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On estimation of the number of zeros of linear combinations of certain zeta-functions

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Let $s=\sigma+it$ be a complex variable. In this note we consider zeta-functions $\varphi_l(s)$, introduced by K. Matsumoto in [3]. Let

$$A_{lm}(x)=\prod_{j=1}^{g_l(m)}\left(1-a_{lm}^{(j)}x^{f_l(j,m)}\right),$$

where $g_l(m)$, $f_l(j,m)\in\mathbb{N}$ and $a_{lm}^{(j)}\in\mathbb{C}$. Here $\mathbb{N}$ and $\mathbb{C}$, as usual, denote the sets of all natural and complex numbers, respectively. Then the functions $\varphi_l(s)$ are defined by

$$\varphi_l(s)=\prod_{m=1}^{\infty}A_{lm}^{-1}(p_m^{-s}) \tag{1}$$

where p_m stands for the mth prime number. If

$$g_l(m)\leqslant c_{1l}p_m^{\alpha_l},\qquad |a_{lm}^{(j)}|\leqslant p_m^{\beta_l}$$

with positive constants c_{1l} and non-negative constants α_l and β_l, then the infinite product in (1) converges absolutely for $\sigma>\alpha_l+\beta_l+1$ and defines a holomorphic function with no zeros.

Now let $r\geqslant 2$, $u_l\in\mathbb{C}$ and

$$Z(s)=\sum_{l=1}^{r}u_l\varphi_l(s).$$

Suppose that at least two of numbers u_l are distinct from zero. Let a real number ϱ_0 be defined by $\max(\alpha_l+\beta_l+1/2)\leqslant\varrho_0<\min(\alpha_l+\beta_l+1)$. Assume that the functions $\varphi_l(s)$ are analytically continuable to the strip $D=\{s\in\mathbb{C}:\ \varrho_0<\sigma<\min(\alpha_l+\beta_l+1)\}$, and that in the half-plane $\sigma>\varrho_0$ the following estimates are satisfied

$$\varphi_l(\sigma+it)=B|t|^{c_1},\qquad c_1>0,$$

$$\int_0^T|\varphi_l(\sigma+it)|^2\,dt=BT,\qquad T\to\infty.$$

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Here B denotes a number (not always the same) bounded by a constant.

Let

$$B_l(m) = \frac{1}{p_m^{\alpha+\beta}} \sum_{\substack{j=1 \\ f_l(j,m)=1}}^{g_l(m)} a_{lm}^{(j)}.$$

In [2] we assumed that $B_l(m)$ is constant for $m \in M_j = \{m : \ p_m \in P_j\}$, where P_j, $j = 1, \dots, k \geqslant r$, are sets of prime numbers such that $P_j \bigcap P_l = \varnothing$ for $j \neq l$, $P = \bigcup_{j=1}^{k} P_j$,, ($P$ denotes the set of all primes) and

$$\sum_{\substack{p \leqslant x \\ p \in P_j}} = b_j \log\log x + d_j + \varrho_j(x),$$

where $b_1 + \ldots + b_k = 1$, $b_j > 0$, $\varrho_j(x) = B(\log x)^{-\theta_j}$ with $\theta_j > 1$, and d_j are some real numbers. In this note we replace the last condition by the following: the sets P_j have a positive density.

Let, for $m \in M_j$,

$$B_1(m) = B_{1j},$$
$$\dots\dots\dots\dots$$
$$B_r(m) = B_{rj},$$

and let

$$G_{kr} = \begin{pmatrix} B_{11} & \dots & B_{1r} \\ \dots & \dots & \dots \\ B_{k1} & \dots & B_{kr} \end{pmatrix}.$$

THEOREM. *Let all above conditions be satisfied. Suppose that rank* $(G_{kr}) = r$. *Then for all* σ_1, σ_2, $\varrho < \sigma_1 < \sigma_2 < \min(\alpha_l + \beta_l + 1)$, *there exists a constant* $c = c(\sigma_1, \sigma_2) > 0$ *such that, for sufficiently large* T, *the function* $Z(s)$ *has more than* cT *zeros lying in the region* $\sigma_1 < \sigma < \sigma_2$, $0 \leqslant t \leqslant T$.

Proof of the theorem coinsides with that of Theorem 1 from [2], however, it uses one new lemma on entire functions of exponential type obtained by the second author. The theorem is stated by the first author.

Let $\mathcal{P}$ be a set of prime numbers having a positive density, i.e.,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \in \mathcal{P} : \ p \leqslant x\} = d > 0, \tag{2}$$

where, as usual,

$$\pi(x) = \sum_{p \leqslant x} 1.$$

LEMMA. *Let $f(s)$ be an entire function of exponential type, and let*

$$\limsup_{r\to\infty} \frac{\log|f(r)|}{r} > -1.$$

Then

$$\sum_{p\in\mathcal{P}} |f(\log p)| = \infty.$$

Proof. Let $\alpha > 0$ be such that

$$\limsup_{y\to\infty} \frac{\log|f(\pm iy)|}{y} \leqslant \alpha.$$

Let us fix a positive number β such that $\alpha\beta < \pi$. Suppose, on the contrary, that

$$\sum_{p\in\mathcal{P}} |f(\log p)| < \infty. \tag{3}$$

Consider the set $A = \{m \in \mathbb{N} : \exists r \in \big((m - 1/4)\beta,\ (m + 1/4)\beta\big) \text{ and } |f(r)| \leqslant e^{-r}\}$. Let, for brevity,

$$\pi_{\mathcal{P}}(x) = \sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}} 1.$$

We have that

$$\sum_{p\in\mathcal{P}} |f(\log p)| \geqslant \sum_{m\notin A} {\sum_m}' |f(\log p)| \geqslant \sum_{m\notin A} {\sum_m}' \frac{1}{p}, \tag{4}$$

where ${\sum_m}'$ denotes the sum running over all prime numbers $p \in \mathcal{P}$ and satisfying $(m - 1/4)\beta < \log p < (m + 1/4)\beta$. Therefore, putting

$$a = \exp\left\{\left(m - \frac{1}{4}\right)\beta\right\}, \qquad b = \exp\left\{\left(m + \frac{1}{4}\right)\beta\right\},$$

we can write

$${\sum_m}' \frac{1}{p} = \sum_{\substack{p\in\mathcal{P}\\ a<p<b}} \frac{1}{p}.$$

By partial summation

$$\sum_{\substack{p\in\mathcal{P}\\ a<p<b}} \frac{1}{p} = \frac{1}{b} \sum_{\substack{p\in\mathcal{P}\\ a<p<b}} 1 + \int_a^b \Big(\sum_{\substack{p\in\mathcal{P}\\ a<p<u}} 1 \Big) \frac{du}{u^2}. \tag{5}$$

Clearly,

$$\sum_{\substack{p\in\mathcal{P}\\ a<p<u}} 1 = \pi_{\mathcal{P}}(u) - \pi_{\mathcal{P}}(a), \tag{6}$$

and, by (2),

$$\pi_{\mathcal{P}}(x) = d\pi(x)(1+o(1)), \qquad x \to \infty.$$

Therefore, for any $\varepsilon > 0$, there exists a number $x_0 = x_0(\varepsilon)$ such that

$$\pi_{\mathcal{P}}(u) \geqslant d\pi(u)(1-\varepsilon),$$
$$\pi_{\mathcal{P}}(a) \leqslant d\pi(a)(1+\varepsilon)$$

if $x \geqslant x_0$. Hence and from (6)

$$\sum_{\substack{p\in\mathcal{P}\\ a<p<u}} 1 \geqslant d\big((\pi(u)-\pi(a)) - \varepsilon(\pi(a)+\pi(u))\big). \tag{7}$$

If x_0 is large enough, then, for $a \geqslant x_0$,

$$\pi(u)-\pi(a) \geqslant \frac{u}{\log u} - \frac{a}{\log a} - \varepsilon\left(\frac{u}{\log u} + \frac{a}{\log a}\right),$$
$$\pi(u)+\pi(a) \leqslant \frac{u}{\log u}(1+\varepsilon) + \frac{a}{\log a}(1+\varepsilon).$$

Hence

$$\frac{1}{2}\big(\pi(u)-\pi(a)\big) - \varepsilon\big(\pi(u)+\pi(a)\big) \geqslant \frac{u}{2\log u} - \frac{a}{2\log a} - \varepsilon\left(\frac{3}{2}+\varepsilon\right)\left(\frac{u}{\log u}+\frac{a}{\log a}\right). \tag{8}$$

Obviously, $\frac{b}{a} = \exp\{\beta/2\}$. Let η be a positive number satisfying $1 < \eta < \exp\{\beta/2\}$, and consider the case $u \geqslant \eta a$. Then

$$\frac{a}{\log a} \leqslant \frac{1+\varepsilon}{\eta}\frac{u}{\log u}$$

if $u \geqslant x_0$ is sufficiently large. Therefore, by (8),

$$\frac{1}{2}\big(\pi(u)-\pi(a)\big) - \varepsilon\big(\pi(u)+\pi(a)\big) \geqslant \left(\frac{1}{2} - \frac{1+\varepsilon}{2\eta} - 2\varepsilon\left(\frac{3}{2}+\varepsilon\right)\right)\frac{u}{\log u} > 0$$

if we choose ε sufficiently small. Hence from (7) we obtain

$$\sum_{\substack{p\in\mathcal{P}\\ a<p<u}} 1 \geqslant \frac{d}{2}\big(\pi(u)-\pi(a)\big), \qquad u \geqslant \eta a.$$

Therefore, from (5) we get by partial summation

$$\sum_{\substack{p\in\mathcal{P}\\ a<p<b}} \frac{1}{p} \geqslant \frac{d}{2b}\big(\pi(b)-\pi(a)\big)+\frac{d}{2}\int_{\eta a}^{b}\big(\pi(u)-\pi(a)\big)\frac{du}{u^2} \geqslant \frac{d}{2}\sum_{\eta a<p<b}\frac{1}{p}. \tag{9}$$

It is not difficult to see that

$$\sum_{\eta a<p<b}\frac{1}{p} = \log\log b - \log\log(\eta a) + B\exp\left\{-c_2\sqrt{\log(\eta a)}\right\} = \left(\frac{1}{2}-\frac{\log\eta}{\beta}\right)\frac{1}{m}+\frac{B}{m^2}.$$

Here β is a fixed positive number. Therefore, if we choose η sufficiently close to 1, we have

$$\left(\frac{1}{2}-\frac{\log\eta}{\beta}\right) \stackrel{def}{=} c_3 > 0.$$

Now (9) yields

$$\sum_{\substack{p\in\mathcal{P}\\ a<p<b}} \frac{1}{p} \geqslant \frac{d}{2}\left(\frac{c_3}{m}+\frac{B}{m^2}\right) = \frac{c_3 d}{2}\frac{1}{m}+\frac{B}{m^2}.$$

This and (4) imply

$$\sum_{m\notin A}\left(\frac{c_3 d}{2}\frac{1}{m}+\frac{B}{m^2}\right) \leqslant \sum_{p\in\mathcal{P}}|f(\log p)| < \infty,$$

hence

$$\sum_{m\notin A}\frac{1}{m} < \infty. \tag{10}$$

Let $A=\{a_m : a_1 < a_2 < \ldots\}$. Then, by (10), we obtain that

$$\lim_{m\to\infty}\frac{a_m}{m} = 1. \tag{11}$$

By the definition of the set A, there exists a sequence $\{\lambda_m\}$ such that

$$\left(a_m-\frac{1}{4}\right)\beta \leqslant \lambda_m \leqslant \left(a_m+\frac{1}{4}\right)\beta,$$

and

$$|f(\lambda_m)| \leqslant e^{-\lambda_m}.$$

Hence, in view of (11),

$$\lim_{m\to\infty}\frac{\lambda_m}{m} = \beta,$$

and

$$\limsup_{m\to\infty}\frac{\log|f(\lambda_m)|}{\lambda_m} \leqslant -1.$$

Applying a version of the Bernstein theorem (Theorem 6.4.12 of [1]), we find that

$$\limsup_{r\to\infty}\frac{\log|f(r)|}{r} \leqslant -1.$$

This contradicts to the assumption of the lemma. Hence the assertion (3) is false, and the lemma is proved.

REFERENCES

[1] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1996.

[2] A. Laurinčikas, On zeros of linear combinations of the Matsumoto zeta-functions, *Liet. Matem. Rink.*, (to appear).

[3] K. Matsumoto, Value-distribution of zeta-functions, *Lecture Notes in Math.*, **1434** (1990), 178–187.

Apie kai kurių dzeta funkcijų tiesinių kombinacijų nulių skaičiaus įvertį

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Straipsnyje nagrinėjamos dzeta funkcijos, apibrėžiamos polinomine Eulerio sandauga, ir gaunamas šių funkcijų tiesinių kombinacijų nulių skaičiaus įvertis iš apačios.