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One functional property of the Lerch zeta-function

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Let $s = \sigma + it$ be a complex variable. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and otherwise by analytic continuation. Here λ and α, $0 < \alpha \leqslant 1$, are real parameters. If λ is not an integer, then $L(\lambda, \alpha, s)$ is an entire function.

This note is devoted to the functional independence of the Lerch zeta-function with rational parameters. The problem of the functional independence of Dirichlet series was formulated by D. Hilbert in 1900, and it was solved for different functions by D.D. Mordukhai-Boltovskoi, A. Ostrowski, A.G. Postnikov, S.M. Voronin and the author.

Let $\lambda = l/r$, $1 \leqslant l < r$, $(l, r) = 1$, and $\alpha = a/q$, $1 \leqslant a < q$, $(a, q) = 1$, be rational numbers. Moreover, for brevity, let $k = rq$, $d = (k, m)$, $\beta_m = lm/k$. Define numbers

$$\eta_v = \sum_{\substack{m-1 \\ m \equiv a (\mathrm{mod} q)}}^{k} e^{2\pi i \beta_m} \overline{\chi}_v(m), \quad v = 0, 1, \ldots, \varphi(k) - 1,$$

where χ_v denotes the Dirichlet character modulo k, and $\varphi(k)$ stands for the Euler function.

THEOREM. *Suppose that there exists at last two primitive charakters modulo k such that the corresponding numbers η_v are distinct from zero. Let F_l, $l = 0, \ldots, n$, be continuous functions, and let the equality*

$$\sum_{l=0}^{n} s^l F_l\Big(q^{-s} L(\lambda, \alpha, s), \big(q^{-s} L(\lambda, \alpha, s)\big)', \ldots, \big(q^{-s} L(\lambda, \alpha, s)\big)^{(N-1)}\Big) = 0,$$

be valid identically for s. Then $F_l \equiv 0$ for $l = 0, 1, \ldots, n$.

The proof of the theorem is based on the universality property of the Lerch zeta-function obtained in [1]. Denote by meas$\{A\}$ the Lebesgue measure of the set A.

LEMMA 1. *Suppose there exist at least two primitive characters modulo k such that the corresponding numbers η_v are distinct from zero. Let $0 < R < 1/4$, and let $f(s)$ be a continuous function on the disc $|s| \leqslant R$ and analytic in the interior of this disc. Then*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas}\left\{\tau \in [0, T], \max_{|s| \leqslant R} \left|q^{-s-3/4-i\tau} L(\lambda, \alpha, s + 3/4 + i\tau) - f(s)\right| < \varepsilon\right\} > 0.$$

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Proof of the lemma is given in [1].

Proof of the theorem. Denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of all real and all complex numbers, respectively. Let a function $h : \mathbb{R} \to \mathbb{C}^N$ be defined by the formula

$$h(t) = \Big(\big(q^{-\sigma-it}L(\lambda, \alpha, \sigma + it)\big), \big(q^{-\sigma-it}L(\lambda, \alpha, \sigma + it)\big)', \dots, \\ \big(q^{-\sigma-it}L(\lambda, \alpha, \sigma + it)\big)^{(N-1)}\Big), \quad \frac{1}{2} < \sigma < 1.$$

At first we will show that the image of $\mathbb{R}$ is dense in $\mathbb{C}^N$.

It is sufficient to prove that for each $\varepsilon > 0$ and arbitrary complex numbers $s_0, s_1, \dots, s_{N-1}$ there exists a real number τ such that

$$\big|L^{(j)}(\lambda, \alpha, \sigma + i\tau) - s_j\big| < \varepsilon \tag{1}$$

for $j = 0, 1, \dots, N-1$. We consider a polynomial

$$p_N(s) = \frac{s_{N-1}s^{N-1}}{(N-1)!} + \dots + \frac{s_1 s}{1!} + \frac{s_0}{0!}.$$

Then we have that

$$p_N^{(j)}(0) = s_j$$

for $j = 0, 1, \dots, N-1$. Let $\hat{\sigma}$, $1/2 < \hat{\sigma} < 1$, be a fixed number and let K be a disc of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ of radius $R < 1/4$ centered at $3/4$. Denote by δ the distance of $\hat{\sigma}$ from the boundary of K. Then by Lemma 1 there exists a number τ such that

$$\max_{s \in K} \big|q^{-s-i\tau}L(\lambda, \alpha, s + i\tau) - p_N(s - \hat{\sigma})\big| < \frac{\varepsilon\delta^N}{2^N N!}. \tag{2}$$

By the Cauchy integral formula

$$\big(q^{-s-i\tau}L(\lambda, \alpha, \hat{\sigma} + i\tau)\big)^{(j)} - s_j \\ = \frac{j!}{2\pi i} \int\limits_{|s-\hat{\sigma}|=\delta/2} \frac{q^{-s-i\tau}L(\lambda, \alpha, s + i\tau) - p_N(s - \hat{\sigma})}{(s - \hat{\sigma})^{j+1}}\, ds.$$

Therefore (1) is a simple consequence of the inequality (2).

To prove the theorem it is sufficient to show that $F_n \equiv 0$.

Suppose that $F_n \not\equiv 0$. Then there exists a bounded region D in $\mathbb{C}^N$ such that the inequality

$$\big|F_n(s_0, s_1, \dots, s_{N-1})\big| > c > 0 \tag{3}$$

holds for all points $(s_0, s_1, \dots, s_{N-1}) \in D$. By the first part of the proof there exists a sequence $\{t_k\}$, $\lim_{k\to\infty} t_k = \infty$, such that

$$\Big(q^{-\sigma-it_k}L(\lambda,\alpha,\sigma+it_k), \big(q^{-\sigma-it_k}L(\lambda,\alpha,\sigma+it_k)\big)', \dots, \big(q^{-\sigma-it_k}L(\lambda,\alpha,\sigma+it_k)\big)^{(N-1)}\Big) \in D.$$

However, this and (3) contradict the hypothesis of the theorem. Hence we obtain that $F_n \equiv 0$, and the theorem is proved.

REFERENCES

[1] A. Laurinčikas, On the Lerch zeta-function with rational parameters, *Liet. Matem. Rink.*, **38**(1) (1998), 113–124.

Viena funkcinė Lercho dzeta funkcijos savybė

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Straipsnyje nagrinėjama Lercho dzeta funkcijos funkcinė nepriklausomybė. Yra įrodoma, jog Lercho dzeta funkcija netenkina jokios algebrinės-diferencialinės lygties.