } \sigma \geq 1. \end{cases}$$

Proof. In [20], it is proved that

$$L(\lambda, \alpha, \frac{1}{2} + it) \ll t^{\frac{32}{205}+\varepsilon} \qquad (t \to \infty),$$

and from the approximation of the Lerch zeta-function by a finite sum (see Garunkštis [47, Theorem 1.2 in Chapter 3]) we see that $L(\lambda, \alpha, \sigma + it) \ll t^{\varepsilon}$, for $\sigma \geq 1$. Now, the lemma follows by the Phragmén-Lindelöf theorem (see Titchmarsh [68, §5.65]) and by the functional equation (2.13) given Stirling's formula (3.1)

$$|\Gamma(1-s)| = \sqrt{2\pi}|t|^{\frac{1}{2}-\sigma} e^{-\frac{\pi|t|}{2}}(1 + O(|t|^{-1})) \qquad (|t| \to \infty),$$

uniformly for $\sigma_0 < \sigma \leq \frac{1}{2}$. □

Proof of Proposition 7.2. To prove the proposition we choose $f(s) = L(\lambda, \alpha, s)$, $s_0 = 3 + it$, and a sufficiently large but fixed radius r in Lemma 7.6. In view of Lemma 7.7 we take $M = b \log T$, where $b = b(r)$. The function $1/L(\lambda, \alpha, s_0)$ is bounded. By the formula (7.2) for some nontrivial zeros, we have the number of zeros in the disc $|s - s_0| < \frac{1}{2}r$ is $< c \log \Im s_0$. This proves Proposition 7.2.

□

7.5 Proof of Theorem 7.3

Proof of Theorem 7.3. If $\lambda = \frac{1}{2}, 1$, then given equalities (2.14) it is well known that the non-real complex number ρ is a zero of $L(\lambda, \lambda, s)$ if and only if $1 - \overline{\rho}$ is also a zero of $L(\lambda, \lambda, s)$.

Next we assume that $0 < \lambda < 1$ and $\lambda \neq \frac{1}{2}$. By the formula (7.2), we see that the number of zeros in the disc $|s - \rho| < \exp(-A\gamma/\log^2 \gamma)$ is $< c \log \Im \rho$. Let $r_k = k \exp\left(-A\gamma/\log \gamma\right)$, $k = 1, \dots, [c \log \gamma] + 1$. By Dirichlet's box principle, there is $1 \leq \ell \leq [c \log \gamma]$ such that $L(\lambda, \lambda, s)$ has no zeros for the ring

$$r_\ell < |s - \rho| \leq r_{\ell+1}.$$

Let

$$r = \frac{r_\ell + r_{\ell+1}}{2} = \left(\ell + \frac{1}{2}\right) \exp\left(-\frac{A\gamma}{\log \gamma}\right).$$

Suppose $G(s)$ and $P(s)$ are defined by the functional equation (7.1). Let $C_R = \{s : |s - \rho| = r\}$ and $C_L = \{s : |s - (1 - \overline{\rho})| = r\}$. The steps of the proof are the following. If $L(\lambda, \lambda, s)$ has N zeros inside of C_R, then by Rouché's theorem 3.2 we expect that $G(s)L(\lambda, \lambda, s) + P(s)$ has N zeros inside C_R. Then by the functional equation (7.1), the function $L(1 - \lambda, \lambda, 1 - s)$ has N zeros inside C_R, then by conjugation $L(\lambda, \lambda, s)$ has N zeros inside C_L. Next, we need to justify the step involving Rouché's theorem.

Notice that $G(s)$ has no zeros. By Rouché's theorem 3.2, the functions $G(s)L(\lambda, \lambda, s)$ and $G(s)L(\lambda, \lambda, s) + P(s)$ have the same number of zeros inside of the circle C_r if on this circle the inequality

$$|P(s)| < |G(s)L(\lambda, \lambda, s)| \tag{7.5}$$

is valid.

In view of the growth of the Lerch zeta-function (see Lemma 7.7) we get that, for sufficiently large t and $-1.4 \leq \sigma \leq 2$,

$$|P(s)| < |\Gamma(s)|t^2 e^{-\pi t/2} \quad \text{and} \quad |G(s)| \geq (2\pi)^{-2}|\Gamma(s)|e^{\pi t/2}.$$

Proposition 7.2 gives, for $s \in C_R$,

$$L(\lambda, \lambda, s) \gg \exp\left(-\left(AC + o(1)\right)\gamma\right),$$

where $AC < \pi$. Thus the inequality (7.5) is valid. By this Theorem 7.3 is proved.

□

Notice that from this proof we have the quantity $\log^2 \gamma$ in the inequality (7.3) of Theorem 7.3 can be replaced (at the expense of more complicated notations) by the smaller quantity $c \log \gamma + 1$, where c is from the proof of Theorem 7.3.

7.6 Proof of Theorem 7.4

The structure of the proof is similar to the proof of the formula (10.28.2) in Section 10.28 of Titchmarsh [69], see also the original proof in Levinson and Montgomery [53]. The main difference is Proposition 7.9 below.

Lemma 7.8. *Let $0 < \lambda \le 1$, $T < t < T + U$, and $0 < U \le T$. If $\sigma_2 < -1$ then*

$$\Re\frac{L'}{L}(\lambda, \lambda, \sigma_2 + it) = -\log t + O_{\sigma_2}(1) \qquad (T \to \infty). \tag{7.6}$$

Moreover, assume that $A > 0$ is such that $4AC < \pi$, where $C = C(\lambda, \lambda, \frac{1}{2})$ is a constant from Proposition 7.2. If the distance from $\frac{1}{2} + it$ to the nearest zero of $L(\lambda, \lambda, s)$ is greater than $\exp(-AT/\log T)$, then

$$\Re\frac{L'}{L}\left(\lambda, \lambda, \frac{1}{2} + it\right) = -\frac{1}{2}\log t + O(1) \qquad (T \to \infty). \tag{7.7}$$

Proof. By the functional equation (2.13), we have

$$\begin{aligned} L(\lambda, \lambda, s) =& (2\pi)^{s-1}\Gamma(1-s)e^{\pi i\frac{1-s}{2} - 2\pi i\lambda^2}\overline{L(\lambda, \lambda, 1-\overline{s})} \\ &\times\left(1 + \frac{e^{-\pi i(s-1)+2\pi i\lambda}L(\lambda, 1-\{\lambda\}, 1-s)}{\overline{L(\lambda, \lambda, 1-\overline{s})}}\right). \end{aligned}$$

The logarithmic derivative gives

$$\frac{L'}{L}(\lambda, \lambda, s) = \log 2\pi - \frac{\Gamma'}{\Gamma}(1-s) - \frac{\pi i}{2} - \overline{\frac{L'}{L}(\lambda, \lambda, 1-\overline{s})} + E(\lambda, s), \tag{7.8}$$

where

$$E(\lambda, s) = \frac{\left(\frac{e^{-\pi i(1-s)+2\pi i\lambda}L(\lambda, 1-\{\lambda\}, 1-s)}{L(1-\lambda, \lambda, 1-s)}\right)'_s}{1 + \frac{e^{-\pi i(1-s)+2\pi i\lambda}L(\lambda, 1-\lambda, 1-s)}{L(1-\lambda, \lambda, 1-s)}}.$$

For $0 < \lambda, \alpha \le 1$, we know that $L(\lambda, \alpha, 1-s) \ne 0$, if $\sigma < -1$, moreover, $L(\lambda, \alpha, 1-s)$ and its derivative have absolutely convergent Dirichlet series, if $\sigma < 0$. Thus

$$E(\lambda, \sigma_2 + it) \ll e^{-\pi t} \qquad (t \to \infty).$$

By Stirling's formula (3.1), we get that

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(|s|^{-1}\right) \quad (\Re(s) \geqslant 0,\ |s| \to \infty). \tag{7.9}$$

This proves the formula (7.6).

We turn to the second part of Lemma 7.8. The expression (7.8) together with the formula (7.9) gives

$$2\Re\frac{L'}{L}\left(\lambda, \lambda, \frac{1}{2} + it\right) = -\log t + \Re E\left(\lambda, \frac{1}{2} + it\right) + O(1) \qquad (T \to \infty).$$

Next we consider the growth of $E\left(\lambda, \frac{1}{2}+it\right)$. For $0<\lambda, \alpha \leq 1$, by Lemma 7.7 and by Cauchy's integral formula for the derivative, there is $B>0$ such that

$$L\left(\lambda, \alpha, \frac{1}{2}-iT\right) = O\left(T^B\right) \quad \text{and} \quad L'\left(\lambda, \alpha, \frac{1}{2}-iT\right) = O\left(T^B\right). \tag{7.10}$$

In view of the conditions of the lemma and the asymptotic formula (7.2) we have the distance d from $\frac{1}{2}+it$ to the nearest zero of $L(\lambda, \lambda, s)$ satisfies the inequalities

$$\exp\left(-AT/\log T\right) < d \ll 1.$$

Then Lemma 7.2 yields

$$E\left(\lambda, \frac{1}{2}+it\right) \ll T^B \exp((-\pi+4CA)T + \log(3T)).$$

This finishes the proof of Lemma 7.8.

□

The following proposition will be important in the proof of Theorem 7.4.

Proposition 7.9. *Let $0<\lambda\leq 1$, $T<t<T+U$, and $0<U\leq T$. Let $A>0$ be such that $AC<\pi$, where the constant C is from Lemma 7.2. Let ρ' be a zero of $L(\lambda, \lambda, s)$ such that $\left|\Re\rho' - \frac{1}{2}\right| < \exp\left(-\frac{AT}{\log T}\right)$ and $T<\Im\rho'<T+U$. Assume that there are $0<\varepsilon<1$ and $\delta>0$ such that the function $L(\lambda, \lambda, s)$ has less than*

$$\frac{\varepsilon}{\log(4+\delta)}\log T \tag{7.11}$$

zeros in the disc $|s-\rho'| \leq \exp(-AT^{1-\varepsilon}/\log T)$. Then, for sufficiently large T, there is a radius r,

$$\exp(-AT/\log T) \leq r \leq \exp(-AT^{1-\varepsilon}/\log T),$$

such that $L(\lambda, \lambda, s) \neq 0$ in the ring

$$r^{(4+\delta)/(2+\delta/3)} \leq |s-\rho'| \leq r^{1/(2+\delta/3)} \tag{7.12}$$

and, for $|s-\rho'|=r$, $\sigma\leq\frac{1}{2}$,

$$\Re\frac{L'}{L}(\lambda, \lambda, s) \leq -\frac{1}{2}\log T + O(1)$$

Proof. Let $r_k = \exp\big(-AT/((4+\delta)^k \log T)\big)$, $k = 0, 1, \ldots, [\frac{\varepsilon}{\log(4+\delta)} \log T]$. By the condition (7.11) and Dirichlet's box principle, there is $j \in \{1, 2, \ldots, [\frac{\varepsilon}{\log(4+\delta)} \log T]\}$ such that the ring

$$r_{j-1} < |s - \rho'| \le r_j \tag{7.13}$$

has no zeros of $L(\lambda, \lambda, s)$. Notice that $r_j^{4+\delta} = r_{j-1}$. The function

$$f(s) = \frac{L'}{L}(\lambda, \lambda, s) - \sum_{\rho\,:\,|\rho-\rho'| \le r_j^{2+\delta}} \frac{1}{s-\rho}$$

is analytic in the disc $|s - \rho'| \le r_j$ and in this disc it has the Taylor expansion

$$f(s) = \sum_{n=0}^{\infty} a_n (s-\rho')^n. \tag{7.14}$$

We bound the coefficients a_n. Cauchy's integral formula for the derivative yields

$$a_n = \frac{1}{2\pi i} \int\limits_{|s-\rho'|=r_j} \frac{f(s)ds}{(s-\rho')^{n+1}}. \tag{7.15}$$

Lemma 7.7 gives that, for any σ_0, there is a positive constant B such that $L(\lambda, \lambda, s) = O(t^B)$ if $\sigma \ge \sigma_0$. By the proof of Theorem 1 in [22] we see that, for any t the modulus $|L(\lambda, \lambda, 3+it)|$ is greater than some positive absolute constant. Therefore by Lemma α from Titchmarsh [69, §3.9] and by the formula (7.2) we obtain, for $|s-\rho'| \le r_j$,

$$\frac{L'}{L}(\lambda, \lambda, s) = \sum_{\rho\,:\,|\rho-(3+i\gamma')| \le 3} \frac{1}{s-\rho} + O(\log T).$$

Then, in view of the definition of $f(s)$ using the zero-free region (7.13), it follows that

$$f(s) = \sum_{\substack{\rho\,:\,|\rho-(3+i\gamma')| \le 3 \\ \text{and } |\rho-\rho'| > r_j}} \frac{1}{s-\rho} + O(\log T).$$

Thus by the formula (7.15)

$$\begin{aligned} a_n = &- \sum_{\substack{\rho\,:\,|\rho-(3+i\gamma')| \le 1 \\ \text{and } |\rho-\rho'| > r_j}} \frac{1}{(\rho-\rho')^{n+1}} + \frac{1}{2\pi i} \int\limits_{|s-\rho'|=r_j} \frac{O(\log T) ds}{(s-\rho')^{n+1}} \\ \ll & r_j^{-n-1} \log T. \end{aligned} \tag{7.16}$$

Now, we choose $r = r_j^{2+\delta/3}$. Then expressions (7.14) and (7.16) yield, for $|s-\rho'| = r$,

$$f(s) - a_0 = \sum_{n=1}^{\infty} a_n (s-s_0)^n \ll r_j^{-1} \log T \sum_{n=1}^{\infty} r_j^{n(1+\delta/3)}$$
$$\ll r_j^{\delta/3} \log T.$$

By this we have, for $|s-\rho'| = r$,

$$\frac{L'}{L}(\lambda, \lambda, s) = a_0 + \sum_{\rho \,:\, |\rho-\rho'| \le r_j^{4+\delta}} \frac{1}{s-\rho} + O\left(r_j^{\delta/3} \log T\right).$$

Taking real parts we obtain, for $|s-\rho'| = r$,

$$\Re \frac{L'}{L}(\lambda, \lambda, s) = \Re a_0 + \sum_{\rho \,:\, |\rho-\rho'| \le r_j^{4+\delta}} \frac{\sigma-\beta}{|s-\rho|^2} + O\left(r_j^{\delta/3} \log T\right). \tag{7.17}$$

We will get an asymptotic formula for $\Re a_0$. We consider the sum over zeros in the formula (7.17). By inequalities $|\rho-\rho'| \le r_j^{4+\delta}$ and $\left|\Re\rho' - \frac{1}{2}\right| < r_0$ we see that, for $|s-\rho'| = r$, $\frac{1}{2} - \left(\left|\Re\rho' - \frac{1}{2}\right| + r_j^{4+\delta}\right) \le \sigma \le \frac{1}{2}$, and large T,

$$|\sigma - \beta| \le \left|\Re\rho' - \frac{1}{2}\right| + r_j^{4+\delta} < r_0 + r_j^{4+\delta} \le 2r_j^{4+\delta}$$

and

$$|s-\rho|^2 \ge (|s-\rho'| - |\rho-\rho'|)^2 = (r_j^{2+\delta/3} - r_j^{4+\delta})^2 > r_j^{4+2\delta/3}/2.$$

The asymptotic formula (7.2) for the number of nontrivial zeros gives that there are $\ll \log T$ zeros in the disc $|\rho-\rho'| \le r_j^{4+\delta}$. Thus, for $|s-\rho'| = r$ and $\frac{1}{2} - \left(\left|\Re\rho' - \frac{1}{2}\right| + r_j^{4+\delta}\right) \le \sigma \le \frac{1}{2}$, we get

$$\sum_{\rho \,:\, |\rho-\rho'| \le r_j^{4+\delta}} \frac{\sigma-\beta}{|s-\rho|^2} \ll r_j^{\delta/3} \log T$$

and

$$\Re \frac{L'}{L}(\lambda, \lambda, s) = \Re a_0 + O\left(r_j^{\delta/3} \log T\right). \tag{7.18}$$

By (7.13) we have the ring $\{z : r_j^{4+\delta} < |z-\rho'| \le r_j\}$ has no zeros. Recall that $|s-\rho'| = r = r_j^{2+\delta/3}$. In view of this the distance from $s = \frac{1}{2} + it$ to the nearest zero is

$$\ge \min(r_j - r_j^{2+\delta/3}, r_j^{2+\delta/3} - r_j^{4+\delta}) > r_0 = \exp\left(-AT/\log T\right).$$

Then the equality (7.7) together with (7.18) gives

$$\Re a_0 = -\frac{1}{2}\log T + O(1). \tag{7.19}$$

By expressions (7.18) and (7.19) we obtain that, for $|s-\rho'| = r$ and $\left|\Re\rho' - \frac{1}{2}\right| + r_j^{2+\delta} \leq \sigma \leq \frac{1}{2}$,

$$\Re\frac{L'}{L}(\lambda,\lambda,s) = -\frac{1}{2}\log T + O\left(r_j^{\delta/3}\log T\right). \tag{7.20}$$

If $|s-\rho'| = r$ and $\sigma < \frac{1}{2} - \left(\left|\Re\rho' - \frac{1}{2}\right| + r_j^{4+\delta}\right)$, then we have

$$\sum_{\rho\,:\,|\rho-\rho'|\leq r_j^{4+\delta}} \frac{\sigma-\beta}{|s-\rho|^2} \leq 0$$

and, in view of formulas (7.17), (7.19),

$$\Re\frac{L'}{L}(\lambda,\lambda,s) \leq -\frac{1}{2}\log T + O\left(r_j^{\delta/3}\log T\right). \tag{7.21}$$

The expressions (7.20) and (7.21) together with the zero-free region (7.13) prove Lemma 7.9.

□

Proof of Theorem 7.4. Let

$$R = \left\{s \in \mathbb{C} : T < t < T+U, -2 < \sigma < \frac{1}{2}\right\}.$$

All the nontrivial zeros of $L(\lambda,\lambda,s)$ and $L'(\lambda,\lambda,s)$ lie to the right-hand side of the line $\sigma = -2$. To start with, the idea is to consider the change of the argument of $L'/L(\lambda,\lambda,s)$ around the boundary of the region R. However, a problem occurs if $\frac{1}{2} + it$ is near to a zero of $L(\lambda,\lambda,s)$. Next, our goal is to exclude the zeros ρ, for which

$$\left|\beta - \frac{1}{2}\right| < \exp\left(-\frac{AT}{\log T}\right) \quad \text{and} \quad T < \gamma < T+U \tag{7.22}$$

from the region R using certain arcs which lie to the left-hand side of the line $\sigma = \frac{1}{2}$. We will use Lemma 7.9.

In this proof we always assume that the zero ρ satisfies inequalities (7.22). By the condition (7.3) of Theorem 7.4 there is $\delta > 0$ such that the function $L(\lambda,\lambda,s)$ has less than

$$\frac{\varepsilon}{\log(2+\delta)}\log T$$

zeros in the disc $|s-\rho| \leq \exp(-AT^{1-\varepsilon}/\log T)$. Then, in view of Lemma 7.9, for each such zero ρ we define an arc of a circle which lies to the left-hand side of the line $\sigma = \frac{1}{2}$ in the following way:

$$C(\rho, r) = \left\{ s : |s-\rho| = r,\, \sigma \leq \frac{1}{2} \right\},$$

where the radius r is from Lemma 7.9. Thus, for $s \in C(\rho, r)$,

$$\Re \frac{L'}{L}(\lambda, \lambda, s) \leq -\frac{1}{2}\log T + O(1).$$

Let S be the set of all such arcs, i.e.

$$S = \{C(\rho, r) : \rho \text{ satisfies inequalities (7.22)}\}.$$

Clearly S is finite. It could be the case that some arcs of S intersect each other or lie inside of other arcs. We say that an arc $C(\rho, r)$ lie inside of the arc $C(\rho', r')$ if $C(\rho, r)$ is in the disc $|s-\rho'| < r'$. Similarly, a zero ρ lies inside of the arc $C(\rho', r')$ if ρ is in the disc $|s-\rho'| < r'$. We will construct a subset S' of S such that each zero ρ, with $\left|\beta - \frac{1}{2}\right| < \exp\left(-\frac{AT}{\log T}\right)$ and $T < \gamma < T+U$, lies inside of some arc from S' and any two arcs of S' do not intersect each other and do not lie inside of each other.

Let $C(\rho_0, r_0) \in S$ be an arc with the largest radius r_0 of all arcs from S. If $C(\rho, r) \in S$ and ρ is not inside of $C(\rho_0, r_0)$, then by the zero-free region (7.12), for sufficiently large T, the arcs $C(\rho_0, r_0)$ and $C(\rho, r)$ do not intersect. Let S_1 be a subset of S defined by

$$S_1 = \{C(\rho, r) \in S \setminus C(\rho_0, r_0) : \rho \text{ is not inside of } C(\rho_0, r_0)\}.$$

Let $C(\rho_1, r_1) \in S_1$ be an arc with the largest radius r_1 of all arcs from S_1. Again, $C(\rho_1, r_1)$ does not intersect any $C(\rho, r) \in S_1$, if ρ is not inside of $C(\rho_1, r_1)$. Let

$$S_2 = \{C(\rho, r) \in S_1 \setminus C(\rho_1, r_1) : \rho \text{ is not inside of } C(\rho_1, r_1)\}.$$

Continuing this way, we construct the desired set of arcs

$$S' = \{C(\rho_0, r_0), C(\rho_1, r_1), C(\rho_2, r_2), \cdots\}.$$

Without loss of generality, we assume that $L(\lambda, \lambda, \sigma+iT) \neq 0$ and $L'(\lambda, \lambda, \sigma+iT) \neq 0$ for $-2 \leq \sigma \leq \frac{1}{2}$. Further, we consider the change of $\arg L'/L(\lambda, \lambda, s)$

along the appropriately indented boundary R' of the region R. More precisely, the upper, left, and lower sides of R' coincide with the upper, left, and lower boundaries of R. To obtain the right-hand side of the contour R', we take the right-hand side boundary of R and deform it by arcs $C(\rho, r)$ from the set S'. We define a continuous function $f : [T, T+U] \to \mathbb{C}$, where $f(x)$ is equal to some complex number at the intersection of the horizontal line $t = x$ with the right-hand side of the contour R'.

To prove the theorem, we will show that the change of $\arg L'/L(\lambda, \lambda, s)$ along the the contour R' is $\ll \log T$. Let R'_1, R'_2, R'_3 and R'_4 denote the right, upper, left, and lower sides of the contour R' accordingly.

We start from $\arg_{R'_1} L'/L(\lambda, \lambda, s)$, where $\arg_{R'_1} L'/L(\lambda, \lambda, s)$ denotes the change of argument of $L'/L(\lambda, \lambda, s)$ along the right-hand side R'_1 of the contour R'. The equality (7.7) from Lemma 7.8 together with Lemma 7.9 gives that

$$\left| \arg_{R'_1} \frac{L'}{L}(\lambda, \lambda, s) \right| < \pi.$$

By analogy, the equality (7.6) from Lemma 7.8 gives

$$\left| \arg_{R'_3} \frac{L'}{L}(\lambda, \lambda, s) \right| < \pi.$$

Next we turn to horizontal sides R'_2 and R'_4. By standard arguments using Jensen's theorem together with the bounds (7.10) it is possible to show that (cf. [25, inequality (7) and below] or Titchmarsh [69, Section 9.4]) $\arg_{R'_2} L(\lambda, \lambda, s) \ll \log T$, $\arg_{R'_2} L'(\lambda, \lambda, s) \ll \log T$, $\arg_{R'_4} L(\lambda, \lambda, s) \ll \log T$, and $\arg_{R'_4} L'(\lambda, \lambda, s) \ll \log T$. This finishes the proof of Theorem 7.4.

□

7.7 Ending notes

We discuss the function $f(t)$ from Theorem 7.4. Let $T < t \le T + U$. In the proof of Theorem 7.4, for $0 < \lambda \le 1$, we construct the function $f(t)$ in a such way that the zero ρ of $L(\lambda, \lambda, s)$ lying near the critical line, more precisely satisfying the inequality

$$\left| \beta - \frac{1}{2} \right| < \exp\left(-\frac{AT}{\log T} \right),$$

must also lie to the right of the point $f(\gamma)+i\gamma$, i.e. $\beta > f(\gamma)$. We expect that the location of the curve $f(t)+it$, $t \in [T, T+U]$, is not accidental and reflects interesting properties of the zeros of the Lerch zeta-function. By 7.3 we see that if ρ is a nontrivial zero of $L(\lambda,\lambda,s)$, then there is a radius $\exp(-A\gamma/\log\gamma) \le r \le \exp(-A\gamma/\log\gamma)\log^2\gamma$ such that the discs

$$|s-\rho| < r \quad \text{and} \quad |s-(1-\overline{\rho})| < r \tag{7.23}$$

contain the same number of zeros. On the other hand, calculations in Section 7.2 suggest that if $0<\lambda<1$, $\lambda \neq \frac{1}{2}$, and ρ is a nontrivial zero of $L(\lambda,\lambda,s)$, then the symmetry described by the formula (7.23) is not strict, namely, $1-\overline{\rho}$ is not a zero of $L(\lambda,\lambda,s)$. Moreover if discs in the expression (7.23) intersect, then both discs possibly contain the same zero(s). From this we expect that nontrivial zeros of $L(\lambda,\lambda,s)$, for $0<\lambda<1$, $\lambda \neq \frac{1}{2}$, can be classified into two classes, heuristically described as follows. One class contains zeros which are relatively far from the critical line. These zeros appear in almost symmetric pairs according to (7.23). Another class consists of zeros which are relatively near the critical line. They are almost symmetric to themselves (in view of (7.23)). We expect that the curve $f(t)+it$, $t \in [T, T+U]$, (or the appropriate version of this curve lying nearest to the critical line) from Theorem 7.4 separates these two classes of zeros.

Notice that in Theorem 7.4, for $\lambda = \frac{1}{2}, 1$, the function $f(t)$ can be constructed at least in two ways. One way is as described in the proof of Theorem 7.4. Another way is to choose $f(t) = \frac{1}{2}$ (see Levinson and Montgomery [53] for the Riemann zeta-function $L(1,1,s)$ and Yıldırım [75] for the Dirichlet L-function $L(\frac{1}{2},\frac{1}{2},s)$). Clearly, if the generalized Riemann hypothesis is true and $\lambda = \frac{1}{2}, 1$, then in Theorem 7.4 for $f(t)$ we can choose any continuous function which is not greater than $\frac{1}{2}$.

8 Conclusions

Results obtained in the previous chapters lead to the following conclusions:

1. The zeros and a-values of the periodic Hurwitz zeta-function are mostly clustered around the critical line $\sigma = \frac{1}{2}$. The same holds for the zeros of the derivative of the Lerch zeta-function.

2. The number of the zeros and a-values of the periodic Hurwitz zeta-function till the given size T does depend only on T, parameter α and properties of the periodical sequence. Notice, that if $a \neq 0$, then the number of a-values of the periodic Hurwitz zeta-function till the given size mainly does not depend on a itself.

3. The number of trivial zeros till the given size T of the Lerch zeta-function is approximately the same as the number of trivial zeros of its derivative.

4. Nontrivial zeros of the Lerch zeta-function with equal parameters on average are symmetrically distributed with a small error term. For this special case, there is the Speiser type relation between zeros of the Lerch zeta-function and its derivative.

Thesis results found in Chapter 6 could be extended by analyzing k-th derivate. This would provide an extension of Berndt results about the Riemann zeta-function [6], but will require more complicated machinery.

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